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Some fixed point theorems for (α,θ,k)-contractive multi-valued mappings with some applications

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Abstract

In this paper, we introduce the notion of (α,θ,k)-contraction multi-valued mappings and establish some fixed point results for such mappings by using some control functions due to Jleli *et al.* (J. Inequal. Appl. 2014:439, 2014) in metric spaces and furnish some interesting examples to illustrate our main results. Also, we give some fixed point results in metric spaces endowed with a graph. Our results generalize and extend recent results given by some authors.

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1 Introduction and preliminaries

Now, we recall some notations and primary results which are needed in the sequel.

Let (X,d) be a metric space. We denote by $N(X)$ the class of all nonempty subsets of X, by $\mathrm{CL}(X)$ the class of all nonempty closed subsets of X, by $\mathrm{CB}(X)$ the class of all nonempty closed bounded subsets of X and by $K(X)$ the class of all nonempty compact subsets of X. For any $A,B\in \mathrm{CL}(X)$, let the mapping $H:\mathrm{CL}(X)\times\mathrm{CL}(X)\to\mathbb{R}_+\cup\{\infty\}$ defined by

$$H(A,B)=\begin{cases}\max\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(b,A)\}, & \text{if the maximum exists;}\\ \infty, & \text{otherwise,}\end{cases}$$

be the *generalized Pompeiu-Hausdorff metric* induced by d, where $d(a,B)=\inf\{d(a,b): b\in B\}$ is the distance from a to $B\subseteq X$.

In 1969, Nadler [1] extended Banach's contraction principle to the class of multi-valued mappings in metric spaces as follows.

Theorem 1.1 [1] *Let (X,d) be a complete metric space and $T:X\to\mathrm{CB}(X)$ be a multi-valued mapping such that*

$$H(Tx,Ty)\le kd(x,y) \tag{1.1}$$

for all $x,y\in X$, where $k\in[0,1)$. Then T has at least one fixed point.

Since Nadler's fixed point theorem, a number of authors have published many interesting fixed point theorems in several ways (see [2–4] and references therein).

In 2012, Samet *et al.* [5] introduced the concept of α-admissible mapping as follows.

Definition 1.2 [5] Let T be a self-mapping on a nonempty set X and $\alpha : X \times X \to [0,\infty)$ be a mapping. The mapping T is said to be α*-admissible* if the following condition holds:

$$x,y \in X, \quad \alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx,Ty) \geq 1.$$

They also proved some fixed point theorems for such mappings under the generalized contractive conditions in complete metric spaces and showed that these results can be utilized to derive some fixed point theorems in partially ordered metric spaces.

Afterward, Asl *et al.* [6] introduced the concept of an α_*-admissible mapping which is the multi-valued version of α-admissible single-valued mapping provided in [5].

Definition 1.3 [6] Let X be a nonempty set, $T : X \to N(X)$ and $\alpha : X \times X \to [0,\infty)$ be two given mappings. The mapping T is said to be α_**-admissible* if the following condition holds:

$$x,y \in X, \quad \alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha_*(Tx,Ty) \geq 1,$$

where $\alpha_*(Tx,Ty) := \inf\{\alpha(a,b) : a \in Tx, b \in Ty\}$.

Asl *et al.* [6] also established a fixed point result for multi-valued mappings in complete metric spaces satisfying some generalized contractive condition.

In 2013, Mohammadi *et al.* [7] extended the concept of an α_*-admissible mapping to the class of α-admissible mappings as follows.

Definition 1.4 [7] Let X be a nonempty set, $T : X \to N(X)$ and $\alpha : X \times X \to [0,\infty)$ be two given mappings. The mapping T is said to be α*-admissible* whenever, for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$, we have $\alpha(y,z) \geq 1$ for all $z \in Ty$.

Remark 1.5 It is clear that an α_*-admissible mapping is also α-admissible, but the converse may not be true.

