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Partial permanence and extinction on stochastic Lotka-Volterra competitive systems

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Abstract

This paper discusses an autonomous competitive Lotka-Volterra model in random environments. The contributions of this paper are as follows. (a) Some sufficient conditions for partial permanence and extinction on this system are established; (b) By using some novel techniques, the conditions imposed on permanence and extinction of one-species are weakened. Finally, a numerical experiment is conducted to validate the theoretical findings.

Keywords: Lotka-Volterra model; random environments; Brownian motions; Itô formula; persistence in mean; extinction

1 Introduction

It is a usual phenomenon for two or more species who live in proximity share the same basic requirements and compete for resources, habitat, food, or territory. It is therefore very important to study the competitive models for multi-species. As we know, the well-known Lotka-Volterra model concerning ecological population modeling has received great attention and has been studied extensively owing to its theoretical and practical significance. A deterministic, competitive Lotka-Volterra system with n interacting species is described by the n-dimensional differential equation

$$\frac{dx_i(t)}{dt} = x_i(t)\left[b_i - \sum_{j=1}^{n} a_{ij}x_j(t)\right], \quad i = 1, 2, \ldots, n, \tag{1.1}$$

where $x_i(t)$ represents the population size of species i at time t, b_i is the growth rate of species i, and a_{ij} represents the effect of interspecific (if $i \neq j$) or intraspecific (if $i = j$) interaction.

On the other hand, from the biological point of view, population systems in the real world are inevitably affected by environmental noise. In practice, the growth rates are often subject to environmental noise. To obtain a more accurate description of such systems, we usually consider the stochastic perturbation of the growth rate b_i by an average value plus an error term. Then the intrinsic growth rate depending on time becomes

$$b_i \rightarrow b_i + \sigma_i \dot{B}_i(t),$$

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where $\dot{B}_i(t)$ is a white noise. As a result, system (1.1) becomes a stochastic Lotka-Volterra competition system with n interacting components as follows:

$$dx_i(t) = x_i(t)\left[\left(b_i - \sum_{j=1}^{n} a_{ij}x_j(t)\right)dt + \sigma_i\, dB_i(t)\right], \quad i = 1, 2, \ldots, n, \tag{1.2}$$

where b_i, a_{ij}, σ_i are non-negative for $i, j = 1, 2, \ldots, n$, and σ_i^2 will be called the noise intensity matrix. Throughout this paper we always assume that the following hypothesis holds:

$$b_i > 0, \qquad a_{ii} > 0, \qquad a_{ij} \geq 0 \quad (i \neq j). \tag{1.3}$$

It is therefore necessary to reveal how the noise affects the population systems. As a matter of fact, stochastic Lotka-Volterra competitive systems have recently been studied by many authors, for example, [1–6].

In the study of population systems, permanence and extinction are two important and interesting topics, respectively meaning that the population system will survive or die out in the future, which have received much attention (see [7–18]). Luo and Mao [15] revealed that a large white noise will force the stochastic Lotka-Volterra systems to become extinct while the population may be persistent under a relatively small white noise. Li and Mao [9] investigated a non-autonomous stochastic Lotka-Volterra competitive system, and some sufficient conditions on stochastic permanence and extinction were obtained. Li *et al.* [11] showed that both stochastic permanence and extinction have close relationships with the stationary probability distribution of the Markov chain. Tran and Yin [16] investigated stochastic permanence and extinction for stochastic competitive Lotka-Volterra systems using feedback controls.

However, most of the existing criteria are established for stochastic general Lotka-Volterra system. Hence, one natural question arises: How to derive some criteria with less conservatism for stochastic Lotka-Volterra competitive systems? This issue constitutes the first motivation of this paper.

Moreover, most of the existing criteria are established for total permanence and total extinction. To the best of our knowledge, partial permanence and partial extinction have scarcely been investigated, which are very important properties. Is it feasible to obtain some partial permanence and extinction conditions for stochastic Lotka-Volterra competitive systems? Thus, the second purpose of this paper is to solve this interesting problem.

The rest of the paper is arranged as follows. The main results of this paper are stated in Sections 3 and 4. In Section 2, some preliminaries, definitions and lemmas are given. Sufficient conditions on persistence in mean and extinction for one-species are obtained in Section 3. Based on these sufficient conditions on one-species, sufficient criteria on partial permanence and extinction on system (1.2) are established in Section 4. Section 5 provides some numerical examples to check the effectiveness of the derived results.

2 Notation

Throughout this paper, unless otherwise specified, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (*i.e.*, it is increasing and right continuous, while $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Let $B(t) = (B_t^1, \ldots, B_t^m)$ be an m-dimensional Brownian motion defined on the probability space. Let $R_+^n = \{x \in R^n : x_i > 0 \text{ for all } 1 \leq i \leq n\}$.

Lemma 2.1 ([19]) *Assume that condition* (1.3) *holds. Then, for any given initial value* $x(0) \in R_+^n$, *there is a unique solution* $x(t)$ *to system* (1.2) *and the solution will remain in* R_+^n *with probability* 1, *namely*

$$P\{x(t) \in R_+^n, \forall t \geq 0\} = 1.$$

Lemma 2.2 ([19]) *Assume that condition* (1.3) *holds. Then, for any given initial value* $x(0) \in R_+^n$, *the solution* $x_i(t)$ *to system* (1.2) *obeys*

$$\sup_{0 \leq t < +\infty} \sum_{i=1}^{n} E x_i^p(t) \leq K_p, \quad i = 1, \dots, n.$$

Definition 2.1 System (1.2) is said to be persistent in mean if there exist positive constants α_i, β_i such that the solution to system (1.2) has the following property:

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t x_i(s) \, ds \leq \beta_i \quad \text{a.s. } i = 1, \dots, n,$$

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t x_i(s) \, ds \geq \alpha_i \quad \text{a.s. } i = 1, \dots, n.$$

To proceed with our study, we consider two auxiliary stochastic differential equations

$$\begin{cases} dy_i(t) = y_i(t)[(b_i - a_{ii}y_i(t)) \, dt + \sigma_i \, dB_i(t)], \\ y_i(0) = x_i(0), \quad i = 1, \dots, n, \end{cases} \tag{2.1}$$

$$\begin{cases} dz_i(t) = z_i(t)[(b_i - \sum_{j \neq i} a_{ij} y_j(t) - a_{ii} z_i(t)) \, dt + \sigma_i \, dB_i(t)], \\ z_i(0) = x_i(0), \quad i = 1, \dots, n, \end{cases} \tag{2.2}$$

where

$$y(t) = \big(y_1(t), \dots, y_n(t)\big)^T, \qquad z(t) = \big(z_1(t), \dots, z_n(t)\big)^T.$$

