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General decay for a wave equation of Kirchhoff type with a boundary control of memory type

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Abstract

A nonlinear wave equation of Kirchhoff type with memory condition at the boundary in a bounded domain is considered. We establish a general decay result which includes the usual exponential and polynomial decay rates. Furthermore, our results allow certain relaxation functions which are not necessarily of exponential and polynomial decay. This improves earlier results in the literature.
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1 Introduction

In this article, we study the asymptotic behavior of the energy function related to a nonlinear wave equation of Kirchhoff type subject to memory condition at the boundary as follows:

$$u_{tt} - M\left(||\nabla u||_2^2\right)\Delta u + l(t)h(\nabla u) - \Delta u_t + a(x)f(u) = 0 \text{ in } \Omega \times (0, \infty), \tag{1.1}$$

$$u = 0 \text{ on } \Gamma_0 \times (0, \infty), \tag{1.2}$$

$$u + \int_0^t g(t-s)\left(M\left(||\nabla u(s)||_2^2\right)\frac{\partial u}{\partial \nu}(s) + \frac{\partial u_t}{\partial \nu}(s)\right)ds = 0 \text{ on } \Gamma_1 \times (0, \infty), \tag{1.3}$$

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } \Omega, \tag{1.4}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. The partition Γ_0 and Γ_1 are closed and disjoint, with $meas(\Gamma_0) > 0$, ν represents the unit normal vector directed towards the exterior of Ω, u is the transverse displacement, and g is the relaxation function considered positive and nonincreasing belonging to $W^{1,2}(\Omega)$.

From the physical point of view, we know that the memory effect described in integral equation (1.3) can be caused by the interaction with another viscoelastic element. In fact, the boundary condition (1.3) signifies that Ω is composed of a material which is clamped in a rigid body in the portion Γ_0 of its boundary and is clamped in a body with viscoelastic properties in the portion of Γ_1.

When $\Gamma_1 = \varphi$, problem (1.1) has its origin in describing the nonlinear vibrations of an elastic string. More precisely, we have

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$$\rho h \frac{\partial^2 u}{\partial t^2} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f \tag{1.5}$$

for $0 < x < L$, $t \geq 0$; where u is the lateral deflection, x the space coordinate, t the time, E the Young modulus, ρ the mass density, h the cross section area, L the length, p_0 the initial axial tension and f the external force. Kirchhoff [1] was the first one who introduced (1.5) to study the oscillations of stretched strings and plates, so that (1.5) is called the wave equation of Kirchhoff type after him. In this direction, problem (1.1) with $\partial\Omega = \Gamma_0$ and $l(t) = 0$ has been investigated by many authors in recent years, and many results concerning existence, nonexistence and asymptotic behavior have been established, see [2-13].

On the other hand, regarding the viscoelastic wave equations with memory term acting in the boundary or in the domain, there are numerous results related to asymptotic behavior of solutions. For example, in the case where $M(s) = 1$, Cavalcanti et al. [14] investigated the existence and uniform decay of strong solutions of wave equation (1.1) with a nonlinear boundary damping of memory type and a nonlinear boundary source when $l(t) = 0$. Cavalcanti and Guesmia [15] considered the following system:

$$u_{tt} - \Delta u + F(x, t, u, \nabla u) = 0 \text{ in } \Omega \times (0, \infty), \tag{1.6}$$

$$u = 0 \text{ on } \Gamma_0 \times (0, \infty), \tag{1.7}$$

$$u + \int_0^t g(t - s) \frac{\partial u}{\partial \nu}(s)ds = 0 \text{ on } \Gamma_1 \times (0, \infty), \tag{1.8}$$

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \text{ in } \Omega, \tag{1.9}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$. They obtained the general decay result which depends on the relaxation function g. In particular, if the relaxation function g decays exponentially (or polynomially), then the solution also decays exponentially (or polynomially) and with the same decay rate. Moreover, when $u_0 = 0$ on Γ_1, they obtained exponential or polynomial decay of solutions, even if the relaxation function g does not converge to 0 at ∞. Later, Messaoudi and Soufyane [16] generalized this result to the case of a system of Timoshenko type. They established general decay rate results, from which the usual exponential and polynomial decay rates are only special cases. Recently, Messaoudi and Soufyane [17] studied the following problem:

$$u_{tt} - \Delta u + f(u) = 0$$

in a bounded domain with boundary conditions (1.7)-(1.9). They improved the results of [15] by applying the multiplier techniques. Indeed, they obtained not only a general decay result, but their works also allowed certain relaxation functions which are not necessarily of exponential or polynomial decay. For other related works, we refer the reader to [18-20] and references therein.

Conversely, in the case where M is not a constant function, Santos [21] considered

$$u_{tt} - \mu(t)u_{xx} = 0, \quad (x, t) \in (0, 1) \times R^+,$$

$$u(0, t) = 0, \quad u(1, t) = - \int_0^t g(t - s)\mu(s)u_x(1, s)ds, \quad \forall t > 0,$$

$$u(0) = u_0, \quad u_t(0) = u_1, \quad x \in (0, 1),$$

where $\mu(t)$ is a nonincreasing function satisfying $\mu(t) \geq \mu_0 > 0$. By denoting k the resolvent kernel of g', he showed that the solution decays exponentially (or polynomially) to zero provided k decays exponentially (or polynomially) to zero. Later on, Santos et al. [22] generalized this result to a nonlinear n-dimensional equation of Kirchhoff type of the form

$$u_{tt} - M\left(||\nabla u||_2^2\right)\Delta u - \Delta u_t + f(u) = 0 \tag{1.10}$$

in a bounded domain with boundary conditions (1.2)-(1.3). In that article, they proved that the energy decays with the same rate of decay of the relaxation function. This latter result improved an earlier one by Park et al. [23], where the authors considered (1.10) in a bounded domain with nonlinear boundary damping and memory term and $M(s) = 1 + s$ and $f = 0$.