Recently, Hussain *et al.* [8] introduced the concept of the α-completeness of metric spaces which is a weaker than the concept of the completeness.

Definition 1.6 [8] Let (X,d) be a metric space and $\alpha : X \times X \to [0,\infty)$ be a mapping. The metric space X is said to be α*-complete* if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ converges in X.

Remark 1.7 If X is a complete metric space, then X is also an α-complete metric space, but the converse is not true.

Example 1.8 Let $X = (0,\infty)$ and the metric $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Define a mapping $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} e^{\frac{xy}{x+y}}, & \text{if } x,y \in [2,5], \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

It is easy to see that (X, d) is not a complete metric space, but (X, d) is an α-complete metric space. Indeed, if $\{x_n\}$ is a Cauchy sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then $x_n \in [2, 5]$ for all $n \in \mathbb{N}$. Since $[2, 5]$ is a closed subset of $\mathbb{R}$, it follows that $([2, 5], d)$ is a complete metric space and so there exists $x^* \in [2, 5]$ such that $x_n \to x^*$ as $n \to \infty$.

Recently, Kutbi and Sintunavarat [9] introduced the concept of the α-continuity for multi-valued mappings in metric spaces as follows.

Definition 1.9 [9] Let (X, d) be a metric space, $\alpha : X \times X \to [0, \infty)$ and $T : X \to \mathrm{CL}(X)$ be two given mappings. The mapping $T : X \to \mathrm{CL}(X)$ is called an *α-continuous multi-valued mapping* if, for all sequence $\{x_n\}$ with $x_n \xrightarrow{d} x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $Tx_n \xrightarrow{H} Tx$ as $n \to \infty$, that is, for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} d(x_n, x) = 0, \quad \alpha(x_n, x_{n+1}) \geq 1 \quad \Longrightarrow \quad \lim_{n\to\infty} H(Tx_n, Tx) = 0.$$

Note that the continuity of T implies the α-continuity of T for all mappings α. In general, the converse is not true (see Example 1.10).

Example 1.10 [9] Let $X = [0, \infty)$, $\lambda \in [10, 20]$ and the metric $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Define two mappings $T : X \to \mathrm{CL}(X)$ and $\alpha : X \times X \to [0, \infty)$ by

$$Tx = \begin{cases} \{\lambda x^2\}, & \text{if } x \in [0, 1], \\ \{x\}, & \text{if } x > 1 \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} \cosh(x^2 + y^2), & \text{if } x, y \in [0, 1], \\ \tanh(x + y), & \text{otherwise.} \end{cases}$$

Clearly, T is not a continue multi-valued mapping on $(\mathrm{CL}(X), H)$. Indeed, for a sequence $\{x_n\}$ in X defined by $x_n = 1 + \frac{1}{n}$ for each $n \geq 1$, we see that $x_n = 1 + \frac{1}{n} \xrightarrow{d} 1$, but $Tx_n = \{1 + \frac{1}{n}\} \xrightarrow{H} \{1\} \neq \{\lambda\} = T1$.

Next, we show that T is an α-continue multi-valued mapping on $(\mathrm{CL}(X), H)$. Let $\{x_n\}$ be a sequence in X such that $x_n \xrightarrow{d} x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Then we have $x, x_n \in [0, 1]$ for all $n \in \mathbb{N}$. Therefore, $Tx_n = \{\lambda x_n^2\} \xrightarrow{H} \{\lambda x^2\} = Tx$. This shows that T is an α-continue multi-valued mapping on $(\mathrm{CL}(X), H)$.

In this paper, we introduce new type of multi-valued mappings so called (α, θ, k)-contraction multi-valued mappings and prove some new fixed point results for such mappings in α-complete metric spaces by using the idea of α-admissible multi-valued mapping due to Mohammadi *et al.* [7] and furnish some interesting examples to illustrate the main results in this paper. Also, we obtain some fixed point results in metric spaces endowed with graph.