Lemma 2.3 *Assume that condition* (1.3) *holds. Let* $x(t)$ *be a solution to system* (1.2) *with* $x(0) \in R_+^n$, *then we have*

$$z(t) \leq x(t) \leq y(t),$$

i.e.,

$$z_i(t) \leq x_i(t) \leq y_i(t), \quad i = 1, \dots, n.$$

Proof By the Itô formula, we derive that

$$\frac{1}{x_i(t)} = \frac{1}{x_i(0)} \exp\left[\left(\frac{\sigma_i^2}{2} - b_i\right)t + \sum_{j \neq i} a_{ij} \int_0^t x_j(s) \, ds - \sigma_i B_i(t)\right]$$

$$+ a_{ii} \int_0^t \exp\left[\left(\frac{\sigma_i^2}{2} - b_i\right)(t - s) + \sum_{j \neq i} a_{ij} \int_s^t x_j(\iota) \, d\iota\right.$$

$$\left. - \sigma_i\big(B_i(t) - B_i(s)\big)\right] ds, \quad i = 1, \dots, n, \tag{2.3}$$

$$\frac{1}{y_i(t)} = \frac{1}{x_i(0)} \exp\left[\left(\frac{\sigma_i^2}{2} - b_i\right)t - \sigma_i B_i(t)\right] + a_{ii} \int_0^t \exp\left[\left(\frac{\sigma_i^2}{2} - b_i\right)(t-s)\right.$$

$$\left. - \sigma_i\big(B_i(t) - B_i(s)\big)\right] ds, \quad i = 1, \ldots, n, \tag{2.4}$$

which means

$$x_i(t) \le y_i(t), \quad i = 1, \ldots, n. \tag{2.5}$$

Applying the Itô formula to equation (2.2) yields

$$\frac{1}{z_i(t)} = \frac{1}{x_i(0)} \exp\left[\left(\frac{\sigma_i^2}{2} - b_i\right)t + \sum_{j \ne i} a_{ij} \int_0^t y_j(s)\,ds - \sigma_i B_i(t)\right]$$

$$+ a_{ii} \int_0^t \exp\left[\left(\frac{\sigma_i^2}{2} - b_i\right)(t-s) + \sum_{j \ne i} a_{ij} \int_s^t y_i(\iota)\,d\iota\right.$$

$$\left. - \sigma_i\big(B_i(t) - B_i(s)\big)\right] ds, \quad i = 1, \ldots, n. \tag{2.6}$$

From the representations of $y_i(t)$ and $z_i(t)$, and by (2.6) we have

$$z_i(t) \le x_i(t), \quad i = 1, \ldots, n. \qquad \square \tag{}$$

3 Persistence in mean and extinction of one-species
3.1 Persistence of one-species

In this section, we investigate persistence in mean and extinction of one-species for system (1.2). Now, let us present some lemmas which are essential to the proof of Theorem 3.1.

Lemma 3.1 ([20]) *Let condition* (1.3) *hold. The solution* $y_i(t)$ *to equation* (2.1) *has the following property*:

$$\lim_{t \to \infty} \frac{\log y_i(t)}{t} = \left(b_i - \frac{\sigma_i^2}{2}\right) \wedge 0 \quad a.s.$$

With the help of Lemma 3.1, we slightly improve Lemma 3.1 of [8] by weakening hypotheses posed on the coefficients of equation (2.2) as follows.

Lemma 3.2 *Let condition* (1.3) *hold, and assume that* $b_i - \frac{\sigma_i^2}{2} > 0$, $b_j - \frac{\sigma_j^2}{2} \ge 0$ $(i \ne j)$ *and* $b_i - \frac{\sigma_i^2}{2} - \sum_{j \ne i} \frac{a_{ij}}{a_{jj}}((b_j - \frac{\sigma_j^2}{2}) \wedge 0) > 0$. *Then the solution to equation* (2.2) *has the property*

$$\lim_{t \to \infty} \frac{\log z_i(t)}{t} = 0 \quad a.s. \tag{3.1}$$

Theorem 3.1 *Let condition* (1.3) *and assumptions in Lemma* 3.2 *hold. Then the solution to system* (1.2) *has the following property*:

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t x_i(s)\,ds \le \frac{1}{a_{ii}}\left(b_i - \frac{\sigma_i^2}{2}\right) \quad a.s., \tag{3.2}$$

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t x_i(s)\,ds \geq \frac{1}{a_{ii}} \left[\left(b_i - \frac{\sigma_i^2}{2} \right) - \sum_{j\neq i} \frac{a_{ij}}{a_{jj}} \left(\left(b_j - \frac{\sigma_j^2}{2} \right) \wedge 0 \right) \right] \quad a.s., \tag{3.3}$$

which means the species i of system (1.2) is persistent in mean.

Proof Applying the Itô formula to equation (2.1) yields

$$\log y_i(t) = \log y_i(0) + \left(b_i - \frac{\sigma_i^2}{2} \right) t - a_{ii} \int_0^t y_i(s)\,ds + \sigma_i B_i(t). \tag{3.4}$$

Then we have

$$\int_0^t y_i(s)\,ds = \frac{(b_i - \frac{\sigma_i^2}{2})}{a_{ii}} t + \frac{\sigma_i}{a_{ii}} B_i(t) - \frac{1}{a_{ii}} \big(\log y_i(t) - \log y_i(0) \big). \tag{3.5}$$

Dividing both sides of (3.5) by t yields

$$\frac{1}{t} \int_0^t y_i(s)\,ds = \frac{1}{a_{ii}} \left[\frac{\log y_i(0)}{t} - \frac{\log y_i(t)}{t} + \left(b_i - \frac{\sigma_i^2}{2} \right) + \frac{1}{t} \int_0^t \sigma_i\,dB_i(s) \right]. \tag{3.6}$$

Note that

$$\lim_{t\to\infty} \frac{\log y_i(0)}{t} = 0, \qquad \frac{1}{t} \int_0^t \sigma_i\,dB_i(s) = 0 \quad a.s.$$

This implies

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t y_i(s)\,ds = -\frac{1}{a_{ii}} \lim_{t\to\infty} \frac{\log y_i(t)}{t} + \frac{1}{a_{ii}} \left(b_i - \frac{\sigma_i^2}{2} \right).$$

By Lemma 3.1, letting $t \to \infty$ on both sides of (3.6) yields

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t y_j(s)\,ds = \frac{1}{a_{jj}} \left(\left(b_j - \frac{\sigma_j^2}{2} \right) \vee 0 \right) \quad a.s. \tag{3.7}$$

Combining Lemma 3.2 and (3.7), we can claim that

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t x_i(s)\,ds \leq \lim_{t\to\infty} \frac{1}{t} \int_0^t y_i(s)\,ds = \frac{1}{a_{ii}} \left(b_i - \frac{\sigma_i^2}{2} \right) \quad a.s. \tag{3.8}$$

