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Research Article

Synchronization of Time Delayed Fractional Order Chaotic Financial System

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The research on a time delayed fractional order financial chaotic system is a hot issue. In this paper, synchronization of time delayed fractional order financial chaotic system is studied. Based on comparison principle of linear fractional equation with delay, by using a fractional order inequality, a sufficient condition is obtained to guarantee the synchronization of master-slave systems. An example is exploited to show the feasibility of the theoretical results.

1. Introduction

In micro-macroeconomics, the study on the dynamics of financial and economical systems is an interesting and important topic [1]. The features of economic data were presented in view of the dynamical behaviors of systems. Many nonlinear continuous models have been introduced to study complex economic dynamics, such as the IS-LM model [2], Goodwin's accelerate model [3], the forced Vander-Pol model [4], and Behrens-Feichtinger model [5]. And the same as the other systems in world, financial system, as a nonlinear system, displays many complex dynamical behaviors, such as depending on initial value sensitivity, the complex phase portraits, positive Lyapunov exponents, and fractal properties. Chaotic phenomena in financial systems mean the systems will have inherent indefiniteness; it is difficult to make effective decision-making by makers, which threats the safety of investment. Therefore, to study the dynamical behaviors in economical and financial systems is indispensable.

Compared with classical integer calculus, the merit of fractional calculus is that it provides an excellent instrument for the description of memory and hereditary properties of dynamical processes [6–8]. Meanwhile, the financial variables such as interest rates, stock market prices, and foreign exchange rates possess long memories, which make it more appropriate to use fractional models compared with integer order ones in financial systems [9–12]. Furthermore, the author found that there existed many attractors in fractional order financial systems, such as fixed points, limit cycles, periodic motions, and chaotic attractors [9].

It is known that time delay can affect oscillation and instability behavior of dynamical systems. Time delay means that the policy from being made to taking effect will have to need some time in financial system, and its influence cannot be neglected. It dominates the decision which makes the policy intervene the economy. Since the pioneering work [13] that time delay was introduced to economic dynamics, the research on the delayed financial system has become one of the hot issues which has received more attention [14–17]. The synchronization of fractional order financial chaotic system with time delay is worth discussing. Recently, there are some works about synchronization on fractional order financial system without delay [18–21]; for example, synchronization and antisynchronization of fractional chaotic financial system via active control strategy were investigated in [18]; control and synchronization of fractional order financial system based on linear control were studied in [19]. However, the problem of synchronization for fractional order chaotic financial system with time delay has not been investigated in the literature.

Given the above discussions, in this paper, based on comparison principle of linear fractional equation with delay, by applying a fractional inequality, a sufficient condition is achieved to ensure the synchronization of fractional order time delayed chaotic financial systems. The result is simple and extremely effective.

The remainder of this paper is organized as follows. In Section 2, some necessary definitions and useful lemmas are introduced, and the model description is given. In Section 3, the synchronization schemes are presented, and sufficient conditions for synchronization are obtained. Numerical simulations are presented in Section 4. Some conclusions are drawn in Section 5.

2. Preliminaries and Model Description

There are some definitions of the fractional order derivatives. Riemann-Liouville fractional derivative and Caputo fractional derivative are mostly used. The main advantage of the Caputo derivative is that its Laplace transform only requires integer order derivatives for the initial conditions; the definition of Caputo derivative is given in this paper.

Definition 1 (see [6]). The fractional integral with noninteger order $\alpha > 0$ of function $x(t)$ is defined as follows:

$$I^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} x(\tau)\, d\tau, \tag{1}$$

where $\Gamma(\cdot)$ is the Gamma function and $\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$.

Definition 2 (see [6]). The Caputo derivative of fractional order α of function $x(t)$ is defined as follows:

$$D^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} x^{(n)}(\tau)\, d\tau, \tag{2}$$

where $n-1 < \alpha < n \in Z^{+}$.

In this paper, we consider a fractional order financial system with time delay, which is described by

$$\begin{aligned} D^{\alpha} x(t) &= z(t) + \left(y(t-\tau) - a\right) x(t), \\ D^{\alpha} y(t) &= 1 - by(t) - x^{2}(t-\tau), \\ D^{\alpha} z(t) &= -x(t-\tau) - cz(t), \end{aligned} \tag{3}$$

where $0 < \alpha < 1$, x, y, z are three state variables, x stands for the interest rate, y represents the investment demand, z denotes the price index, a is the saving amount, b is the cost per investment, c is the elasticity of demand of the commercial markets and parameters, a, b, c are nonnegative real constants, and $\tau > 0$ is time delay of the system. When $\alpha = 0.9$, $a = 3$, $b = 0.1$, $c = 1$, and $\tau = 0.1$, system (3) displays chaotic attractors with the initial value $x(0) = 0.1$, $y(0) = 4$, and $z(0) = 0.5$, which is shown in Figure 1.

