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A weighted $W^{2,p}$-*a priori* bound for a class of elliptic operators

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Abstract

We prove a weighted $W^{2,p}$-*a priori* bound, $p > 1$, for a class of uniformly elliptic second-order differential operators on unbounded domains. We deduce a uniqueness and existence result for the solution of the related Dirichlet problem.

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1 Introduction

In the last years the interest in weighted Sobolev spaces has been rapidly increasing, both for what concerns the structure of the spaces themselves as well as for their suitability in the theory of PDEs with prescribed boundary conditions. In particular, these spaces find a natural field of application in the case of unbounded domains. Indeed, in this setting, there is the need to impose not only conditions on the boundary of the set, but also conditions that control the behavior of the solution at infinity.

In this paper, we study, in an unbounded open subset Ω of $\mathbb{R}^n$, $n \geq 2$, the uniformly elliptic second-order linear differential operator with discontinuous coefficients:

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a,$$

and the related Dirichlet problem

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega), \\ Lu = f, \quad f \in L_s^p(\Omega), \end{cases} \tag{1.1}$$

where $p > 1$, $s \in \mathbb{R}$, and $W_s^{2,p}(\Omega)$, $\mathring{W}_s^{1,p}(\Omega)$ and $L_s^p(\Omega)$ are certain weighted Sobolev and Lebesgue spaces recently introduced in [1]. To be more precise, the considered weight ρ^s is a power of a function ρ of class $C^2(\bar{\Omega})$ such that $\rho : \bar{\Omega} \to \mathbb{R}_+$ and

$$\sup_{x \in \Omega} \frac{|\partial^\alpha \rho(x)|}{\rho(x)} < +\infty, \quad \forall |\alpha| \leq 2,$$

$$\lim_{|x| \to +\infty} \left(\rho(x) + \frac{1}{\rho(x)} \right) = +\infty \quad \text{and} \quad \lim_{|x| \to +\infty} \frac{\rho_x(x) + \rho_{xx}(x)}{\rho(x)} = 0.$$

To fix the ideas, one can think of the function

$$\rho(x) = \left(1 + |x|^2\right)^t, \quad t \in \mathbb{R} \setminus \{0\}.$$

Among the various hypotheses involving discontinuous coefficients, we consider here those of Miranda type, referring to the classical paper [2] where the a_{ij} have derivatives in $L^n(\Omega)$. Namely, we suppose that the $(a_{ij})_{x_h}$ belong to suitable Morrey-type spaces that extend to unbounded domains the classical notion of Morrey spaces (see Section 3 for the details).

Always in the framework of unbounded domains, no-weighted problems weakening the hypotheses of [2] have been studied. We quote here, for instance, [3–6] for $p = 2$ and [7] for $p > 1$. A very general case, where the a_{ij} have vanishing mean oscillation (VMO), has been taken into account in [8] (for the pioneer works considering VMO assumptions in the framework of bounded domains, we refer to [9–11]) and in [12, 13] in a weighted contest. Variational problems can be found in [14–16]. Quasilinear elliptic equations with quadratic growth have been considered in [17].

The main result of this work consists in a weighted $W^{2,p}$-bound, $p > 1$, having the only term $\|Lu\|_{L^p_s(\Omega)}$ on the right-hand side,

$$\|u\|_{W^{2,p}_s(\Omega)} \leq c\|Lu\|_{L^p_s(\Omega)}, \quad \forall u \in W^{2,p}_s(\Omega) \cap \mathring{W}^{1,p}_s(\Omega), \tag{1.2}$$

where the dependence of the constant c is completely described. Estimate (1.2) allows us to deduce the solvability of the related Dirichlet problem. This work generalizes to all $p > 1$ a previous result of [1] where a no-weighted and a weighted case have been analyzed for $p = 2$. In [18] we considered an analogous problem, with $p > 1$, but without weight. Related variational results were studied in [19–21].

2 Weight functions and weighted spaces

We start recalling the definitions of our specific weight functions. Then we will focus on certain classes of related weighted Sobolev spaces recently introduced in [1], where a detailed description and the proofs of all the properties below can be found.

