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Uniform Lorentz norm estimates for convolution operators

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Abstract

Uniform endpoint Lorentz norm improving estimates for convolution operators with affine arclength measure supported on simple plane curves are established. The estimates hold for a wide class of simple curves, and the condition is stated in terms of averages of the square of the affine arclength weight, extending previously known results.

MSC: Primary 44A35; secondary 42B35

Keywords: affine arclength; convolution; Lorentz space

1 Introduction

Let $\phi : (a,b) \to \mathbb{R}$ be a C^2 function such that $\phi''(t) \geq 0$ for all $t \in (a,b)$. In this paper, we consider the convolution operator $\mathcal{T}$ given by

$$\mathcal{T}f(x_1,x_2) = \int_a^b f\bigl(x_1 - t, x_2 - \phi(t)\bigr)\omega(t)\,dt \tag{1.1}$$

for $f \in C_0^\infty(\mathbb{R}^2)$. Here and in what follows, we denote $\omega(t) := (\phi''(t))^{1/3}$. Curves of the form $(t,\phi(t))$ are said to be simple according to Drury and Marshall [1]. The measure $\omega(t)\,dt$ supported on the curve $(t,\phi(t))$ is known as the affine arclength measure, which is based on the affine arclength parameter as in [2], and was introduced by Drury and Marshall [1] in dealing with the Fourier restriction problem related to curves, and later by Drury [3] in studying convolution operators with measures supported on curves. We refer interested readers to [2–4] for the relevance of affine geometry in this subject. One big benefit of using the affine arclength measure in place of the Euclidean arclength measure $\sqrt{1+\phi'(t)^2}\,dt$ has been its effect of mitigating degeneracies and it is believed that various uniform sharp estimates hold for a wide class of curves.

As is well known, the typeset $\mathcal{S} = \{(p^{-1},q^{-1}) : \mathcal{T}$ is bounded from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)\}$ of $\mathcal{T}$ is contained in the convex hull of $\{(0,0),(1,1),(2/3,1/3)\}$ and uniform estimates in a, b, and ϕ are expected only for $(1/p,1/q) = (2/3,1/3)$. Many conditions to guarantee optimal uniform $L^{3/2}$-L^3 estimates have been known so far. See [3, 5–12] for example. Among other things, the author proved the following.

Theorem 1.1 (Choi [12]) *Let J be an open interval in $\mathbb{R}$, and $\phi : J \to \mathbb{R}$ be a C^2 function such that $\phi'' \geq 0$. Suppose that there exists a positive constant A such that*

$$\omega^{1/2}(t_1)\omega^{1/2}(t_2) \leq \left(\frac{A}{t_2 - t_1}\int_{t_1}^{t_2} \omega^3(t)\,dt\right)^{1/3}$$

holds whenever $t_1 < t_2$ *and* $[t_1, t_2] \subset J$. *Let* $\mathcal{T}$ *be the operator defined as in* (1.1). *Then there exists a constant C that depends only on A such that*

$$\|\mathcal{T}f\|_{L^3(\mathbb{R}^2)} \le C\|f\|_{L^{3/2}(\mathbb{R}^2)}$$

holds uniformly in $f \in C_0^\infty(\mathbb{R}^2)$.

Under somewhat stronger assumptions on $\phi(t)$ or $\omega(t)$, the endpoint Lebesgue norm estimate aforementioned can be improved to optimal Lorentz norm estimates, namely from $L^{3/2}(\mathbb{R}^2)$ into $L^{3,3/2}(\mathbb{R}^2)$ and $L^{3/2,3}(\mathbb{R}^2)$ into $L^3(\mathbb{R}^2)$. We refer interested readers to [6, 8, 10, 11] for known sufficient conditions for optimal and nearly optimal Lorentz norm estimates. Most importantly, Oberlin established the following uniform optimal Lorentz norm improving estimates.