2 The main results

Recently, Jleli *et al.* [10] introduced the class Θ of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(θ_1) θ is non-decreasing;
(θ_2) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
(θ_3) there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$;
(θ_4) θ is continuous;

and proved the following result.

Theorem 2.1 (Corollary 2.1 of [10]) *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x,y \in X, \quad d(Tx,Ty) \neq 0 \quad \Longrightarrow \quad \theta\bigl(d(Tx,Ty)\bigr) \leq \theta\bigl(d(x,y)\bigr)^k. \tag{2.1}$$

Then T has a unique fixed point.

Observe that Banach's contraction principle follows immediately from the above theorem (see [10]).

In this section, we introduce the concept of (α,θ,k)-contraction multi-valued mappings and prove fixed point results for such mappings without the assumption of the completeness of domain of mappings and the continuity of mappings.

Definition 2.2 Let (X,d) be a metric space. A multi-valued mapping $T : X \to K(X)$ is said to be an *(α,θ,k)-contraction* if there exist $\alpha : X \times X \to [0,\infty)$, $\theta \in \Theta$, and $k \in (0,1)$ such that

$$x,y \in X, \quad H(Tx,Ty) \neq 0 \quad \Longrightarrow \quad \alpha(x,y)\theta\bigl(H(Tx,Ty)\bigr) \leq \theta\bigl(M(x,y)\bigr)^k, \tag{2.2}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Now, we give the first main result in this paper.

Theorem 2.3 *Let (X,d) be a metric space and $T : X \to K(X)$ be an (α,θ,k)-contraction mapping. Suppose that the following conditions hold*:

(S_1) *(X,d) is an α-complete metric space*;
(S_2) *T is an α-admissible multi-valued mapping*;
(S_3) *there exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$*;
(S_4) *T is an α-continuous multi-valued mapping.*

Then T has a fixed point in X.

Proof Starting from x_0 and $x_1 \in Tx_0$ in (S_3), then we have $\alpha(x_0,x_1) \geq 1$. If $x_0 = x_1$, then x_0 is a fixed point of T. Assume that $x_0 \neq x_1$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and so we have nothing to prove. Let $x_1 \notin Tx_1$, that is, $d(x_1, Tx_1) > 0$. Since $H(Tx_0, Tx_1) \geq d(x_1, Tx_1) > 0$, it follows from the (α, θ, k)-contractive condition that

$$\begin{aligned}
&\theta\big(H(Tx_0, Tx_1)\big) \\
&\quad \leq \alpha(x_0,x_1)\theta\big(H(Tx_0, Tx_1)\big) \\
&\quad \leq \theta\left(\max\left\{d(x_0,x_1), d(x_0, Tx_0), d(x_1, Tx_1), \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2}\right\}\right)^k \\
&\quad = \theta\left(\max\left\{d(x_0,x_1), d(x_1, Tx_1), \frac{d(x_0, Tx_1)}{2}\right\}\right)^k \\
&\quad \leq \theta\left(\max\left\{d(x_0,x_1), d(x_1, Tx_1), \frac{d(x_0,x_1) + d(x_1, Tx_1)}{2}\right\}\right)^k \\
&\quad = \theta\big(\max\{d(x_0,x_1), d(x_1, Tx_1)\}\big)^k. \qquad (2.3)
\end{aligned}$$

If $\max\{d(x_0,x_1), d(x_1, Tx_1)\} = d(x_1, Tx_1)$, then we have

$$\begin{aligned}
1 &< \theta\big(d(x_1, Tx_1)\big) \\
&\leq \theta\big(H(Tx_0, Tx_1)\big) \\
&\leq \theta\big(\max\{d(x_0,x_1), d(x_1, Tx_1)\}\big)^k \\
&= \theta\big(d(x_1, Tx_1)\big)^k, \qquad (2.4)
\end{aligned}$$

which is a contradiction. Therefore, $\max\{d(x_0,x_1), d(x_1, Tx_1)\} = d(x_0,x_1)$. From (2.3), it follows that

