Luo *Boundary Value Problems* 2011, **2011**:39
http://www.boundaryvalueproblems.com/content/2011/1/39

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Global attractor of the extended Fisher-Kolmogorov equation in H^k spaces

Hong Luo[1,2]

Correspondence: lhscnu@163.com
[1]College of Mathematics, Sichuan University, Chengdu, Sichuan 610041, PR China
Full list of author information is available at the end of the article

Abstract

The long-time behavior of solution to extended Fisher-Kolmogorov equation is considered in this article. Using an iteration procedure, regularity estimates for the linear semigroups and a classical existence theorem of global attractor, we prove that the extended Fisher-Kolmogorov equation possesses a global attractor in Sobolev space H^k for all $k > 0$, which attracts any bounded subset of $H^k(\Omega)$ in the H^k-norm.

2000 Mathematics Subject Classification: 35B40; 35B41; 35K25; 35K30.

Keywords: semigroup of operator, global attractor, extended Fisher-Kolmogorov equation, regularity

1 Introduction

This article is concerned with the following initial-boundary problem of extended Fisher-Kolmogorov equation involving an unknown function $u = u(x, t)$:

$$\begin{cases} \frac{\partial u}{\partial t} = -\beta\Delta^2 u + \Delta u - u^3 + u & in \quad \Omega \times (0, \infty), \\ u = 0, \quad \Delta u = 0, & in \quad \partial\Omega \times (0, \infty), \\ u(x, 0) = \varphi, & in \quad \Omega, \end{cases} \tag{1.1}$$

where $\beta > 0$ is given, Δ is the Laplacian operator, and Ω denotes an open bounded set of $R^n (n = 1, 2, 3)$ with smooth boundary $\partial\Omega$.

The extended Fisher-Kolmogorov equation proposed by Dee and Saarloos [1-3] in 1987-1988, which serves as a model in studies of pattern formation in many physical, chemical, or biological systems, also arises in the theory of phase transitions near Lifshitz points. The extended Fisher-Kolmogorov equation (1.1) have extensively been studied during the last decades. In 1995-1998, Peletier and Troy [4-7] studied spatial patterns, the existence of kinds and stationary solutions of the extended Fisher-Kolmogorov equation (1.1) in their articles. Van der Berg and Kwapisz [8,9] proved uniqueness of solutions for the extended Fisher-Kolmogorov equation in 1998-2000. Tersian and Chaparova [10], Smets and Van den Berg [11], and Li [12] catch Periodic and homoclinic solution of Equation (1.1).

The global asymptotical behaviors of solutions and existence of global attractors are important for the study of the dynamical properties of general nonlinear dissipative dynamical systems. So, many authors are interested in the existence of global attractors such as Hale, Temam, among others [13-23].

In this article, we shall use the regularity estimates for the linear semigroups, combining with the classical existence theorem of global attractors, to prove that the extended Fisher-Kolmogorov equation possesses, in any kth differentiable function spaces $H^k(\Omega)$, a global attractor, which attracts any bounded set of $H^k(\Omega)$ in H^k-norm. The basic idea is an iteration procedure which is from recent books and articles [20-23].

2 Preliminaries

Let X and X_1 be two Banach spaces, $X_1 \subset X$ a compact and dense inclusion. Consider the abstract nonlinear evolution equation defined on X, given by

$$\begin{cases} \frac{du}{dt} = Lu + G(u), \\ u(x,0) = u_0. \end{cases} \tag{2.1}$$

where $u(t)$ is an unknown function, L: $X_1 \to X$ a linear operator, and G: $X_1 \to X$ a nonlinear operator.

A family of operators $S(t)$: $X \to X (t \geq 0)$ is called a semigroup generated by (2.1) if it satisfies the following properties:

(1) $S(t)$: $X \to X$ is a continuous map for any $t \geq 0$,
(2) $S(0) = id$: $X \to X$ is the identity,
(3) $S(t + s) = S(t) \cdot S(s)$, $\forall t, s \geq 0$. Then, the solution of (2.1) can be expressed as

$$u(t, u_0) = S(t)u_0.$$

Next, we introduce the concepts and definitions of invariant sets, global attractors, and ω-limit sets for the semigroup $S(t)$.

Definition 2.1 Let $S(t)$ be a semigroup defined on X. A set $\Sigma \subset X$ is called an invariant set of $S(t)$ if $S(t)\Sigma = \Sigma$, $\forall t \geq 0$. An invariant set Σ is an attractor of $S(t)$ if Σ is compact, and there exists a neighborhood $U \subset X$ of Σ such that for any $u_0 \in U$,

$$\inf_{v\in\Sigma} \| S(t)u_0 - v\|_X \to 0, \quad \text{as} \quad t \to \infty.$$

In this case, we say that Σ attracts U. Especially, if Σ attracts any bounded set of X, Σ is called a global attractor of $S(t)$ in X.