Theorem 1.2 (Oberlin [11]) *Let J be an open interval. Suppose that* $\omega(t)$ *is monotone increasing and that there exists a positive constant A such that*

$$\sqrt{\omega(t_1)\omega(t_2)} \le A\omega\bigl((t_1 + t_2)/2\bigr) \tag{1.2}$$

holds whenever $t_1 < t_2$ *and* $[t_1, t_2] \subset J$. *Then the operator* $\mathcal{T}$ *given by* (1.1) *satisfies*

$$\|\mathcal{T}f\|_{L^{3,3/2}(\mathbb{R}^2)} \le C\|f\|_{L^{3/2}(\mathbb{R}^2)},$$
$$\|\mathcal{T}f\|_{L^3(\mathbb{R}^2)} \le C\|f\|_{L^{3/2,3}(\mathbb{R}^2)}$$

for all $f \in C_0^\infty(\mathbb{R}^2)$, *where C is a constant depending only on A.*

For the proof of the optimality, see [13] by Stovall along with [8] by Bak *et al.* It is interesting to ask if the condition in Theorem 1.2 can be relaxed to cover more general curves. Based on an ingenious argument of Oberlin in [11], the author aims to establish a uniform optimal Lorentz norm improving estimate under a condition on averages of the square of $\omega(t)$. The average condition is a slightly stronger version of that in Theorem 1.1, and yet covers most simple plane curves studied up to now including those in Theorem 1.2.

This paper is organized as follows: in the following section, conditions on $\omega(t)$ are introduced and the main theorem is stated. The last section is devoted to the proof of the main theorem. As usual, absolute constants may grow from line to line.

2 Statement of the main theorem

Before we state our main result, we introduce certain conditions on functions defined on intervals.

Definition 2.1 Let $0 < p < \infty$. For an interval J_1 in $\mathbb{R}$, a locally L^p function $\Phi : J_1 \to \mathbb{R}^+$, and a positive real number A, we let

$$\mathfrak{G}_p(\Phi, A) := \biggl\{ F : J_1 \to \mathbb{R}^+ \biggm| \sqrt{F(t_1)F(t_2)} \le A\biggl(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \Phi^p(t)\,dt\biggr)^{1/p} \text{ whenever } t_1 < t_2 \text{ and } [t_1, t_2] \subset J_1 \biggr\}$$

and we let

$$\mathcal{E}_p(A) := \{\Phi : J \to \mathbb{R}^+ \mid J \text{ is an interval and } \Phi \in \mathfrak{G}_p(\Phi, A)\}.$$

An interesting subclass of $\mathcal{E}_p(2^{1/p}A)$, $0 < p < \infty$, was introduced by Bak *et al.* [14] in studying Fourier restriction estimates related to degenerate curves.

Definition 2.2 For an interval J and a positive real number A, a function $\Phi : J \to \mathbb{R}^+$ is said to be a member of $\tilde{\mathcal{E}}(A)$ if

- Φ is monotone; and
- whenever $t_1 < t_2$ and $[t_1, t_2] \subset J$,

  $$\sqrt{\Phi(t_1)\Phi(t_2)} \le A\Phi\big((t_1 + t_2)/2\big)$$

  holds.

The condition (1.2) can be rewritten as $\omega \in \tilde{\mathcal{E}}(A)$.

Remark 2.3 It seems appropriate to mention some properties of $\mathcal{E}_p(A)$ and $\tilde{\mathcal{E}}(A)$ mentioned above.