$$1 < \theta\big(d(x_1, Tx_1)\big) \leq \theta\big(H(Tx_0, Tx_1)\big) \leq \theta\big(d(x_0,x_1)\big)^k. \qquad (2.5)$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that

$$d(x_1,x_2) = d(x_1, Tx_1). \qquad (2.6)$$

From (2.5) and (2.6), it follows that

$$1 < \theta\big(d(x_1,x_2)\big) < \theta\big(d(x_0,x_1)\big)^k. \qquad (2.7)$$

If $x_1 = x_2$ or $x_2 \in Tx_2$, then it follows that x_2 is a fixed point of T and so we have nothing to prove. Therefore, we may assume that $x_1 \neq x_2$ and $x_2 \notin Tx_2$. Since $x_1 \in Tx_0$, $x_2 \in Tx_1$, $\alpha(x_0,x_1) \geq 1$ and T is an α-admissible multi-valued mapping, we have $\alpha(x_1,x_2) \geq 1$. Applying the (α, θ, k)-contractive condition, we have

$$\begin{aligned}
&\theta\big(H(Tx_1, Tx_2)\big) \\
&\quad \leq \alpha(x_1,x_2)\theta\big(H(Tx_1, Tx_2)\big) \\
&\quad \leq \theta\left(\max\left\{d(x_1,x_2), d(x_1, Tx_1), d(x_2, Tx_2), \frac{d(x_1, Tx_2) + d(x_2, Tx_1)}{2}\right\}\right)^k
\end{aligned}$$

$$
\begin{aligned}
&= \theta\left(\max\left\{d(x_1,x_2), d(x_2,Tx_2), \frac{d(x_1,Tx_2)}{2}\right\}\right)^k \\
&\le \theta\left(\max\left\{d(x_1,x_2), d(x_2,Tx_2), \frac{d(x_1,x_2)+d(x_2,Tx_2)}{2}\right\}\right)^k \\
&= \theta\bigl(\max\bigl\{d(x_1,x_2), d(x_2,Tx_2)\bigr\}\bigr)^k. \qquad (2.8)
\end{aligned}
$$

Suppose that $\max\{d(x_1,x_2), d(x_2,Tx_2)\} = d(x_2,Tx_2)$. From (2.8), it follows that

$$
\begin{aligned}
1 &< \theta\bigl(d(x_2,Tx_2)\bigr) \\
&\le \theta\bigl(H(Tx_1,Tx_2)\bigr) \\
&\le \theta\bigl(\max\bigl\{d(x_1,x_2), d(x_2,Tx_2)\bigr\}\bigr)^k \qquad (2.9) \\
&= \theta\bigl(d(x_2,Tx_2)\bigr)^k,
\end{aligned}
$$

which is a contradiction. Therefore, we may let $\max\{d(x_1,x_2), d(x_2,Tx_2)\} = d(x_1,x_2)$. From (2.8), it follows that

$$
1 < \theta\bigl(d(x_2,Tx_2)\bigr) \le \theta\bigl(H(Tx_1,Tx_2)\bigr) \le \theta\bigl(d(x_1,x_2)\bigr)^k. \qquad (2.10)
$$

Since Tx_2 is compact, there exists $x_3 \in Tx_2$ such that

$$
d(x_2,x_3) = d(x_2,Tx_2). \qquad (2.11)
$$

From (2.10) and (2.11), it follows that

$$
1 < \theta\bigl(d(x_2,x_3)\bigr) \le \theta\bigl(d(x_1,x_2)\bigr)^k \le \theta\bigl(d(x_0,x_1)\bigr)^{k^2}. \qquad (2.12)
$$