We note that stability of problems with the nonlinear term $h(\nabla u)$ requires a careful treatment because we do not have any information about the influence of the integral $\int_\Omega h(\nabla u)u_t dx$ about the sign of the derivative $E'(t)$. Although the subject is important, there are few mathematical results in the presence of the nonlinearity given by $h(\nabla u)$, see [24-26]. In light of this and previous articles [17,22], it is interesting to investigate whether we still have the similar general decay result as in [17] for nondissipative distributed system (1.1) with the memory-type damping acting on a part of the boundary. Hence, the main purpose of this article is to answer the above question for system (1.1)-(1.4). Consequently, by following the arguments close to the one in [17] with necessary modification required the nature of our problem, we establish a general decay result which includes the usual exponential and polynomial decay rates. Furthermore, our results allow a larger class of relax functions which are not necessarily of exponential and polynomial decay. Therefore, this improves earlier results in the literature [22,27].

In order to obtain our results, we consider system (1.1)-(1.4), under some assumptions on $a(x)$, $l(t)$, M and f. Precisely, we state the general assumptions:

(A1) $a(x): \Omega \to R^+$ is a function.

(A2) $f \in C^1(R)$ is a function and satisfies

$$uf(u) \geq \beta F(u) \geq 0 \quad for \quad \beta > 2, \tag{1.11}$$

where $F(u) = \int_0^u f(s)ds$ with

$$F(u) \leq d|u|^p \quad \text{for all } u \in R,$$

$d > 0$ and $1 \leq p \leq \frac{n}{n-2}$.

(A3) M is a C^1 function on $[0, \infty)$ satisfying

$$M(\lambda) \geq m_0 > 0 \quad \text{and} \quad M(\lambda)\lambda \geq \widehat{M}(\lambda) \quad \text{for all} \quad \lambda \geq 0, \tag{1.12}$$

Where $\widehat{M}(\lambda) = \int_0^\lambda M(s)ds$.

(A4) $h : R^n \to R$ is a C^1 function such that ∇h is bounded and there exists $\beta_1 > 0$ such that

$$|h(\xi)| \leq \beta_1|\xi| \quad \text{for all } \xi \in R^n, \tag{1.13}$$

and $l(t)$ is a positive and nonincreasing function.

The remainder of this article is organized as follows. In Section 2, we introduce some notations, present Lemma 2.1 to describe more general relations between the relaxation function g and the corresponding resolvent kernel k and state the existence result to system (1.1)-(1.4). In Section 3, we give the proof of our main result Theorem 3.5.

2 Preliminaries

In this section, we introduce some notations and establish the existence of solutions of the problem (1.1)-(1.4). In what follows, let $||\cdot||_p$ denote the usual $L^p(\Omega)$ norm $||\cdot||_{L^p(\Omega)}$, for $1 \le p \le \infty$. We define the convolution product operator by

$$(g * u)(t) = \int_0^t g(t-s)u(s)ds, \tag{2.1}$$

and set

$$(g \circ \phi)(t) = \int_0^t g(t-s)||\phi(t) - \phi(s)||_2^2 ds, \tag{2.2}$$

$$(g \diamond \phi)(t) = \int_0^t g(t-s)(\phi(t) - \phi(s))ds. \tag{2.3}$$

Using Hölder's inequality, we observe that

$$|g \diamond \phi(t)|^2 \le \int_0^t |g(s)|ds(|g| \circ \phi)(t). \tag{2.4}$$

Next, we shall use Equation 1.3 to estimate the boundary term $M\left(||\nabla u(s)||_2^2\right)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu}$. Differentiating (1.3), we obtain

$$M\left(||\nabla u(t)||_2^2\right)\frac{\partial u}{\partial \nu}(t) + \frac{\partial u_t}{\partial \nu}(t) + \frac{1}{g(0)}\int_0^t g'(t-s)\left(M\left(||\nabla u(s)||_2^2\right)\frac{\partial u}{\partial \nu}(s) + \frac{\partial u_t}{\partial \nu}(s)\right)ds$$
$$= -\frac{1}{g(0)}u_t.$$

Assume the function k is the resolvent kernel of the relaxation function g, then

$$k + \frac{1}{g(0)}k * g' = -\frac{1}{g(0)}g'.$$

Applying Volterra's inverse operator yields

$$M\left(||\nabla u(t)||_2^2\right)\frac{\partial u}{\partial \nu}(t) + \frac{\partial u_t}{\partial \nu}(t) = -\frac{1}{g(0)}(u_t + k * u_t),$$

which implies that

$$M\left(||\nabla u(t)||_2^2\right)\frac{\partial u}{\partial \nu}(t) + \frac{\partial u_t}{\partial \nu}(t)$$
$$= -\tau\{u_t + k(0)u - k(t)u_0 + k' * u\} \quad \text{on } \Gamma_1 \times (0, \infty), \tag{2.5}$$

where $\tau = \frac{1}{g(0)}$. Reciprocally, taking $u_0 = 0$ on Γ_1, identity (2.5) implies (1.3). As we are interested in relaxation functions of more general decay and the resolvent k

appeared in Equation 2.5, we want to know if the resolvent k has the same property with the relaxation function g involved in (1.3). The following lemma answers this question. Let h be a relaxation function and k its resolvent kernel, that is

$$k(t) = h(t) + (k * h)(t).$$

Lemma 2.1. [15,17,22]*If $h : [0, \infty) \rightarrow R^+$ is continuous, then k is also a positive continuous function. Moreover,*

(1) If there exists a positive constant c_0 such that

$$h(t) \leq c_0 e^{-\int_0^t \gamma(s)ds},$$

where $\gamma : [0, \infty) \rightarrow R^+$, is a nonincreasing function satisfying, for some positive constant $\varepsilon < 1$,

$$c_1 = \int_0^\infty e^{-\int_0^t (1-\epsilon)\gamma(s)ds} dt < \frac{1}{c_0}.$$

Then, k satisfies

$$k(t) \leq \frac{c_0}{1 - c_0 c_1} e^{-\int_0^t \varepsilon \gamma(s)ds}.$$

(2) Suppose that

$$h(t) \leq \frac{c_0}{(1 + t)^p}$$

for $c_0 < p - 1$. Then, there exists a positive constant $\varepsilon < 1$ such that

$$k(t) \leq \frac{\beta}{(1 + t)^{\varepsilon p}},$$

where $\beta > 0$ is a constant.