Let Ω be an open subset of $\mathbb{R}^n$, not necessarily bounded, $n \geq 2$. We consider a weight $\rho : \bar{\Omega} \to \mathbb{R}_+$ such that $\rho \in C^2(\bar{\Omega})$ and

$$\sup_{x\in\Omega} \frac{|\partial^\alpha \rho(x)|}{\rho(x)} < +\infty, \quad \forall |\alpha| \leq 2. \tag{2.1}$$

An example is given by

$$\rho(x) = \left(1 + |x|^2\right)^t, \quad t \in \mathbb{R}.$$

For $k \in \mathbb{N}_0$, $p \in [1, +\infty[$ and $s \in \mathbb{R}$, and given a function ρ satisfying (2.1), we define $W^{k,p}_s(\Omega)$ as the space of distributions u on Ω such that

$$\|u\|_{W^{k,p}_s(\Omega)} = \sum_{|\alpha|\leq k} \left\|\rho^s \partial^\alpha u\right\|_{L^p(\Omega)} < +\infty, \tag{2.2}$$

equipped with the norm given in (2.2). Furthermore, we denote the closure of $C^\infty_\circ(\Omega)$ in $W^{k,p}_s(\Omega)$ by $\mathring{W}^{k,p}_s(\Omega)$, and put $W^{0,p}_s(\Omega) = L^p_s(\Omega)$.

Let us quote the following fundamental tool, which will allow us to exploit no-weighted results in order to pass to the weighted case.

Lemma 2.1 *Let $k \in \mathbb{N}_0$, $p \in [1, +\infty[$ and $s \in \mathbb{R}$. If assumption* (2.1) *is satisfied, then there exist two constants $c_1, c_2 \in \mathbb{R}_+$ such that*

$$c_1 \|u\|_{W^{k,p}_s(\Omega)} \le \|\rho^t u\|_{W^{k,p}_{s-t}(\Omega)} \le c_2 \|u\|_{W^{k,p}_s(\Omega)}, \quad \forall t \in \mathbb{R}, \forall u \in W^{k,p}_s(\Omega), \tag{2.3}$$

with $c_1 = c_1(t)$ and $c_2 = c_2(t)$.

Moreover, if Ω has the segment property, then the map

$$u \to \rho^s u$$

defines a topological isomorphism from $W^{k,p}_s(\Omega)$ to $W^{k,p}(\Omega)$ and from $\mathring{W}^{k,p}_s(\Omega)$ to $\mathring{W}^{k,p}(\Omega)$.

From now on, we also suppose that the weight ρ satisfies the further assumptions

$$\lim_{|x| \to +\infty} \left(\rho(x) + \frac{1}{\rho(x)} \right) = +\infty \quad \text{and} \quad \lim_{|x| \to +\infty} \frac{\rho_x(x) + \rho_{xx}(x)}{\rho(x)} = 0. \tag{2.4}$$

As an example, we can then consider

$$\rho(x) = \left(1 + |x|^2\right)^t, \quad t \in \mathbb{R} \setminus \{0\}.$$

Let us associate to ρ the function σ defined by

$$\begin{cases} \sigma = \rho & \text{if } \rho \to +\infty \text{ for } |x| \to +\infty, \\ \sigma = \frac{1}{\rho} & \text{if } \rho \to 0 \text{ for } |x| \to +\infty. \end{cases} \tag{2.5}$$

It is easily seen that σ verifies (2.1) too, and, moreover,

$$\lim_{|x| \to +\infty} \sigma(x) = +\infty, \qquad \lim_{|x| \to +\infty} \frac{\sigma_x(x) + \sigma_{xx}(x)}{\sigma(x)} = 0. \tag{2.6}$$

Now, we fix a cutoff function $f \in C^\infty_\circ(\bar{\mathbb{R}}_+)$ such that

$$0 \le f \le 1, \quad f(t) = 1 \quad \text{if } t \in [0,1], \qquad f(t) = 0 \quad \text{if } t \in [2, +\infty[,$$

and we set

$$\zeta_k : x \in \bar{\Omega} \to f\left(\frac{\sigma(x)}{k}\right), \quad k \in \mathbb{N},$$

and

$$\Omega_k = \left\{ x \in \Omega : \sigma(x) < k \right\}, \quad k \in \mathbb{N}. \tag{2.7}$$

Let us finally introduce the sequence

$$\eta_k : x \in \bar{\Omega} \to 2k\zeta_k(x) + \left(1 - \zeta_k(x)\right)\sigma(x), \quad k \in \mathbb{N}.$$

One has that, for any $k \in \mathbb{N}$, σ and η_k are equivalent, namely

$$c_1' \sigma \le \eta_k \le c_2' \sigma \quad \text{in } \bar{\Omega}. \tag{2.8}$$

