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A fixed point theorem for set-valued quasi-contractions in b-metric spaces

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Abstract

In this article, we give a fixed point theorem for set-valued quasi-contraction maps in *b*-metric spaces. This theorem extends, unifies and generalizes several well known comparable results in the existing literature.

1 Introduction

The Banach contraction principle [1] is a very popular tool in solving existence problems in many branches of mathematical analysis. This famous theorem can be stated as follows.

Theorem 1.1. *Let* (X, d) *be a complete metric space and* T *be a mapping of* X *into itself satisfying:*

$$d(Tx, Ty) \le kd(x, y), \quad \forall x, y \in X, \tag{1.1}$$

where k *is a constant in* [0, 1). *Then,* T *has a unique fixed point* $x^* \in X$.

A mapping $T : X \to X$ is said to be a quasi-contraction if there exists $0 \le q < 1$ such that for any $x, y \in X$,

$$d(Tx, Ty) \le q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\},$$

In 1974, Ćirić [2] introduced these maps and proved an existence and uniqueness fixed point theorem.

The study of fixed points for multi-valued contraction maps using the Hausdorff metric was initiated by Nadler [3] in 1969, who extended the Banach contraction principle to set-valued mappings. Since then many authors have studied fixed points for set-valued maps. The theory of set-valued maps has many applications in control theory, convex optimization, differential equations and economics.

Definition 1.1. *Let* X *be any nonempty set. An element* x *in* X *is said to be a a fixed point of a multi-valued mapping* $T : X \to 2^X$ *if* $x \in Tx$, *where* 2^X *denotes the collection of all nonempty subsets of* X.

Let (X, d) be a metric space. Let $\mathfrak{CB}(X)$ be the collection of all nonempty closed bounded subsets of X. For $A, B \in \mathfrak{CB}(X)$, define

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\}, \tag{1.2}$$

where

$$\delta(A, B) = \sup\{d(a, B), \quad a \in A\}, \quad \delta(B, A) = \sup\{d(b, A), \quad b \in B\} \tag{1.3}$$

with

$$d(a, C) = \inf\{d(a, x), x \in C\}, \quad C \in \mathfrak{CB}(X). \tag{1.4}$$

Note that H is called the Hausdorff metric induced by the metric d.

Definition 1.2. *Let (X, d) be a metric space. The set-valued map $T : X \to \mathfrak{CB}(X)$ is said to be a q-set-valued quasi-contraction if there exists $0 \le q < 1$ such that for any x, $y \in X$,*

$$H(Tx, Ty) \le q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Recently, Amini-Harandi [4] proved a set-valued version of the above mentioned Ćirić's theorem [2] as follows:

Theorem 1.2. *Let (X, d) be a complete metric space. Suppose that $T : X \to \mathfrak{CB}(X)$ is said to be a q-set-valued quasi-contraction. Assume that $q < \frac{1}{2}$, then T has a fixed point in X, that is, there exists $u \in X$ such that $u \in Tu$.*

In the sequel, the letters $\mathbb{R}_+, \mathbb{N}$ and $\mathbb{N}^*$ will denote the set of all nonnegative real numbers, the set of all natural numbers and the set of all positive natural numbers, respectively.

Some problems, particularly the problem of the convergence of measurable functions with respect to a measure, lead to a generalization of notion of a metric. Using this idea, Czerwik [5] presented a generalization of the well known Banachs's fixed point theorem [1] in so-called b-metric spaces. Consistent with [5,6], we use the following notations and definitions.

Definition 1.3. [6]*Let X be a nonempty set and $s \ge 1$ a given real number. A function $d : X \times X \to \mathbb{R}_+$ is called a b-metric provided that, for all $x, y, z \in X$,*

(bm-1) $d(x, x) = 0$,

(bm-2) $d(x, y) = d(y, x)$,

(bm-3) $d(x, y) \le s(d(x, z) + d(z, y))$.

Note that a (usual) metric space is evidently a b-metric space. However, Czerwik [5,6] has shown that a b-metric on X need not be a metric on X (see also [7-11]. The following example of Singh and Prasad [12] shows that a b-metric on X need not be a metric on X.