For a set $D \subset X$, we define the ω-limit set of D as follows:

$$\omega(D) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)D},$$

where the closure is taken in the X-norm. Lemma 2.1 is the classical existence theorem of global attractor by Temam [17].

Lemma 2.1 Let $S(t)$: $X \to X$ be the semigroup generated by (2.1). Assume the following conditions hold:

(1) $S(t)$ has a bounded absorbing set $B \subset X$, i.e., for any bounded set $A \subset X$ there exists a time $t_A \geq 0$ such that $S(t)u_0 \in B$, $\forall u_0 \in A$ and $t > t_A$;

(2) $S(t)$ is uniformly compact, i.e., for any bounded set $U \subset X$ and some $T > 0$ sufficiently large, the set $\overline{\bigcup_{t\geq T} S(t)U}$ is compact in X.

Then the ω-limit set $\mathcal{A} = \omega(B)$ of B is a global attractor of (2.1), and $\mathcal{A}$ is connected providing B is connected.

Note that we used to assume that the linear operator L in (2.1) is a sectorial operator which generates an analytic semigroup e^{tL}. It is known that there exists a constant $\lambda \geq 0$ such that $L - \lambda I$ generates the fractional power operators $\mathcal{L}^\alpha$ and fractional order spaces X_α for $\alpha \in R^1$, where $\mathcal{L} = -(L - \lambda I)$. Without loss of generality, we assume that L generates the fractional power operators $\mathcal{L}^\alpha$ and fractional order spaces X_α as follows:

$$\mathcal{L}^\alpha = (-L)^\alpha : X_\alpha \to X, \alpha \in R^1,$$

where $X_\alpha = D(\mathcal{L}^\alpha)$ is the domain of $\mathcal{L}^\alpha$. By the semigroup theory of linear operators [24], we know that $X_\beta \subset X_\alpha$ is a compact inclusion for any $\beta > \alpha$.

Thus, Lemma 2.1 can equivalently be expressed in Lemma 2.2 [20-23].

Lemma 2.2 Let $u(t, u_0) = S(t)u_0(u_0 \in X, t \geq 0)$ be a solution of (2.1) and $S(t)$ be the semigroup generated by (2.1). Let X_α be the fractional order space generated by L. Assume:

(1) for some $\alpha \geq 0$, there is a bounded set $B \subset X_\alpha$ such that for any $u_0 \in X_\alpha$ there exists $t_{u_0} > 0$ with

$$u(t, u_0) \in B, \quad \forall t > t_{u_0};$$

(2) there is a $\beta > \alpha$, for any bounded set $U \subset X_\beta$ there are $T > 0$ and $C > 0$ such that

$$\| u(t, u_0)\|_{X_\beta} \leq C, \quad \forall t > T, \quad u_0 \in U.$$

Then, Equation (2.1) has a global attractor $\mathcal{A} \subset X_\alpha$ which attracts any bounded set of X_α in the X_α-norm.

For Equation (2.1) with variational characteristic, we have the following existence theorem of global attractor [20,22].

Lemma 2.3 Let L: $X_1 \to X$ be a sectorial operator, $X_\alpha = D((-L)^\alpha)$ and G: $X_\alpha \to X(0 < \alpha < 1)$ be a compact mapping. If

(1) there is a functional F: $X_\alpha \to R$ such that $DF = L + G$ and $F(u) \leq -\beta_1 \| u \|^2_{X_\alpha} + \beta_2$,
(2) $< Lu + Gu, u >_X \leq -C_1 \| u \|^2_{X_\alpha} + C_2$,
then
(1) Equation (2.1) has a global solution

$$u \in C([0, \infty), X_\alpha) \cap H^1([0, \infty), X) \cap C([0, \infty), X),$$

(2) Equation (2.1) has a global attractor $\mathcal{A} \subset X$ which attracts any bounded set of X, where DF is a derivative operator of F, and β_1, β_2, C_1, C_2 are positive constants.

For sectorial operators, we also have the following properties which can be found in [24].