Furthermore, concerning the derivatives, we have also, for any $k \in \mathbb{N}$,

$$\frac{(\eta_k)_x}{\eta_k} \le c_3' \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x}{\sigma} \quad \text{in } \bar{\Omega}, \tag{2.9}$$

$$\frac{(\eta_k)_{xx}}{\eta_k} \le c_4' \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x^2 + \sigma \sigma_{xx}}{\sigma^2} \quad \text{in } \bar{\Omega}. \tag{2.10}$$

Moreover,

$$\lim_{k \to +\infty} \sup_{\Omega \setminus \Omega_k} \frac{\sigma_x(x) + \sigma_{xx}(x)}{\sigma(x)} = 0. \tag{2.11}$$

3 A class of spaces of Morrey type

Here we recall the definitions and the main properties of a class of spaces of Morrey type where the coefficients of our operator will be chosen. These spaces are a generalization to unbounded domains of the classical Morrey spaces and were introduced for the first time in [22]; see also [23] for more details.

Thus, in the sequel let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \ge 2$. The σ-algebra of all Lebesgue measurable subsets of Ω is denoted by $\Sigma(\Omega)$. Given $E \in \Sigma(\Omega)$, $|E|$ is its Lebesgue measure, χ_E its characteristic function and $E(x,\tau) = E \cap B(x,\tau)$ ($x \in \mathbb{R}^n$, $\tau \in \mathbb{R}_+$), where $B(x,\tau)$ is the open ball centered in x and with radius τ.

For $\lambda \in [0, n[$, $q \in [1, +\infty[$, the space of Morrey type $M^{q,\lambda}(\Omega, t)$ ($t \in \mathbb{R}_+$) is the set of all functions g in $L^q_{\mathrm{loc}}(\bar{\Omega})$ such that

$$\|g\|_{M^{q,\lambda}(\Omega,t)} = \sup_{\substack{\tau \in]0,t] \\ x \in \Omega}} \tau^{-\lambda/q} \|g\|_{L^q(\Omega(x,\tau))} < +\infty, \tag{3.1}$$

endowed with the norm defined in (3.1). One can easily check that, for any arbitrary $t_1, t_2 \in \mathbb{R}_+$, a function g belongs to $M^{q,\lambda}(\Omega, t_1)$ if and only if it belongs to $M^{q,\lambda}(\Omega, t_2)$ and the norms of g in these two spaces are equivalent. Hence, we limit our attention to the space $M^{q,\lambda}(\Omega) = M^{q,\lambda}(\Omega, 1)$.

The closures of $C_\circ^\infty(\Omega)$ and $L^\infty(\Omega)$ in $M^{q,\lambda}(\Omega)$ are denoted by $M_\circ^{q,\lambda}(\Omega)$ and $\tilde{M}^{q,\lambda}(\Omega)$, respectively.

The following inclusions (algebraic and topological) hold true:

$$M_\circ^{q,\lambda}(\Omega) \subset \tilde{M}^{q,\lambda}(\Omega).$$

Moreover, one has

$$M^{q,\lambda}(\Omega) \subseteq M^{q_0,\lambda_0}(\Omega) \quad \text{if } q_0 \le q \text{ and } \frac{\lambda_0 - n}{q_0} \le \frac{\lambda - n}{q}. \tag{3.2}$$

We put $M^q(\Omega) = M^{q,0}(\Omega)$, $\tilde{M}^q(\Omega) = \tilde{M}^{q,0}(\Omega)$ and $M_\circ^q(\Omega) = M_\circ^{q,0}(\Omega)$.

Now, let us define the moduli of continuity of functions belonging to $\tilde{M}^{q,\lambda}(\Omega)$ or $M_{\circ}^{q,\lambda}(\Omega)$. For $h \in \mathbb{R}_+$ and $g \in M^{q,\lambda}(\Omega)$, we set

$$F[g](h) = \sup_{\substack{E\in\Sigma(\Omega) \\ \sup_{x\in\Omega}|E(x,1)|\le\frac{1}{h}}} \|g\chi_E\|_{M^{q,\lambda}(\Omega)}.$$