Example 1.4. *Let $X = \{0, 1, 2\}$ and $d(2, 0) = d(0, 2) = m \ge 2$, $d(0, 1) = d(1, 2) = d(0, 1) = d(2, 1) = 1$ and $d(0, 0) = d(1, 1) = d(2, 2) = 0$. Then,*

$$d(x, y) \le \frac{m}{2}[d(x, z) + d(z, y)],$$

for all $x, y, z \in X$. If $m > 2$, the ordinary triangle inequality does not hold.

An example of a b-metric space was given in [13].

Example 1.5. *Let E be a Banach space and 0_E be the zero vector of E. Let P be a cone in E with $int(P) \ne \varnothing$ and $\le$ be a partial ordering with respect to P. A mapping $d : X \times X \to E$ is called a cone metric on the nonempty set X if the following axioms are satisfied:*

(1) $0_E \le d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0_E$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

The pair (X, d), where X is a nonempty set and d is a cone metric, is called a cone metric space.

Notice that in [[14], Lemma 5], if the cone P is normal with a constant K, then the cone metric $d : X \times X \to E$ is continuous, i.e. if $\{x_n\}$, $\{y_n\}$ are sequences in X with $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$, as $n \to \infty$.

Let E be a Banach space and P be a normal cone in E with the coefficient of normality denoted by K. Let $D : X \times X \to \mathbb{R}$ be defined by $D(x, y) = ||d(x, y)||$, where $d : X \times X \to E$ is a cone metric space. Then (X, D) is a b-metric space with constant $s := K \geq 1$.

Moreover, since the topology τ_d generated by the cone metric d coincides with the topology τ_D generated by the b-metric D, (see [[15], Theorem 2.4]), the b-metric D is continuous.

Let (X, d) be a b-metric space. From now on, we keep the same notations given by (1.2)-(1.4), except that d is a b-metric on X. We cite the following lemmas from Czerwik [5,6,9] and Singh et al. [11].

Lemma 1.3. *Let (X, d) be a b-metric space. For any $A, B, C \in \mathfrak{CB}(X)$ and any $x, y \in X$, we have the following:*

(i) $d(x, B) \leq d(x, b)$ for any $b \in B$,

(ii) $\delta(A, B) \leq H(A, B)$,

(iii) $d(x, B) \leq H(A, B)$ for any $x \in A$,

(iv) $H(A, A) = 0$,

(v) $H(A, B) = H(B, A)$,

(vi) $H(A, C) \leq s(H(A, B) + H(B, C))$,

(vii) $d(x, A) \leq s(d(x, y) + d(y, A))$.

Lemma 1.4. *Let (X, d) be a b-metric space. Let A and B be in $\mathfrak{CB}(X)$. Then for each $\alpha > 0$ and for all $b \in B$ there exists $a \in A$ such that $d(a, b) \leq H(A, B) + \alpha$.*

Lemma 1.5. *Let (X, d) be a b-metric space. For $A \in \mathfrak{CB}(X)$ and $x \in X$, we have*

$$d(x, A) = 0 \Leftrightarrow x \in \bar{A} = A.$$

In this article, we establish the analogous of Theorem 1.2 on a complete b-metric space. The main theorem extends several well known comparable results in the existing literature.

2 Main results

We start with the following preliminary lemma which we need in the sequel.

Lemma 2.1. *[16]Let (X, d) be a b-metric space and $\{y_n\}$ a sequence in X such that*

$$d(\gamma_{n+1}, \gamma_{n+2}) \leq \gamma d(\gamma_n, \gamma_{n+1}), \quad n = 0, 1, \ldots,$$

where $0 \leq \gamma < 1$. Then, $\{y_n\}$ is a Cauchy sequence in X provided that $s\gamma < 1$.

Let (X, d) be a b-metric space. Again as in [4], the set-valued map $T : X \to \mathfrak{CB}(X)$ is said to be a q-set-valued quasi-contraction if for any $x, y \in X$,

$$H(Tx, T\gamma) \leq qM(x, \gamma), \tag{2.1}$$

where $0 \leq q < 1$ and

$$M(x, \gamma) = \max\left\{d(x, \gamma), \ d(x, Tx), \ d(\gamma, T\gamma), \ d(x, T\gamma), \ d(\gamma, Tx)\right\}.$$

Our main result is the following.

Theorem 2.2. *Let* (X, d) *be a complete* b−*metric space. Suppose that* T *is a* q-*set-valued quasi-contraction. Assume that* $q < \frac{1}{s^2+s}$, *then* T *has a fixed point in* X, *that is, there exists* $u \in X$ *such that* $u \in Tu$.