1. It is a simple matter to check:
   - $\tilde{\mathcal{E}}(A) \subset \mathcal{E}_p(2^{1/p}A)$ for all $p \in (0, \infty)$;
   - $\Phi \in \mathcal{E}_p(A)$ if and only if $\Phi^p \in \mathcal{E}_1(A^p)$;
   - $\Phi \in \mathcal{E}_p(A)$ implies $\lambda\Phi \in \mathcal{E}_p(A)$ for all $\lambda > 0$; and
   - $\Phi \in \mathcal{E}_p(A)$ implies $\Phi(a \cdot +b) \in \mathcal{E}_p(A)$ for all $(a, b) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$.
2. If $0 < p_1 < p_2 < \infty$, $\Phi : J \to \mathbb{R}^+ \in \mathcal{E}_{p_1}(A)$, and $\Phi \in L^{p_2}_{\mathrm{loc}}(J)$, then $\Phi \in \mathcal{E}_{p_2}(A)$ by Hölder's inequality.
3. The class $\tilde{\mathcal{E}}(1)$ is essentially the class of logarithmically concave functions, which already encompasses many useful examples. Simplest examples are the exponential function and $\Phi(t) = t^\alpha$, $t > 0$, for $\alpha \ge 0$. More interesting example is the function $\Phi(t) = e^{-1/t}$, $t > 0$, which models a curve 'flat' at the origin. A hierarchy of flatter functions that belong to $\tilde{\mathcal{E}}(1)$ was constructed by Bak *et al.* [14].
4. For a polynomial $p(t)$ of degree N, $|p(t)|$ belongs to $\tilde{\mathcal{E}}(2^{N/2})$ after (possibly) decomposing the real line into at most $3^{N/2}$ intervals.
5. Nevertheless, there are functions that belong to $\mathcal{E}_p(A)$ but do not belong to $\tilde{\mathcal{E}}(A')$ for any $A' > 0$. Two examples of curves that our result covers that are not covered in [11] can be constructed with the aid of the examples given below.

Example 2.4 Consider $\Phi_\beta(t) = t^{-\beta}$, $t > 0$, for $\beta \ge 2$. Then, for given $0 < t_1 < t_2 < \infty$, we have by a change of variable

$$\begin{aligned}\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \Phi_\beta(t)\,dt &= \frac{1}{(\lambda - 1)t_1^\beta}\int_1^\lambda t^{-\beta}\,dt \\ &= \frac{1}{(\lambda - 1)t_1^\beta}\int_{\lambda^{-1}}^1 t^{\beta-2}\,dt,\end{aligned}$$

where $\lambda := t_2/t_1 > 1$. Since $t^{\beta-2}$ is logarithmically concave, we see

$$\begin{aligned}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Phi_\beta(t)\,dt &\geq \frac{1}{2}\frac{1-\lambda^{-1}}{(\lambda-1)t_1^\beta}\lambda^{(-\beta+2)/2}\\ &= \frac{1}{2}\frac{1}{t_1^\beta\lambda^{\beta/2}} = \frac{1}{2}\sqrt{\Phi_\beta(t_1)\Phi_\beta(t_2)},\end{aligned}$$

which implies $\Phi_\beta \in \mathcal{E}_1(2)$. In view of Remark 2.3, given $\beta > 0$, $\Phi_\beta \in \mathcal{E}_p(2^{1/p})$ if $p \geq 2/\beta$. One can easily see $\Phi_\beta \notin \tilde{\mathcal{E}}(A')$ for any $A' > 0$ and $\beta > 0$.

Example 2.5 Consider $\Phi : (0,\infty) \to \mathbb{R}^+$ given by $\Phi(t) = (2t)^{1/2}e^{t^2}$. Then we have $\sqrt{\Phi(t)\Phi(1)} \sim t^{1/4}e^{t^2/2}$ and $\Phi((t+1)/2) = O(t^{1/2}e^{t^2/3})$ as $t \to \infty$, which clearly implies $\Phi \notin \tilde{\mathcal{E}}(A)$ for all $A > 0$. On the other hand, $\Phi \in \mathcal{E}_2(1)$ by the following.

