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Research Article

On the Stability Problem in Fuzzy Banach Space

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We investigate the generalized Ulam-Hyers stability of the Cauchy functional equation and pose two open problems in fuzzy Banach space.

1. Introduction and Preliminaries

In 1940, Ulam [1] asked the first question on the stability problem. In 1941, Hyers [2] solved the problem of Ulam. This result was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an *unbounded Cauchy difference*.

Theorem 1.1 (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality:*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p < 1$. Then, the limit $L(x) = \lim_{n\to\infty}(1/2^n)f(2^n x)$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

In 1990, Th. M. Rassias [5] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [6] gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [6], as well as by Th. M. Rassias and Šemrl [7], that one cannot prove a Th. M. Rassias type theorem when $p = 1$. Găvruţa [8] proved that the function $f(x) = x \ln|x|$, if $x \neq 0$ and $f(0) = 0$ satisfies (1.1) with $\epsilon = p = 1$ but

$$\sup_{x\neq 0}\frac{|f(x)-A(x)|}{|x|} \geq \sup_{n\in\mathbb{N}}\frac{|n\ln n - A(n)|}{n} = \sup_{n\in\mathbb{N}}|\ln n - A(1)| = \infty \tag{1.3}$$

for any additive function $A : \mathbb{R} \to \mathbb{R}$. J. M. Rassias [9] replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^{p_1}\|y\|^{p_2}$ for $p_1, p_2 \in \mathbb{R}$ with $p_1 + p_2 \neq 1$ (see also [10, 11]) and has obtained the following theorem.

Theorem 1.2. *Let X be a real normed linear space and Y a real complete normed linear space. Assume that $f : X \to Y$ is an approximately additive mapping for which there exist constants $\theta \geq 0$ and $p = p_1 + p_2 \neq 1$ such that f satisfies the inequality:*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\|x\|^{p_1}\|y\|^{p_2} \tag{1.4}$$

for all $x, y \in X$. Then, there exists a unique additive mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \leq \frac{\theta}{|2^p - 2|}\|x\|^p \tag{1.5}$$

for all $x \in X$. If, in addition, $f : X \to Y$ is a mapping such that the transformation $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is an $\mathbb{R}$-linear mapping.

In the case $p = 1$, we do not have stability [12]. In 1994, a further generalization of Th. M. Rassias' Theorem was obtained by Găvruţa [13], in which he replaced the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Isac and Th. M. Rassias [14] replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^{p_1} + \|y\|^{p_2}$ in Theorem 1.1 and solved stability problem when $p_2 \leq p_1 < 1$ or $1 < p_2 \leq p_1$, also they asked the question whether such a theorem can be proved for $p_2 < 1 < p_1$. Găvruţa [8] gave a negative answer to this question. Isac and Th. M. Rassias [15] applied the Ulam-Hyers-Rassias stability theory to prove fixed point theorems and study some new applications in nonlinear analysis. During the last two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Ulam-Hyers stability to a number of functional equations and mappings (see [16–40]). We also refer the readers to the books of Czerwik [41] and Hyers et al. [42].

Th. M. Rassias [43] has obtained the following theorem and posed a problem.

Theorem 1.3. *Let E_1 and E_2 be two Banach spaces, and let $f : E_1 \to E_2$ be a mapping such that $f(tx)$ is continuous in t for each fixed x. Assume that there exist $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.6}$$

for all $x, y \in X$. Let k be a positive integer $k > 2$. Then, there exists a unique linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{k\theta}{k - k^p}\|x\|^p s(k,p) \tag{1.7}$$

for all $x \in X$, where

$$s(k,p) = 1 + \frac{1}{k}\sum_{m=2}^{k-1} m^p. \tag{1.8}$$

Th. M. Rassias Problem

What is the best possible value of k in Theorem 1.3?

Găvruţa et al. have given a generalization of [13] and have answered to Th. M. Rassias problem [44].

In [45], J. M. Rassias et al. have investigated the generalized Ulam-Hyers "product-sum" stability of functional equations and have obtained the following theorem.

Theorem 1.4 (see [45]). *Let $f : E \to F$ be a mapping which satisfies the inequality*

$$\begin{aligned}&\left\| f(mx + y) + f(mx - y) - 2f(x + y) - 2f(x - y) - 2\left(m^2 - 2\right)f(x) + 2f(y)\right\|_F \\ &\quad \leq \epsilon\left(\|x\|_E^p\|y\|_E^p + \|x\|_E^{2p} + \|y\|_E^{2p}\right)\end{aligned} \tag{1.9}$$

for all $x, y \in E$ with $x \perp y$, where ϵ and p are constants with $\epsilon, p > 0$ and either $m > 1, p < 1$ or $m < 1$, $p > 1$ with $m \neq 0$, $m \neq \pm 1, m \neq \sqrt{\pm 2}$, and $-1 \neq |m|^{p-1} < 1$. Then, the limit $\lim_{n\to\infty} m^{-2n} f(m^n x)$ exists for all $x \in E$ and $Q : E \to F$ is the unique orthogonally Euler-Lagrange quadratic mapping such that

$$\|f(x) - Q(x)\|_F \leq \frac{\epsilon}{2\left|m^2 - m^{2p}\right|}\|x\|_E^{2p} \tag{1.10}$$

for all $x \in E$.