In order to get main results, we give some lemmas as follows.

Lemma 3 (see [22]). *Suppose $x(t) \in R^{n}$ is a continuous and differentiable vector-value function. Then, for any time instant $t \geq t_0$, we have*

$$\frac{1}{2} D^{\alpha} x^{T}(t)\, x(t) \leq x^{T}(t)\, D^{\alpha} x(t), \tag{4}$$

where $0 < \alpha < 1$.

Lemma 4 (see [23]). *Suppose $V(t) \in R^{1}$ is a continuously differentiable and nonnegative function, satisfying*

$$\begin{aligned} D^{\alpha} V(t) &\leq -aV(t) + bV(t-\tau), \quad 0 < \alpha < 1 \\ V(t) &= \varphi(t) \geq 0, \quad t \in [-\tau, 0], \end{aligned} \tag{5}$$

where $t \in [0, +\infty)$. If $a > b > 0$, for all $\varphi(t) \geq 0$, $\tau > 0$, then $\lim_{t \to +\infty} V(t) = 0$.

3. Main Results

In this section, we discuss the synchronization of fractional order delayed financial system. The main aim is to design a proper controller to achieve synchronization between master system and slave system. Without loss of generally, the master system is chosen as

$$\begin{aligned} D^{\alpha} x_m(t) &= z_m(t) + \left(y_m(t-\tau) - a\right) x_m(t), \\ D^{\alpha} y_m(t) &= 1 - by_m(t) - x_m^{2}(t-\tau), \\ D^{\alpha} z_m(t) &= -x_m(t-\tau) - cz_m(t). \end{aligned} \tag{6}$$

The slave system is selected as

$$\begin{aligned} D^{\alpha} x_s(t) &= z_s(t) + \left(y_s(t-\tau) - a\right) x_s(t) + u_1(t), \\ D^{\alpha} y_s(t) &= 1 - by_s(t) - x_s^{2}(t-\tau) + u_2(t), \\ D^{\alpha} z_s(t) &= -x_s(t-\tau) - cz_s(t) + u_3(t), \end{aligned} \tag{7}$$

where $u_i(t)$ $(i = 1, 2, 3)$ are the external control inputs to be designed.

Let $e_1(t) = x_s - x_m$, $e_2(t) = y_s - y_m$, and $e_3(t) = z_s - z_m$ be the synchronization errors.

Subtracting the master system (4) from the slave system (5), the error system is obtained as follows:

$$\begin{aligned} D^{\alpha} e_1(t) &= e_3(t) + e_2(t-\tau)\, x_s(t) + y_m(t-\tau)\, e_1 \\ &\quad - ae_1(t) + u_1(t), \\ D^{\alpha} e_2(t) &= -be_2(t) - e_1(t-\tau)\left(x_s(t-\tau) + x_m(t-\tau)\right) \\ &\quad + u_2(t), \\ D^{\alpha} e_3(t) &= -e_1(t-\tau) - ce_3(t) + u_3(t). \end{aligned} \tag{8}$$

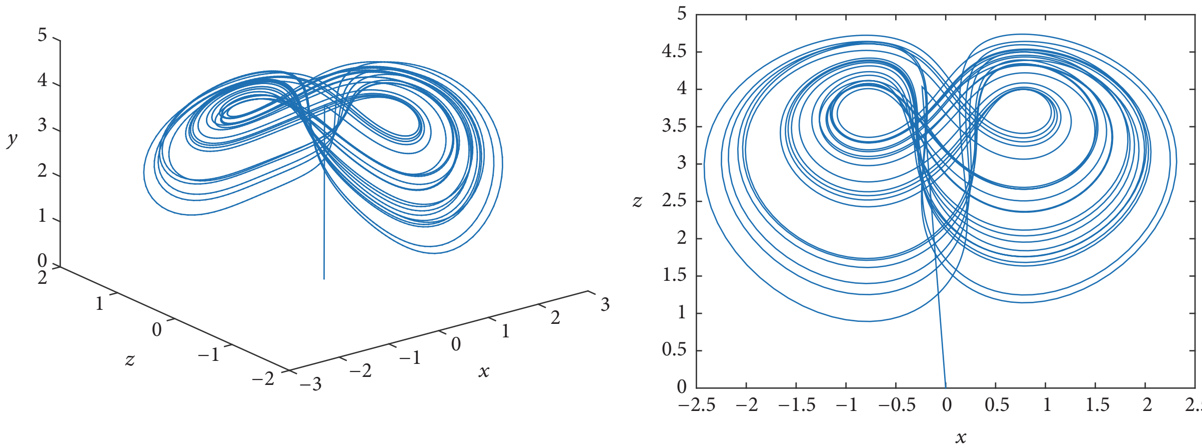

Figure 1: Chaotic attractors of system (3).