Now we process to show assertion (3.3). Applying the Itô formula to $\log z_i(t)$ yields

$$\log z_i(t) = \log z_i(0) - \int_0^t \left(\frac{\sigma_i^2}{2} - b_i \right) ds - a_{ii} \int_0^t z_i(s)\,ds$$

$$+ \sum_{j\neq i} \int_0^t a_{ij} z_j(s)\,ds + \int_0^t \sigma_i\,dB_i(s). \tag{3.9}$$

Dividing both sides of (3.9) by t yields

$$\frac{\log z_i(t)}{t} = \frac{\log z_i(0)}{t} - \frac{1}{t}\int_0^t \left(\frac{\sigma_i^2}{2} - b_i\right)ds - \frac{a_{ii}}{t}\int_0^t z_i(s)\,ds$$

$$+ \frac{1}{t}\sum_{j\neq i}\int_0^t a_{ij}y_j(s)\,ds + \frac{1}{t}\int_0^t \sigma_i\,dB_i(s). \tag{3.10}$$

Using Lemma 3.2 and the law of strong large number for martingale, we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \sigma_i\,dB_i(s) = 0, \qquad \lim_{t\to\infty}\frac{\log z_i(t)}{t} = 0 \quad \text{a.s.} \tag{3.11}$$

Combining Lemma 3.2 and (3.11), letting $t \to +\infty$ on both sides of (3.9) yields

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t z_i(s)\,ds = \frac{1}{a_{ii}}\left[\left(b_i - \frac{\sigma_i^2}{2}\right) + \sum_{j\neq i}\lim_{t\to\infty}\frac{a_{ij}}{t}\int_0^t y_j(s)\,ds\right]$$

$$= \frac{1}{a_{ii}}\left[\left(b_i - \frac{\sigma_i^2}{2}\right) - \sum_{j\neq i}\frac{a_{ij}}{a_{jj}}\left(\left(b_j - \frac{\sigma_j^2}{2}\right)\wedge 0\right)\right] > 0. \tag{3.12}$$

Since $x_i(t) \geq z_i(t)$, assertion (3.3) is true. Therefore this theorem is proved. $\qquad\square$

Remark 3.1 Compared with the existing literature [8], the conditions imposed on the permanence of one-species are weaker.

Applying Lemma 3.1 to system (1.2), we have the following corollary, which coincides with Theorem 3.1 in [8].

Corollary 3.1 *Let condition* (1.3) *hold and assume that* $b_i - \frac{\sigma_i^2}{2} > 0$, $b_i - \frac{\sigma_i^2}{2} - \sum_{j\neq i}\frac{a_{ij}}{a_{jj}}(b_j - \frac{\sigma_j^2}{2}) > 0$ *for all* $i = 1,\dots,n$. *System* (1.2) *is persistent in mean.*

3.2 Extinction of one-species

Theorem 3.2 *Let condition* (1.3) *hold and* $x_i(t)$ *be the solution to system* (1.2) *with positive initial value* $x_i(0)$. *Then we have the following assertions*:

(i) *If* $\sigma_i^2 > 2b_i$, *the solution* $x_i(t)$ *to system* (1.2) *has the property that*

$$\limsup_{t\to\infty}\frac{\log x_i(t)}{t} \leq b_i - \frac{\sigma_i^2}{2} \quad \text{a.s.} \tag{3.13}$$

That is, the species i *of system* (1.2) *will become extinct.*

(ii) *If* $\sigma_i^2 = 2b_i$, *the solution* $x_i(t)$ *to system* (1.2) *has the property that*

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s.} \tag{3.14}$$

That is, the species i *of system* (1.2) *still become extinct with probability one.*

Proof The proof is rather technical, so we will divide it into two steps. The first step is to show the exponential extinction of species i when $\sigma_i^2 > 2b_i$. The second step is to show the extinction in the case of $\sigma_i^2 = 2b_i$.

Step 1. Applying the Itô formula to $\log x_i(t)$ yields

$$\log x_i(t) = \log x_i(0) + \int_0^t \left(b_i - \frac{\sigma_i^2}{2} \right) ds - \sum_{j=1}^n a_{ij} \int_0^t x_j(s)\, ds + \int_0^t \sigma_i\, dB_i(s). \tag{3.15}$$

Dividing both sides of (3.15) by t yields

$$\frac{\log x_i(t)}{t} = \frac{\log x_i(0)}{t} + \frac{1}{t} \int_0^t \left(b_i - \frac{\sigma_i^2}{2} \right) ds$$

$$- \frac{1}{t} \sum_{j=1}^n a_{ij} \int_0^t x_j(s)\, ds + \frac{1}{t} \int_0^t \sigma_i\, dB_i(s). \tag{3.16}$$

Using the law of strong large number for martingales, we can claim that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma_i\, dB_i(s) = 0 \quad \text{a.s.}$$

Letting $t \to \infty$ yields

$$\limsup_{t \to \infty} \frac{\log x_i(t)}{t} \le b_i - \frac{\sigma_i^2}{2} \quad \text{a.s.}$$

Step 2. Now, let us finally show assertion (3.14). Decompose the sample space into three mutually exclusive events as follows:

$$\Omega_{i1} = \left\{ \omega : \limsup_{t \to \infty} x_i(t) \ge \liminf_{t \to \infty} x_i(t) = \gamma_i > 0 \right\};$$

$$\Omega_{i2} = \left\{ \omega : \limsup_{t \to \infty} x_i(t) > \liminf_{t \to \infty} x_i(t) = 0 \right\};$$

$$\Omega_{i3} = \left\{ \omega : \lim_{t \to \infty} x_i(t) = 0 \right\}.$$

When $\sigma_i^2 = 2b_i$, equation (3.16) has the following form:

$$\frac{\log x_i(t)}{t} = \frac{\log x_i(0)}{t} - \frac{1}{t} \sum_{j=1}^n a_{ij} \int_0^t x_i(s)\, ds + \frac{1}{t} \int_0^t \sigma_i\, dB_i(s). \tag{3.17}$$

Furthermore, we decompose the sample space into the following two mutually exclusive events according to the convergence of $\int_0^\infty x_i(s)\, ds$:

$$E_{i1} = \left\{ \omega : \int_0^\infty x_i(s)\, ds < \infty \right\}, \qquad E_{i2} = \left\{ \omega : \int_0^\infty x_i(s)\, ds = \infty \right\}. \tag{3.18}$$

The proof of $\lim_{t \to \infty} x_i(t) = 0$ a.s. is equivalent to showing $E_{i1} \subset \Omega_{i3}$, $E_{i2} \subset \Omega_{i3}$ a.s. The strategy of the proof is as follows.