Proposition 2.6 *Let $\psi : J \to \mathbb{R}$. Suppose that $\psi' \in \mathcal{E}_1(A)$ for some $A > 0$. Then the function Φ given by $\Phi(t) = (\psi')^{1/p}(t)\exp(\psi(t))$ belongs to $\mathcal{E}_p(A^{1/p})$ for $0 < p < \infty$.*

Proof Let $t_1 < t_2$. Since $\psi' \in \mathcal{E}_1(A)$, we have

$$\psi(t_2) - \psi(t_1) = \int_{t_1}^{t_2}\psi'(t)\,dt \geq A^{-1}(t_2-t_1)\sqrt{\psi'(t_1)\psi'(t_2)} > 0$$

by the fundamental theorem of calculus and the assumption on $\psi'(t)$. A change of variable gives

$$\begin{aligned}\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Phi^p(t)\,dt &= \frac{1}{t_2-t_1}\int_{t_1}^{t_2}e^{p\psi(t)}\psi'(t)\,dt\\ &= \frac{1}{p(t_2-t_1)}\int_{p\psi(t_1)}^{p\psi(t_2)}e^t\,dt\\ &= \frac{\psi(t_2)-\psi(t_1)}{t_2-t_1}\times\frac{e^{p\psi(t_2)}-e^{p\psi(t_1)}}{p(\psi(t_2)-\psi(t_1))}.\end{aligned}$$

From

$$\begin{aligned}\frac{e^b-e^a}{b-a} &= e^{(b+a)/2}\times\frac{e^{(b-a)/2}-e^{-(b-a)/2}}{2\times(b-a)/2}\\ &= e^{(b+a)/2}\times\frac{\sinh((b-a)/2)}{(b-a)/2} \geq e^{(b+a)/2}\end{aligned}$$

for all $a < b$, we see

$$\frac{e^{p\psi(t_2)}-e^{p\psi(t_1)}}{p(\psi(t_2)-\psi(t_1))} \geq e^{p(\psi(t_1)+\psi(t_2))/2}.$$

Altogether, we obtain

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\Phi^p(t)\,dt \geq A^{-1}e^{p(\psi(t_1)+\psi(t_2))/2}\sqrt{\psi'(t_1)\psi'(t_2)} = A^{-1}\big(\Phi(t_1)\Phi(t_2)\big)^{p/2}.$$

By taking the pth root we obtain the desired estimate. $\square$

We are now ready to state the main theorem of this paper.

Theorem 2.7 *Let $-\infty \le a < b \le \infty$, and let $\phi : (a,b) \to \mathbb{R}$ be a C^2 function such that $\phi'' \ge 0$ on the interval. Suppose that there exists a positive constant A such that $\omega \in \mathcal{E}_2(A)$, i.e.*

$$\omega^{1/2}(t_1)\omega^{1/2}(t_2) \le A\left(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \omega^2(t)\,dt\right)^{1/2}$$

holds whenever $a < t_1 < t_2 < b$. Let $\mathcal{T}$ be the operator given by (1.1). *Then there exists a constant C that depends only on A such that*

$$\|\mathcal{T}f\|_{L^{3,3/2}(\mathbb{R}^2)} \le C\|f\|_{L^{3/2}(\mathbb{R}^2)}, \tag{2.1}$$

$$\|\mathcal{T}f\|_{L^{3}(\mathbb{R}^2)} \le C\|f\|_{L^{3/2,3}(\mathbb{R}^2)} \tag{2.2}$$

holds uniformly in $f \in C_0^\infty(\mathbb{R}^2)$.

Remark 2.8 Some remarks are in order.

- In view of Remark 2.3, Proposition 2.6, Example 2.4 and Example 2.5, the condition $\omega \in \tilde{\mathcal{E}}(A)$ is strictly stronger than the condition $\omega \in \mathcal{E}_2(\sqrt{2}A)$ in Theorem 2.7, and therefore our result improves Theorem 1.2.
- An explicit example is also available. Consider $\phi(t) = t^{-1/2}\exp(t^2)$ defined for $t \in (c,\infty)$, where c is a large constant. A simple calculation shows $\omega(t) \sim t^{1/2}\exp(t^2/3)$. By Proposition 2.6, $\omega \in \mathcal{E}_2(A)$ for some $A > 0$. Thus, the corresponding operator $\mathcal{T}$ satisfies endpoint Lorentz estimates (2.1) and (2.2) by Theorem 2.7, but Theorem 1.2 is not directly applicable.
- It is not known whether $\omega \in \mathcal{E}_2(A)$ in Theorem 2.7 can be further relaxed to $\omega \in \mathcal{E}_p(A)$ for some $p > 2$. More generally, one can ask for the optimal p such that $\omega \in \mathcal{E}_p(A)$ guarantees the boundedness of $\mathcal{T}$ from $L^{\frac{3}{2},q}(\mathbb{R}^2)$ to $L^{3,r}(\mathbb{R}^2)$ for given $q \le r$.