Control schemes $u_i(t)$ $(i = 1, 2, 3)$ are defined as follows:

$$
\begin{aligned}
u_1(t) &= -e_2(t-\tau)x_s(t) - y_m(t-\tau)e_1(t) + k_1 e_1(t),\\
u_2(t) &= e_1(t-\tau)\left(x_s(t-\tau) + x_m(t-\tau)\right) + k_2 e_2(t),\\
u_3(t) &= k_3 e_3(t),
\end{aligned}
\tag{9}
$$

where k_1, k_2, k_3 are feedback gains.

Combining (8) with (9), the error system is given by

$$
\begin{aligned}
D^\alpha e_1(t) &= (-a + k_1)e_1(t) + e_3(t),\\
D^\alpha e_2(t) &= (-b + k_2)e_2(t),\\
D^\alpha e_3(t) &= (-c + k_3)e_3(t) - e_1(t-\tau).
\end{aligned}
\tag{10}
$$

Theorem 5. *If the feedback gains k_1, k_2, k_3 satisfy the following condition: $l > 1$, the synchronization between system (6) and system (7) is achieved, where $l = \min\{a-k_1-1/2, b-k_2, c-k_3-1\}$ denotes the minimal value of $\{a-k_1-1/2, b-k_2, c-k_3-1\}$.*

Proof. Construct a Lyapunov function:

$$
V(t) = \frac{1}{2}\left(e_1^2(t) + e_2^2(t) + e_3^2(t)\right).
\tag{11}
$$

From Lemma 3,

$$
\begin{aligned}
D^\alpha V(t) &= D^\alpha\left[\frac{1}{2}\left(e_1^2(t) + e_2^2(t) + e_3^2(t)\right)\right]\\
&\le e_1(t)D^\alpha e_1(t) + e_2(t)D^\alpha e_2(t)\\
&\quad + e_3(t)D^\alpha e_3(t)\\
&= e_1(t)\left[(-a+k_1)e_1(t) + e_3(t)\right]\\
&\quad + e_2(t)\left[(-b+k_2)e_2(t)\right]\\
&\quad + e_3(t)\left[(-c+k_3)e_3(t) - e_1(t-\tau)\right]\\
&\le e_1(t)(-a+k_1)e_1(t) + \left|e_1(t)\right|\left|e_3(t)\right|\\
&\quad + e_2(t)(-b+k_2)e_2(t)\\
&\quad + e_3(t)(-c+k_3)e_3(t) + \left|e_3(t)\right|\left|e_1(t-\tau)\right|\\
&\le (-a+k_1)e_1^2(t) + \frac{1}{2}\left(e_1^2(t) + e_3^2(t)\right)\\
&\quad + (-b+k_2)e_2^2(t) + (-c+k_3)e_3^2(t)\\
&\quad + \frac{1}{2}\left(e_3^2(t) + e_1^2(t-\tau)\right)\\
&= \left(-a + k_1 + \frac{1}{2}\right)e_1^2(t) + (-b+k_2)e_2^2(t)\\
&\quad + (-c + k_3 + 1)e_3^2(t) + \frac{1}{2}e_1^2(t-\tau)\\
&\le -2lV(t) + 2V(t-\tau).
\end{aligned}
\tag{12}
$$

According to Lemma 4, when $l > 1$, then system (6) synchronizes system (7). □

Remark 6. Recently, there are a few works about the synchronization of fractional order chaotic financial systems [18–21], but these results are without considering time delay. Here, we consider a fractional order delayed chaotic financial systems.

Remark 7. Compared with [18, 19], in this paper, comparison principle of linear fractional equation with delay is used, and the sufficient condition of the synchronization between master-slave systems is achieved; from Figures 3–5, the controllers are effective.

4. Numerical Simulations

In this section, the numerical simulations are given to show the effectiveness of the theoretical results. The step-length $h = 0.01$ in the Adams–Bashforth–Moulton predictor-corrector scheme is taken [24]. The parameters of the time delayed fractional order financial chaotic system are chosen as $\alpha = 0.9$, $a = 3$, $b = 0.1$, $c = 1$, and $\tau = 0.1$. The initial values of the master system and slave system are selected as $x_m(0) = 2$, $y_m(0) = 1$, $z_m(0) = 4$, $x_s(0) = 0.2$, $y_s(0) = 0.3$, and $z_s(0) = 0.1$, respectively. The synchronization errors are