Continuing this process, we can construct a sequence $\{x_n\}$ in X such that

$$
x_n \neq x_{n+1} \in Tx_n, \qquad (2.13)
$$

$$
\alpha(x_n,x_{n+1}) \ge 1 \qquad (2.14)
$$

and

$$
1 < \theta\bigl(d(x_{n+1},x_{n+2})\bigr) \le \theta\bigl(d(x_0,x_1)\bigr)^{k^n} \qquad (2.15)
$$

for all $n \in \mathbb{N} \cup \{0\}$. This shows that $\lim_{n\to\infty} \theta(d(x_n,x_{n+1})) = 1$ and so

$$
\lim_{n\to\infty} d(x_n,x_{n+1}) = 0 \qquad (2.16)
$$

by our assumptions about θ. From similar arguments as in the proof of Theorem 2.1 of [10], it follows that there exist $n_1 \in \mathbb{N}$ and $r \in (0,1)$ such that

$$
d(x_n,x_{n+1}) \le \frac{1}{n^{\frac{1}{r}}}
$$

for all $n \geq n_1$. Now, for $m > n > n_1$ we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{r}}}.$$

Since $0 < r < 1$, $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ converges. Therefore, $d(x_n, x_m) \to 0$ as $m, n \to \infty$. Thus we proved that $\{x_n\}$ is a Cauchy sequence in X. From (2.14) and the α-completeness of (X, d), there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

By the α-continuity of multi-valued mapping T, we have

$$\lim_{n \to \infty} H\bigl(Tx_n, Tx^*\bigr) = 0, \tag{2.17}$$

which implies that

$$d\bigl(x^*, Tx^*\bigr) = \lim_{n \to \infty} d\bigl(x_{n+1}, Tx^*\bigr) \leq \lim_{n \to \infty} H\bigl(Tx_n, Tx^*\bigr) = 0.$$

Therefore, $x^* \in Tx^*$ and hence T has a fixed point. This completes the proof. □

Next, we give the second main result in this paper.

Theorem 2.4 *Let (X, d) be a metric space and $T : X \to K(X)$ be an (α, θ, k)-contraction mapping. Suppose that the following conditions hold*:

- (S_1) *(X, d) is an α-complete metric space*;
- (S_2) *T is an α-admissible multi-valued mapping*;
- (S_3) *there exist x_0 and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$*;
- (S_4') *if $\{x_n\}$ is a sequence in X with $x_n \to x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, then we have*

$$\theta\bigl(H(Tx_n, Tx)\bigr) \leq \theta\bigl(M(x_n, x)\bigr)^k, \tag{2.18}$$

where

$$M(x_n, x) = \max\left\{d(x_n, x), d(x_n, Tx_n), d(x, Tx), \frac{d(x_n, Tx) + d(x, Tx_n)}{2}\right\}$$

for all $n \in \mathbb{N}$.

Then T has a fixed point in X.

Proof Following the proof of Theorem 2.3, we know that $\{x_n\}$ is a Cauchy sequence in X such that $x_n \to x^*$ as $n \to \infty$ and

$$\alpha(x_n, x_{n+1}) \geq 1 \tag{2.19}$$

for all $n \in \mathbb{N}$. Suppose that $d(x^*, Tx^*) > 0$. By using (2.18), we have

$$\begin{aligned} &\theta\bigl(d\bigl(x_{n+1}, Tx^*\bigr)\bigr) \\ &\quad \leq \theta\bigl(H\bigl(Tx_n, Tx^*\bigr)\bigr) \end{aligned}$$

$$\leq \theta\left(\max\left\{d(x_n,x^*),d(x_n,Tx_n),d(x^*,Tx^*),\frac{d(x_n,Tx^*)+d(x^*,Tx_n)}{2}\right\}\right)^k$$

$$\leq \theta\left(\max\left\{d(x_n,x^*),d(x_n,x_{n+1}),d(x^*,Tx^*),\frac{d(x_n,Tx^*)+d(x^*,x_{n+1})}{2}\right\}\right)^k \tag{2.20}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.20), we have

$$\theta\big(d(x^*,Tx^*)\big) \leq \theta\big(d(x^*,Tx^*)\big)^k.$$

This implies that $\theta(d(x^*,Tx^*)) = 0$, which is a contradiction. Therefore, we have $d(x^*, Tx^*) = 0$, that is, $x^* \in Tx^*$. This completes the proof. □

Remark 2.5 From Remark 1.5, the conclusion in Theorems 2.3 and 2.4 are still hold if we replace condition (S_2) by the following condition:

(S_2') T is an α_*-admissible multi-valued mapping.