3 Proof of the main theorem

Before we prove the theorem, we begin with a couple of lemmas.

Lemma 3.1 *Let J be an interval in $\mathbb{R}$, and let $\omega : J \to \mathbb{R}_+$ be a continuous function such that $\omega \in \mathcal{E}_2(A)$ for some $A > 0$, i.e.*

$$\omega^{1/2}(t_1)\omega^{1/2}(t_2) \le A\left(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \omega^2(t)\,dt\right)^{1/2}$$

holds whenever $t_1 < t_2$ and $[t_1,t_2] \subset J$. Then the following holds:

$$\omega(t_1)^{1/3}\omega(t_2)^{1/3}\omega(t^*)^{1/3} \le 6^{1/3}A\left(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \omega^3(t)\,dt\right)^{1/3} \tag{3.1}$$

whenever $t_1 < t_2$ and $t^* \in [t_1,t_2] \subset J$.

Proof of Lemma 3.1 Let $t^* \in [t_1,t_2] \subset J$. From

$$\omega^{1/2}(t_1)\omega^{1/2}(t_2) \le A\left(\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \omega^2(t)\,dt\right)^{1/2},$$

we obtain

$$
\begin{aligned}
&\omega^{1/2}(t_1)\omega^{1/2}(t_2)\omega^{1/2}(t^*) \\
&\quad \le A\left(\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\omega^2(t)\omega(t^*)\,dt\right)^{1/2} \\
&\quad \le A^{3/2}\left(\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\omega(t)\left|\frac{1}{t^*-t}\int_t^{t^*}\omega^2(s)\,ds\right|dt\right)^{1/2} \\
&\quad \le A^{3/2}\left(\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\omega^3(t)\,dt\right)^{1/6} \\
&\qquad \times\left(\frac{1}{t_2-t_1}\int_{t_1}^{t_2}\left|\frac{1}{t^*-t}\int_t^{t^*}\omega^2(s)\,ds\right|^{3/2}dt\right)^{1/3}
\end{aligned}
$$

by hypothesis and Hölder's inequality. Applying Hardy's inequality twice gives us

$$
\left(\int_{t_1}^{t_2}\left|\frac{1}{t^*-t}\int_t^{t^*}\omega^2(s)\,ds\right|^{3/2}dt\right)^{2/3} \le 6\left(\int_{t_1}^{t_2}\omega^3(t)\,dt\right)^{2/3},
$$

and so we obtain

$$
\omega^{1/2}(t_1)\omega^{1/2}(t_2)\omega^{1/2}(t^*) \le \frac{6^{1/2}A^{3/2}}{(t_2-t_1)^{1/2}}\left(\int_{t_1}^{t_2}\omega^3(t)\,dt\right)^{1/2}.
$$

By taking the 2/3th power, we obtain the desired estimate. □

The following lemma, which is nearly a triviality, generalizes a version of Lemma 2.2 in [11].