Note that the mixed "product-sum" function was introduced by J. M. Rassias in 2008-2009 [46–48].

We recall some basic facts concerning fuzzy normed space.

Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0,1]$ (so-called fuzzy subset) is said to be a fuzzy norm on X if for all $x, y \in X$ and all $c, t \in \mathbb{R}$,

($N1$) $N(x, c) = 0$ for $c \leq 0$;

($N2$) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;

($N3$) $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$;

($N4$) $N(x + y, \text{t}) \geq \min\{N(x, t), N(y, t)\}$;

$(N5)$ $N(x,\cdot)$ is a nondecreasing function of $\mathbb{R}$ and

$$\lim_{t\to\infty} N(x,t) = 1. \tag{1.11}$$

The pair (X, N) is called a fuzzy normed linear space. The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [49–51].

Let (X, N) be a fuzzy normed space and let $\{x_n\}$ be a sequence in X. Then, $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we denote it by $\lim_{n\to\infty} x_n = x$.

A sequence $\{x_n\}$ in a fuzzy normed space (X, N) is called Cauchy if, for each $\epsilon > 0$ and $\delta > 0$, one can find some n_0 such that

$$N(x_m - x_n, \delta) > 1 - \epsilon \tag{1.12}$$

for all $n, m \geq n_0$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy. If, in a fuzzy-normed space, each Cauchy sequence is convergent, then the fuzzy-norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Stability of Cauchy, Jensen, quadratic, and cubic function equation in fuzzy normed spaces have first been investigated in [50–53].

In this paper, we give a generalization of the results from [13] and pose two open problems in fuzzy Banach space. For convenience, we use the following abbreviation for a given mapping f:

$$Df(x,y) =: f(x+y) - f(x) - f(y). \tag{1.13}$$

2. Stability of the Cauchy Functional Equation

Hereafter, unless otherwise stated, we will assume that X is real vector space, (Y, N) is a complete fuzzy norm space and k is a fixed integer greater than 1.

Theorem 2.1. *Let (Z, N') be a fuzzy normed space and $\varphi : X \times X \to Z$ be a mapping such that, $\varphi(kx, ky) = \alpha\varphi(x, y)$ for some α with $0 < \alpha < k$. Suppose that $f : X \to Y$ be mapping such that*

$$N(Df(x,y),t) \geq N'(\varphi(x,y),t) \tag{2.1}$$

for all $x, y \in X$ and all positive real number t. Then, there is a unique additive mapping $T_k : X \to Y$ such that $T_k(x) = \lim_{n\to\infty} f(k^n x)/k^n$ and

$$N(T_k(x) - f(x), t) \geq M_k(x, (k-\alpha)t), \tag{2.2}$$

where $M_k(x,t) := \min\{N'(\varphi(x, ix), t) : 1 \leq i < k\}$.

Proof. By induction on k, we show that

$$N(f(kx) - kf(x), t) \geq M_k(x,t) := \min\{N'(\varphi(x, ix), t) : 1 \leq i < k\} \tag{2.3}$$

for all $x \in X$ and all positive real number t. Letting $y = x$ in (2.1), we get

$$N(f(2x) - 2f(x), t) \geq N'(\varphi(x, x), t). \tag{2.4}$$

So we get (2.3) for $k = 2$.

Assume that (2.3) holds for k with $k > 2$. Letting $y = kx$ in (2.1), we get

$$N(f((k+1)x) - f(x) - f(kx), t) \geq N'(\varphi(x, kx), t). \tag{2.5}$$

for all $x \in X$. By using (2.3) and (2.5), we get (2.3) for $k + 1$ and this completes the induction argument. Replacing x by $k^n x$ in (2.3), we get

$$N\left(f\left(k^{n+1}x\right) - kf(k^n x), t\right) \geq M_k(k^n x, t). \tag{2.6}$$