Given a function $g \in M^{q,\lambda}(\Omega)$, the following characterization holds:

$$g \in \tilde{M}^{q,\lambda}(\Omega) \quad \Longleftrightarrow \quad \lim_{h\to+\infty} F[g](h) = 0,$$

while

$$g \in M_{\circ}^{q,\lambda}(\Omega) \quad \Longleftrightarrow \quad \lim_{h\to+\infty} \bigl(F[g](h) + \bigl\|(1-\zeta_h)g\bigr\|_{M^{q,\lambda}(\Omega)}\bigr) = 0,$$

where ζ_h denotes a function of class $C_{\circ}^{\infty}(\mathbb{R}^n)$ such that

$$0 \le \zeta_h \le 1, \qquad \zeta_h|_{\overline{B(0,h)}} = 1, \qquad \operatorname{supp}\zeta_h \subset B(0,2h).$$

Thus, if g is a function in $\tilde{M}^{q,\lambda}(\Omega)$, a *modulus of continuity* of g in $\tilde{M}^{q,\lambda}(\Omega)$ is a map $\tilde{\sigma}^{q,\lambda}[g]: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) \le \tilde{\sigma}^{q,\lambda}[g](h), \qquad \lim_{h\to+\infty} \tilde{\sigma}^{q,\lambda}[g](h) = 0.$$

While if g belongs to $M_{\circ}^{q,\lambda}(\Omega)$, a *modulus of continuity* of g in $M_{\circ}^{q,\lambda}(\Omega)$ is an application $\sigma_{\circ}^{q,\lambda}[g]: \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$F[g](h) + \bigl\|(1-\zeta_h)g\bigr\|_{M^{q,\lambda}(\Omega)} \le \sigma_{\circ}^{q,\lambda}[g](h), \qquad \lim_{h\to+\infty} \sigma_{\circ}^{q,\lambda}[g](h) = 0.$$

We finally recall a result of [18], obtained adapting to our framework a more general embedding theorem proved in [24].

Lemma 3.1 *Let $p > 1$ and $r, t \in [p, +\infty[$. If Ω is an open subset of $\mathbb{R}^n$ having the cone property and $g \in M^r(\Omega)$, with $r > p$ if $p = n$, then*

$$u \to gu \tag{3.3}$$

is a bounded operator from $W^{1,p}(\Omega)$ to $L^p(\Omega)$. Moreover, there exists a constant $c \in \mathbb{R}_+$ such that

$$\|gu\|_{L^p(\Omega)} \le c\|g\|_{M^r(\Omega)}\|u\|_{W^{1,p}(\Omega)}, \tag{3.4}$$

with $c = c(\Omega, n, p, r)$.

If $g \in M^t(\Omega)$, with $t > p$ if $p = n/2$, then the operator in (3.3) *is bounded from $W^{2,p}(\Omega)$ to $L^p(\Omega)$. Moreover, there exists a constant $c' \in \mathbb{R}_+$ such that*

$$\|gu\|_{L^p(\Omega)} \le c'\|g\|_{M^t(\Omega)}\|u\|_{W^{2,p}(\Omega)}, \tag{3.5}$$

with $c' = c'(\Omega, n, p, t)$.

4 An *a priori* bound

Let $p > 1$ and assume that

$$\Omega \text{ has the uniform } C^{1,1}\text{-regularity property} \tag{h_0}$$

(we refer the reader, for instance, to [25] for the definition).

Consider the differential operator

$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a, \tag{4.1}$$

with the following conditions on the leading coefficients:

$$\begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega), \quad i,j = 1,\ldots,n, \\ \quad \exists \nu > 0 : \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \text{ a.e. in } \Omega, \forall \xi \in \mathbb{R}^n, \\ (a_{ij})_{x_h} \in M_o^{q,\lambda}(\Omega), \quad i,j,h = 1,\ldots,n, \text{ with} \\ \quad q > 2 \text{ and } \lambda = 0 \text{ for } n = 2, \\ \quad q \in]2,n] \text{ and } \lambda = n - q \text{ for } n > 2. \end{cases} \tag{h_1}$$

For lower-order terms, we suppose that

$$\begin{cases} a_i \in M_o^r(\Omega), \quad i = 1,\ldots,n, \text{ with} \\ \quad r > 2 \text{ if } p \leq 2 \text{ and } r = p \text{ if } p > 2 \text{ for } n = 2, \\ \quad r \geq p \text{ and } r \geq n, \text{ with } r > p \text{ if } p = n \text{ for } n > 2, \end{cases} \tag{h_2}$$

$$\begin{cases} a \in \tilde{M}^t(\Omega), \quad \text{with} \\ \quad t = p \text{ for } n = 2, \\ \quad t \geq p \text{ and } t \geq \frac{n}{2}, \text{ with } t > p \text{ if } p = \frac{n}{2} \text{ for } n > 2, \\ \operatorname{ess\,inf}_\Omega a = a_0 > 0. \end{cases} \tag{h_3}$$

Let us start observing that, in view of Lemma 3.1, under the assumptions (h_0)-(h_3), the operator $L : W^{2,p}(\Omega) \to L^p(\Omega)$ is bounded.