Proof. Obviously, $M(x, y) = 0$ if and only if $x = y$ is a fixed point of T. For the rest, assume that $M(x, y) > 0$ for all $x, y \in X$.

Take

$$\varepsilon = \frac{1}{2}\left(\frac{1}{s^2+s} - q\right) \quad \text{and} \quad \beta = q + \varepsilon = \frac{1}{2}\left(\frac{1}{s^2+s} + q\right).$$

Since we assumed that $q < \frac{1}{s^2+s}$, so $\varepsilon > 0$ and $0 < \beta < 1$.

Let $x_0 \in X$ and $x_1 \in Tx_0$. By Lemma 1.4, there must exist $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq H(Tx_0, Tx_1) + \varepsilon M(x_0, x_1) \leq qM(x_0, x_1) + \varepsilon M(x_0, x_1) = \beta M(x_0, x_1).$$

Similarly, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) \leq H(Tx_1, Tx_2) + \varepsilon M(x_1, x_2) \leq qM(x_1, x_2) + \varepsilon M(x_1, x_2) = \beta M(x_1, x_2).$$

Thus, by induction there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) \leq H(Tx_{n-1}, Tx_n) + \varepsilon M(x_{n-1}, x_n) \leq \beta M(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N}^*.$$

Set $d_n = d(x_n, x_{n+1})$. We have

$$d_n = d(x_n, x_{n+1}) \leq \beta M(x_{n-1}, x_n). \tag{2.3}$$

Assume for some $n \in \mathbb{N}$, $x_n = x_{n+1}$, then $x_n \in Tx_n$, so the proof is completed. For the rest, for each n assume that $d_n \neq 0$.

On the other hand, for any $n \in \mathbb{N}^*$

$$
\begin{aligned}
M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), \; d(x_{n-1}, Tx_{n-1}), \; d(x_n, Tx_n), \; d(x_{n-1}, Tx_n), \; d(x_n, Tx_{n-1})\} \\
&\leq \max\{d(x_{n-1}, x_n), \; d(x_{n-1}, x_n), \; d(x_n, x_{n+1}), \; d(x_{n-1}, x_{n+1}), \; d(x_n, x_n)\} \\
&= \max\{d_{n-1}, d_{n-1}, d_n, \; d(x_{n-1}, x_{n+1}), \; 0\} \\
&\leq \max\{d_{n-1}, d_n, s(d_{n-1} + d_n)\}.
\end{aligned}
$$

If for some $n \in \mathbb{N}^*$, $\max\{d_{n-1}, d_n, s(d_{n-1} + d_n)\} = d_n$, then from (2.3) we find that $0 < d_n \leq \beta d_n$, which is a contradiction with respect to $0 < \beta < 1$. We deduce

$$\max\{d_{n-1}, d_n, s(d_{n-1} + d_n)\} = \max\{d_{n-1}, s(d_{n-1} + d_n)\}.$$

Therefore, (2.3) becomes

$$d_n \leq \beta \max\{d_{n-1}, s(d_{n-1} + d_n)\}.$$

Put

$$\gamma = \max\left\{\beta, \frac{s\beta}{1 - s\beta}\right\}.$$

Thus

$$d_n \leq \gamma d_{n-1}, \quad \forall n \in \mathbb{N}^*. \tag{2.4}$$

Having in mind $s \geq 1$, $\beta = \frac{1}{2}\left(\frac{1}{s^2+s} + q\right)$ and $q < \frac{1}{s^2+s}$, so it is easy that

$$\gamma s < 1. \tag{2.5}$$

Therefore, by (2.4), (2.5) and Lemma 2.1, the sequence $\{x_n\}$ is Cauchy in (X, d). Since the b-metric space (X, d) is complete, so there exists $u \in X$ such that

$$\lim_{n \to +\infty} d(x_n, u) = 0. \tag{2.6}$$

We claim that $u \in Tu$, that is, u is a fixed point of T. From (2.1), we have

$$H(Tx_n, Tu) \leq qM(x_n, u),$$

where

$$\begin{aligned} M(x_n, u) &= \max\left\{d(x_n, u), d(x_n, Tu), d(u, Tu), d(x_n, Tu), d(u, Tx_n)\right\} \\ &\leq \max\left\{d(x_n, u), d(x_n, Tu), d(u, Tu), d(x_n, Tu), d(u, x_{n+1})\right\}. \end{aligned}$$

