Improved expressions for performance parameters for complex filters

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Improved expressions are given for the performance parameters for transverse and axial gains for complex pupil filters. These expressions can be used to predict the behavior of filters that give a small axial shift in the focal intensity maximum and also predict the changes in gain for different observation planes. © 2007 Optical Society of America

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Sheppard and Hegedus [1] introduced transverse and axial gain factors describing the focusing properties of rotationally symmetric pupil filters or masks in the paraxial regime. These factors are expressed simply in terms of the moments of the pupil and avoid the necessity to calculate the diffracted field of the lens. The treatment holds for real filters, which includes the class of amplitude filters, but also the important class of binary phase-only filters with a phase change of π. De Juana *et al.* [2] extended the gain parameters to the case of general-phase filters, for the case when the intensity maximum is shifted only a small distance from the geometrical focus. Ledesma *et al.* [3] introduced an alternative approach for any complex filter, in which the plane of best focus is calculated first, and generalized gain parameters in the surroundings of the shifted focus are then calculated. This approach is much more flexible and is preferable for many phase filters, as the intensity peaks on the axis can be situated far from the geometrical focal plane, with the filter acting like a zone plate. But, unfortunately, it does not lead to analytic expressions for the filter parameters since numerical calculation of the plane of best focus is needed. Phase-only filters have advantages when the Strehl ratio is an important property (e.g., in astronomy), but for many applications when efficiency is not important the performance of amplitude filters is better, as the relative strength of the sidelobes is decreased [4,5].

The transverse and axial gains are calculated from the second derivatives of the transverse or axial intensity variations, normalized by the intensity. To obtain a second derivative to second order, and thus to get an expression for the gains as a function of axial position, an expression for intensity accurate to fourth order must be used. De Juana *et al.*'s [2] Eq. (8) for transverse gain is calculated from an expression containing a third-order term, while their Eq. (9) for axial gain is calculated from an expansion of intensity to only second order.

In the paraxial Debye regime, the amplitude in the focal region of a lens is [6]

$$U(v,u) = 2\int_0^1 P(\rho)J_0(v\rho)\exp\left(-\frac{1}{2}iu\rho^2\right)\rho \mathrm{d}\rho, \qquad (1)$$

where the optical coordinates for cylindrical coordinates r,z are $v=(2\pi r/\lambda)\sin\alpha$, $u=(8\pi z/\lambda)\sin^2(\alpha/2)$, with α being the semiangle of convergence of the lens. We obtain for the intensity to fourth order in u for the case of a complex pupil:

$$\begin{aligned} I(v,u) = {}& |I_0|^2 - u\,\mathrm{Im}(I_0I_1^*) - \frac{v^2}{2}\mathrm{Re}(I_0I_1^*) - \frac{u^2}{4}[\mathrm{Re}(I_0I_2^*) \\ & - |I_1|^2] + \frac{uv^2}{4}\mathrm{Im}(I_0I_2^*) + \frac{u^3}{24}\mathrm{Im}(I_0I_3^* - 3I_1I_2^*) \\ & + \frac{v^4}{32}\mathrm{Re}(I_0I_2^* + 2|I_1|^2) + \frac{u^2v^2}{16}\mathrm{Re}(I_0I_3^* - I_1I_2^*) \\ & + \frac{u^4}{192}\mathrm{Re}(I_0I_4^* - 4I_1I_3^* + 3|I_2|^2). \end{aligned} \qquad (2)$$

De Juana *et al.* [2] include the first five terms of this expansion in their treatment. Equating to zero the partial derivative with respect to u on axis, the position of the intensity peak u_F for small u is approximately

$$u_F = -2\frac{\mathrm{Im}(I_0I_1^*)}{\mathrm{Re}(I_0I_2^*) - |I_1|^2}, \qquad (3)$$

agreeing with de Juana *et al.* [2]. At the intensity peak, $u=u_F$, the Strehl ratio is

$$S = |I_0|^2 - \frac{u_F}{2}\mathrm{Im}(I_0I_1^*) \qquad (4)$$

to the second order.