Thus

$$N\left(\frac{f(k^{n+1}x)}{k^{n+1}} - \frac{f(k^n x)}{k^n}, \frac{t}{k^{n+1}}\right) \geq M_k\left(x, \frac{t}{\alpha^n}\right) \tag{2.7}$$

for all $x \in X$ and all positive real number t. Hence,

$$\begin{aligned}
&N\left(\frac{1}{k^{n+1}} f\left(k^{n+1}x\right) - \frac{1}{k^m} f(k^m x), \sum_{i=m}^{n} \frac{\alpha^i}{k^{i+1}} t\right) \\
&\quad \geq N\left(\sum_{i=m}^{n} \frac{1}{k^{i+1}} f\left(k^{i+1}x\right) - \frac{1}{k^i} f\left(k^i x\right), \sum_{i=m}^{n} \frac{\alpha^i}{k^{i+1}} t\right) \\
&\quad \geq \min \bigcup_{i=m}^{n} \left\{ N\left(\frac{1}{k^{i+1}} f\left(k^{i+1}x\right) - \frac{1}{k^i} f\left(k^i x\right), \frac{\alpha^i}{k^{i+1}} t\right)\right\} \\
&\quad \geq M_k(x, t).
\end{aligned} \tag{2.8}$$

Let $\epsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} M_k(x, t) = 1$, there is some $t_0 > 0$ such that $M_k(x, t_0) > 1-\epsilon$. Since $\sum_{n=0}^{\infty} (\alpha^n / k^n) t_0 < \infty$, there is some $n_0 \in N$ such that $\sum_{i=m}^{n} (\alpha^i / k^i) t_0 < k\delta$ for all $n > m \geq n_0$. It follows that

$$\begin{aligned}
&N\left(\frac{1}{k^{n+1}} f\left(k^{n+1}x\right) - \frac{1}{k^m} f(k^m x), \delta\right) \\
&\quad \geq N\left(\frac{1}{k^{n+1}} f\left(k^{n+1}x\right) - \frac{1}{k^m} f(k^m x), \sum_{i=m}^{n} \frac{\alpha^i}{k^{i+1}} t_0\right) \\
&\quad \geq M_k(x, t_0) > 1 - \epsilon
\end{aligned} \tag{2.9}$$

for all $x \in X$ and all nonnegative integers n and m with $n > m \geq n_0$. Therefore, the sequence $\{(1/k^n) f(k^n x)\}$ is a Cauchy sequence in (Y, N) for all $x \in X$. Since (Y, N) is complete, the sequence $\{(1/k^n)f(k^n x)\}$ converges in Y for all $x \in X$. So one can define the mapping $T_k : X \to Y$ by

$$T_k(x) := \lim_{n\to\infty} \frac{1}{k^n} f(k^n x) \tag{2.10}$$

for all $x \in X$. Now, we show that T_k is an additive mapping. It follows from (2.1) and (2.10) that

$$\begin{aligned} N(DT_k(x,y),t) &= \lim_{n\to\infty} N\left(\frac{Df(k^n x, k^n y)}{k^n}, t\right) \\ &\geq \lim_{n\to\infty} N'\left(\frac{\varphi(k^n x, k^n y)}{k^n}, t\right) \\ &= \lim_{n\to\infty} N'\left(\varphi(x,y), \frac{k^n}{\alpha^n} t\right) \\ &= 1 \end{aligned} \tag{2.11}$$

for all $x, y \in X$ and all positive real number t. Therefore, the mapping T_k is additive. Moreover, if we put $m = 0$ in (2.8), we observe that

$$N\left(\frac{1}{k^{n+1}} f\left(k^{n+1}x\right) - f(x), \sum_{i=0}^{n} \frac{\alpha^i}{k^{i+1}} t\right) \geq M_k(x,t). \tag{2.12}$$

Therefore,

$$N\left(\frac{1}{k^{n+1}} f\left(k^{n+1}x\right) - f(x), t\right) \geq M_k\left(x, \frac{t}{\sum_{i=0}^{n} (\alpha^i / k^{i+1})}\right). \tag{2.13}$$

It follows from (2.13), for large enough n, that

$$\begin{aligned} N(T_k(x) - f(x), t) &\geq \min\left\{N\left(\frac{f(k^{n+1}x)}{k^{n+1}} - f(x), t\right), N\left(T_k(x) - \frac{f(k^{n+1}x)}{k^{n+1}}, t\right)\right\} \\ &\geq M_k\left(x, \frac{t}{\sum_{i=0}^{n} (\alpha^i / k^{i+1})}\right) \\ &\geq M_k(x, (k-\alpha)t). \end{aligned} \tag{2.14}$$

Now, we show that T_k is unique. Let T' be another additive mapping from X into Y, which satisfies the required inequality. Then, for each $x \in X$ and $t > 0$, we have

$$\begin{aligned} N(T_k(x) - T'(x), t) &\geq \min\{N(T_k(x) - f(x), t), N(f(x) - T'(x), t)\} \\ &\geq M_k(x, (k-\alpha)t). \end{aligned} \tag{2.15}$$