★ First, by using the techniques proposed in [21], we show that $E_{i1} \subset \Omega_{i3}$. It is sufficient to show $P(E_{i1} \cap \Omega_{i1}) = 0$ and $P(E_{i1} \cap \Omega_{i2}) = 0$.

★ Second, using some novel techniques, we prove that $P(E_{i2} \cap \Omega_{i1}) = 0$ and $P(E_{i2} \cap \Omega_{i2}) = 0$, which means $E_{i2} \subset \Omega_{i3}$ a.s.

Now we realize this strategy as follows.

Case 1. Let us now show $E_{i1} \subset \Omega_{i3}$. Clearly, $x_i(t) \in C(R_+, R)$ a.s. It is straightforward to see from E_{i1} that $\liminf_{t\to\infty} x_i(t) = 0$ a.s. Therefore, we have obtained that $P(E_{i1} \cap \Omega_{i1}) = 0$. Now we only need to prove that $P(E_{i1} \cap \Omega_{i2}) = 0$. We prove it by contradiction.

If $P(E_{i1} \cap \Omega_{i2}) > 0$, there exists a number $\epsilon > 0$ such that

$$P(J_1 \cap E_{i1}) \geq 2\epsilon, \tag{3.19}$$

where $J_1 = \{\limsup_{t\to\infty} x_i(t) > 2\epsilon\}$. Let us now define a sequence of stopping times

$$\tau_1 = \inf\{t \geq 0 : x_i(t) \geq 2\epsilon\}, \qquad \tau_{2k} = \inf\{t \geq \tau_{2k-1} : x_i(t) \leq \epsilon\},$$

$$\tau_{2k+1} = \inf\{t \geq \tau_{2k} : x_i(t) \geq 2\epsilon\}, \quad k = 1, 2, \ldots.$$

From E_{i1}, we also have $E(I_{E_{i1}} \int_0^\infty x_i(s)\,ds) < \infty$, then we compute it

$$E\left(I_{E_{i1}} \int_0^\infty x_i(s)\,ds\right) \geq \sum_{k=1}^\infty E\left[I_{\{\tau_{2k-1}<\infty,\tau_{2k}<\infty\}\cap E_{i1}} \int_{\tau_{2k-1}}^{\tau_{2k}} x_i(s)\,ds\right]$$

$$\geq \epsilon \sum_{k=1}^\infty E\left[I_{\{\tau_{2k-1}<\infty\}\cap E_{i1}}(\tau_{2k} - \tau_{2k-1})\right],$$

where I_A is the indicator function for all sets A. Since $\tau_{2k} < \infty$ whenever $\tau_{2k-1} < \infty$, by the above formula, so we have

$$\epsilon \sum_{k=1}^\infty E\left[I_{\{\tau_{2k-1}<\infty\}\cap E_{i1}}(\tau_{2k-1} - \tau_{2k})\right] < \infty. \tag{3.20}$$

On the other hand, integrating equation (1.2) from 0 to t yields

$$x_i(t) = x_i(0) + \int_0^t x_i(s)\left(b_i - \sum_{j=1}^n a_{ij}x_j(s)\right)ds + \int_0^t \sigma_i x_i(s)\,dB_i(s). \tag{3.21}$$

A simple computation shows that

$$E\left[x_i^2(s) \cdot \left(b_i - \sum_{j=1}^n a_{ij}x_j(s)\right)^2\right]$$

$$\leq \frac{1}{2}E(x_i^4(s)) + \frac{1}{2}E\left[\left(b_i - \sum_{j=1}^n a_{ij}x_j(s)\right)^4\right]$$

$$\leq \frac{1}{2}E(x_i^4(s)) + \frac{1}{2}E\left(b_i^4 + 6b_i^2 \sum_{j=1}^n a_{ij}^2 x_j^2(s) + \sum_{j=1}^n a_{ij}^4 x_j^4(s)\right)$$

$$\leq \frac{1}{2}K_4 + \frac{1}{2}\left(b_i^4 + 6b_i^2 \sum_{j=1}^n a_{ij}^2 K_2 + \sum_{j=1}^n a_{ij}^4 K_4\right)$$

$$=: M_i^2,$$

and

$$E\big(\sigma_i^2 \cdot x_i^2(s)\big) = \sigma_i^2 \cdot E\big(x_i^2(s)\big) \le \sigma_i^2 \cdot K_2 =: N_i^2,$$

where K_2 and K_4 are defined in Lemma 2.2. Using the Hölder inequality and Burkholder-Davis-Gundy inequality (see [19]), we compute

$$E\Big[I_{\{\tau_{2k-1}<\infty\}\cap E_{i1}} \sup_{0\le t\le T} \big|x_i(\tau_{2k-1}+t) - x_i(\tau_{2k-1})\big|^2\Big]$$

$$\le 2E\Big[I_{\{\tau_{2k-1}<\infty\}\cap E_{i1}} \sup_{0\le t\le T}\Big|\int_{\tau_{2k-1}}^{\tau_{2k-1}+t}\Big(x_i(s)\cdot\Big(b_i - \sum_{j=1}^{n} a_{ij}x_j(s)\Big)\Big)\,ds\Big|^2\Big]$$

$$+ 2E\Big[I_{\{\tau_{2k-1}<\infty\}\cap E_{i1}} \sup_{0\le t\le T}\Big|\int_{\tau_{2k-1}}^{\tau_{2k-1}+t}\big(\sigma_i\cdot x_i(s)\big)\,dB_i(s)\Big|^2\Big]$$

$$\le 2TE\Big[I_{\{\tau_{2k-1}<\infty\}\cap E_{i1}}\int_{\tau_{2k-1}}^{\tau_{2k-1}+T}\Big(x_i^2(s)\cdot\Big(b_i - \sum_{j=1}^{n} a_{ij}x_j(s)\Big)^2\Big)(s)\,ds\Big]$$

$$+ 8E\Big[I_{\{\tau_{2k-1}<\infty\}\cap E_{i1}}\int_{\tau_{2k-1}}^{\tau_{2k-1}+T}\big(\sigma_i^2\cdot x_i^2(s)\big)\,ds\Big]$$

$$\le 2T\big(M_i^2 + 4N_i^2\big). \tag{3.22}$$

Furthermore, we choose $T = T(\epsilon) > 0$ sufficiently small for

$$2T\big(M_i^2 + 4N_i^2\big) \le \epsilon^3.$$

It then follows from (3.22) that

$$P\big(\{\tau_{2k-1}<\infty\}\cap\{H_k\cap E_{i1}\}\big) \le \frac{2(T+4)T(M_i^2 + N_i^2)}{\epsilon^2} \le \epsilon, \tag{3.23}$$

where

$$H_k = \Big\{\sup_{1\le t\le T}\big|x_i(\tau_{2k-1}+t) - x_i(\tau_{2k-1})\big| \ge \epsilon\Big\}, \quad k = 1, 2, \ldots, n.$$

