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# A regularization projection algorithm for various problems with nonlinear mappings in Hilbert spaces

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Full list of author information is available at the end of the article**Abstract**

In this paper, a regularization projection algorithm is investigated for solving common elements of an equilibrium problem, a variational inequality problem and a fixed point problem of a strictly pseudocontractive mapping. Strong convergence theorems are established in the framework of real Hilbert spaces.

**Keywords:** Hilbert space; equilibrium problem; variational inequality; nonexpansive mapping; fixed point

## 1 Introduction and preliminaries

Let  $H$  be a real Hilbert space with the inner product  $\langle \cdot, \cdot \rangle$  and the norm  $\| \cdot \|$ . Let  $C$  be a nonempty closed convex subset of  $H$ .  $P_C$  stands for the metric projection from  $H$  onto  $C$ . Let  $F$  be a bifunction of  $C \times C$  into  $\mathbb{R}$ , where  $\mathbb{R}$  denotes the set of real numbers. Consider the following equilibrium problem in the terminology of Blum and Oettli [1]:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0, \quad \forall y \in C. \quad (1.1)$$

In this paper, the solution set of problem (1.1) is denoted by  $\text{FP}(F)$ .

To study problem (1.1), we may assume that  $F$  satisfies the following conditions:

- (A1)  $F(x, x) = 0$  for all  $x \in C$ ;
- (A2)  $F$  is monotone, i.e.,  $F(x, y) + F(y, x) \leq 0$  for all  $x, y \in C$ ;
- (A3) for each  $x, y, z \in C$ ,

$$\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

- (A4) for each  $x \in C$ ,  $y \mapsto F(x, y)$  is convex and lower semi-continuous.

Let  $S$  be a self-mapping of  $C$ .  $F(S)$  stands for the fixed point set of  $S$ . Recall that  $S$  is said to be  $\beta$ -contractive if there exists a constant  $\beta \in [0, 1)$  such that

$$\|Sx - Sy\| \leq \beta \|x - y\|, \quad \forall x, y \in C.$$

$S$  is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

$S$  is said to be  $\kappa$ -strictly pseudocontractive if there exists a constant  $\kappa \in [0, 1)$  such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa \|(x - Sx) - (y - Sy)\|^2, \quad \forall x, y \in C.$$

The class of  $\kappa$ -strictly pseudocontractive mappings was introduced by Browder and Petryshyn [2]. It is clear that every nonexpansive mapping is a 0-strictly pseudocontractive mapping.

Let  $A : C \rightarrow H$  be a mapping. Recall that  $A$  is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

$A$  is said to be strongly monotone if there exists a constant  $\alpha > 0$  such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

For such a case, we say that  $A$  is  $\alpha$ -strongly monotone.  $A$  is said to be inverse-strongly monotone if there exists a constant  $\alpha > 0$  such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, we say that  $A$  is  $\alpha$ -inverse-strongly monotone. It is clear that  $A$  is  $\alpha$ -inverse-strongly monotone if and only if  $A^{-1}$  is  $\alpha$ -strongly monotone.

Recall that the classical variational inequality is to find  $x \in C$  such that

$$\langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

It is known that  $x \in C$  is a solution of (1.2) if and only if  $x$  is a fixed point of  $P_C(I - rA)$ , where  $r > 0$  is a constant and  $I$  is an identity mapping. From now on, the solution set of (1.2) is denoted by  $VI(C, A)$ .

A set-valued mapping  $T : H \rightarrow 2^H$  is said to be monotone if for all  $x, y \in H, f \in Tx$  and  $g \in Ty$  imply  $\langle x - y, f - g \rangle \geq 0$ . A monotone mapping  $T : H \rightarrow 2^H$  is maximal if the graph  $G(T)$  of  $T$  is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping  $T$  is maximal if and only if, for any  $(x, f) \in H \times H, \langle x - y, f - g \rangle \geq 0$  for all  $(y, g) \in G(T)$  implies  $f \in Tx$ .

Recently, Iiduka and Takahshi [3] investigated variational inequality (1.2) and fixed points of a nonexpansive mapping based on a Halpern-like algorithm. To be more clear, they proved the following result.

**Theorem 1T** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $A$  be an  $\alpha$ -inverse-strongly monotone mapping of  $C$  into  $H$ , and let  $S$  be a nonexpansive mapping of  $C$  into itself such that  $F(S) \cap VI(C, A) \neq \emptyset$ . Suppose  $x_1 = x \in C$  and  $\{x_n\}$  is given by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n)$$

*for every  $n = 1, 2, \dots$ , where  $\{\alpha_n\}$  is a sequence in  $[0, 1)$  and  $\{\lambda_n\}$  is a sequence in  $[0, 2\alpha]$ . If  $\{\alpha_n\}$  and  $\{\lambda_n\}$  are chosen so that  $\lambda_n \in [a, b]$  for some  $a, b$  with  $0 < a < b < 2\alpha, \lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$  and  $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ , then  $\{x_n\}$  converges strongly to  $P_{F(S) \cap VI(C, A)} x$ .*

Since equilibrium problem (1.1) provides a unified model of several problems such as variational inequalities, fixed point problems and inclusion problems. In [4], Takahashi and Takahashi further studied fixed points of a nonexpansive mapping and equilibrium problem (1.1) based on the viscosity approximation method, which was introduced by Moudafi [5]. To be more clear, they proved the following result.