So,

$$\begin{aligned}N(T_k(x)-T'(x),t) &= N\left(\frac{T_k(k^nx)}{k^n}-\frac{T'(k^nx)}{k^n},t\right)\\ &= N(T_k(k^nx)-T'(k^nx),k^nt)\\ &\geq M_k(k^nx,(k-\alpha)k^nt)\\ &\geq M_k\left(x,(k-\alpha)\frac{k^n}{\alpha^n}t\right).\end{aligned} \tag{2.16}$$

Hence, the right-hand side of the above inequality tends to 1 as $n \to \infty$. It follows that $T_k(x)=T'(x)$ for all $x \in X$. □

Theorem 2.2. *Let (Z,N') be a fuzzy normed space and, $\Phi : X \times X \to Z$ be a mapping such that $\Phi(k^{-1}x,k^{-1}y)=\alpha^{-1}\Phi(x,y)$ for some α with $\alpha > k$. Suppose that $f : X \to Y$ be mapping such that*

$$N(Df(x,y),t) \geq N'(\Phi(x,y),t) \tag{2.17}$$

for all $x,y \in X$ and all positive real number t. Then, there is a unique additive mapping $T_k : X \to Y$ such that $T_k(x)=\lim_{n\to\infty}k^nf(x/k^n)$ and

$$N(T_k(x)-f(x),t) \geq M_k(x,(\alpha-k)t), \tag{2.18}$$

where $M_k(x,t) := \min\{N'(\Phi(x,ix),t) : 1 \leq i < k\}$.

Proof. Similarly to the proof of Theorem 2.1, we have

$$N(f(kx)-kf(x),t) \geq M_k(x,t) \tag{2.19}$$

for all $x \in X$ and all positive real number t. Replacing x by x/k^{n+1} in (2.19), we get

$$N\left(f\left(\frac{x}{k^n}\right)-kf\left(\frac{x}{k^{n+1}}\right),t\right) \geq M_k\left(\frac{x}{k^{n+1}},t\right). \tag{2.20}$$

Thus,

$$N\left(k^nf\left(\frac{x}{k^n}\right)-k^{n+1}f\left(\frac{x}{k^{n+1}}\right),k^nt\right) \geq M_k\left(x,\alpha^{n+1}t\right) \tag{2.21}$$

for all $x \in X$ and all positive real number t. Hence,

$$\begin{aligned}
N\left(k^{n+1} f\left(\frac{x}{k^{n+1}}\right)-k^{m} f\left(\frac{x}{k^{m}}\right), \sum_{i=m}^{n} \frac{k^{i}}{\alpha^{i+1}} t\right) &\geq N\left(\sum_{i=m}^{n} k^{i+1} f\left(\frac{x}{k^{i+1}}\right)-k^{i} f\left(\frac{x}{k^{i}}\right), \sum_{i=m}^{n} \frac{k^{i}}{\alpha^{i+1}} t\right) \\
&\geq \min \bigcup_{i=m}^{n}\left\{N\left(k^{i+1} f\left(\frac{x}{k^{i+1}}\right)-k^{i} f\left(\frac{x}{k^{i}}\right), \frac{k^{i}}{\alpha^{i+1}} t\right)\right\} \\
&\geq M_{k}(x, t).
\end{aligned} \tag{2.22}$$

Let $\epsilon>0$ and $\delta>0$ be given. Since $\lim_{t \rightarrow \infty} M_{k}(x, t)=1$, there is some $t_{0}>0$ such that $M_{k}(x, t_{0})>1-\epsilon$. Since $\sum_{n=0}^{\infty}(k^{n} / \alpha^{n}) t_{0}<\infty$, there is some $n_{0} \in N$ such that $\sum_{i=m}^{n}(k^{i} / \alpha^{i}) t_{0}<\alpha \delta$ for all $n>m \geq n_{0}$. It follows from (2.22) that

$$\begin{aligned}
N\left(k^{n+1} f\left(\frac{x}{k^{n+1}}\right)-k^{m} f\left(\frac{x}{k^{m}}\right), \delta\right) &\geq N\left(k^{n+1} f\left(\frac{x}{k^{n+1}}\right)-k^{m} f\left(\frac{x}{k^{m}}\right), \sum_{i=m}^{n} \frac{k^{i}}{\alpha^{i+1}} t_{0}\right) \\
&\geq M_{k}(x, t_{0})>1-\epsilon
\end{aligned} \tag{2.23}$$

for all $x \in X$ and all nonnegative integers n and m with $n>m \geq n_{0}$. Therefore, the sequence $\{k^{n} f(x / k^{n})\}$ is a Cauchy sequence in (Y, N) for all $x \in X$. Since (Y, N) is complete, the sequence $\{k^{n} f(x / k^{n})\}$ converges in Y for all $x \in X$. So one can define the mapping $T_{k}: X \rightarrow Y$ by

$$T_{k}(x):=\lim_{n \rightarrow \infty} k^{n} f\left(\frac{x}{k^{n}}\right) \tag{2.24}$$

for all $x \in X$. The rest of the proof is similar to the proof of Theorem 2.1 □

Theorem 2.3. *Let X be a normed space, let (Z, N') be a fuzzy normed space, and let $\psi:[0, \infty) \rightarrow [0, \infty)$ be a function such that*

(1) $\psi(ts)=\psi(t) \psi(s)$,

(2) $\psi(t)<t$ *for all* $t>1$.