Based on this lemma, we will use (2.5) instead of (1.3), i.e., we can consider system (1.1)-(1.4) as follows:

$$u_{tt} - M\left(\|\nabla u\|_2^2\right) \Delta u + l(t)h(\nabla u) - \Delta u_t + a(x)f(u) = 0 \text{ in } \Omega \times (0, \infty),$$

$$u = 0 \text{ on } \Gamma_0 \times (0, \infty),$$

$$M\left(\|\nabla u(t)\|_2^2\right) \frac{\partial u}{\partial \nu}(t) + \frac{\partial u_t}{\partial \nu}(t) = -\tau \{u_t + k(0)u - k(t)u_0 + k' * u\} \text{ on } \Gamma_1 \times (0, \infty),$$

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x) \text{ in } \Omega.$$

We notice that, due to the condition (1.2), the solution of system (1.1)-(1.4) must belong to the following space:

$$V = \{v \in H^1(\Omega); \quad v = 0 \text{ on } \Gamma_0\},$$

which endowed with the norm $\|\nabla \cdot\|_2$ is a Hilbert space. Now, we are ready to give the well-posedness of system (1.1)-(1.4).

Theorem 2.2. *Let $k \in W^{2,1}(R^+) \cap W^{1,\infty}(R^+)$, $(u_0, u_1) \in (H^2(\Omega) \cap V)^2$ and satisfy the compatibility condition*

$$M\left(\|\nabla u_0\|_2^2\right) \frac{\partial u_0}{\partial \nu} + \frac{\partial u_1}{\partial \nu} + \tau u_1 = 0 \text{ on } \Gamma_1.$$

Assume further that (A1)-(A4) *hold. Then, there exists a unique solution u of system* (1.1)-(1.4) *such that*

$$u \in L^{\infty}(0, \infty; H^2(\Omega) \cap V), \quad u_t \in L^{\infty}(0, \infty; V),$$
$$u_{tt} \in L^{\infty}(0, \infty; L^2(\Omega)).$$

Proof. Using the Galerkin method and procedures similar to that of [22,28], we can obtain the result. □

3 Decay of solutions

In this section, we study the asymptotic behavior of the solutions of system (1.1)-(1.4) when the resolvent kernel k satisfies

$$k(0) > 0, \quad k(t) \geq 0, \quad k'(t) \leq 0, \quad k''(t) \geq -\gamma(t)k'(t), \tag{3.1}$$

where $\gamma : [0, \infty) \rightarrow R^+$ is a function satisfying the following condition:

$$\gamma(t) > 0, \quad \gamma'(t) \leq 0 \quad \text{and} \quad \int_0^\infty \gamma(s)ds = \infty. \tag{3.2}$$

To get our result, we further assume that

$$0 < l(t) \leq \gamma(t) \quad \text{for all } t \geq 0. \tag{3.3}$$

Let x_0 be a fixed point in R^n. Set

$$m = m(x) = x - x_0, \quad R(x_0) = \max\{||m(x)||_2; x \in \bar{\Omega}\}$$

and partition the boundary $\partial\Omega$ into two sets

$$\Gamma_0 = \{x \in \partial\Omega; \ m(x) \cdot \nu \leq 0\}, \quad \Gamma_1 = \{x \in \partial\Omega; \ m(x) \cdot \nu > 0\}. \tag{3.4}$$

Define the first-order energy function of system (1.1)-(1.4) by

$$\begin{aligned} E(t) \quad &= \frac{1}{2}\left(||u_t||_2^2 + \widehat{M}\left(||\nabla u||_2^2\right)\right) + \int_\Omega a(x)F(u)dx \\ &+ \frac{\tau}{2}k(t)\int_{\Gamma_1}|u|^2 d\Gamma - \frac{\tau}{2}\int_{\Gamma_1}k' \circ u d\Gamma. \end{aligned} \tag{3.5}$$

The following lemma is associated with the property of the convolution operator, which is used to estimate the energy identity.

Lemma 3.1. *If g, $\varphi \in C^1(R^+)$, then*

$$\begin{aligned} (g * \phi)\phi_t \quad &= \quad -\frac{1}{2}g(t)|\phi(t)|^2 + \frac{1}{2}g' \circ \phi \\ &\quad -\frac{1}{2}\frac{d}{dt}\left(g \circ \phi - \left(\int_0^t g(s)ds\right)|\phi(t)|^2\right). \end{aligned} \tag{3.6}$$

Proof. Our conclusion is followed by differentiating the term $g \circ \varphi$. □

Lemma 3.2. *Under the assumptions of* (A1)-(A4), *the energy function E(t) satisfies*

$$\begin{aligned} \frac{d}{dt}E(t) \leq \ &\frac{-\tau}{2}\int_{\Gamma_1}|u_t|^2 d\Gamma + \frac{\tau}{2}k^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma + \frac{\tau}{2}k'(t)\int_{\Gamma_1}|u|^2 d\Gamma \\ &-\frac{\tau}{2}\int_{\Gamma_1}k'' \circ u d\Gamma - \int_\Omega |\nabla u_t|^2 dx - \int_\Omega l(t)h(\nabla u)u_t dx. \end{aligned} \tag{3.7}$$

Proof. Multiplying Equation 1.1 by u_t, and integrating by parts over Ω, we get

$$\frac{d}{dt}\left[\frac{1}{2}\left(||u_t||_2^2 + \widehat{M}\left(||\nabla u||_2^2\right)\right) + \int_\Omega a(x)F(u)dx\right]$$

$$= \int_{\Gamma_1}\left(M\left(||\nabla u||_2^2\right)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu}\right)u_t d\Gamma - \int_\Omega |\nabla u_t|^2 dx - \int_\Omega \Delta l(t)h(\nabla u)u_t dx.$$