From (2.4), we may write $d_n \leq \gamma^n d_0$ for each n. Since $\gamma < 1$, we have

$$\lim_{n \to +\infty} d_n =: d(x_n, x_{n+1}) = 0.$$

The condition (rm-3) yields $d(x_{n+1}, u) \leq s(d(x_{n+1}, x_n) + d(x_n, u))$, so

$$\lim_{n \to +\infty} d(x_{n+1}, u) = 0. \tag{2.7}$$

Again, by Lemma 1.3, $d(x_n, Tu) \leq s(d(x_n, u) + d(u, Tu))$, then letting $n \to +\infty$ and using (2.4), we get

$$\limsup_{n \to +\infty} d(x_n, Tu) \leq s\, d(u, Tu). \tag{2.8}$$

By (2.6)-(2.8), letting $n \to +\infty$, we get

$$\limsup_{n \to +\infty} M(x_n, u) \leq s\, d(u, Tu), \tag{2.9}$$

Moreover, since $x_{n+1} \in Tx_n$, hence $d(x_{n+1}, Tu) \leq H(Tx_n, Tu)$. Again, because $d(u, Tu) \leq s(d(u, x_{n+1}) + d(x_{n+1}, Tu))$, so

$$\begin{aligned} \frac{1}{s}d(u, Tu) &\leq d(x_{n+1}, u) + d(x_{n+1}, Tu) \\ &\leq d(x_{n+1}, u) + H(Tx_n, Tu) \\ &\leq d(x_{n+1}, u) + qM(x_n, u). \end{aligned}$$

Letting $n \to +\infty$ and using (2.4) and (2.9), we find

$$\frac{1}{s}d(u, Tu) \leq qsd(u, Tu).$$

This leads to

$$d(u, Tu) \leq qs^2 d(u, Tu). \tag{2.10}$$

But, since $q < \frac{1}{s^2+s}$, so $qs^2 < 1$, hence (2.10) is true unless $d(u, Tu) = 0$. From Lemma 1.5, we deduce that $u \in Tu$ because that Tu is a closed subset in X. This completes the proof of Theorem 2.2.

Remark 2.1. *Taking $s = 1$ in Theorem 2.2 (it corresponds to the case of metric spaces), the condition on q becomes $q < \frac{1}{2}$, so we find Theorem 1.2 of Amini-Harandi.*

Hence, *Theorem* 2.2 *is an extension of the result of Amini-Harandi* [4], *which itself improves Theorem* 3.3 *of Daffer and Kaneko* [17]*and Corollary* 3.3 *of Rouhani and Moradi* [18].

In 2008, Singh et al. [16] obtained the following result.

Theorem 2.3. *Let* (X, d) *be a complete b-metric space. Take* $0 < q < 1$ *and* $0 < k < 1$ *with* $\alpha = \max\left\{q^{1-k}, \frac{sq^{1-k}}{2-sq^{1-k}}\right\}$. *Suppose that* $T : X \to \mathfrak{CB}(X)$ *such that*

$$H(Tx, Ty) \leq q \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\} \qquad (2.11)$$

for all $x, y \in X$. *Assume that* $sq^{1-k} < 1$ *and* $\alpha s < 1$. *Then, T has a fixed point in X, that is, there exists* $u \in X$ *such that* $u \in Tu$.

Remark 2.2. *Note that*

$$\frac{1}{2}[d(x, Ty) + d(y, Tx)] \leq \max\{d(x, Ty), d(y, Tx)\},$$

so each mapping satisfying (2.11) *is also a q-set-valued quasi-contraction. Note that Ćirić* [19]*was the first who studied the contraction* (2.11) *for single-valued mappings in a metric space. If we take the parameters* $0 < k < 1$ *and* $0 < q < 1$ *such that*

$$k < \frac{\ln 2}{\ln(s^2 + s)} \quad and \quad q < \frac{1}{s^2 + s},$$

it is easy to get $sq^{1-k} < 1$ *and* $\alpha s < 1$ *where* α *is given by Theorem* 2.3. *Therefore, in this case Theorem* 2.2 *improves Theorem* 2.3.