Recalling the fact that $\tau_k < \infty$, for $k = 1, 2, \ldots$, whenever $\omega \in J_1$, we further compute

$$P\big(\{\tau_{2k-1}<\infty\}\cap\big\{H_k^c\cap E_{i1}\big\}\big)$$

$$= P\big(\{\tau_{2k-1}<\infty\}\cap E_{i1}\big) - P\big(\{\tau_{2k-1}<\infty\}\cap\{H_k\cap E_{i1}\}\big)$$

$$\ge 2\epsilon - \epsilon = \epsilon.$$

If $\omega \in \{\tau_{2k-1}<\infty\}\cap\{H_k^c\cap E_{i1}\}$, note that

$$\tau_{2k}(\omega) - \tau_{2k-1}(\omega) \ge T. \tag{3.24}$$

We derive from (3.20) and (3.24) that

$$\infty > \epsilon \sum_{k=1}^{\infty} E\big[I_{\{\tau_{2k-1}<\infty\}\cap E_{i1}}(\tau_{2k}-\tau_{2k-1})\big]$$

$$\geq \epsilon \sum_{k=1}^{\infty} E\big[I_{\{\tau_{2k-1}<\infty\}\cap\{H_k^c\cap E_{i1}\}}(\tau_{2k-1}-\tau_{2k})\big]$$

$$\geq \epsilon T \sum_{k=1}^{\infty} P\big(\{\tau_{2k-1}<\infty\}\cap\{H_k^c\cap E_{i1}\}\big)$$

$$\geq \epsilon T \sum_{k=1}^{\infty} \epsilon = \infty, \tag{3.25}$$

which is a contraction. So that $P(E_{i1}\cap\Omega_{i2}) = 0$ holds. Therefore, we obtain that $E_{i1}\subset\Omega_{i3}$.

Case 2. Now, we turn to prove that $E_{i2}\subset\Omega_{i3}$ a.s. It is sufficient to show $P(E_{i2}\cap\Omega_{i1}) = 0$ and $P(E_{i2}\cap\Omega_{i2}) = 0$. We prove it by contradiction.

If $P(E_{i2}\cap\Omega_{i1}) > 0$ holds, for any $\omega \in E_{i2}\cap\Omega_{i1}$, $\epsilon_0 \in (0, \frac{\gamma_i}{2})$, there exists $T = (\epsilon_0, \omega)$ such that

$$x_i(t) > \gamma_i - \epsilon_0 > \frac{\gamma_i}{2} \quad \forall t > T \text{ a.s.}$$

It then follows from (3.17) that

$$\frac{1}{t}\int_0^t x_i(s)\,ds = \frac{1}{t}\int_0^T x_i(s)\,ds + \frac{1}{t}\int_T^t x_i(s)\,ds \geq \frac{1}{t}\int_0^T x_i(s)\,ds + \frac{t-T}{t}\frac{\gamma_i}{2} \quad \text{a.s.}$$

Letting $t \to \infty$, we obtain that

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_i(s)\,ds > \frac{\gamma_i}{2} > 0 \quad \text{a.s.}$$

This implies

$$\limsup_{t\to\infty}\frac{\log x_i(t)}{t} \leq -\sum_{j=1}^n a_{ij}\frac{\gamma_i}{2} < 0 \quad \text{a.s.},$$

which contradicts the definition of E_{i2} and Ω_{i1}. So $P(E_{i2}\cap\Omega_{i1}) = 0$ must hold.

Now we process to show $P(E_{i2}\cap\Omega_{i2}) > 0$ is false. For this purpose, we need more notations as follows:

$$\Pi_t^\epsilon(i) := \big\{0 \leq s \leq t : x_i(s) \geq \epsilon\big\}, \qquad \delta_t^\epsilon(i) := \frac{m(\Pi_t^\epsilon(i))}{t},$$

$$\delta^\epsilon(i) := \liminf_{t\to\infty}\delta_t^\epsilon i, \qquad \Delta^\epsilon(i) := \big\{\omega \in E_{i2}\cap\Omega_{i2} : \delta^\epsilon(i) > 0\big\},$$

where $m(\Pi_t^\epsilon(i))$ indicates the length of $\Pi_t^\epsilon(i)$. It is easy to see that $\Delta^0(i) = E_{i2}\cap\Omega_{i2}$. For any $\epsilon_1 < \epsilon_2$, simple computations show that

$$\Pi_t^{\epsilon_1}(i) \supset \Pi_t^{\epsilon_2}(i), \qquad m\big(\Pi_t^{\epsilon_1}(i)\big) \geq m\big(\Pi_t^{\epsilon_2}(i)\big),$$

$$\delta_t^{\epsilon_1}(i) = \frac{m(\Pi_t^{\epsilon_1})(i)}{t} \geq \delta_t^{\epsilon_2}(i) = \frac{m(\Pi_t^{\epsilon_2})(i)}{t},$$

which implies

$$\delta^{\epsilon_2}(i) \le \delta^{\epsilon_1}(i), \qquad \Delta^{\epsilon_2}(i) \subset \Delta^{\epsilon_1}(i) \quad \forall \epsilon_1 < \epsilon_2.$$

It is easy to observe from the continuity of probability that

$$P\big(\Delta^\epsilon(i)\big) \to P\big(\Delta^0(i)\big) = P(E_2 \cap \Omega_2) \quad \text{as } \epsilon \to 0.$$

If $P(E_{i2} \cap \Omega_{i2}) > 0$, there exists $\epsilon > 0$ such that $P(D^\epsilon) > 0$. For any $\omega \in \Delta^\epsilon(i)$, simple computations show that

$$\frac{1}{t}\int_0^t x_i(s)\,ds = \frac{1}{t}\int_{\Pi_t^\epsilon(i)} x_i(s)\,ds + \frac{1}{t}\int_{[0,t]\setminus \Pi_t^\epsilon} x_i(s)\,ds \ge \frac{1}{t}\int_{\Pi_t^\epsilon(i)} x_i(s)\,ds \quad \text{a.s.}$$

By letting $t \to \infty$, we have

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t x_i(s)\,ds \ge \liminf_{t\to\infty} \frac{1}{t}\int_{\Pi_t^\epsilon} x_i(s)\,ds \ge \delta^\epsilon(i)\epsilon \quad \text{a.s.} \tag{3.26}$$

Substituting (3.26) into (3.17), we obtain that

$$\limsup_{t\to\infty} \frac{\log x_i(t)}{t} \le -\sum_{j=1}^n a_{ij}\delta^\epsilon(i)\epsilon < 0 \quad \text{a.s.}$$

This contradicts the definition of E_{i2} and Ω_{i2}. It yields the desired assertion $P(E_{i2} \cap \Omega_{i2}) = 0$ immediately. Combining the fact $E_{i1} \subset \Omega_{i3}$, $P(E_{i2} \cap \Omega_{i1}) = 0$ and $P(E_{i2} \cap \Omega_{i2}) = 0$, we can claim that

$$\lim_{t\to\infty} x_i(t) = 0 \quad \text{a.s.}$$

The proof is completed. □

Remark 3.2 In comparison with [8] and [11], we point out that species i is still extinct when $\sigma_i^2 = 2b_i$ by using some novel stochastic analysis techniques.