**Theorem TT** *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  satisfying (A1)-(A4), and let  $S$  be a nonexpansive mapping of  $C$  into  $H$  such that  $F(S) \cap \text{EP}(F) \neq \emptyset$ . Let  $f$  be a contraction of  $H$  into itself, and let  $\{x_n\}$  and  $\{u_n\}$  be sequences generated by  $x_1 \in H$  and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S u_n \end{cases}$$

for every  $n = 1, 2, \dots$ , where  $\{\alpha_n\} \subset [0, 1]$  and  $\{r_n\} \subset (0, \infty)$  satisfy  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,  $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ,  $\liminf_{n \rightarrow \infty} r_n > 0$  and  $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ . Then  $\{x_n\}$  and  $\{u_n\}$  converge strongly to  $z \in F(S) \cap \text{EP}(F)$ , where  $z = P_{F(S) \cap \text{EP}(F)} f(z)$ .

Subsequently, many authors investigated common solution problems based on hybrid projection methods due to real world applications, for example, image restoration and digital signal processing; see [6–20] and the references therein. In view of the complexity of convex, hybrid projection methods, they are not easy to implement. In this paper, we study a regularization projection algorithm for solving equilibrium problem (1.1), variational inequality (1.2) and a fixed point problem of a  $\kappa$ -strictly pseudocontractive mapping. Possible computation errors are taken into account. Strong convergence theorems are established in the framework of real Hilbert spaces.

In order to prove our main results, we also need the following lemmas.

**Lemma 1.1** [21] *Let  $C$  be a nonempty closed convex subset of  $H$ . Let  $A$  be a monotone mapping of  $C$  into  $H$ , and let  $N_C v$  be a normal cone to  $C$  at  $v \in C$ , i.e.,*

$$N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}$$

and define a mapping  $T$  on  $C$  by

$$T v = \begin{cases} A v + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then  $T$  is maximal monotone and  $0 \in T v$  if and only if  $\langle A v, u - v \rangle \geq 0$  for all  $u \in C$ .

**Lemma 1.2** [2] *Let  $C$  be a nonempty closed convex subset of  $H$ , and let  $S : C \rightarrow H$  be a  $\kappa$ -strictly pseudocontractive mapping. Then  $I - S$  is demi-closed at zero.*

**Lemma 1.3** [1] *Let  $C$  be a nonempty closed convex subset of  $H$ , and let  $F : C \times C \rightarrow \mathbb{R}$  be a bifunction satisfying (A1)-(A4). Then, for any  $r > 0$  and  $x \in H$ , there exists  $z \in C$  such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all  $r > 0$  and  $x \in H$ . Then the following hold:

- (a)  $T_r$  is single-valued;
- (b)  $T_r$  is firmly nonexpansive, i.e., for any  $x, y \in H$ ,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

- (c)  $F(T_r) = EP(F)$ ;
- (d)  $EP(F)$  is closed and convex.

**Lemma 1.4** [2] *Let  $C$  be a nonempty closed convex subset of  $H$ , and let  $S : C \rightarrow H$  be a  $\kappa$ -strictly pseudocontractive mapping. Define a mapping  $T$  by  $T = \delta I + (1 - \delta)S$ , where  $\delta$  is a constant in  $(0, 1)$ . If  $\delta \in [\kappa, 1)$  then  $T$  is nonexpansive and  $F(T) = F(S)$ .*

**Lemma 1.5** [22] *Let  $\{x_n\}$  and  $\{y_n\}$  be bounded sequences in  $H$ , and let  $\{\beta_n\}$  be a sequence in  $[0, 1]$  with  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ . Suppose  $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$  for all integers  $n \geq 0$  and*

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then  $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$ .

**Lemma 1.6** [23] *Assume that  $\{\alpha_n\}$  is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n + \mu_n,$$

where  $\{\gamma_n\}$  is a sequence in  $(0, 1)$  and  $\{\mu_n\}, \{\delta_n\}$  are real sequences such that

- (i)  $\sum_{n=1}^{\infty} \gamma_n = \infty$  and  $\sum_{n=1}^{\infty} \mu_n < \infty$ ;
- (ii)  $\limsup_{n \rightarrow \infty} \delta_n / \gamma_n \leq 0$  or  $\sum_{n=1}^{\infty} |\delta_n| < \infty$ .

Then  $\lim_{n \rightarrow \infty} \alpha_n = 0$ .