We define the axial and transverse gains as

$$G_A(u) = -24 \left. \frac{\partial^2 I(v,u)/\partial u^2}{I(v,u)} \right|_{v=0},$$

$$G_T(u) = -2 \left. \frac{\partial^2 I(v,u)/\partial v^2}{I(v,u)} \right|_{v=0}. \quad (5)$$

The gains are unity for a plain circular aperture, and the parabolic width of the focal spot is inversely proportional to the square root of the gain parameter. We can obtain expressions for the axial and transverse gains as a function of axial position close to the focal plane to second order in u. Both the second derivative and the intensity vary with axial position, exhibiting maxima that do not in general occur at exactly the same axial position. Putting $u=u_F$, we find at the intensity maximum

$$G_A = 12 \frac{\mathrm{Re}(I_0 I_2^*) - |I_1|^2 - \frac{u_F}{2}\mathrm{Im}(I_0 I_3^* - 3I_1 I_2^*) - \frac{u_F^2}{8}\mathrm{Re}(I_0 I_4^* - 4I_1 I_3^* + 3|I_2|^2)}{|I_0|^2 - \frac{u_F}{2}\mathrm{Im}(I_0 I_1^*)}, \quad (6)$$

$$G_T = 2 \frac{\mathrm{Re}(I_0 I_1^*) - \frac{u_F}{2}\mathrm{Im}(I_0 I_2^*) - \frac{u_F^2}{8}\mathrm{Re}(I_0 I_3^* - I_1 I_2^*)}{|I_0|^2 - \frac{u_F}{2}\mathrm{Im}(I_0 I_1^*)}, \quad (7)$$

which are different from de Juana and co-workers's [2] Eqs. (8) and (9). As the gains exhibit approximately a parabolic behavior with maximum at u_F, terms to order u_F^2 should be included to get accurate expressions for the gains near the peak of the parabola.

To test these expressions we considered a circular aperture with a pupil function $\exp(iu_0\rho^2/2)$, corresponding to a simple defocus. According to an exact treatment, the Strehl ratio and gains at the intensity peak should be independent of the value of u_0. The exact expression for the intensity is $\{\sin[(u-u_0)/4]/[(u-u_0)/4]\}^2$, which exhibits only a shift due to defocus. Figure 1(a) shows the axial gain factor G_A, calculated by using the fourth-order expansion in intensity, compared with those calculated by using a third- or second-order expansion for the case when $u_0=1$, corresponding to a phase shift of 0.5 rad at the edge of the pupil. The exact behavior is also shown. It is seen that the using the full fourth-order expression gives the correct value of unity at the focus position $u=1$, whereas the second-order approximation gives a gain factor of 0.95. Keeping terms in intensity to third order gives a gain factor of 1.05. The fourth-order expansion gives a good prediction of the behavior as a function of u. We see that the fourth order and exact expressions show a gain factor that decreases with distance from the focus, whereas the second-order approximation shows an increase in gain factor. Figure 1(b) shows the behavior for $u=u_0$, corresponding to the true focus. For the second-order treatment, the axial gain factor at the focus is substantially in error even for small values of u_0 ($\sim$10% for $|u_0|=1.6$), whereas the result from Eq. (9) is correct to within 1% for $|u_0|<1.6$. In the fourth-order approximation, the variation in axial gain with u_0 exhibits a flat, quartic behavior for small u_0. The third-order expansion is actually no more accurate than the second-order expansion for small u_0, the difference being the sign of the error.

Figure 2 shows analogous results for the transverse gain. Figure 2(a) shows the behavior with axial position of the transverse gain for $u_0=1$. Again the third-order treatment is no more accurate than the second-order treatment at the axial position of maxi-

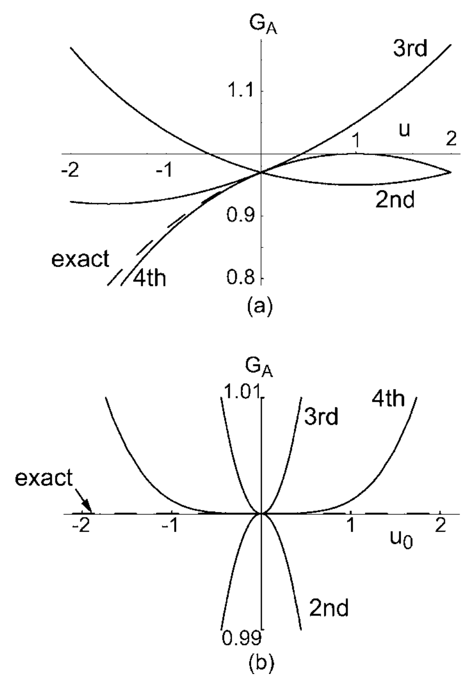

Fig. 1. (a) Axial gain factors G_A along the axis calculated from a fourth-order expansion of intensity with a defocus filter $u_0=1$. The exact behavior and that predicted using second- and third-order expansions are also shown. (b) Axial gain factors G_A at $u=u_0$ corresponding to the true focus position calculated from Eq. (6) (fourth order). The exact behavior and that predicted using second- and third-order expansions are also shown.