Exploiting (2.5), (3.6) and the definition of $E(t)$ by (3.5), we have

$$\frac{d}{dt}E(t) \leq -\tau \int_{\Gamma_1}|u_t|^2 d\Gamma + \tau \int_{\Gamma_1}k(t)u_0 u_t d\Gamma + \frac{\tau}{2}k'(t)\int_{\Gamma_1}|u|^2 d\Gamma$$

$$-\frac{\tau}{2}\int_{\Gamma_1}k'' \circ u d\Gamma - \int_\Omega |\nabla u_t|^2 dx - \int_\Omega l(t)h(\nabla u)u_t dx.$$

Then, using Hölder's inequality and Young's inequality, the inequality (3.7) is obtained. □

Next, we construct a Lyapunov functional which is equivalent to $E(t)$. To do so, for $N > 0$ large enough, let

$$L(t) = NE(t) + \psi(t), \tag{3.8}$$

where

$$\psi(t) = \int_\Omega \left(m \cdot \nabla u(t) + \left(\frac{n}{2} - \theta\right)u\right)u_t dx \tag{3.9}$$

for $0 < \theta < 1$.

For the purpose of achieving our main result, we need the following lemmas.

Lemma 3.3. *There exist two positive constants α_1 and α_2 such that the relation*

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t)$$

holds for all $t \geq 0$.

Proof. From (3.9) and using Young's inequality, we get

$$|\psi(t)| \leq \left(R(x^0) + B_1\left(\frac{n}{2} - \theta\right)\right)||u_t||_2||\nabla u||_2$$

$$\leq \frac{2}{\sqrt{m_0}}\left(R(x^0) + B_1\left(\frac{n}{2} - \theta\right)\right)E(t),$$

where we have used the fact that $||u_t||_2^2 \leq 2E(t)$ by (3.5) and

$$||\nabla u||_2^2 \leq \frac{2}{m_0}E(t) \tag{3.10}$$

due to $\hat{M}(\lambda) \geq m_0\lambda > 0$ by (A3) and (3.5). Here $B_1 > 0$ is the smallest constant such that

$$||u||_2 \leq B_1||\nabla u||_2, \quad \forall u \in V. \tag{3.11}$$

Thus, from (3.8), we deduce that

$$|L(t) - NE(t)| = |\psi(t)| \leq \frac{2}{\sqrt{m_0}}\left(R(x^0) + B_1\left(\frac{n}{2} - \theta\right)\right)E(t).$$

Hence, selecting

$$N > \frac{2}{\sqrt{m_0}} \left(R(x^0) + B_1 \left(\frac{n}{2} - \theta \right) \right),$$ (3.12)

there exist two positive constants α_1 and α_2 such that the relation

$$\alpha_1 E(t) \leq L(t) \leq \alpha_2 E(t)$$

holds. □

Lemma 3.4. *Let* (A1)-(A4) *and* (3.1)-(3.3) *hold, with* β_1 *(given by* (A4)) *small enough and*

$$\lim_{t \to \infty} k(t) = 0.$$ (3.13)

Then, for some t_0 *large enough, the functional* $L(t)$ *verifies, along the solution u of* (1.1)-(1.4),

$$\begin{aligned}
L'(t) \quad \leq \quad &- \frac{\alpha}{2} E(t) + c_4 k^2(t) \int_{\Gamma_1} |u_0|^2 d\Gamma - c_5 \int_{\Gamma_1} k' \circ u d\Gamma \\
&+ \int_{\Omega} \left[\left(n + \alpha - \left(\frac{n}{2} - \theta \right) \beta \right) a(x) + m \cdot \nabla a \right] F(u) dx
\end{aligned}$$ (3.14)

for all $t \geq t_0$, *where* $\alpha = \min \{ 2\theta, 1 - \theta \}$ *and* c_i *are positive constants given in the proof,* $i = 4, 5$.

Proof. First, we are going to estimate the derivative of $\psi(t)$. From (3.9) and using Equation 1.1, we have

$$\begin{aligned}
\frac{d}{dt} \psi(t) \quad = \quad &\frac{1}{2} \int_{\Gamma_1} (m \cdot v) |u_t|^2 d\Gamma - \theta \int_{\Omega} |u_t|^2 dx \\
&+ \int_{\Omega} \left(m \cdot \nabla u(t) + \left(\frac{n}{2} - \theta \right) u \right) M \left(||\nabla u||_2^2 \right) \Delta u dx \\
&+ \int_{\Omega} \left(m \cdot \nabla u(t) + \left(\frac{n}{2} - \theta \right) u \right) \Delta u_t dx \\
&- \int_{\Omega} l(t) h(\nabla u) \left(m \cdot \nabla u(t) + \left(\frac{n}{2} - \theta \right) u \right) dx \\
&- \int_{\Omega} \left(m \cdot \nabla u(t) + \left(\frac{n}{2} - \theta \right) u \right) a(x) f(u) dx.
\end{aligned}$$

Performing integration by parts and using Young's inequality, we obtain

$$\begin{aligned}
\frac{d}{dt} \psi(t) \quad \leq \quad &\frac{1}{2} \int_{\Gamma_1} (m \cdot v) |u_t|^2 d\Gamma - \theta \int_{\Omega} |u_t|^2 dx \\
&+ \int_{\Gamma_1} \left(M \left(||\nabla u||_2^2 \right) \frac{\partial u}{\partial v} + \frac{\partial u_t}{\partial v} \right) \left(m \cdot \nabla u(t) + \left(\frac{n}{2} - \theta \right) u \right) d\Gamma \\
&- \frac{M \left(||\nabla u||_2^2 \right)}{2} \int_{\Gamma_1} (m \cdot v) |\nabla u|^2 d\Gamma - (1 - \theta) M(||\nabla u||_2^2) ||\nabla u||_2^2 \\
&+ \varepsilon c_0 M \left(||\nabla u||_2^2 \right) ||\nabla u||_2^2 + C_\varepsilon \int_{\Omega} |\nabla u_t|^2 dx \\
&- \int_{\Omega} l(t) h(\nabla u) \left(m \cdot \nabla u(t) + \left(\frac{n}{2} - \theta \right) u \right) dx \\
&- \int_{\Omega} \left(m \cdot \nabla u(t) + \left(\frac{n}{2} - \theta \right) u \right) a(x) f(u) dx,
\end{aligned}$$ (3.15)