Corollary 3.2 *Let condition* (1.3) *hold and* $x(t)$ *be a solution to system* (1.2) *with positive initial value* $x(0)$. *Assume that there exists an integer* m, $1 \le m < n$, *such that*

$$\sigma_i^2 > 2b_i, \quad i = 1,\ldots,m, \tag{3.27}$$

$$\sigma_i^2 = 2b_i, \quad i = m+1,\ldots,n. \tag{3.28}$$

Then we have the following assertions:
(i) *For all* $i = 1,\ldots,n$, *the solution* $x_i(t)$ *to system* (1.2) *has the property that*

$$\lim_{t\to\infty} \frac{\log x_i(t)}{t} = b_i - \frac{\sigma_i^2}{2} \quad \text{a.s. } i = 1,\ldots,m. \tag{3.29}$$

(ii) *For all $i = m + 1, \ldots, n$, the solution $x_i(t)$ to system (1.2) has the property that*

$$\lim_{t \to \infty} \frac{\log x_i(t)}{t} = 0 \quad a.s. \ i = m + 1, \ldots, n. \tag{3.30}$$

Proof By virtue of Theorem 3.2, for all $\sigma_i^2 > 2b_i$, $i = 1, \ldots, m$, we obtain that

$$\limsup_{t \to \infty} \frac{\log x_i(t)}{t} \leq b_i - \frac{\sigma_i^2}{2} \quad a.s. \ i = 1, \ldots, m.$$

This shows that for any $\epsilon_i \in (0, \frac{\sigma_i^2}{2} - b_i)$ there is a positive random variable $T(\epsilon_i)$ such that

$$x_i(t) \leq \exp\left\{ \left(b_i - \frac{\sigma_i^2}{2} \right)t + \epsilon_i t \right\} \quad \forall t > T(\epsilon_i), a.s. \ i = 1, \ldots, m,$$

which means

$$\int_0^\infty x_i(s) \, ds < \infty \quad a.s. \ i = 1, \ldots, m.$$

Then letting $t \to \infty$ on both sides of (3.16) yields

$$\lim_{t \to \infty} \frac{\log x_i(t)}{t} = b_i - \frac{\sigma_i^2}{2} \quad a.s. \ i = 1, \ldots, m,$$

which is the required assertion (3.29).

Now we process to show assertion (3.30). By utilizing Theorem 3.1 and conditions (3.28), we derive

$$\lim_{t \to \infty} x_i(t) = 0 \quad a.s. \ i = m + 1, \ldots, n.$$

This implies

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t x_i(s) \, ds = 0 \quad a.s. \ i = m + 1, \ldots, n. \tag{3.31}$$

By the law of strong large numbers for martingales and (3.31), letting $t \to \infty$ on both sides of (3.17) yields

$$\lim_{t \to \infty} \frac{\log x_i(t)}{t} = 0 \quad a.s. \ i = m + 1, \ldots, n.$$

The proof is completed. $\qquad \square$

4 Partial permanence and extinction

Now in this section we present conditions for system (1.2) to be partially permanent and extinct. To proceed with our study, we consider the following auxiliary stochastic equation:

$$\begin{cases} d\Phi_i(t) = \Phi_i(t)(b_i - \sum_{j=1}^m a_{ij} \Phi_j(t)) \, dt + \sigma_i \Phi_i(t) \, dB_i(t), \\ \Phi_i(0) = x_i(0), \quad i = 1, \ldots, m. \end{cases} \tag{4.1}$$

Theorem 4.1 *Let condition* (1.3) *hold. Assume that there exists an integer* m, $1 \le m < n$, *such that*

$$b_i > \frac{\sigma_i^2}{2}, \qquad a_{ii} - \sum_{j \ne i}^{m} a_{ji} > 0,$$

$$b_i - \frac{\sigma_i^2}{2} - \sum_{k \ne i}^{m} \frac{a_{ik}}{a_{kk}} \left(b_k - \frac{\sigma_k^2}{2} \right) > 0, \quad i = 1, \dots, m, \tag{4.2}$$

$$b_i < \frac{\sigma_i^2}{2}, \quad i = m+1, \dots, n. \tag{4.3}$$

Then we have the following assertions:

(i) *For all* $i = 1, \dots, m$, *the solution* $x(t)$ *to system* (1.2) *has the property that*

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t x_i(s)\, ds \le \frac{1}{a_{ii}} \left(b_i - \frac{\sigma_i^2}{2} \right) \quad a.s. \ i = 1, \dots, m. \tag{4.4}$$

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t x_i(s)\, ds$$

$$\ge \frac{1}{a_{ii}} \left[\left(b_i - \frac{\sigma_i^2}{2} \right) - \sum_{k \ne i}^{m} \frac{a_{ik}}{a_{kk}} \left(b_k - \frac{\sigma_k^2}{2} \right) \right] \quad a.s. \ i = 1, \dots, m. \tag{4.5}$$

That is, for each $i = 1, \dots, m$, *the species* i *of system* (1.2) *is persistent in mean*;

(ii) *For all* $i = m+1, \dots, n$, *the solution* $x(t)$ *to system* (1.2) *has the property that*

$$\limsup_{t \to \infty} \frac{\log x_i(t)}{t} \le b_i - \frac{\sigma_i^2}{2} - \sum_{j=1}^{m} \frac{a_{ij}}{a_{jj}} \left[\left(b_j - \frac{\sigma_j^2}{2} \right) \right.$$

$$\left. - \sum_{k \ne j}^{m} \frac{a_{jk}}{a_{kk}} \left(b_k - \frac{\sigma_k^2}{2} \right) \right] \quad a.s. \ i = m+1, \dots, n. \tag{4.6}$$

That is, for each $i = m+1, \dots, n$, *the species* i *will become extinct.*

Proof We will divide the proof into two steps. The first step is to show the permanence of the top m species of system (1.2). The second step is to show the extinction for the bottom $n - m$ species of system (1.2).