Suppose that a mapping $f: X \rightarrow Y$ satisfies the inequality:

$$N(D f(x, y), t) \geq N'((\psi(\|x\|)+\psi(\|y\|)) z_{0}, t) \tag{2.25}$$

for all $x, y \in X$ and all positive real number t, where z_{0} is a fixed vector of Z. Then, there exists a unique additive mapping $T_{k}: X \rightarrow Y$ satisfying $T_{k}(x):=\lim_{n \rightarrow \infty}(f(k^{n} x) / k^{n})$ and

$$N(T_{k}(x)-f(x), t) \geq N'\left(\psi(\|x\|) z_{0}, \frac{k-\psi(k)}{\sigma_{k}(\psi)} t\right) \tag{2.26}$$

for all $x \in X$, where $\sigma_{k}(\psi)=\max\{1+\psi(i): 1 \leq i<k\}$. Moreover, $T_{k}=T_{2}$ for all $k \geq 2$.

Proof. Let

$$\varphi(x,y) = (\psi(\|x\|) + \psi(\|y\|))z_0 \tag{2.27}$$

for all $x, y \in X$. So,

$$\varphi(kx,ky) = \psi(k)\varphi(x,y). \tag{2.28}$$

where $\psi(k) < k$. By using Theorem 2.1, we can get (2.26). Now, we show that $T_k = T_2$. It follows from (1) that $\psi(k^n) = (\psi(k))^n$. Replacing x by $2^n x$ in (2.26), we get

$$N(T_k(2^n x) - f(2^n x), t) \geq N'\left(\psi(\|2^n x\|)z_0, \frac{k-\psi(k)}{\sigma_k(\psi)}t\right) \tag{2.29}$$

for all $x \in X$. So we have

$$N\left(T_k(x) - \frac{f(2^n x)}{2^n}, t\right) \geq N'\left(\psi(\|x\|)z_0, \frac{k-\psi(k)}{\sigma_k(\psi)\psi(2^n)}2^n t\right) \tag{2.30}$$

Using (2) and passing the limit $n \to \infty$ in (2.30), we get $T_k = T_2$. □

Theorem 2.4. *Let X be a normed space, let (Z, N') be a fuzzy normed space, and let $\psi : [0,\infty) \to [0,\infty)$ be a function such that*

(1) $\psi(ts) = \psi(t)\psi(s)$,

(2) $\psi(t) > t$ *for all* $t > 1$.

Suppose that a mapping $f : X \to Y$ satisfies the inequality:

$$N(Df(x,y),t) \geq N'((\psi(\|x\|) + \psi(\|y\|))z_0, t) \tag{2.31}$$

for all $x, y \in X$ and all positive real number t, where z_0 is a fixed vector of Z. Then, there exists a unique additive mapping $T_k : X \to Y$ satisfying $T_k(x) := \lim_{n\to\infty} k^n f(x/k^n)$ and

$$N(T_k(x) - f(x), t) \geq N'\left(\psi(\|x\|)z_0, \frac{\psi(k)-k}{\sigma_k(\psi)}t\right) \tag{2.32}$$

for all $x \in X$, where

$$\sigma_k(\psi) = \max\{1 + \psi(i) : 1 \leq i < k\}. \tag{2.33}$$

Moreover, $T_k = T_2$ for all $k \geq 2$.

Proof. Let

$$\Phi(x,y) = (\psi(\|x\|) + \psi(\|y\|))z_0 \tag{2.34}$$

for all $x, y \in X$. So, we have

$$\Phi\left(k^{-1}x, k^{-1}y\right) = \psi\left(k^{-1}\right)\Phi(x,y), \tag{2.35}$$

where $\psi(k^{-1}) = \psi(k)^{-1} < k^{-1}$. It follows from (1) that $\psi(k^{-n}) = (\psi(k))^{-n}$. By using Theorem 2.2, we can get (2.32). Now, we show that $T_k = T_2$. Replacing x by $x/2^n$ in (2.32), we get

$$N\left(T_k\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right), t\right) \geq N'\left(\psi\left(\left\|\left(\frac{x}{2^n}\right)\right\|\right)z_0, \frac{\psi(k) - k}{\sigma_k(\psi)}t\right). \tag{2.36}$$

for all $x \in X$. So we have

$$N\left(T_k(x) - 2^n f\left(\frac{x}{2^n}\right), t\right) \geq N'\left(\psi(\|x\|)z_0, \frac{\psi(k) - k}{2^n \sigma_k(\psi)\psi(2^{-n})}t\right). \tag{2.37}$$