where $\varepsilon > 0$, c_ε and c_0 are some positive constants. In the following, we will estimate the last two terms on the right-hand side of (3.15). It follows from (1.13), Hölder's inequality, (3.11), (3.3) and (3.10) that

$$
\begin{aligned}
\int_\Omega l(t) h(\nabla u) & \left(m \cdot \nabla u(t) + \left(\frac{n}{2} - \theta \right) u \right) dx \\
& \le \gamma(0) \beta_1 \left(R(x^0) + B_1 \left(\frac{n}{2} - \theta \right) \right) ||\nabla u||_2^2 \\
& \le \frac{2\gamma(0)\beta_1 c_1}{m_0} E(t),
\end{aligned}
\tag{3.16}
$$

where $c_1 = R(x^0) + B_1(\frac{n}{2} - \theta)$. Taking (1.11) and (3.4) into account, we have

$$
\begin{aligned}
- \int_\Omega & \left(m \cdot \nabla u(t) + \left(\frac{n}{2} - \theta \right) u \right) a(x) f(u) dx \\
& = - \int_\Omega a(x) m \cdot \nabla F(u) dx - \left(\frac{n}{2} - \theta \right) \int_\Omega a(x) u f(u) dx \\
& \le \int_\Omega (n a(x) + m \cdot \nabla a) F(u) dx - \int_{\Gamma_1} a(x)(m \cdot v) F(u) d\Gamma \\
& \quad - \left(\frac{n}{2} - \theta \right) \beta \int_\Omega a(x) F(u) dx \\
& \le \int_\Omega \left[\left(n - \left(\frac{n}{2} - \theta \right) \beta \right) a(x) + m \cdot \nabla a \right] F(u) dx.
\end{aligned}
\tag{3.17}
$$

A substitution of (3.16)-(3.17) into (3.15), we obtain

$$
\begin{aligned}
\frac{d}{dt} \psi(t) \le & \frac{1}{2} \int_{\Gamma_1} (m \cdot v) |u_t|^2 d\Gamma - \theta ||u_t||_2^2 - (1 - \theta - \varepsilon c_0) M \left(||\nabla u||_2^2 \right) ||\nabla u||_2^2 \\
& + \int_{\Gamma_1} \left(M \left(||\nabla u||_2^2 \right) \frac{\partial u}{\partial v} + \frac{\partial u_t}{\partial v} \right) \left(m \cdot \nabla u(t) + \left(\frac{n}{2} - \theta \right) u \right) d\Gamma \\
& + C_\varepsilon \int_\Omega |\nabla u_t|^2 dx - \frac{M \left(||\nabla u||_2^2 \right)}{2} \int_{\Gamma_1} (m \cdot v) |\nabla u|^2 d\Gamma + \frac{2\gamma(0)\beta_1 c_1}{m_0} E(t) \\
& + \int_\Omega \left[(n - \left(\frac{n}{2} - \theta \right) \beta) a(x) + m \cdot \nabla a \right] F(u) dx.
\end{aligned}
\tag{3.18}
$$

Now, we analyze the boundary term on the right-hand side of (3.18). Applying Young's inequality and $M(\lambda) \ge m_0 > 0$ by (1.12), we have, for $\varepsilon_1 > 0$,

$$
\begin{aligned}
\int_{\Gamma_1} & \left(M \left(||\nabla u||_2^2 \right) \frac{\partial u}{\partial v} + \frac{\partial u_t}{\partial v} \right) (m \cdot \nabla u(t) + \left(\frac{n}{2} - \theta \right) u) d\Gamma \\
& \le \varepsilon_1 \int_{\Gamma_1} \left[|m \cdot \nabla u(t)|^2 + \left(\frac{n}{2} - \theta \right)^2 |u|^2 \right] d\Gamma + C_{\varepsilon_1} \int_{\Gamma_1} \left| M \left(||\nabla u||_2^2 \right) \frac{\partial u}{\partial v} + \frac{\partial u_t}{\partial v} \right|^2 d\Gamma \\
& \le \varepsilon_1 \int_{\Gamma_1} (m \cdot v) |\nabla u|^2 d\Gamma + \left(\frac{n}{2} - \theta \right)^2 B_* \varepsilon_1 ||\nabla u||_2^2 + C_{\varepsilon_1} \int_{\Gamma_1} \left| M \left(||\nabla u||_2^2 \right) \frac{\partial u}{\partial v} + \frac{\partial u_t}{\partial v} \right|^2 d\Gamma \\
& \le \varepsilon_1 \int_{\Gamma_1} (m \cdot v) |\nabla u|^2 d\Gamma + \left(\frac{n}{2} - \theta \right)^2 \frac{B_* \varepsilon_1}{m_0} M \left(||\nabla u||_2^2 \right) ||\nabla u||_2^2 \\
& \quad + C_{\varepsilon_1} \int_{\Gamma_1} \left| M \left(||\nabla u||_2^2 \right) \frac{\partial u}{\partial v} + \frac{\partial u_t}{\partial v} \right|^2 d\Gamma,
\end{aligned}
$$

where C_{ε_1} is a positive constant and $B_* > 0$ is the constant such that

$$
\int_{\Gamma_1} |u|^2 d\Gamma \le B_* ||\nabla u||_2^2, \quad \forall u \in V.
\tag{3.19}
$$