Step 1. Applying the Itô formula to (4.1) yields

$$d \log \Phi_i(t) = \left(b_i - \frac{\sigma_i^2}{2} - \sum_{j=1}^{m} a_{ij} \Phi_j(t) \right) dt + \sigma_i\, dB_i(t), \quad i = 1, \dots, m. \tag{4.7}$$

Simple computations show that

$$d\left(\log x_i(t) - \log \Phi_i(t) \right) = - \sum_{j=1}^{m} a_{ij} \left(x_j(t) - \Phi_j(t) \right) dt$$

$$- \sum_{l=m+1}^{n} a_{il} x_l(t)\, dt, \quad i = 1, \dots, m. \tag{4.8}$$

Applying the Itô formula to $V(t) = \sum_{i=1}^{m} |\log x_i(t) - \log \Phi_i(t)|$ yields

$$D^+ V(t) = \sum_{i=1}^{m} \mathrm{sgn}(x_i(t) - \Phi_i(t)) \cdot [d \log x_i(t) - d \log \Phi_i(t)]$$

$$= -\sum_{i=1}^{m} \mathrm{sgn}(x_i(t) - \Phi_i(t)) \cdot \left[\sum_{j=1}^{m} a_{ij}(x_j(t) - \Phi_j(t)) + \sum_{l=m+1}^{n} a_{il} x_l(t) \right] dt$$

$$\leq -\sum_{i=1}^{m} a_{ii} |x_i(t) - \Phi_i(t)| \, dt + \sum_{i=1}^{m} \sum_{j \neq i}^{m} a_{ij} |x_j(t) - \Phi_j(t)| \, dt + \sum_{i=1}^{m} \sum_{l=m+1}^{n} a_{il} x_l(t) \, dt$$

$$= -\sum_{i=1}^{m} a_{ii} |x_i(t) - \Phi_i(t)| \, dt + \sum_{j=1}^{m} \sum_{i \neq j}^{m} a_{ji} |x_i(t) - \Phi_i(t)| \, dt + \sum_{i=1}^{m} \sum_{l=m+1}^{n} a_{il} x_l(t) \, dt$$

$$= -\sum_{i=1}^{m} a_{ii} |x_i(t) - \Phi_i(t)| \, dt + \sum_{i=1}^{m} \sum_{j \neq i}^{m} a_{ji} |x_i(t) - \Phi_i(t)| \, dt + \sum_{i=1}^{m} \sum_{l=m+1}^{n} a_{il} x_l(t) \, dt$$

$$\leq -\sum_{i=1}^{m} \left(a_{ii} - \sum_{j \neq i}^{m} a_{ji} \right) |x_i(t) - \Phi_i(t)| \, dt + \sum_{i=1}^{m} \sum_{l=m+1}^{n} a_{il} x_l(t) \, dt.$$

Hence we get

$$D^+ V(t) \leq -\mu \sum_{i=1}^{m} |x_i(t) - \Phi_i(t)| \, dt + \sum_{l=m+1}^{n} \theta_l x_l(t) \, dt, \quad i = 1, \ldots, m, \tag{4.9}$$

where $\mu = \min_{1 \leq i \leq m}(a_{ii} - \sum_{j \neq i}^{m} a_{ji}) > 0$, $\theta_l = \sum_{i=1}^{m} a_{il} \geq 0$. We therefore have

$$V(t) + \mu \int_0^t \sum_{i=1}^{m} |x_i(s) - \Phi_i(s)| \, ds$$

$$\leq V(0) + \sum_{l=m+1}^{n} \theta_l \int_0^t x_l(s) \, ds, \quad i = 1, \ldots, m. \tag{4.10}$$

Letting $t \to \infty$ on both sides of (4.10) yields

$$\int_0^\infty |x_i(s) - \Phi_i(s)| \, ds \leq \int_0^\infty \sum_{i=1}^{m} |x_i(s) - \Phi_i(s)| \, ds$$

$$\leq \frac{1}{\mu} \left[V(0) + \sum_{l=m+1}^{n} \theta_l \int_0^\infty x_l(s) \, ds \right]. \tag{4.11}$$

By Theorem 3.2 and condition (4.3), we have

$$\int_0^\infty x_l(s) \, ds < +\infty \quad \text{a.s. } l = m+1, \ldots, n. \tag{4.12}$$

Substituting (4.12) into (4.11) yields

$$\int_0^\infty |x_i(s) - \Phi_i(s)| \, ds < +\infty \quad \text{a.s. } i = 1, \ldots, m. \tag{4.13}$$

By virtue of the similar techniques proposed in Step 2 of Theorem 3.2, we have

$$\lim_{t \to \infty} |x_i(t) - \Phi_i(t)| = 0 \quad \text{a.s. } i = 1, \ldots, m. \tag{4.14}$$

When condition (4.2) is satisfied, by applying Corollary 3.1 to system (4.1), we have

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t \Phi_i(s) \, ds \le \frac{1}{a_{ii}} \left(b_i - \frac{\sigma_i^2}{2} \right) \quad \text{a.s. } i = 1, \ldots, m,$$

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t \Phi_i(s) \, ds \ge \frac{1}{a_{ii}} \left[\left(b_i - \frac{\sigma_i^2}{2} \right) - \sum_{k \ne i}^m \frac{a_{ik}}{a_{kk}} \left(b_k - \frac{\sigma_k^2}{2} \right) \right] \quad \text{a.s. } i = 1, \ldots, m.$$

A simple computation shows that

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t x_i(s) \, ds \le \limsup_{t \to \infty} \frac{1}{t} \int_0^t \left(x_i(s) - \Phi_i(s) \right) ds + \limsup_{t \to \infty} \frac{1}{t} \int_0^t \Phi_i(s) \, ds$$

$$\le \frac{1}{a_{ii}} \left(b_i - \frac{\sigma_i^2}{2} \right) \quad \text{a.s. } i = 1, \ldots, m, \tag{4.15}$$

$$\liminf_{t \to \infty} \frac{1}{t} \int_0^t x_i(s) \, ds$$

$$\ge \liminf_{t \to \infty} \frac{1}{t} \int_0^t \left(x_i(s) - \Phi_i(s) \right) ds + \liminf_{t \to \infty} \frac{1}{t} \int_0^t \Phi_i(s) \, ds$$

$$\ge \frac{1}{a_{ii}} \left[\left(b_i - \frac{\sigma_i^2}{2} \right) - \sum_{k \ne i}^m \frac{a_{ik}}{a_{kk}} \left(b_k - \frac{\sigma_k^2}{2} \right) \right] \quad \text{a.s. } i = 1, \ldots, m. \tag{4.16}$$

Therefore, we obtain that $x_i(t)$ is persistent in mean, for all $i = 1, \ldots, m$.