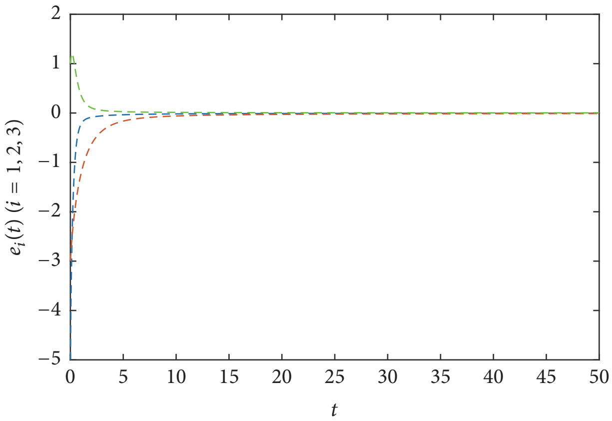

FIGURE 2: The errors state of e_1, e_2, e_3.

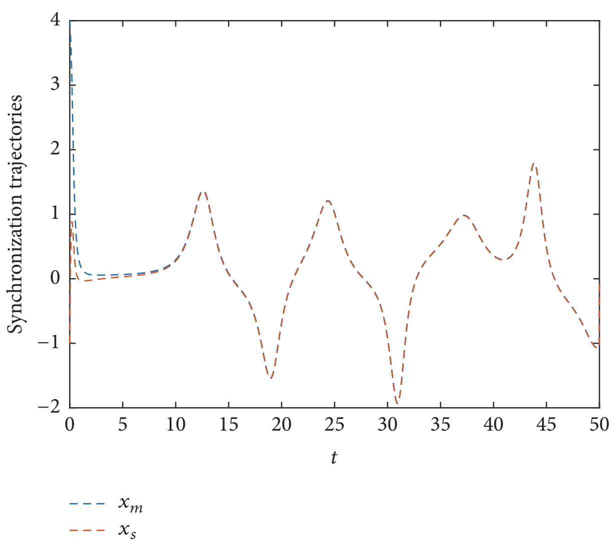

FIGURE 3: The synchronization trajectories of x_m, x_s.

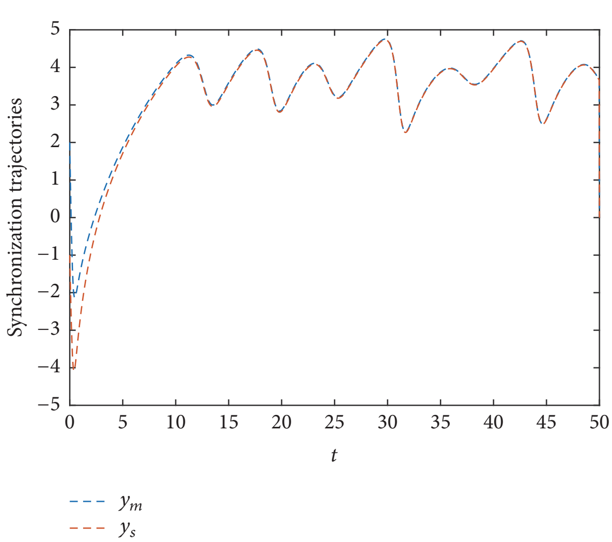

FIGURE 4: The synchronization trajectories of y_m, y_s.

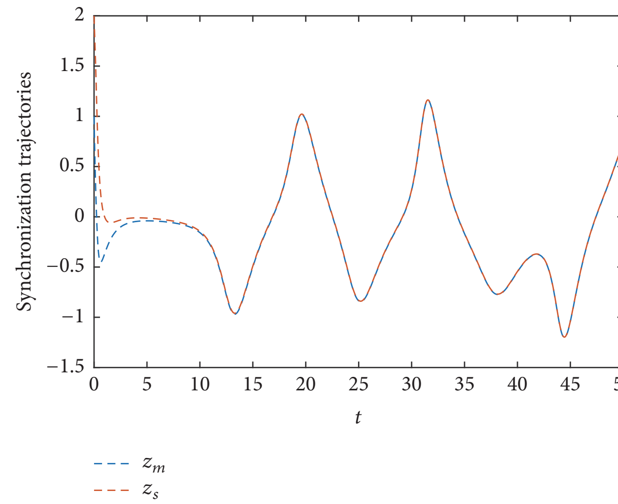

FIGURE 5: The synchronization trajectories of z_m, z_s.

depicted in Figure 2. The state synchronization trajectories of the master-slave systems are shown in Figures 3–5.

5. Conclusions

Time delay is a sensitive factor for the financial system. In this paper, we study the synchronization of a generalized financial system which takes time delay into consideration. The sufficient condition for synchronization is established according to comparison principle of linear fractional equation with delay. The numerical simulations indicate that the method designed for synchronization is effective.

Conflicts of Interest

The authors declare no conflicts of interest.

Acknowledgments

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