Using (2) and passing the limit $n \to \infty$ in (2.37), we get $T_k = T_2$. □

Theorem 2.5. *Let X be a normed space, let p be a nonnegative real number such that $p \neq 1$, and let $H : [0,\infty) \times [0,\infty) \to [0,\infty)$ be a homogeneous function of degree p. Suppose that (Z, N') be a fuzzy normed space and let $f : X \to Y$ be mapping such that*

$$N(Df(x,y), t) \geq N'(H(\|x\|, \|y\|)z_0, t) \tag{2.38}$$

for all $x, y \in X$ and all positive real number t, where z_0 is a fixed vector of Z. Then, there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq M_k(x, |k^p - k|t), \tag{2.39}$$

where $M_k(x,t) := \min\{N'(\|x\|^p H(1,i)z_0, t) : 1 \leq i < k\}$.

Proof. The proof follows from Theorems 2.1 and 2.2. □

For the particular cases $H(x,y) = \theta(x^p + y^p)$, $H(x,y) = x^r y^s$, $H(x,y) = x^r y^s + x^{r+s} + y^{r+s} (r+s = p)$, and $H(x,y) = \min\{x^p, y^p\}$, we have the following corollaries.

Corollary 2.6. *Let X be a normed space, let p be a nonnegative real number such that $p \neq 1$. Suppose that (Z, N') be a fuzzy normed space and $f : X \to Y$ be mapping such that*

$$N(Df(x,y), t) \geq N'((\|x\|^p + \|y\|^p)\theta, t) \tag{2.40}$$

for all $x, y \in X$ and all positive real number t, where θ is a fixed vector of Z. Then, there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq N'\left(\|x\|^p\theta, \frac{|k^p - k|}{1 + (k-1)^p}t\right). \tag{2.41}$$

Corollary 2.7. *Let X be a normed space, r, s be non-negative real numbers such that $p := r + s \neq 1$. Suppose that (Z, N') be a fuzzy normed space and $f : X \to Y$ be mapping such that*

$$N(Df(x,y),t) \geq N'(\|x\|^r \|y\|^s \theta, t) \tag{2.42}$$

for all $x, y \in X$ and all positive real number t, where θ is a fixed vector of Z. Then there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq N'\left(\|x\|^p \theta, \frac{|k^p - k|}{(k-1)^s} t\right). \tag{2.43}$$

Corollary 2.8. *Let X be a normed space, and let r, s be nonnegative real numbers such that $p := r + s \neq 1$. Suppose that (Z, N') be a fuzzy normed space and let $f : X \to Y$ be mapping such that*

$$N(Df(x,y),t) \geq N'\left(\theta\|x\|^r \|y\|^s + \theta\|x\|^{r+s} + \theta\|y\|^{r+s}, t\right) \tag{2.44}$$

for all $x, y \in X$ and all positive real number t, where θ is a fixed vector of Z. Then, there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq N'\left(\|x\|^p \theta, \frac{|k^p - k|}{(k-1)^s + (k-1)^p + 1} t\right). \tag{2.45}$$

Corollary 2.9. *Let X be a normed space, let p be a nonnegative real number such that $p \neq 1$. Suppose that (Z, N') be a fuzzy normed space and let $f : X \to Y$ be mapping such that*

$$N(Df(x,y),t) \geq N'(\min\{\|x\|^p, \|y\|^p\}\theta, t) \tag{2.46}$$

for all $x, y \in X$ and all positive real number t, where θ is a fixed vector of Z. Then, there exists a unique additive mapping $T_k : X \to Y$ such that

$$N(T_k(x) - f(x), t) \geq N'(\|x\|^p \theta, |k^p - k| t). \tag{2.47}$$

Problem 1. Whether Theorem 2.5 and/or such Corollaries can be proved for $p = 1$?

Problem 2. What is the best possible value of k in Corollaries 2.6 and 2.7?

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References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York, NY, USA, 1960.

[2] D. H. Hyers, “On the stability of the linear functional equation,” *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] T. Aoki, “On the stability of the linear transformation in Banach spaces,” *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] Th. M. Rassias, “On the stability of the linear mapping in Banach spaces,” *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[5] Th. M. Rassias, “Problem 16; 2, report of the 27th International Symposium on functional equations,” *Aequationes Mathematicae*, vol. 39, pp. 292–293, 1990.