Now, we give an example to illustrate Theorem 2.4.

Example 2.6 Let $X = (-10,10)$ and the metric $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Define $T : X \to K(X)$ and $\alpha : X \times X \to [0,\infty)$ by

$$Tx = \begin{cases} [-|x|,|x|], & \text{if } x \in (-10,0),\\ [0,\frac{x}{4}], & \text{if } x \in [0,2],\\ [\frac{x+10}{4},6], & \text{if } x \in (2,10) \end{cases}$$

and

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x,y \in [0,2],\\ 0, & \text{otherwise.} \end{cases}$$

Clearly, (X,d) is not a complete metric space. Many fixed point results are not applicable here.

Next, we show that Theorem 2.4 can be guarantee the existence of fixed point of T. Define a function $\theta : (0,\infty) \to (1,\infty)$ by

$$\theta(t) = e^{\sqrt{te^t}}$$

for all $t \in (0,\infty)$. It is easy to see that $\theta \in \Theta$ (see also [10]).

Firstly, we show that T is an (α,θ,k)-contraction multi-valued mappings with $k = \frac{1}{2}$. For all $x,y \in [0,2]$ with $H(Tx,Ty) \neq 0$, we have $x \neq y$ and then

$$\begin{aligned}\alpha(x,y)\theta\big(H(Tx,Ty)\big) &= \theta\left(\frac{|x-y|}{4}\right)\\ &= e^{\sqrt{\frac{|x-y|}{4}e^{\frac{|x-y|}{4}}}}\\ &\leq e^{\frac{1}{2}\sqrt{|x-y|e^{|x-y|}}}\end{aligned}$$

$$= e^{\frac{1}{2}\sqrt{d(x,y)e^{d(x,y)}}}$$
$$\leq e^{\frac{1}{2}\sqrt{M(x,y)e^{M(x,y)}}}$$
$$= \theta\big(M(x,y)\big)^k,$$

that is, the condition (2.2) holds. Therefore, T is an (α,θ,k)-contraction multi-valued mappings with $k = \frac{1}{2}$. Moreover, it is easy to see that T is an α-admissible multi-valued mapping and there exists $x_0 = 1 \in X$ and $x_1 = 1/4 \in Tx_0$ such that

$$\alpha(x_0, x_1) = \alpha(1, 1/4) \geq 1.$$

Finally, for each sequence $\{x_n\}$ in X with $x_n \to x \in X$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, we have $x, x_n \in [0, 2]$ for all $n \in \mathbb{N}$. Then we obtain

$$\begin{aligned} \theta\big(H(Tx_n, Tx)\big) &= \theta\left(\frac{|x_n - x|}{4}\right) \\ &= e^{\sqrt{\frac{|x_n-x|}{4}e^{\frac{|x_n-x|}{4}}}} \\ &\leq e^{\frac{1}{2}\sqrt{|x_n-x|e^{|x_n-x|}}} \\ &= e^{\frac{1}{2}\sqrt{d(x_n,x)e^{d(x_n,x)}}} \\ &\leq e^{\frac{1}{2}\sqrt{M(x_n,x)e^{M(x_n,x)}}} \\ &= \theta\big(M(x_n,x)\big)^k \end{aligned}$$

for all $n \in \mathbb{N}$. Thus the condition (S_4') in Theorem 2.4 holds. Therefore, by using Theorem 2.4, it follows that T has a fixed point in X. In this case, T has infinitely fixed points such as -8, -2, and 0.