Then let us recall some known results contained in Theorem 3.2 and Corollary 3.3 of [18].

Theorem 4.1 *Let L be defined in* (4.1). *If hypotheses* (h_0)-(h_3) *are satisfied, then there exists a constant* $c \in \mathbb{R}_+$ *such that*

$$\|u\|_{W^{2,p}(\Omega)} \leq c\|Lu\|_{L^p(\Omega)}, \quad \forall u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega), \tag{4.2}$$

with $c = c(\Omega, n, \nu, p, r, t, \|a_{ij}\|_{L^\infty(\Omega)}, \sigma_o^{q,\lambda}[(a_{ij})_{x_h}], \sigma_o^r[a_i], \tilde{\sigma}^t[a], a_0)$.

Moreover, the problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega), \\ Lu = f, \quad f \in L^p(\Omega), \end{cases} \tag{4.3}$$

is uniquely solvable.

Now we prove the claimed weighted $W^{2,p}$-bound.

Theorem 4.2 *Let L be defined in* (4.1). *Under hypotheses* (h_0)-(h_3), *there exists a constant* $c \in \mathbb{R}_+$ *such that*

$$\|u\|_{W_s^{2,p}(\Omega)} \leq c\|Lu\|_{L_s^p(\Omega)}, \quad \forall u \in W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega), \tag{4.4}$$

with $c = c(\Omega, n, s, \nu, p, r, t, \|a_{ij}\|_{L^\infty(\Omega)}, \|a_i\|_{M^r(\Omega)}, \sigma_o^{q,\lambda}[(a_{ij})_{x_h}], \sigma_o^r[a_i], \tilde{\sigma}^t[a], a_0)$.

Proof Fix $u \in W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega)$. If $\rho \to +\infty$ for $|x| \to +\infty$, $\sigma = \rho$. Thus, in view of the isomorphism of Lemma 2.1, one has that $\sigma^s u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$.

Now if we write, $\eta_k = \eta$, for a fixed $k \in \mathbb{N}$, since η and σ are equivalent, one also has that $\eta^s u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$. Hence, the estimate in Theorem 4.1 applies giving that there exists $c_0 \in \mathbb{R}_+$ such that

$$\|\eta^s u\|_{W^{2,p}(\Omega)} \leq c_0 \|L(\eta^s u)\|_{L^p(\Omega)}, \tag{4.5}$$

with $c_0 = c_0(\Omega, n, \nu, p, r, t, \|a_{ij}\|_{L^\infty(\Omega)}, \sigma_o^{q,\lambda}[(a_{ij})_{x_h}], \sigma_o^r[a_i], \tilde{\sigma}^t[a], a_0)$.

Simple computations give then

$$\begin{aligned} L(\eta^s u) = \eta^s Lu - s\sum_{i,j=1}^n a_{ij}\big((s-1)\eta^{s-2}\eta_{x_i}\eta_{x_j}u + \eta^{s-1}\eta_{x_ix_j}u \\ + 2\eta^{s-1}\eta_{x_i}u_{x_j}\big) + s\sum_{i=1}^n a_i\eta^{s-1}\eta_{x_i}u. \end{aligned} \tag{4.6}$$

Using (4.5) and (4.6), we deduce that

$$\begin{aligned} \|\eta^s u\|_{W^{2,p}(\Omega)} \leq c_1 \Bigg(\|\eta^s Lu\|_{L^p(\Omega)} + \sum_{i,j=1}^n \big(\|\eta^{s-2}\eta_{x_i}\eta_{x_j}u\|_{L^p(\Omega)} \\ + \|\eta^{s-1}\eta_{x_ix_j}u\|_{L^p(\Omega)} + \|\eta^{s-1}\eta_{x_i}u_{x_j}\|_{L^p(\Omega)}\big) \\ + \sum_{i=1}^n \|a_i\eta^{s-1}\eta_{x_i}u\|_{L^p(\Omega)} \Bigg), \end{aligned} \tag{4.7}$$

where $c_1 \in \mathbb{R}_+$ depends on the same parameters as c_0 and on s.