Thus, (3.18) becomes

$$
\begin{aligned}
\frac{d}{dt}\psi(t) \leq {} & \frac{1}{2}\int_{\Gamma_1}(m\cdot v)|u_t|^2 d\Gamma - \theta\|u_t\|_2^2 \\
& - \left(1 - \theta - \varepsilon c_0 - \left(\frac{n}{2}-\theta\right)^2\frac{B_*\varepsilon_1}{m_0}\right)M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 \\
& - \left(\frac{M(\|\nabla u\|_2^2)}{2}-\varepsilon_1\right)\int_{\Gamma_1}(m\cdot v)|\nabla u|^2 d\Gamma + C_\varepsilon\int_{\Omega}|\nabla u_t|^2 dx \\
& + C_\varepsilon\int_{\Gamma_1}\left|M\left(\|\nabla u\|_2^2\right)\frac{\partial u}{\partial v}+\frac{\partial u_t}{\partial v}\right|^2 d\Gamma + \frac{2\gamma(0)\beta_1 c_1}{m_0}E(t) \\
& + \int_{\Omega}\left[\left(n-\left(\frac{n}{2}-\theta\right)\beta\right)a(x)+m\cdot\nabla a\right]F(u)dx.
\end{aligned}
\tag{3.20}
$$

By rewriting the boundary condition (2.5) as

$$
M\left(\|\nabla u\|_2^2\right)\frac{\partial u}{\partial v}+\frac{\partial u_t}{\partial v} = -\tau\{u_t + k(t)u(t) - k(t)u0 - k'\diamond u\},
$$

and, then, combining (3.7) and (3.20), we deduce that

$$
\begin{aligned}
L'(t) = {} & NE'(t) + \psi'(t) \\
\leq {} & -\left(\frac{N\tau}{2}-\frac{(m\cdot v)}{2}-8\tau^2 c_{\varepsilon_1}\right)\int_{\Gamma_1}|u_t|^2 d\Gamma - (N-c_\varepsilon)\|\nabla u_t\|_2^2 \\
& -\theta\|u_t\|_2^2 - \left(1 - \theta - \varepsilon c_0 - \left(\frac{n}{2}-\theta\right)^2\frac{B_*\varepsilon_1}{m_0}\right)M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 \\
& +8\tau^2 c_{\varepsilon_1}k^2(t)\int_{\Gamma_1}|u|^2 d\Gamma + \left(\frac{N\tau}{2}+8\tau^2 c_{\varepsilon_1}\right)k^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma \\
& -\frac{N\tau}{2}\int_{\Gamma_1}k''\circ u\, d\Gamma - \left(\frac{M\left(\|\nabla u\|_2^2\right)}{2}-\varepsilon_1\right)\int_{\Gamma_1}(m\cdot v)|\nabla u|^2 d\Gamma \\
& + 8\tau^2 c_{\varepsilon_1}\int_{\Gamma_1}|k'\diamond u|^2 d\Gamma + \frac{2\gamma(0)\beta_1 c_1}{m_0}E(t) - Nl(t)\int_{\Omega}h(\nabla u)u_t dx \\
& + \int_{\Omega}\left[\left(n-\left(\frac{n}{2}-\theta\right)\beta\right)a(x)+m\cdot\nabla a\right]F(u)dx.
\end{aligned}
$$

Similarly as in deriving (3.16), we note that

$$
\begin{aligned}
l(t)\int_{\Omega}h(\nabla u)u_t dx &\leq \gamma(t)\beta_1\left(\frac{1}{2}\|u_t\|_2^2+\frac{1}{2}\|\nabla u\|_2^2\right) \\
&\leq \gamma(t)\beta_1 c_3 E(t) \leq \gamma(0)\beta_1 c_3 E(t),
\end{aligned}
\tag{3.21}
$$

where $c_3 = 1 + \frac{1}{m_0}$. This implies that

$$
\begin{aligned}
L'(t) \leq {} & -\left(\frac{N\tau}{2}-\frac{(m\cdot v)}{2}-8\tau^2 c_{\varepsilon_1}\right)\int_{\Gamma_1}|u_t|^2 d\Gamma \\
& -\theta\|u_t\|_2^2 - (N-c_\varepsilon)\|\nabla u_t\|_2^2 \\
& - \left(1 - \theta - \varepsilon c_0 - \left(\frac{n}{2}-\theta\right)^2\frac{B_*\varepsilon_1}{m_0}\right)M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 \\
& +8\tau^2 c_{\varepsilon_1}k^2(t)\int_{\Gamma_1}|u|^2 d\Gamma + \left(\frac{N\tau}{2}+8\tau^2 c_{\varepsilon_1}\right)k^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma \\
& -\frac{N\tau}{2}\int_{\Gamma_1}k''\circ u\, d\Gamma - \left(\frac{M\left(\|\nabla u\|_2^2\right)}{2}-\varepsilon_1\right)\int_{\Gamma_1}(m\cdot v)|\nabla u|^2 d\Gamma \\
& +8\tau^2 c_{\varepsilon_1}\int_{\Gamma_1}|k'\diamond u|^2 d\Gamma + \beta_1\gamma(0)\left(Nc_3+\frac{2c_1}{m_0}\right)E(t) \\
& + \int_{\Omega}\left[\left(n-\left(\frac{n}{2}-\theta\right)\beta\right)a(x)+m\cdot\nabla a\right]F(u)dx.
\end{aligned}
$$

At this point, we choose

$$\varepsilon = \varepsilon_1 < \min\left\{\frac{m_0}{4}, \frac{(1-\theta)}{2\left(c_0 + \left(\frac{n}{2} - \theta\right)^2 B_*\right)}\right\}.$$