## 2 Main results

**Theorem 2.1** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $A : C \rightarrow H$  be an  $\alpha$ -inverse-strongly monotone mapping, and let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  which satisfies (A1)-(A4). Let  $S : C \rightarrow H$  be a  $\kappa$ -strictly pseudocontractive mapping, and let  $f$  be a  $\beta$ -contraction on  $H$ . Assume that  $\Omega = F(S) \cap VI(C, A) \cap EP(F) \neq \emptyset$ . Let  $\{r_n\}$  and  $\{s_n\}$  be positive real number sequences. Let  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  and  $\{\delta_n\}$  be real number sequences in  $(0, 1)$  such that  $\alpha_n + \beta_n + \gamma_n = 1$ . Let  $\{x_n\}$  be a sequence generated in the following process:*

$$\begin{cases} x_1 \in H, \\ F(z_n, z) + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0, \quad \forall z \in C, \\ y_n = P_C(z_n - s_n A z_n + e_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n (\delta_n y_n + (1 - \delta_n) S y_n), \end{cases}$$

where  $\{e_n\}$  is a sequence in  $H$ . Assume that the control sequences satisfy the following conditions:

- (a)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
- (b)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ ;
- (c)  $\sum_{n=1}^{\infty} \|e_n\| < \infty$ ,  $\lim_{n \rightarrow \infty} |\delta_{n+1} - \delta_n| = 0$  and  $\kappa \leq \delta_n \leq \delta < 1$ ;
- (d)  $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ ;
- (e)  $\lim_{n \rightarrow \infty} |s_{n+1} - s_n| = 0$ ,  $0 < s \leq s_n \leq s' < 2\alpha$ ,

where  $\delta, s, s'$  are real constants. Then  $\{x_n\}$  converges strongly to  $q$ , which is also a unique solution to the variational inequality

$$\langle f(x) - x, x - y \rangle \geq 0, \quad \forall y \in C.$$

*Proof* For any  $x, y \in C$ , we see that

$$\begin{aligned} & \| (I - s_n A)x - (I - s_n A)y \|^2 \\ &= \| x - y \|^2 - 2s_n \langle x - y, Ax - Ay \rangle + s_n^2 \| Ax - Ay \|^2 \\ &\leq \| x - y \|^2 - s_n(2\alpha - s_n) \| Ax - Ay \|^2. \end{aligned}$$

By using condition (e), we see that  $\| (I - s_n A)x - (I - s_n A)y \| \leq \| x - y \|$ . This proves that  $I - s_n A$  is nonexpansive. Put  $S_n = \delta_n I + (1 - \delta_n)S$ . It follows from Lemma 1.4 that  $S_n$  is nonexpansive and  $F(S_n) = F(S)$ . Let  $p \in \Omega$  be fixed arbitrarily. Hence, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|S_n y_n - p\| \\ &\leq \alpha_n \beta \|x_n - p\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\ &\leq (1 - \alpha_n(1 - \beta)) \|x_n - p\| + \alpha_n \|f(p) - p\| + \|e_n\| \\ &\leq \max \left\{ \|x_n - p\|, \frac{\|f(p) - p\|}{1 - \beta} \right\} + \|e_n\|. \end{aligned}$$

It follows that  $\|x_n - p\| \leq \max \{ \|x_1 - p\|, \frac{\|f(p) - p\|}{1 - \beta} \} + \sum_{n=1}^{\infty} \|e_n\|$ . This shows that  $\{x_n\}$  is bounded, so are  $\{y_n\}$  and  $\{z_n\}$ . Let  $\lambda_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$ . It follows that

$$\begin{aligned} \lambda_{n+1} - \lambda_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} S_{n+1} y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n S_n y_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1} (f(x_{n+1}) - S_{n+1} y_{n+1}) + (1 - \beta_{n+1}) S_{n+1} y_n}{1 - \beta_{n+1}} \\ &\quad - \frac{\alpha_n (f(x_n) - S_n y_n) + (1 - \beta_n) S_n y_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1} (f(x_{n+1}) - S_{n+1} y_{n+1})}{1 - \beta_{n+1}} - \frac{\alpha_n (f(x_n) - S_n y_n)}{1 - \beta_n} + S_{n+1} y_{n+1} - S_n y_n. \end{aligned}$$

Hence, we have

$$\begin{aligned} \|\lambda_{n+1} - \lambda_n\| &\leq \frac{\alpha_{n+1} \|f(x_{n+1}) - S_{n+1} y_{n+1}\|}{1 - \beta_{n+1}} + \frac{\alpha_n \|f(x_n) - S_n y_n\|}{1 - \beta_n} \\ &\quad + \|S_{n+1} y_{n+1} - S_n y_n\|. \end{aligned} \tag{2.1}$$