On the other hand, from Lemma 3.1 and (2.9), we get

$$\|a_i\eta^{s-1}\eta_{x_i}u\|_{L^p(\Omega)} \leq c_2 \sup_{\Omega\setminus\Omega_k} \frac{\sigma_x}{\sigma}\|a_i\|_{M^r(\Omega)}\|\eta^s u\|_{W^{1,p}(\Omega)}, \tag{4.8}$$

with $c_2 = c_2(\Omega, n, p, r)$.

Putting together (2.9), (2.10), (4.7) and (4.8), we obtain the bound

$$\|\eta^s u\|_{W^{2,p}(\Omega)} \leq c_3 \left[\|\eta^s Lu\|_{L^p(\Omega)} + \left(\sup_{\Omega\setminus\Omega_k} \frac{\sigma_x^2 + \sigma\sigma_{xx}}{\sigma^2} + \sup_{\Omega\setminus\Omega_k} \frac{\sigma_x}{\sigma} \right) \|\eta^s u\|_{W^{2,p}(\Omega)} \right], \tag{4.9}$$

where c_3 depends on the same parameters as c_1 and on $\|a_i\|_{M^r(\Omega)}$.

Observe that by (2.11) it follows that there exists $k_o \in \mathbb{N}$ such that

$$\left(\sup_{\Omega \setminus \Omega_{k_o}} \frac{\sigma_x^2 + \sigma \sigma_{xx}}{\sigma^2} + \sup_{\Omega \setminus \Omega_{k_o}} \frac{\sigma_x}{\sigma} \right) \le \frac{1}{2c_3}. \tag{4.10}$$

Therefore, if we still denote by η the function η_{k_o}, combining (4.9) and (4.10), we obtain

$$\|\eta^s u\|_{W^{2,p}(\Omega)} \le 2c_3 \|\eta^s Lu\|_{L^p(\Omega)}. \tag{4.11}$$

This together with the fact that σ and η are equivalent and in view of Lemma 2.1 (applied with $t = s$ and considering σ as weight function) gives

$$\sum_{|\alpha| \le 2} \|\sigma^s \partial^\alpha u\|_{L^p(\Omega)} \le c_4 \|\sigma^s Lu\|_{L^p(\Omega)}, \tag{4.12}$$

with c_4 depending on the same parameters as c_3 and on k_o, that is (4.4).

If $\rho \to 0$ for $|x| \to +\infty$, then $\sigma = \rho^{-1}$, thus, always in view of the isomorphism of Lemma 2.1, one has that $\sigma^{-s} u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$. Therefore arguing as to get (4.12), one obtains

$$\sum_{|\alpha| \le 2} \|\sigma^{-s} \partial^\alpha u\|_{L^p(\Omega)} \le c_5 \|\sigma^{-s} Lu\|_{L^p(\Omega)}. \tag{4.13}$$

This concludes the proof of Theorem 4.2. □

5 Some uniqueness and existence results

In this last section we exploit our weighted estimate in order to deduce the solvability of the related Dirichlet problem.

A preliminary result is needed.

Lemma 5.1 *If hypothesis* (h_0) *is satisfied, then the Dirichlet problem*

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega), \\ -\Delta u + bu = f, \quad f \in L_s^p(\Omega), \end{cases} \tag{5.1}$$

is uniquely solvable, with

$$b = 1 + \left| -s(s+1) \sum_{i=1}^n \frac{\sigma_{x_i}^2}{\sigma^2} + s \sum_{i=1}^n \frac{\sigma_{x_i x_i}}{\sigma} \right|, \tag{5.2}$$

if $\rho \to +\infty$ *for* $|x| \to +\infty$, *or*

$$b = 1 + \left| -s(s-1) \sum_{i=1}^n \frac{\sigma_{x_i}^2}{\sigma^2} - s \sum_{i=1}^n \frac{\sigma_{x_i x_i}}{\sigma} \right|, \tag{5.3}$$

if $\rho \to 0$ *for* $|x| \to +\infty$.

Proof Let us first consider the case $\rho \to +\infty$ for $|x| \to +\infty$. Since $\sigma = \rho$, the function u is a solution of problem (5.1) if and only if $w = \sigma^s u$ is a solution of

$$\begin{cases} w \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ -\Delta(\sigma^{-s} w) + b\sigma^{-s} w = f, \quad f \in L^p_s(\Omega), \end{cases} \tag{5.4}$$

with b given by (5.2).