Once $\varepsilon = \varepsilon_1$ is fixed (hence c_ε and c_{ε_1} are also fixed), we pick N large satisfying (3.12) and

$$N > \max\left\{\frac{\max_{\Gamma_1}|m \cdot \nu| + 16\tau^2 c_{\varepsilon_1}}{\tau}, c_\varepsilon\right\} \tag{3.22}$$

at the same time. Then, from the properties of $k(t)$ by (3.1) and noting that $|g \diamond \phi(t)|^2 \le \int_0^t |g(s)| ds (|g| \circ \phi)(t)$ by (2.4), we see that

$$\begin{aligned}
L'(t) \le\ & -\theta\|u_t\|_2^2 - \frac{1-\theta}{2}M\left(\|\nabla u\|_2^2\right)\|\nabla u\|_2^2 \\
& - k(0)\int_{\Gamma_1} k' \circ u d\Gamma + c_4 k^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma \\
& + 8\tau^2 c_{\varepsilon_1} k^2(t)\int_{\Gamma_1}|u|^2 d\Gamma + \beta_1 \gamma(0)\left(Nc_3 + \frac{2c_1}{m_0}\right)E(t) \\
& + \int_\Omega\left[\left(n - \left(\frac{n}{2} - \theta\right)\beta\right)a(x) + m \cdot \nabla a\right]F(u)dx.
\end{aligned}$$

Utilizing the inequality $M(\lambda)\lambda \ge \widehat{M}(\lambda)$ by (1.12) and the definition of $E(t)$ by (3.5), we obtain

$$\begin{aligned}
L'(t) \le\ & -\left(\alpha - \beta_1 \gamma(0)\left(Nc_3 + \frac{2c_1}{m_0}\right)\right)E(t) + \left(\frac{\tau\alpha}{2}k(t) + 8\tau^2 c_{\varepsilon_1} k^2(t)\right)\int_{\Gamma_1}|u|^2 d\Gamma \\
& - \left(\frac{\tau\alpha}{2} + k(0)\right)\int_{\Gamma_1} k' \circ u d\Gamma + c_4 k^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma \\
& + \int_\Omega\left[\left(n + \alpha - \left(\frac{n}{2} - \theta\right)\beta\right)a(x) + m \cdot \nabla a\right]F(u)dx,
\end{aligned}$$

which together with (3.19) and (3.10) infers that

$$\begin{aligned}
L'(t) \le\ & -\left(\alpha - \beta_1 \gamma(0)\left(Nc_3 + \frac{2c_1}{m_0}\right)\right)E(t) + \frac{2B_*}{m_0}\left(\frac{\tau\alpha}{2}k(t) + 8\tau^2 c_{\varepsilon_1} k^2(t)\right)E(t) \\
& - \left(\frac{\tau\alpha}{2} + k(0)\right)\int_{\Gamma_1} k' \circ u d\Gamma + c_4 k^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma \\
& + \int_\Omega\left[\left(n + \alpha - \left(\frac{n}{2} - \theta\right)\beta\right)a(x) + m \cdot \nabla a\right]F(u)dx,
\end{aligned}$$

where $\alpha = \min\{2\theta, 1 - \theta\}$. Besides, we note that there exists t_0 large enough satisfying

$$k(t) \le \frac{m_0}{2B_*}\min\left\{\sqrt{\frac{\alpha}{64\tau^2 c_{\varepsilon_2}}}, \frac{1}{4\tau}\right\} \quad \text{for } t \ge t_0, \tag{3.23}$$

because of $\lim_{t\to\infty} k(t) = 0$ by (3.13). Therefore, taking β_1 small enough such that

$$0 < \beta_1 < \frac{\alpha}{4\gamma(0)\left(Nc_3 + \frac{2c_1}{m_0}\right)}, \tag{3.24}$$

then,

$$
\begin{aligned}
L'(t) \leq\ & -\frac{\alpha}{2}E(t) + c_4 k^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma - c_5\int_{\Gamma_1} k' \circ u d\Gamma \\
& + \int_\Omega\left[\left(n+\alpha-\left(\frac{n}{2}-\theta\right)\beta\right)a(x) + m\cdot\nabla a\right]F(u)dx
\end{aligned}
\tag{3.25}
$$

for all $t \geq t_0$, where c_i are positive constants, $i = 4, 5$. This completes the proof. □

Theorem 3.5. *Given that $(u_0, u_1) \in (H^2(\Omega) \cap V)^2$, assume that (A1)-(A4), (3.1)-(3.3) and (3.13)hold, with β_1 (given by (A4)) small enough. Assume further that*

$$
\left(n+\alpha-\left(\frac{n}{2}-\theta\right)\beta\right)a(x) + m\cdot\nabla a < 0, \quad \forall x\in\Omega.
\tag{3.26}
$$

Then, for some t_0 large enough, we have, $\forall t \geq t_0$,

$$
E(t) \leq cE(t_0)e^{-a_1\int_0^t \gamma(s)ds} \quad \text{if } u_0 = 0 \text{ on } \Gamma_1,
\tag{3.27}
$$

otherwise (if $u_0 \neq 0$ on Γ_1),

$$
E(t) \leq c\left\{E(t_0) + \left(\int_{\Gamma_1}|u_0|^2 d\Gamma\right)\int_{t_0}^t k^2(s)e^{a_1\int_{t_0}^s \gamma(\zeta)d\zeta}ds\right\}e^{-a_1\int_0^t \gamma(s)ds},
\tag{3.28}
$$

where a_1 is a fixed positive constant and c is a generic positive constant.

Proof. Multiplying (3.25) by $\gamma(t)$ and exploiting (3.26), (3.1) and (3.7), we derive that

$$
\begin{aligned}
\gamma(t)L'(t) \leq\ & -\frac{\alpha}{2}\gamma(t)E(t) + c_4 k^2(t)\gamma(t)\int_{\Gamma_1}|u_0|^2 d\Gamma - c_5\gamma(t)\int_{\Gamma_1} k'\circ u d\Gamma \\
\leq\ & -\frac{\alpha}{2}\gamma(t)E(t) + c_4 k^2(t)\gamma(t)\int_{\Gamma_1}|u_0|^2 d\Gamma + c_5\int_{\Gamma_1} k''\circ u d\Gamma \\
\leq\ & -\frac{\alpha}{2}\gamma(t)E(t) + c_6 k^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma - c_7 E'(t) \\
& - c_7 l(t)\int_\Omega h(\nabla u)u_t dx,
\end{aligned}
\tag{3.29}
$$