Since  $z_n = T_{r_n}x_n$ , we find that  $F(z_n, z) + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0, \forall z \in C$  and  $F(z_{n+1}, z) + \frac{1}{r_{n+1}} \langle z - z_{n+1}, z_{n+1} - x_{n+1} \rangle \geq 0, \forall z \in C$ . It follows that  $F(z_n, z_{n+1}) + \frac{1}{r_n} \langle z_{n+1} - z_n, z_n - x_n \rangle \geq 0$  and  $F(z_{n+1}, z_n) + \frac{1}{r_{n+1}} \langle z_n - z_{n+1}, z_{n+1} - x_{n+1} \rangle \geq 0, \forall z \in C$ . By using condition (A2), we find that  $\langle z_{n+1} - z_n, \frac{z_n - x_n}{r_n} - \frac{z_{n+1} - x_{n+1}}{r_{n+1}} \rangle \geq 0$ . Hence, we have

$$\left\langle z_{n+1} - z_n, z_{n+1} - x_n - \frac{r_n}{r_{n+1}}(z_{n+1} - x_{n+1}) \right\rangle \geq \|z_{n+1} - z_n\|^2.$$

This implies that  $\langle z_{n+1} - z_n, x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(z_{n+1} - x_{n+1}) \rangle \geq \|z_{n+1} - z_n\|^2$ . Hence, we have

$$\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|T_{r_{n+1}}x_n - x_{n+1}\|.$$

It follows that

$$\begin{aligned} \|y_{n+1} - y_n\| &\leq \|P_C(z_{n+1} - s_{n+1}Az_{n+1} + e_{n+1}) - P_C(z_n - s_{n+1}Az_n + e_{n+1})\| \\ &\quad + \|P_C(z_n - s_{n+1}Az_n + e_{n+1}) - P_C(z_n - s_nAz_n + e_n)\| \\ &\leq \|(I - s_{n+1}A)z_{n+1} - (I - s_{n+1}A)z_n\| \\ &\quad + \|(z_n - s_{n+1}Az_n + e_{n+1}) - (z_n - s_nAz_n + e_n)\| \\ &\leq \|z_{n+1} - z_n\| + |s_{n+1} - s_n| \|Az_n\| + \|e_{n+1}\| + \|e_n\| \\ &\leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|T_{r_{n+1}}x_n - x_n\| \\ &\quad + |s_{n+1} - s_n| \|Az_n\| + \|e_{n+1}\| + \|e_n\|. \end{aligned}$$

This implies that

$$\begin{aligned} &\|S_{n+1}y_{n+1} - S_ny_n\| \\ &\leq \|S_{n+1}y_{n+1} - S_{n+1}y_n\| + \|S_{n+1}y_n - S_ny_n\| \\ &\leq \|y_{n+1} - y_n\| + \|S_{n+1}y_n - S_ny_n\| \\ &\leq \|x_{n+1} - x_n\| + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|T_{r_{n+1}}x_n - x_n\| \\ &\quad + |s_{n+1} - s_n| \|Az_n\| + \|e_{n+1}\| + \|e_n\| + |\delta_{n+1} - \delta_n| \|Sy_n - y_n\|. \end{aligned} \tag{2.2}$$

Substituting (2.2) into (2.1), we find that

$$\begin{aligned} &\|\lambda_{n+1} - \lambda_n\| - \|x_{n+1} - x_n\| \\ &\leq \frac{\alpha_{n+1} \|f(x_{n+1}) - S_{n+1}y_{n+1}\|}{1 - \beta_{n+1}} + \frac{\alpha_n \|f(x_n) - S_ny_n\|}{1 - \beta_n} \\ &\quad + \frac{|r_{n+1} - r_n|}{r_{n+1}} \|T_{r_{n+1}}x_n - x_n\| \\ &\quad + |s_{n+1} - s_n| \|Az_n\| + \|e_{n+1}\| + \|e_n\| + |\delta_{n+1} - \delta_n| \|Sy_n - y_n\|. \end{aligned}$$

It follows from conditions (a)-(e) that

$$\limsup_{n \rightarrow \infty} (\|\lambda_{n+1} - \lambda_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By using Lemma 1.5, we see that  $\lim_{n \rightarrow \infty} \|\lambda_n - x_n\| = 0$ , which in turn implies that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{2.3}$$

Since  $T_{r_n}$  is firmly nonexpansive, we find that

$$\begin{aligned} \|z_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \\ &\leq \langle x_n - p, z_n - p \rangle \\ &= \frac{1}{2} (\|x_n - p\|^2 + \|z_n - p\|^2 - \|x_n - z_n\|^2). \end{aligned}$$

That is,

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - z_n\|^2.$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|S_n y_n - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|z_n - p\|^2 + f_n \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - \gamma_n \|x_n - z_n\|^2 + f_n, \end{aligned}$$

where  $f_n = \|e_n\|(\|e_n\| + 2\|z_n - p\|)$ . This further implies that

$$\begin{aligned} \gamma_n \|x_n - z_n\|^2 &\leq \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + f_n \\ &\leq \alpha_n \|f(x_n) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| + f_n. \end{aligned}$$