Step 2. For all $i = m + 1, \ldots, n$, applying Itô to $\log x_i(t)$ yields

$$\frac{\log x_i(t)}{t} = \frac{\log x_i(0)}{t} + \frac{1}{t} \int_0^t \left(b_i - \frac{\sigma_i^2}{2} \right) ds - \sum_{j=1}^m \frac{a_{ij}}{t} \int_0^t x_j(s) \, ds$$

$$- \sum_{l=m+1}^n \frac{a_{il}}{t} \int_0^t x_l(s) \, ds + \frac{1}{t} \int_0^t \sigma_i \, dB_i(s), \quad i = m + 1, \ldots, n. \tag{4.17}$$

It follows from (4.15) that

$$\liminf_{t \to \infty} \frac{1}{t} \sum_{j=1}^m a_{ij} \int_0^t x_j(s) \, ds$$

$$\ge \sum_{j=1}^m \frac{a_{ij}}{a_{jj}} \left[\left(b_j - \frac{\sigma_j^2}{2} \right) - \sum_{k \ne j}^m \frac{a_{jk}}{a_{kk}} \left(b_k - \frac{\sigma_k^2}{2} \right) \right] \quad \text{a.s. } j = 1, \ldots, m.$$

By letting $t \to \infty$ on both sides of (4.16) yields. We can conclude that

$$\limsup_{t \to \infty} \frac{\log x_i(t)}{t} \le b_i - \frac{\sigma_i^2}{2} - \sum_{j=1}^m \frac{a_{ij}}{a_{jj}} \left[\left(b_j - \frac{\sigma_j^2}{2} \right) \right.$$

$$\left. - \sum_{k \ne j}^m \frac{a_{jk}}{a_{kk}} \left(b_k - \frac{\sigma_k^2}{2} \right) \right] \quad \text{a.s. } i = m + 1, \ldots, n.$$

Finally, we can get $x_i(t)$ will become extinct for all $i = m + 1, \ldots, n$. The proof is completed. □

5 Numerical simulations

In this paper, we have discussed the persistence in mean and extinction of system (1.2). Moreover, sufficient conditions have been established in Theorems 3.1, 3.2 and 4.1. Thus, in this section, we give out the numerical experiment for the case $n = 2$ as follows to support to our results.
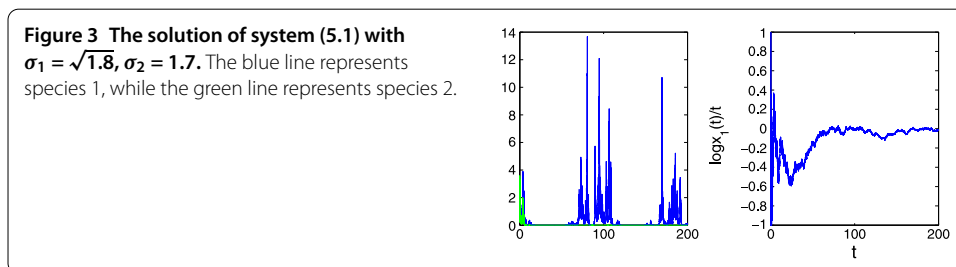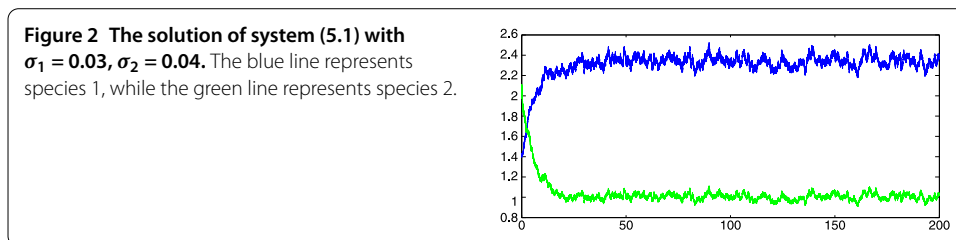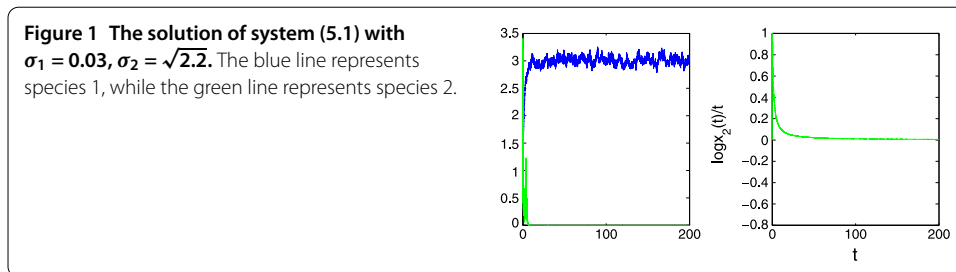
$$\begin{cases} dx_1(t) = x_1(t)[(0.9 - 0.3x_1(t) - 1.2x_2(t))\,dt + \sigma_1\,dB_1(t)], \\ dx_2(t) = x_2(t)[(1.1 - 0.3x_1(t) - 0.4x_2(t))\,dt + \sigma_2\,dB_2(t)]. \end{cases} \tag{5.1}$$

The existence and uniqueness of the solution follows from Lemma 2.1. We consider the solution with initial data $x_1(0) = 1.4$, $x_2(0) = 2.1$. By Matlab software, we simulate the solution to system (5.1) with different values of σ_1 and σ_2.

In Figure 1, $\sigma_1 = 0.03$, $\sigma_2 = \sqrt{2.2}$. By Theorems 3.1, 3.2 and 4.1, species 1 is persistent in mean and species 2 is extinct with zero exponential extinction rate.

In Figure 2, $\sigma_1 = 0.03$, $\sigma_2 = 0.04$. By the conditions of Corollary 3.1, all of the species are persistent in mean.

In Figure 3, $\sigma_1 = \sqrt{1.8}$, $\sigma_2 = 1.7$. By Corollary 3.2, species 1 is extinct with zero exponential extinction rate and species 2 is exponentially extinct.

Figure 1 The solution of system (5.1) with $\sigma_1 = 0.03$, $\sigma_2 = \sqrt{2.2}$. The blue line represents species 1, while the green line represents species 2.

Figure 2 The solution of system (5.1) with $\sigma_1 = 0.03$, $\sigma_2 = 0.04$. The blue line represents species 1, while the green line represents species 2.

Figure 3 The solution of system (5.1) with $\sigma_1 = \sqrt{1.8}$, $\sigma_2 = 1.7$. The blue line represents species 1, while the green line represents species 2.

6 Conclusions

This paper is devoted to partial permanence and extinction on a stochastic Lotka-Volterra competitive model. Firstly, by using some novel techniques, we established some weaker sufficient conditions on the persistence in mean and extinction for one-species. Secondly, based on these sufficient conditions for one-species and some stochastic analysis techniques, sufficient criteria for ensuring the partial permanence and extinction of the populations of the n different species in the ecosystem have been obtained. Finally, numerical experiment is provided to illustrate the effectiveness of our results.

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