[6] Z. Gajda, “On stability of additive mappings,” *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[7] Th. M. Rassias and P. Šemrl, “On the behavior of mappings which do not satisfy Hyers-Ulam stability,” *Proceedings of the American Mathematical Society*, vol. 114, no. 4, pp. 989–993, 1992.

[8] P. Găvruţa, “On a problem of G. Isac and Th. M. Rassias concerning the stability of mappings,” *Journal of Mathematical Analysis and Applications*, vol. 261, no. 2, pp. 543–553, 2001.

[9] J. M. Rassias, “On approximation of approximately linear mappings by linear mappings,” *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.

[10] J. M. Rassias, “On approximation of approximately linear mappings by linear mappings,” *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.

[11] J. M. Rassias, “Solution of a problem of Ulam,” *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.

[12] P. Găvruţă, “An answer to question of John M. Rassias concerning the stability of Cauchy equation,” *Advanced in Equation and Inequality*, pp. 67–71, 1999.

[13] P. Găvruţa, “A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings,” *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[14] G. Isac and Th. M. Rassias, “Functional inequalities for approximately additive mappings,” in *Stability of Mappings of Hyers-Ulam Type*, Hadronic Press Collection of Original Articles, pp. 117–125, Hadronic Press, Palm Harbor, Fla, USA, 1994.

[15] G. Isac and Th. M. Rassias, “Stability of ψ-additive mappings: applications to nonlinear analysis,” *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 2, pp. 219–228, 1996.

[16] C. Baak and M. S. Moslehian, “On the stability of J^*-homomorphisms,” *Nonlinear Analysis*, vol. 63, no. 1, pp. 42–48, 2005.

[17] B. Bouikhalene, E. Elqorachi, and J. M. Rassias, “The superstability of d’Alembert’s functional equation on the Heisenberg group,” *Applied Mathematics Letters*, vol. 23, no. 1, pp. 105–109, 2010.

[18] L. Cădariu and V. Radu, “The fixed points method for the stability of some functional equations,” *Carpathian Journal of Mathematics*, vol. 23, no. 1-2, pp. 63–72, 2007.

[19] G. Z. Eskandani, “On the Hyers-Ulam-Rassias stability of an additive functional equation in quasi-Banach spaces,” *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 405–409, 2008.

[20] G. Z. Eskandani, H. Vaezi, and Y. N. Dehghan, “Stability of a mixed additive and quadratic functional equation in non-Archimedean Banach modules,” *Taiwanese Journal of Mathematics*, vol. 14, no. 4, pp. 1309–1324, 2010.

[21] G. Z. Eskandani, H. Vaezi, and F. Moradlou, “On the Hyers-Ulam-Rassias stability of functional equations in quasi-Banach spaces,” *International Journal of Applied Mathematics & Statistics*, vol. 15, pp. 1–15, 2009.

[22] P. Găvruţă, “On the Hyers-Ulam-Rassias stability of mappings,” in *Recent Progress in Inequalities*, G. V. Milovanovic, Ed., vol. 430, pp. 465–469, Kluwer Academic, Dordrecht, The Netherlands, 1998.

[23] P. Găvruţă and L. Cădariu, “General stability of the cubic functional equation,” *Buletinul Stiintific al Universităťii “Politehnica” din Timişoara. Seria Matematica-Fizica*, vol. 47(61), no. 1, pp. 59–70, 2002.

[24] K. W. Jun and Y. H. Lee, “On the Hyers-Ulam-Rassias stability of a Pexiderized quadratic inequality,” *Mathematical Inequalities & Applications*, vol. 4, no. 1, pp. 93–118, 2001.

[25] K. Jun, H. Kim, and J. Rassias, “Extended Hyers-Ulam stability for Cauchy-Jensen mappings,” *Journal of Difference Equations and Applications*, pp. 1–15, 2007.