By using conditions (a) and (b), we find from (2.3) that

$$\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0. \tag{2.4}$$

Since  $A$  is  $\alpha$ -inverse-strongly monotone, we find that

$$\begin{aligned} \|y_n - p\|^2 &\leq \|(z_n - s_n A z_n) - (p - s_n A p) + e_n\|^2 \\ &\leq \|(z_n - p) - s_n(A z_n - A p)\|^2 + f_n \\ &\leq \|x_n - p\|^2 - s_n(2\alpha_n - s_n) \|A z_n - A p\|^2 + f_n. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|S_n y_n - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - s_n(2\alpha - s_n) \gamma_n \|A z_n - A p\|^2 + f_n. \end{aligned}$$

This yields that

$$s_n(2\alpha - s_n)\gamma_n \|Az_n - Ap\|^2 \leq \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + f_n.$$

By using (2.3), we find from conditions (a), (c) and (d) that

$$\lim_{n \rightarrow \infty} \|Az_n - Ap\| = 0. \tag{2.5}$$

Since  $P_C$  is firmly nonexpansive, we find that

$$\begin{aligned} \|y_n - p\|^2 &\leq \langle (I - s_nA)z_n + e_n - (I - s_nA)p, y_n - p \rangle \\ &= \frac{1}{2} \{ \|(I - s_nA)z_n + e_n - (I - s_nA)p\|^2 + \|y_n - p\|^2 \\ &\quad - \|(I - s_nA)z_n + e_n - (I - s_nA)p - (y_n - p)\|^2 \} \\ &\leq \frac{1}{2} \{ \|z_n - p\|^2 + f_n + \|y_n - p\|^2 - \|z_n - y_n - (s_n(Az_n - Ap) - e_n)\|^2 \} \\ &= \frac{1}{2} \{ \|z_n - p\|^2 + f_n + \|y_n - p\|^2 - \|z_n - y_n\|^2 \\ &\quad + 2\langle z_n - y_n, s_n(Az_n - Ap) - e_n \rangle - \|s_n(Az_n - Ap) - e_n\|^2 \} \\ &\leq \frac{1}{2} \{ \|x_n - p\|^2 + f_n + \|y_n - p\|^2 - \|z_n - y_n\|^2 \\ &\quad + 2\|z_n - y_n\| \|s_n(Az_n - Ap) - e_n\| - \|s_n(Az_n - Ap) - e_n\|^2 \}, \end{aligned}$$

which yields that

$$\begin{aligned} \|y_n - p\|^2 &\leq \|x_n - p\|^2 + f_n - \|z_n - y_n\|^2 + 2s_n \|z_n - y_n\| \|Az_n - Ap\| \\ &\quad + 2\|z_n - y_n\| \|e_n\|. \end{aligned}$$

It follows that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|S_n y_n - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|y_n - p\|^2 \\ &\leq \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 + f_n - \gamma_n \|z_n - y_n\|^2 \\ &\quad + 2s_n \gamma_n \|z_n - y_n\| \|Az_n - Ap\| + 2\|z_n - y_n\| \|e_n\|. \end{aligned}$$

This in turn leads to

$$\begin{aligned} \gamma_n \|z_n - y_n\|^2 &\leq \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + f_n \\ &\quad + 2s_n \gamma_n \|z_n - y_n\| \|Az_n - Ap\| + 2\|z_n - y_n\| \|e_n\|. \end{aligned}$$

By use of (2.3) and (2.5), we find from restrictions (a), (b), (c) and (e) that

$$\lim_{n \rightarrow \infty} \|z_n - y_n\| = 0. \tag{2.6}$$



On the other hand, we have

$$\gamma_n \|S_n y_n - x_n\| \leq \|x_{n+1} - x_n\| + \alpha_n \|x_n - f(x_n)\|.$$

By using conditions (a) and (b), we find from (2.3) that

$$\lim_{n \rightarrow \infty} \|x_n - S_n y_n\| = 0. \tag{2.7}$$

Since  $S_n$  is nonexpansive, we find that

$$\begin{aligned} \|S_n x_n - x_n\| &\leq \|S_n x_n - S_n y_n\| + \|S_n y_n - x_n\| \\ &\leq \|x_n - z_n\| + \|z_n - y_n\| + \|S_n y_n - x_n\|. \end{aligned}$$

It follows from (2.4), (2.6) and (2.7) that

$$\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0. \tag{2.8}$$

Notice that

$$\begin{aligned} \|Sx_n - x_n\| &\leq \|Sx_n - (\delta_n x_n + (1 - \delta_n)Sx_n)\| + \|S_n x_n - x_n\| \\ &\leq \delta_n \|Sx_n - x_n\| + \|S_n x_n - x_n\|. \end{aligned}$$

By using condition (c), we find that

$$\lim_{n \rightarrow \infty} \|x_n - Sx_n\| = 0. \tag{2.9}$$

Now, we are in a position to show  $\limsup_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle \leq 0$ , where  $q = P_\Omega f(q)$ . To show it, we can choose a subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  such that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle = \lim_{i \rightarrow \infty} \langle f(q) - q, x_{n_i} - q \rangle.$$

Since  $\{x_{n_i}\}$  is bounded, we can choose a subsequence  $\{x_{n_{i_j}}\}$  of  $\{x_{n_i}\}$  which converges weakly to some point  $\bar{x}$ . We may assume, without loss of generality, that  $\{x_{n_i}\}$  converges weakly to  $\bar{x}$ .