Clearly, for any $i \in \{1, \dots, n\}$, one has

$$\frac{\partial^2}{\partial x_i^2}(\sigma^{-s} w) = \sigma^{-s} w_{x_i x_i} - 2s\sigma^{-s-1}\sigma_{x_i} w_{x_i} + s(s+1)\sigma^{-s-2}\sigma_{x_i}^2 w - s\sigma^{-s-1}\sigma_{x_i x_i} w,$$

hence (5.4) is equivalent to the problem

$$\begin{cases} w \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ -\Delta w + \sum_{i=1}^n \alpha_i w_{x_i} + \alpha w = g, \quad g \in L^p(\Omega), \end{cases} \tag{5.5}$$

where

$$\alpha_i = 2s\frac{\sigma_{x_i}}{\sigma}, \quad i = 1, \dots, n,$$
$$\alpha = b - s(s+1)\sum_{i=1}^n \frac{\sigma_{x_i}^2}{\sigma^2} + s\sum_{i=1}^n \frac{\sigma_{x_i x_i}}{\sigma}, \qquad g = \sigma^s f.$$

Since σ verifies (2.1) and (2.6), by (1.6) of [3] (which gives that both σ_{x_i}/σ and $\sigma_{x_i x_i}/\sigma$ are in every $M^\tau_\circ(\Omega)$, $\tau > 1$) one has that we are in the hypotheses of Theorem 4.1, therefore (5.5) is uniquely solvable, and then problem (5.1) is uniquely solvable too.

Now assume that $\rho \to 0$ for $|x| \to +\infty$. Since in this case $\sigma = \rho^{-1}$, now the function u solves problem (5.1) if and only if $w = \sigma^{-s} u$ is a solution of

$$\begin{cases} w \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ -\Delta(\sigma^{s} w) + b\sigma^{s} w = f, \quad f \in L^p_s(\Omega), \end{cases} \tag{5.6}$$

with b given by (5.3).

Thus, problem (5.6) is equivalent to

$$\begin{cases} w \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ -\Delta w + \sum_{i=1}^n \alpha_i w_{x_i} + \alpha w = g, \quad g \in L^p(\Omega), \end{cases} \tag{5.7}$$

where

$$\alpha_i = -2s\frac{\sigma_{x_i}}{\sigma}, \quad i = 1, \dots, n,$$
$$\alpha = b - s(s-1)\sum_{i=1}^n \frac{\sigma_{x_i}^2}{\sigma^2} - s\sum_{i=1}^n \frac{\sigma_{x_i x_i}}{\sigma}, \qquad g = \sigma^{-s} f.$$

The thesis follows then arguing as in the previous case. □

Theorem 5.2 *Let L be defined in* (4.1). *Under hypotheses* (h_0)-(h_3), *the problem*

$$\begin{cases} u \in W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega), \\ Lu = f, \quad f \in L_s^p(\Omega), \end{cases} \tag{5.8}$$

is uniquely solvable.

Proof For each $\tau \in [0,1]$ we put

$$L_\tau = \tau(L) + (1-\tau)(-\Delta + b),$$

with b given by (5.2) if $\rho \to +\infty$ for $|x| \to +\infty$, or by (5.3) if $\rho \to 0$ for $|x| \to +\infty$.

By Theorem 4.2 one obtains

$$\|u\|_{W_s^{2,p}(\Omega)} \le c\|L_\tau u\|_{L_s^p(\Omega)}, \quad \forall u \in W_s^{2,p}(\Omega) \cap \mathring{W}_s^{1,p}(\Omega), \forall \tau \in [0,1].$$

Thus, taking into account the result of Lemma 5.1 and using the method of continuity along a parameter (see, *e.g.*, Theorem 5.2 of [26]), we obtain the claimed result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors conceived and wrote this article in collaboration and with the same responsibility. Both of them read and approved the final manuscript.