[26] S. M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[27] S. M. Jung, "Asymptotic properties of isometries," *Journal of Mathematical Analysis and Applications*, vol. 276, no. 2, pp. 642–653, 2002.
[28] Pl. Kannappan, "Quadratic functional equation and inner product spaces," *Results in Mathematics*, vol. 27, no. 3-4, pp. 368–372, 1995.
[29] H. M. Kim, J. M. Rassias, and Y. S. Cho, "Stability problem of Ulam for Euler-Lagrange quadratic mappings," *Journal of Inequalities and Applications*, vol. 2007, Article ID 10725, 15 pages, 2007.
[30] Y. S. Lee and S. Y. Chung, "Stability of an Euler-Lagrange-Rassias equation in the spaces of generalized functions," *Applied Mathematics Letters of Rapid Publication*, vol. 21, no. 7, pp. 694–700, 2008.
[31] D. Miheţ, "The fixed point method for fuzzy stability of the Jensen functional equation," *Fuzzy Sets and Systems*, vol. 160, no. 11, pp. 1663–1667, 2009.
[32] F. Moradlou, H. Vaezi, and G. Z. Eskandani, "Hyers-Ulam-Rassias stability of a quadratic and additive functional equation in quasi-Banach spaces," *Mediterranean Journal of Mathematics*, vol. 6, no. 2, pp. 233–248, 2009.
[33] M. S. Moslehian, "On the orthogonal stability of the Pexiderized quadratic equation," *Journal of Difference Equations and Applications*, vol. 11, no. 11, pp. 999–1004, 2005.
[34] P. Nakmahachalasint, "On the generalized Ulam-Gavruta-Rassias stability of mixed-type linear and Euler-Lagrange-Rassias functional equations," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 63239, 10 pages, 2007.
[35] C. Park and J. M. Rassias, "Stability of the Jensen-type functional equation in C^*-algebras: a fixed point approach," *Abstract and Applied Analysis*, vol. 2009, Article ID 360432, 17 pages, 2009.
[36] A. Pietrzyk, "Stability of the Euler-Lagrange-Rassias functional equation," *Demonstratio Mathematica*, vol. 39, no. 3, pp. 523–530, 2006.
[37] J. M. Rassias, J. Lee, and H. M. Kim, "Refinned Hyers-Ulam stability for Jensen type mappings," *Journal of the Chungcheong Mathematical Society*, vol. 22, pp. 101–116, 2009.
[38] J. M. Rassias, "Complete solution of the multi-dimensional problem of Ulam," *Discussiones Mathematicae*, vol. 14, pp. 101–107, 1994.
[39] K. Ravi and M. Arunkumar, "On the Ulam-Gavruta-Rassias stability of the orthogonally Euler-Lagrange type functional equation," *International Journal of Applied Mathematics & Statistics*, vol. 7, pp. 143–156, 2007.
[40] K. Ravi, J. M. Rassias, M. Arunkumar, and R. Kodandan, "Stability of a generalized mixed type additive, quadratic, cubic and quartic functional equation," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 4, article 114, pp. 1–29, 2009.
[41] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, USA, 2002.
[42] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and their Applications, 34, Birkhäuser, Basle, Switzerland , 1998.
[43] Th. M. Rassias, "On a modified Hyers-Ulam sequence," *Journal of Mathematical Analysis and Applications*, vol. 158, no. 1, pp. 106–113, 1991.
[44] P. Găvruţa, M. Hossu, D. Popescu, and C. Căprău, "On the stability of mappings and an answer to a problem of Th. M. Rassias," *Annales Mathématiques Blaise Pascal*, vol. 2, no. 2, pp. 55–60, 1995.
[45] K. Ravi, M. Arunkumar, and J. M. Rassias, "Ulam stability for the orthogonally general Euler-Lagrange type functional equation," *International Journal of Mathematics and Statistics*, vol. 3, no. A08, pp. 36–46, 2008.
[46] H. X. Cao, J. R. Lv, and J. M. Rassias, "Superstability for generalized module left derivations and generalized module derivations on a Banach module. I," *Journal of Inequalities and Applications*, vol. 2009, Article ID 718020, 10 pages, 2009.
[47] H. X. Cao, J. R. Lv, and J. M. Rassias, "Superstability for generalized module left derivations and generalized module derivations on a Banach module. II," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 10, no. 3, article 85, pp. 1–8, 2009.
[48] M. B. Savadkouhi, M. E. Gordji, J. M. Rassias, and N. Ghobadipour, "Approximate ternary Jordan derivations on Banach ternary algebras," *Journal of Mathematical Physics*, vol. 50, no. 4, Article ID 042303, pp. 1–9, 2009.
[49] T. Bag and S. K. Samanta, "Finite dimensional fuzzy normed linear spaces," *Journal of Fuzzy Mathematics*, vol. 11, no. 3, pp. 687–705, 2003.
[50] A. K. Mirmostafaee and M. S. Moslehian, "Fuzzy versions of Hyers-Ulam-Rassias theorem," *Fuzzy Sets and Systems*, vol. 159, no. 6, pp. 720–729, 2008.
[51] A. K. Mirmostafaee, M. Mirzavaziri, and M. S. Moslehian, "Fuzzy stability of the Jensen functional equation," *Fuzzy Sets and Systems*, vol. 159, no. 6, pp. 730–738, 2008.

[52] A. K. Mirmostafaee and M. S. Moslehian, "Fuzzy approximately cubic mappings," *Information Sciences*, vol. 178, no. 19, pp. 3791–3798, 2008.

[53] A. K. Mirmostafaee and M. S. Moslehian, "Fuzzy almost quadratic functions," *Results in Mathematics*, vol. 52, no. 1-2, pp. 161–177, 2008.