Next, we prove  $\bar{x} \in \Omega$ . First, we show  $x \in EP(F)$ . Notice that

$$F(y_n, y) + \frac{1}{r_n} \langle y - y_n, y_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

By using condition (A2), we see that  $\frac{1}{r_n} \langle y - y_n, y_n - x_n \rangle \geq F(y, y_n)$ ,  $\forall y \in C$ . Replacing  $n$  by  $n_i$ , we arrive at

$$\left\langle y - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, y_{n_i}), \quad \forall y \in C.$$

By using (2.4) and (2.6), we find that  $\{y_{n_i}\}$  converges weakly to  $\bar{x}$ . It follows that  $0 \geq F(y, \bar{x})$ . For each  $t$  with  $0 < t \leq 1$ , let  $z_t = tz + (1 - t)\bar{x}$ , where  $z \in C$ . It follows that  $z_t \in C$ . Hence,

we have  $F(z_t, \bar{x}) \leq 0$ . It follows that

$$0 = F(z_t, z_t) \leq tF(z_t, z) + (1 - t)F(z_t, \bar{x}) \leq tF(z_t, z),$$

which yields that  $F(z_t, z) \geq 0, \forall z \in C$ . Letting  $t \downarrow 0$ , we obtain from condition (A3) that  $F(\bar{x}, z) \geq 0, \forall z \in C$ . This implies that  $\bar{x} \in EP(F)$ .

Next, we show that  $\bar{x} \in VI(C, A)$ . Let  $T$  be a maximal monotone mapping defined by

$$Tx = \begin{cases} Ax + N_Cx, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

For any given  $(x, y) \in \text{Graph}(T)$ , we have  $y - Ax \in N_Cx$ . Since  $y_n \in C$ , we have  $\langle x - y_n, y - Ax \rangle \geq 0$ . Since  $y_n = P_C(z_n - s_nAz_n + e_n)$ , we see that  $\langle x - y_n, y_n - (I - s_nA)z_n - e_n \rangle \geq 0$  and hence

$$\left\langle x - y_n, \frac{y_n - z_n - e_n}{s_n} + Az_n \right\rangle \geq 0.$$

It follows that

$$\begin{aligned} \langle x - y_{n_i}, y \rangle &\geq \langle x - y_{n_i}, Ax \rangle \\ &\geq \langle x - y_{n_i}, Ax \rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i} - e_{n_i}}{s_{n_i}} + Az_{n_i} \right\rangle \\ &= \langle x - y_{n_i}, Ax - Ay_{n_i} \rangle + \langle x - y_{n_i}, Ay_{n_i} - Az_{n_i} \rangle \\ &\quad - \left\langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i} - e_{n_i}}{s_{n_i}} \right\rangle \\ &\geq \langle x - y_{n_i}, Ay_{n_i} - Az_{n_i} \rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i} - e_{n_i}}{s_{n_i}} \right\rangle. \end{aligned}$$

Since  $A$  is Lipschitz continuous, we see that  $\langle x - \bar{x}, y \rangle \geq 0$ . Notice that  $T$  is maximal monotone and hence  $0 \in T\bar{x}$ . This shows that  $\bar{x} \in VI(C, A)$ . By using Lemma 1.2, we find that  $\bar{x} \in F(S)$ . It follows that  $\limsup_{n \rightarrow \infty} \langle f(q) - q, x_n - q \rangle \leq 0$ ,

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\leq \alpha_n \langle f(x_n) - q, x_{n+1} - q \rangle + \beta_n \|x_n - q\| \|x_{n+1} - q\| + \gamma_n \|S_n y_n - q\| \|x_{n+1} - q\| \\ &\leq \alpha_n \langle f(x_n) - f(q), x_{n+1} - q \rangle + \alpha_n \langle f(q) - q, x_{n+1} - q \rangle + \beta_n \|x_n - \bar{x}\| \|x_{n+1} - q\| \\ &\quad + \gamma_n (\|x_n - q\| + e_n) \|x_{n+1} - q\| \\ &\leq \frac{\alpha_n \beta + \beta_n + \gamma_n}{2} (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + \alpha_n \langle f(q) - q, x_{n+1} - q \rangle + e_n \|x_{n+1} - q\|. \end{aligned}$$

It follows that

$$\|x_{n+1} - q\|^2 \leq (1 - \alpha_n(1 - \beta)) \|x_n - q\|^2 + 2\alpha_n \langle f(q) - q, x_{n+1} - q \rangle + 2K_n e_n,$$

where  $K = \sup_{n \geq 1} \{\|x_n - q\|\}$ . By using Lemma 1.6, we find that  $\lim_{n \rightarrow \infty} \|x_n - q\| = 0$ . This completes the proof. □

For nonexpansive mappings, we have the following result.