where $c_6 = c_4\gamma(0) + c_5$ and $c_7 = \frac{2c_5}{\tau}$. Employing (3.21) again, (3.29) becomes

$$
F_1'(t) - \gamma'(t)L(t) \leq -\gamma(t)\left(\frac{\alpha}{2} - \beta_1 c_7 c_3\right)E(t) + c_6 k^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma,
$$

where

$$
F_1(t) = \gamma(t)L(t) + c_7 E(t),
$$

which is equivalent to $E(t)$ due to Lemma 3.3 and $\gamma(t)$ is nonincreasing by (3.2). In addition to (3.24), we further require

$$
0 < \beta_1 < \frac{\alpha}{8c_7 c_3},
$$

then, we have

$$
F_1'(t) \leq -a_1\gamma(t)F_1(t) + c_6 k^2(t)\int_{\Gamma_1}|u_0|^2 d\Gamma, \forall t \geq t_0,
\tag{3.30}
$$

where a_1 is a positive constant.

Case I: If $u_0 = 0$ on Γ_1, then (3.30) reduces to

$$F_1'(t) \leq -a_1 \gamma(t) F_1(t), \quad \forall t \geq t_0.$$

Integrating the above inequality over (t_0, t) to get

$$F_1(t) \leq F_1(t_0) e^{-a_1 \int_{t_0}^{t} \gamma(s) ds}, \quad \forall t \geq t_0.$$

Then, using the fact $F_1(t)$ is equivalent to $E(t)$, we obtain, for some positive constant c,

$$\begin{aligned}
E(t) &\leq cE(t_0) e^{-a_1 \int_{t_0}^{t} \gamma(s) ds} \\
&= cE(t_0) e^{a_1 \int_{0}^{t_0} \gamma(s) ds} e^{-a_1 \int_{0}^{t} \gamma(s) ds}, \quad \forall t \geq t_0.
\end{aligned}$$

Thus, (3.27) is proved.

Case II: If $u_0 \neq 0$ on Γ_1, then (3.30) gives

$$F_1'(t) \leq -a_1 \gamma(t) F_1(t) + c_8 k^2(t), \quad \forall t \geq t_0,$$

where $c_8 = c_6 \int_{\Gamma_1} |u_0|^2 d\Gamma$. Direct computations give

$$\left(e^{a_1 \int_{t_0}^{t} \gamma(s) ds} F_1(t) \right)' \leq c_8 k^2(t) e^{a_1 \int_{t_0}^{t} \gamma(s) ds}.$$

An integration over (t_0, t) yields

$$F_1(t) \leq \left(F_1(t_0) + c_8 \int_{t_0}^{t} k^2(s) e^{a_1 \int_{t_0}^{s} \gamma(\zeta) d\zeta} ds \right) e^{-a_1 \int_{t_0}^{t} \gamma(s) ds}, \quad \forall t \geq t_0.$$

Again using the fact $F_1(t)$ is equivalent to $E(t)$, we obtain, for some positive constant c,

$$E(t) \leq c \left\{ E(t_0) + \left(\int_{\Gamma_1} |u_0|^2 d\Gamma \right) \int_{t_0}^{t} k^2(s) e^{a_1 \int_{t_0}^{s} \gamma(\zeta) d\zeta} ds \right\} e^{a_1 \int_{0}^{t_0} \gamma(s) ds} e^{-a_1 \int_{0}^{t} \gamma(s) ds}, \quad \forall t \geq t_0.$$

This completes the proof of Theorem 3.5. □

4 Conclusion and suggestions

Santos et al. [22] considered problem (1.1)-(1.4) with $a = 1$ and without a function of the gradient term. They showed the solution decays exponentially (or polynomially) to zero provided the kernel decays exponentially (or polynomially) to zero. Recently, Messaoudi and Soufyane in 2010 [17] considered a semi-linear wave equation, in a bounded domain, where the memory-type damping is acting on the boundary. They established a general decay result, from which the usual exponential and polynomial decay rate are only special cases. Motivated by this, we intended to investigate the decay properties of problem (1.1)-(1.4) using the work of Messaaoudi and Soufyane [17]. Since stability of problems with the nonlinear term $h(\nabla u)$ requires a careful treatment, it is interesting to investigate whether we still have the similar general decay result as that of [16] in the presence of a function of the gradient term. This is our motivation to consider problem (1.1)-(1.4). And, this problem is not considered before.

By adopting and modifying the method proposed by Messaoudi and Soufyane in 2010 [17], we establish a general decay result, from which the usual exponential and polynomial decay rate are only special cases. Further, our result allows certain kernels which are not necessarily of exponential or polynomial decay. In this way, we improved the results of Santos et al. [22], in which they considered problem (1.1)-(1.4) with $a = 1$ and in the absence of $l(t)h(\nabla u)$. Moreover, we note that our result also holds for problem (1.1)-(1.4) with $a = 1$ and $l(t) = 0$ and without imposing strong damping term, thus our result improves the one of Bae et al. [27]. More precisely, the estimate (3.27) and (3.28) generalizes the exponential and polynomial decay result given in [22,27]. Indeed, we obtain exponential decay for $\gamma(t) = c$ and polynomial decay for $\gamma(t) = c(1 + t)^{-1}$, where c is a positive constant. Additionally, as in [17], our result allows kernels which satisfy $k''(t) \geq c\,(-k')^{1+q}$, for $0 < q < 1$ instead of the usual assumption $0 < q < \frac{1}{2}$. It suffices to take, for example, $k(t) = (1 + t)^{-\lambda}$, for $\lambda > 0$. Direct computations yield

$$k''(t) = c(-k'(t))^{1+\frac{1}{1+\lambda}}.$$

It is clear that $0 < \frac{1}{1+\lambda} < 1$, for $\lambda > 0$.

Though we consider the conditions on the term involving the gradient are too restrictive and we combine some known ideas to obtain our result, our findings extend those decay results in [22,27] and these findings are interesting to those with closely concerns. For future work, we will consider not necessarily decreasing kernels and relax the condition of $h(\nabla u)$.

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