3 Some applications

In 2008, Jachymski [11] obtained a generalization of Banach's contraction principle for mappings on a metric space endowed with a graph. Afterward, Dinevari and Frigon [12] extended the results of Jachymski [11] to multi-valued mappings.

In this section, we give the existence of fixed point theorems on a metric space endowed with graph. The following notions and definitions are needed.

Let (X,d) be a metric space. A set $\{(x,x) : x \in X\}$ is called a *diagonal* of the Cartesian product $X \times X$, which is denoted by Δ. Consider a graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, *i.e.*, $\Delta \subseteq E(G)$. We assume that G has no parallel edges and so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat G as a weighted graph by assigning to each edge the distance between its vertices.

Definition 3.1 [9] Let X be a nonempty set endowed with a graph G and $T : X \to N(X)$ be a multi-valued mapping, where X is a nonempty set. The mapping T *preserves edges weakly* if, for each $x \in X$ and $y \in Tx$ with $(x,y) \in E(G)$, we have $(y,z) \in E(G)$ for all $z \in Ty$.

Definition 3.2 [9] Let (X,d) be a metric space endowed with a graph G. The metric space X is said to be *$E(G)$-complete* if every Cauchy sequence $\{x_n\}$ in X with $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ converges in X.

Definition 3.3 [9] Let (X,d) be a metric space endowed with a graph G. A mapping $T: X \to \mathrm{CL}(X)$ is called an *$E(G)$-continuous mapping* to $(\mathrm{CL}(X),H)$ if, for any $x \in X$ and any sequence $\{x_n\}$ with $\lim_{n\to\infty} d(x_n,x) = 0$ and $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} H(Tx_n,Tx) = 0.$$

Definition 3.4 Let (X,d) be a metric space endowed with a graph G. A mapping $T: X \to K(X)$ is called an *$(E(G),\theta,k)$-contraction multi-valued mapping* if there exist a function $\theta \in \Theta$ and $k \in (0,1)$ such that

$$x,y \in X, (x,y) \in E(G), \quad H(Tx,Ty) \neq 0 \quad \Longrightarrow \quad \theta\bigl(H(Tx,Ty)\bigr) \leq \theta\bigl(\bigl(M(x,y)\bigr)\bigr)^k, \tag{3.1}$$

where

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}.$$

Theorem 3.5 *Let (X,d) be a metric space endowed with a graph G and $T: X \to K(X)$ be a $(E(G),\theta,k)$-contraction multi-valued mapping. Suppose that the following conditions hold*:

- (S_1) *(X,d) is an $E(G)$-complete metric space*;
- (S_2) *T preserves edges weakly*;
- (S_3) *there exist x_0 and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E(G)$*;
- (S_4) *T is an $E(G)$-continuous multi-valued mapping.*

Then T has a fixed point in X.

Proof This result can be obtained from Theorem 2.3 if we define a mapping $\alpha: X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof. □

By using Theorem 2.4, we get the following result.

Theorem 3.6 *Let (X,d) be a metric space endowed with a graph G and $T: X \to K(X)$ be a $(E(G),\theta,k)$-contraction multi-valued mapping. Suppose that the following conditions hold*:

- (S_1) *(X,d) is an $E(G)$-complete metric space*;
- (S_2) *T preserves edges weakly*;
- (S_3) *there exist x_0 and $x_1 \in Tx_0$ such that $(x_0,x_1) \in E(G)$*;
- (S_4') *if $\{x_n\}$ is a sequence in X with $x_n \to x \in X$ as $n \to \infty$ and $(x_n,x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then we have*

$$\theta\bigl(H(Tx_n,Tx)\bigr) \leq \theta\bigl(M(x_n,x)\bigr)^k, \tag{3.2}$$

where

$$M(x_n, x) = \max\left\{d(x_n, x), d(x_n, Tx_n), d(x, Tx), \frac{d(x_n, Tx) + d(x, Tx_n)}{2}\right\}$$

for all $n \in \mathbb{N}$.

Then T *has a fixed point in* X.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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