Lemma 3.2 *Suppose F is nonnegative and continuous on some interval $[a,b]$. For $t \in [a,b]$, we let $\tilde{F}(t) := \max_{[t,b]} F$, and for $\rho > 0$, we let*

$$
E_\rho = \{t \in [a,b] : \tilde{F}(t)(b-t) \le \rho\}.
$$

Then we have

$$
\int_{E_\rho} F(t)\,dt \le \rho.
$$

Proof of Lemma 3.2 Observe that the function $t \mapsto \tilde{F}(t)(b-t)$ is a monotone decreasing function. Let $\rho > 0$ be given. Since $b \in E_\rho$, E_ρ is nonempty. Let $t_* := \inf E_\rho$. Then we have $\tilde{F}(t_*)(b-t_*) \le \rho$. From this, we obtain

$$
\begin{aligned}
\int_{E_\rho} F(t)\,dt &= \int_{t_*}^{b} F(t)\,dt \\
&\le \tilde{F}(t_*)(b-t_*) = \rho,
\end{aligned}
$$

which finishes the proof. □

Proof of Theorem 2.7 It suffices to prove (2.1) by duality. We may further assume, without loss of generality, $-\infty < a < b < \infty$, since a uniform estimate independent of a and b will allow us a suitable limiting argument. For a measurable subset E of either $\mathbb{R}$ or $\mathbb{R}^2$, we denote the Lebesgue measure and the characteristic function of E by $|E|$ and $\mathbb{1}_E$, respectively. We also write $\gamma(t) = (t, \phi(t))$.

By a well-known interpolation argument [7, 8], it suffices to show that

$$\int_a^b \left(\int_a^b \mathbb{1}_E\big(\gamma(t_2) - \gamma(t_1)\big)\omega(t_1)\,dt_1 \right)^2 \omega(t_2)\,dt_2 \le C|E|$$

holds for measurable sets $E \subset \mathbb{R}^2$. In view of the simple identities

$$\begin{aligned}
&\int_a^b \left(\int_{t_2}^b \mathbb{1}_E\big(\gamma(t_2) - \gamma(t_1)\big)\omega(t_1)\,dt_1 \right)^2 \omega(t_2)\,dt_2 \\
&\quad = \int_a^b \left(\int_a^{a+b-t_2} \mathbb{1}_E\big(\gamma(t_2) - \gamma(a+b-t_1)\big)\omega(a+b-t_1)\,dt_1 \right)^2 \omega(t_2)\,dt_2 \\
&\quad = \int_a^b \left(\int_a^{t_2} \mathbb{1}_E\big(\gamma(a+b-t_2) - \gamma(a+b-t_1)\big)\omega(a+b-t_1)\,dt_1 \right)^2 \omega(a+b-t_2)\,dt_2 \\
&\quad = \int_a^b \left(\int_a^{t_2} \mathbb{1}_{\bar{E}}\big(\bar{\gamma}(t_2) - \bar{\gamma}(t_1)\big)\bar{\omega}(t_1)\,dt_1 \right)^2 \bar{\omega}(t_2)\,dt_2,
\end{aligned}$$

where $\bar{\gamma}(t) := (t, \bar{\phi}(t))$, $\bar{\phi}(t) := \phi(a+b-t)$, $\bar{\omega}(t) := (\bar{\phi}''(t))^{1/3} = \omega(a+b-t) \in \mathcal{E}_2(A)$, and $\bar{E} := \{(x_1, x_2) : (-x_1, x_2) \in E\}$, it is enough to establish that

$$\int_a^b \left(\int_a^{t_2} \mathbb{1}_E\big(\gamma(t_2) - \gamma(t_1)\big)\omega(t_1)\,ds_t \right)^2 \omega(t_2)\,dt_2 \le C|E| \tag{3.2}$$

holds for measurable sets $E \subset \mathbb{R}^2$. To do this, we let $\Delta := \{(t_1, t_2) : a < t_1 < t_2 < b\}$. The mapping $\Phi : \Delta \to \mathbb{R}^2$ given by $\Phi(t_1, t_2) = \gamma(t_2) - \gamma(t_1)$ is one-to-one and the absolute value of the Jacobian determinant $J(t_1, t_2)$ of Φ is given by

$$J(t_1, t_2) = \phi'(t_2) - \phi'(t_1).$$

Given measurable $\Omega \subset \Delta$ and $t_2 \in (a, b)$, we apply Lemma 3.2 with