Remark 2.3. *Theorem* 2.2 *is a partial reply to a question proposed by Singh et al.* [[16], p. 412].

In the case where $T : X \to X$ is a q-single-valued quasi-contraction on a b-metric space, we have the following corollary (it is a consequence of Theorem 2.2).

Corollary 2.4. *Let* (X, d) *be a b-metric space and* $T : X \to X$. *Suppose there exists* $0 \leq q < 1$ *such that*

$$d(Tx, Ty) \leq q \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Assume that $q < \frac{1}{s^2+s}$, *then T has a fixed point, that is, there exists* $u \in X$ *such that* $u = Tu$.

Proof. It follows by applying Theorem 2.2 and the fact that $H(Tx, Ty) = d(Tx, Ty)$ for each $x, y \in X$. □

Remark 2.4. *Taking s = 1 in Corollary* 2.4, *we find the result of Ćirić* [2].

Now, we give some examples illustrating our results.

Example 2.1. *Let* $X = 0[1]$*and* $d(x, y) = |x - y|^2$ *for all* $x, y \in X$. *It is obvious that d is a b-metric on X with s = 2 and* (X, d) *is complete. Also, d is not a metric on X. Define* $T : X \to \mathfrak{CB}(X)$ *by*

$$Tx = \begin{cases} \{\frac{1}{2}, \frac{1}{3}\} & if\, 0 \leq x < 1 \\ \{\frac{1}{3}\} & if \quad x = 1. \end{cases}$$

We shall check that (2.1) *is satisfied for all* $x, y \in X$. *Without loss of generality, take* $x \leq y$. *If* $x = y \in X$ *or* $x, y \in [0, 1)$, *then* $Tx = Ty$, *so* $H(Tx, Ty) = 0$, *that is* (2.1) *holds. Otherwise for all* $x < y$, *(that is* $0 \leq x < 1$ *and* $y = 1$*), standard calculations yield that*

$$H(Tx,\ Ty) = \frac{1}{36} \le \frac{4}{63} = \frac{1}{7}\cdot\frac{4}{9} = qd(y,\ Ty) \le qM(x,\ y),$$

where $q = \frac{1}{7} < \frac{1}{6} = \frac{1}{s^2+s}$ and $M(x, y)$ is given by (2.2).

We deduce that (2.1) holds for all $x, y \in X$. All hypotheses of Theorem 2.2 are satisfied. Here, $\frac{1}{2}$ are $\frac{1}{3}$ are the two fixed points of T.

On the other hand, Nadler's Theorem [3] is not applicable for Euclidian metric $d_0(x, y) = |x - y|$. Indeed, let H_0 be the Hausdoff metric induced by the metric d_0. For $x = \frac{5}{6}$ and $y = 1$, we have

$$H_0(Tx,\ Ty) = \frac{1}{6} > \frac{k}{6} = kd_0(x,\ y) \text{ for all } k \in [0,\ 1).$$

Also, we couldn't apply Theorem 1.2 (the main result of Amini-Harandi [4]). Indeed, take the metric $D : X \times X \to [0, \infty)$ given by

$$D(x,\ y) = \begin{cases} 1 \text{ if } x \ne y \\ 0 \text{ if } x = y. \end{cases}$$

Let H_D be the Hausdorff metric induced by the metric D. Again, for $x = \frac{2}{3}$ and $y = 1$, we get that

$$H_D(Tx,\ Ty) = 1 > \lambda = \lambda \max\{D(x,\ y),\ D(x,\ Tx),\ D(y,\ Ty),\ D(x,\ Ty),\ D(y,\ Tx)\} \text{ for all } \lambda \in \left[0,\ \frac{1}{2}\right).$$

Example 2.2. *Let $X = [[1], \infty)$ be equipped with the complete b-metric $d(x, y) = |x - y|^2$ for all $x, y \in X$, ($s = 2$). Define $T : X \to \mathfrak{CB}(X)$ by $Tx = \left[1, 1 + \frac{x}{3}\right]$ for all $x \in X$. Also, take $q = \frac{1}{9}$. We have*

$$H(Tx,\ Ty) = \frac{1}{9}(x - y)^2 = qd(x,\ y) \le qM(x,\ y),$$

for all $x, y \in X$, that is (2.1) holds. All hypotheses of Theorem 2.2 are satisfied and u = 1 is a fixed point of T .

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Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approve the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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