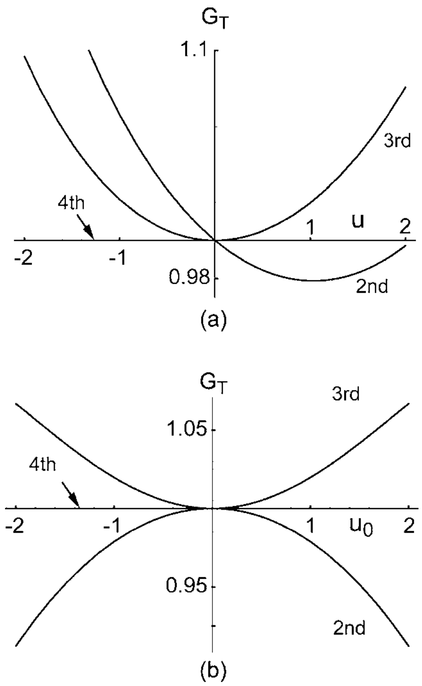

Fig. 2. (a) Transverse gain factors G_T along the axis calculated from a fourth-order expansion of intensity with a defocus filter $u_0=1$. The behavior predicted using second- and third-order expansions is also shown. (b) Axial gain factors G_T at $u=u_0$ corresponding to the true focus position calculated from Eq. (7) (fourth order). The behavior predicted using second- and third-order expansions is also shown.

mum intensity. The transverse gain predicted by the fourth-order theory is independent of u. Figure 2(b) shows the transverse gain factor at $u=u_0$ from Eq. (10). Again we see that the third-order approximation gives errors even for small values of u_0. It gives little improvement over the second-order theory, giving only a sign change in the error for small u_0. In both cases, the error is 5% for approximately $u_0=1.6$. The fourth-order theory gives a transverse gain independent of u_0, agreeing with the exact behavior.

Next we consider further the axial variation in gains for the case of a pupil that is real (but not necessarily positive). We obtain

$$G_A(u) = 12\frac{I_0I_2 - I_1^2 - \dfrac{u^2}{8}(I_0I_4 - 4I_1I_3 + 3I_2^2)}{I_0^2 - \dfrac{u^2}{4}(I_0I_2 - I_1^2)}, \tag{8}$$

$$G_T(u) = 2\frac{I_0I_1 - \dfrac{u^2}{8}(I_0I_3 - I_1I_2)}{I_0^2 - \dfrac{u^2}{4}(I_0I_2 - I_1^2)}, \tag{9}$$

which reduce to the known forms for $u=0$. For a uniform circular pupil, $I_n=1/(n+1)$, and

$$I(v,u) = 1 - \frac{v^2}{4} - \frac{u^2}{48} + \frac{5v^4}{192} + \frac{u^2v^2}{192} + \frac{u^4}{5760} \tag{10}$$

to the fourth order, and the gains can be calculated directly from this expression. The axial gain $G_A(u)$ predicted by Eq. (10) is correct to 1% for $|u|<2$, whereas retaining only second-order terms results in an error as large as 20%. We find that $G_T(u)=1$, independent of u. This corresponds to the tubular focal spot described by Born and Wolf [6]. Defining a parameter F as

$$F = \frac{I_1(I_0I_2 - I_1^2)}{I_0(I_0I_3 - I_1I_2)}, \tag{11}$$

the condition $F \geq 1$ ensures that the transverse resolution is maintained over the range where the intensity is appreciable. For confocal imaging, where the point-spread function of a single lens is squared, the parameter F is equal to 2, consistent with the presence of an optical sectioning property.

Improved expressions have been given for the gain factors of phase filters, valid for small axial displacements of the true focus from the geometrical focus. The allowed axial displacements are larger than those for the expressions due to de Juana [2], $|u_0| \leq 1.6$ instead of $|u_0| \leq 0.4$ for approximately 1% accuracy. These expressions have been validated for the particular case of a phase mask that produces a small axial shift of the focal intensity. The method fails completely, however, if an inflection point in axial intensity occurs between the intensity peak and the geometrical focus ($|u_0|>5$), so that if the intensity peak is distant from the geometrical focal plane, the approach of [3] is necessary.

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