$$\rho = \frac{1}{2} \int_a^{t_2} \mathbb{1}_\Omega(t_1, t_2)\omega(t_1)\,dt_1,$$

to obtain

$$\int_{\tilde{\omega}(t_1;t_2)(t_2-t_1) \le \rho} \mathbb{1}_\Omega(t_1, t_2)\omega(t_1)\,dt_1 \le \frac{1}{2} \int_a^{t_2} \mathbb{1}_\Omega(t_1, t_2)\omega(t_1)\,dt_1,$$

where $\tilde{\omega}(t_1; t_2) := \max_{[t_1, t_2]} \omega$. From this, we get

$$\int_{\tilde{\omega}(t_1;t_2)(t_2-t_1) \ge \rho} \mathbb{1}_\Omega(t_1, t_2)\omega(t_1)\,dt_1 \ge \frac{1}{2} \int_a^{t_2} \mathbb{1}_\Omega(t_1, t_2)\omega(t_1)\,dt_1,$$

and so

$$\begin{aligned}\frac{1}{4}\left(\int_a^{t_2} \mathbb{1}_\Omega(t_1,t_2)\omega(t_1)\,dt_1\right)^2 &\le \rho\int_{\tilde{\omega}(t_1;t_2)(t_2-t_1)\ge\rho} \mathbb{1}_\Omega(t_1,t_2)\omega(t_1)\,dt_1\\ &\le \int_{\tilde{\omega}(t_1;t_2)(t_2-t_1)\ge\rho} \mathbb{1}_\Omega(t_1,t_2)\omega(t_1)\tilde{\omega}(t_1;t_2)(t_2-t_1)\,dt_1\\ &\le \int_a^{t_2} \mathbb{1}_\Omega(t_1,t_2)\omega(t_1)\tilde{\omega}(t_1;t_2)(t_2-t_1)\,dt_1.\end{aligned}$$

Multiplying by $\omega(t_2)$ and integrating with respect to t_2 provides us with

$$\begin{aligned}&\int_a^b\left(\int_a^{t_2} \mathbb{1}_\Omega(t_1,t_2)\omega(t_1)\,dt_1\right)^2\omega(t_2)\,dt_2\\ &\quad\le 4\int_a^b\int_a^{t_2} \mathbb{1}_\Omega(t_1,t_2)\omega(t_1)\omega(t_2)\tilde{\omega}(t_1;t_2)(t_2-t_1)\,dt_1\,dt_2.\end{aligned}$$

Notice that for $a < t_1 < t_2 < b$, there exists $t_* \in [t_1,t_2]$ such that $\tilde{\omega}(t_1;t_2) = \omega(t_*)$. By Lemma 3.1, we have

$$\begin{aligned}\omega(t_1)\omega(t_2)\tilde{\omega}(t_1;t_2)(t_2-t_1) &= \omega(t_1)\omega(t_2)\omega(t_*)(t_2-t_1)\\ &\le 6A^3\int_{t_1}^{t_2}\omega^3(t)\,dt\\ &= 6A^3\int_{t_1}^{t_2}\phi''(t)\,dt\\ &= 6A^3\bigl(\phi'(t_2)-\phi'(t_1)\bigr)\\ &= 6A^3 J(t_1,t_2),\end{aligned}$$

which further implies

$$\int_a^b\left(\int_a^{t_2} \mathbb{1}_\Omega(t_1,t_2)\omega(t_1)\,dt_1\right)^2\omega(t_2)\,dt_2 \le 24A^3\int_a^b\int_a^b \mathbb{1}_\Omega(t_1,t_2)J(t_1,t_2)\,dt_2\,dt_1.$$

Letting $\Omega = \{(t_1,t_2)\in\Delta : \gamma(t_1)-\gamma(t_2)\in E\}$ and making a change of variables, we obtain the desired estimate (3.2). □

Competing interests
The author declares that he has no competing interests.

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