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References

1. Monsurrò, S, Salvato, M, Transirico, M: $W^{2,2}$ A priori bounds for a class of elliptic operators. Int. J. Differ. Equ. (2011). doi:10.1155/2011/572824
2. Miranda, C: Sulle equazioni ellittiche del secondo ordine di tipo non variazionale, a coefficienti discontinui. Ann. Mat. Pura Appl. **63**, 353-386 (1963)
3. Transirico, M, Troisi, M: Equazioni ellittiche del secondo ordine di tipo non variazionale in aperti non limitati. Ann. Mat. Pura Appl. **152**, 209-226 (1988)
4. Transirico, M, Troisi, M: Ulteriori contributi allo studio delle equazioni ellittiche del secondo ordine in aperti non limitati. Boll. Unione Mat. Ital., B **4**, 679-691 (1990)
5. Caso, L, Cavaliere, P, Transirico, M: A priori bounds for elliptic equations. Ric. Mat. **51**, 381-396 (2002)
6. Caso, L, Cavaliere, P, Transirico, M: Existence results for elliptic equations. J. Math. Anal. Appl. **274**, 554-563 (2002)
7. Caso, L, Cavaliere, P, Transirico, M: Solvability of the Dirichlet problem in $W^{2,p}$ for elliptic equations with discontinuous coefficients in unbounded domains. Matematiche **57**, 287-302 (2002)
8. Caso, L, Cavaliere, P, Transirico, M: An existence result for elliptic equations with *VMO*-coefficients. J. Math. Anal. Appl. **325**, 1095-1102 (2007)
9. Chiarenza, F, Frasca, M, Longo, P: Interior $W^{2,p}$ estimates for non divergence elliptic equations with discontinuous coefficients. Ric. Mat. **40**, 149-168 (1991)
10. Chiarenza, F, Frasca, M, Longo, P: $W^{2,p}$-Solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients. Trans. Am. Math. Soc. **336**, 841-853 (1993)
11. Ragusa, MA: Cauchy-Dirichlet problem associated to divergence form parabolic equations. Commun. Contemp. Math. **6**(3), 377-393 (2004)
12. Boccia, S, Monsurrò, S, Transirico, M: Elliptic equations in weighted Sobolev spaces on unbounded domains. Int. J. Math. Math. Sci. (2008). doi:10.1155/2008/582435
13. Boccia, S, Monsurrò, S, Transirico, M: Solvability of the Dirichlet problem for elliptic equations in weighted Sobolev spaces on unbounded domains. Bound. Value Probl. (2008). doi:10.1155/2008/901503
14. Bottaro, G, Marina, ME: Problema di Dirichlet per equazioni ellittiche di tipo varia-zionale su insiemi non limitati. Boll. Unione Mat. Ital. **8**(4), 46-56 (1973)
15. Lions, PL: Remarques sur les équations linéaires elliptiques du second ordre sous forme divergence dans les domaines non bornés. Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. **78**(8), 205-212 (1985)
16. Lions, PL: Remarques sur les équations linéaires elliptiques du second ordre sous forme divergence dans les domaines non bornés II. Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat. **79**(8), 178-183 (1985)

17. Donato, P, Giachetti, D: Quasilinear elliptic equations with quadratic growth on unbounded domains. Nonlinear Anal. **8**(10), 791-804 (1986)
18. Monsurrò, S, Transirico, M: A $W^{2,p}$-estimate for a class of elliptic operators. Int. J. Pure Appl. Math. **83**(3), 489-499 (2013). doi:10.12732/ijpam.v83i3.9
19. Monsurrò, S, Transirico, M: A L^p-estimate for weak solutions of elliptic equations. Abstr. Appl. Anal. (2012). doi:10.1155/2012/376179
20. Monsurrò, S, Transirico, M: Dirichlet problem for divergence form elliptic equations with discontinuous coefficients. Bound. Value Probl. (2012). doi:10.1186/1687-2770-2012-67
21. Monsurrò, S, Transirico, M: A priori bounds in L^p for solutions of elliptic equations in divergence form. Bull. Sci. Math. (2013, in press). doi:10.1016/j.bulsci.2013.02.002
22. Transirico, M, Troisi, M, Vitolo, A: Spaces of Morrey type and elliptic equations in divergence form on unbounded domains. Boll. Unione Mat. Ital., B **9**(7), 153-174 (1995)
23. Caso, L, D'Ambrosio, R, Monsurrò, S: Some remarks on spaces of Morrey type. Abstr. Appl. Anal. (2010). doi:10.1155/2010/242079
24. Cavaliere, P, Longobardi, M, Vitolo, A: Imbedding estimates and elliptic equations with discontinuous coefficients in unbounded domains. Matematiche **51**, 87-104 (1996)
25. Adams, RA: Sobolev Spaces. Academic Press, New York (1975)
26. Gilbarg, D, Trudinger, NS: Elliptic Partial Differential Equations of Second Order, 2nd edn. Springer, Berlin (1983)

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