**Corollary 2.2** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $A : C \rightarrow H$  be an  $\alpha$ -inverse-strongly monotone mapping, and let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  which satisfies (A1)-(A4). Let  $S : C \rightarrow H$  be nonexpansive, and let  $f$  be a  $\beta$ -contraction on  $H$ . Assume that  $\Omega = F(S) \cap VI(C, A) \cap EP(F) \neq \emptyset$ . Let  $\{r_n\}$  and  $\{s_n\}$  be positive real number sequences. Let  $\{\alpha_n\}, \{\beta_n\}$  and  $\{\gamma_n\}$  be real number sequences in  $(0, 1)$  such that  $\alpha_n + \beta_n + \gamma_n = 1$ . Let  $\{x_n\}$  be a sequence generated in the following process:*

$$\begin{cases} x_1 \in H, \\ F(z_n, z) + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0, \quad \forall z \in C, \\ y_n = P_C(z_n - s_n A z_n + e_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S y_n, \end{cases}$$

where  $\{e_n\}$  is a sequence in  $H$ . Assume that the control sequences satisfy the following conditions:

- (a)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
- (b)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ ;
- (c)  $\sum_{n=1}^{\infty} \|e_n\| < \infty$ ;
- (d)  $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ ;
- (e)  $\lim_{n \rightarrow \infty} |s_{n+1} - s_n| = 0, 0 < s \leq s_n \leq s' < 2\alpha$ ,

where  $s, s'$  are real constants. Then  $\{x_n\}$  converges strongly to  $q$ , which is also a unique solution to the variational inequality

$$\langle f(x) - x, x - y \rangle \geq 0, \quad \forall y \in C.$$

Further, if  $S$  is an identity, we have the following result.

**Corollary 2.3** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $A : C \rightarrow H$  be an  $\alpha$ -inverse-strongly monotone mapping, and let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  which satisfies (A1)-(A4). Let  $f$  be a  $\beta$ -contraction on  $H$ . Assume that  $\Omega = VI(C, A) \cap EP(F) \neq \emptyset$ . Let  $\{r_n\}$  and  $\{s_n\}$  be positive real number sequences. Let  $\{\alpha_n\}, \{\beta_n\}$  and  $\{\gamma_n\}$  be real number sequences in  $(0, 1)$  such that  $\alpha_n + \beta_n + \gamma_n = 1$ . Let  $\{x_n\}$  be a sequence generated in the following process:*

$$\begin{cases} x_1 \in H, \\ F(z_n, z) + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0, \quad \forall z \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n P_C(z_n - s_n A z_n + e_n), \end{cases}$$

where  $\{e_n\}$  is a sequence in  $H$ . Assume that the control sequences satisfy the following conditions:

- (a)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
- (b)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ ;
- (c)  $\sum_{n=1}^{\infty} \|e_n\| < \infty$ ;
- (d)  $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ ;
- (e)  $\lim_{n \rightarrow \infty} |s_{n+1} - s_n| = 0, 0 < s \leq s_n \leq s' < 2\alpha$ ,

where  $s, s'$  are real constants. Then  $\{x_n\}$  converges strongly to  $q$ , which is also a unique solution to the variational inequality

$$\langle f(x) - x, x - y \rangle \geq 0, \quad \forall y \in C.$$

Putting  $F(x, y) = 0$  and  $r_n = 1$ , we find the following result.

**Corollary 2.4** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $A : C \rightarrow H$  be an  $\alpha$ -inverse-strongly monotone mapping. Let  $S : C \rightarrow H$  be a  $\kappa$ -strictly pseudocontractive mapping, and let  $f$  be a  $\beta$ -contraction on  $H$ . Assume that  $\Omega = F(S) \cap VI(C, A) \neq \emptyset$ . Let  $\{s_n\}$  be a positive real number sequence. Let  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  and  $\{\delta_n\}$  be real number sequences in  $(0, 1)$  such that  $\alpha_n + \beta_n + \gamma_n = 1$ . Let  $\{x_n\}$  be a sequence generated in the following process:*

$$\begin{cases} x_1 \in H, \\ z_n = P_C x_n, \\ y_n = P_C(z_n - s_n A z_n + e_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n(\delta_n y_n + (1 - \delta_n) S y_n), \end{cases}$$

where  $\{e_n\}$  is a sequence in  $H$ . Assume that the control sequences satisfy the following conditions:

- (a)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
- (b)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ ;
- (c)  $\sum_{n=1}^{\infty} \|e_n\| < \infty, \lim_{n \rightarrow \infty} |\delta_{n+1} - \delta_n| = 0$  and  $\kappa \leq \delta_n \leq \delta < 1$ ;
- (d)  $\lim_{n \rightarrow \infty} |s_{n+1} - s_n| = 0, 0 < s \leq s_n \leq s' < 2\alpha$ ,

where  $\delta, s, s'$  are real constants. Then  $\{x_n\}$  converges strongly to  $q$ , which is also a unique solution to the variational inequality

$$\langle f(x) - x, x - y \rangle \geq 0, \quad \forall y \in C.$$

**Corollary 2.5** *Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $F$  be a bifunction from  $C \times C$  to  $\mathbb{R}$  which satisfies (A1)-(A4). Let  $S : C \rightarrow H$  be a  $\kappa$ -strict pseudocontraction, and let  $f$  be a  $\beta$ -contraction on  $H$ . Assume that  $\Omega = F(S) \cap EP(F) \neq \emptyset$ . Let  $\{r_n\}$  be a positive real number sequence. Let  $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$  and  $\{\delta_n\}$  be real number sequences in  $(0, 1)$  such that  $\alpha_n + \beta_n + \gamma_n = 1$ . Let  $\{x_n\}$  be a sequence generated in the following process:*

$$\begin{cases} x_1 \in H, \\ F(z_n, z) + \frac{1}{r_n} \langle z - z_n, z_n - x_n \rangle \geq 0, \quad \forall z \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n(\delta_n z_n + (1 - \delta_n) S z_n), \end{cases}$$

where  $\{e_n\}$  is a sequence in  $H$ . Assume that the control sequences satisfy the following conditions:

- (a)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
- (b)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ ;
- (c)  $\sum_{n=1}^{\infty} \|e_n\| < \infty, \lim_{n \rightarrow \infty} |\delta_{n+1} - \delta_n| = 0$  and  $\kappa \leq \delta_n \leq \delta < 1$ ;
- (d)  $\lim_{n \rightarrow \infty} |r_{n+1} - r_n| = 0$  and  $\liminf_{n \rightarrow \infty} r_n > 0$ ,

where  $\delta$  is a real constant. Then  $\{x_n\}$  converges strongly to  $q$ , which is also a unique solution to the variational inequality

$$\langle f(x) - x, x - y \rangle \geq 0, \quad \forall y \in C.$$

**Remark 2.6** Comparing Theorem 2.1, Corollaries 2.2-2.5 with Theorems IT, TT and the corresponding results in [6–20], we have the following.

- We extend the nonlinear mapping from the class of nonexpansive mappings to the class of  $\kappa$ -strictly pseudocontractive mappings.
- Possible computation errors are taken into account.
- The conditions imposed on control sequences  $\{\alpha_n\}$  and  $\{s_n\}$  are relaxed.
- The common element is also a unique solution of variational inequality (1.2).

**Theorem 2.7** Let  $C$  be a closed convex subset of a real Hilbert space  $H$ . Let  $A : C \rightarrow H$  be an  $\alpha$ -inverse-strongly monotone mapping, and let  $f$  be a  $\beta$ -contraction on  $H$ . Assume that  $\text{VI}(C, A) \neq \emptyset$ . Let  $\{s_n\}$  be a positive real number sequence. Let  $\{\alpha_n\}$ ,  $\{\beta_n\}$  and  $\{\gamma_n\}$  be real number sequences in  $(0, 1)$  such that  $\alpha_n + \beta_n + \gamma_n = 1$ . Let  $\{x_n\}$  be a sequence generated in  $x_1 \in C$ ,  $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n$ , where  $y_n = P_C(x_n - s_n A x_n + e_n)$ , where  $\{e_n\}$  is a sequence in  $H$ . Assume that the control sequences satisfy the following conditions:

- $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ;
- $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ ;
- $\sum_{n=1}^{\infty} \|e_n\| < \infty$ ,  $\lim_{n \rightarrow \infty} |s_{n+1} - s_n| = 0$ ,  $0 < s \leq s_n \leq s' < 2\alpha$ ,

where  $s$  and  $s'$  are real constants. Then  $\{x_n\}$  converges strongly to  $q = P_{\text{VI}(C, A)} f(q)$ .

**Remark 2.8** To construct a mathematical model which is as close as possible to a real complex problem, we often have to use more than one constraint. Solving such problems, we have to obtain some solution which is simultaneously the solution of two or more subproblems. This is a common problem in diverse areas of mathematics and physical sciences. It consists of trying to find a solution satisfying certain constraints. In this paper, we investigate the problem of solving common solutions of an equilibrium problem, a variational inequality problem and a fixed point problem of a strictly pseudocontractive mapping based on a regularization projection algorithm. Possible computation errors are also taken into account. Strong convergence theorems are established without compact assumptions and additional metric projections in the framework of real Hilbert spaces.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors have contributed equally to the work. All authors read and approved the final version of the manuscript.

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