Lemma 2.4 Let L: $X_1 \to X$ be a sectorial operator which generates an analytic semigroup $T(t) = e^{tL}$. If all eigenvalues λ of L satisfy $Re\lambda < -\lambda_0$ for some real number $\lambda_0 > 0$, then for $\mathcal{L}^\alpha (\mathcal{L} = -L)$ we have

(1) $T(t)$: $X \to X_\alpha$ is bounded for all $\alpha \in R^1$ and $t > 0$,
(2) $T(t)\mathcal{L}^\alpha x = \mathcal{L}^\alpha T(t)x$, $\forall x \in X_\alpha$,

(3) for each $t > 0$, $\mathcal{L}^{\alpha} T(t) : X \to X$ is bounded, and

$$||\mathcal{L}^{\alpha} T(t)|| \le C_{\alpha} t^{-\alpha} e^{-\delta t},$$

where $\delta > 0$ and $C_{\alpha} > 0$ are constants only depending on α,

(4) the X_{α}-norm can be defined by

$$||x||_{X_{\alpha}} = ||\mathcal{L}^{\alpha} x||_{X}, \tag{2.2}$$

(5) if $\mathcal{L}$ is symmetric, for any $\alpha, \beta \in R^1$ we have

$$< \mathcal{L}^{\alpha} u, v>_X = < \mathcal{L}^{\alpha-\beta} u, \mathcal{L}^{\beta} v>_X.$$

3 Main results

Let H and H_1 be the spaces defined as follows:

$$H = L^2(\Omega), \quad H_1 = \{u \in H^4(\Omega) : u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0\}. \tag{3.1}$$

We define the operators L: $H_1 \to H$ and G: $H_1 \to H$ by

$$\begin{cases} Lu = -\beta \Delta^2 u + \Delta u \\ G(u) = -u^3 + u, \end{cases} \tag{3.2}$$

Thus, the extended Fisher-Kolmogorov equation (1.1) can be written into the abstract form (2.1). It is well known that the linear operator L: $H_1 \to H$ given by (3.2) is a sectorial operator and $\mathcal{L} = -L$. The space $D(-L) = H_1$ is the same as (3.1), $H_{\frac{1}{2}}$ is given by $H_{\frac{1}{2}}$ = closure of H_1 in $H^2(\Omega)$ and $H_k = H^{2k}(\Omega) \cap H_1$ for $k \ge 1$.

Before the main result in this article is given, we show the following theorem, which provides the existence of global attractors of the extended Fisher-Kolmogorov equation (1.1) in H.

Theorem 3.1 The extended Fisher-Kolmogorov equation (1.1) has a global attractor in H and a global solution

$$u \in C([0, \infty), H_{\frac{1}{2}}) \cap H^1([0, \infty), H).$$

Proof. Clearly, $L = -\beta\Delta^2 + \Delta$: $H_1 \to H$ is a sectorial operator, and $G : H_{\frac{1}{2}} \to H$ is a compact mapping.

We define functional $I : H_{\frac{1}{2}} \to R$, as

$$I(u) = \frac{1}{2} \int_{\Omega} (-\beta|\Delta u|^2 - |\nabla u|^2 + u^2 - \frac{1}{2} u^4) dx,$$

which satisfies $DI(u) = Lu + G(u)$.

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{\Omega} (-\beta|\Delta u|^2 - |\nabla u|^2 + u^2 - \frac{1}{2} u^4) dx \\ &\le \frac{1}{2} \int_{\Omega} (-\beta|\Delta u|^2 + u^2 - \frac{1}{2} u^4) dx \\ &\le \frac{1}{2} \int_{\Omega} (-\beta|\Delta u|^2 + 1) dx, \\ I(u) &\le -\beta_1 ||u||^2_{H_{\frac{1}{2}}} + \beta_2, \end{aligned} \tag{3.3}$$

which implies condition (1) of Lemma 2.3.

$$
\begin{aligned}
< Lu + G(u), u >&= \int_{\Omega} (-\beta u \Delta^2 u + u\Delta u + u^2 - u^4)dx \\
&= \int_{\Omega} (-\beta |\Delta u|^2 - |\nabla u|^2 + u^2 - u^4)dx \\
&\le \int_{\Omega} (-\beta |\Delta u|^2 + u^2 - u^4)dx \\
&\le \int_{\Omega} (-\beta |\Delta u|^2 + 1)dx,
\end{aligned}
$$

$$< Lu + G(u), u >\le -C_1 ||u||^2_{H_{\frac{1}{2}}} + C_2, \tag{3.4}$$

which implies condition (2) of Lemma 2.3.

This theorem follows from (3.3), (3.4), and Lemma 2.3.

The main result in this article is given by the following theorem, which provides the existence of global attractors of the extended Fisher-Kolmogorov equation (1.1) in any kth-order space H_k.

Theorem 3.2 For any $\alpha \ge 0$ the extended Fisher-Kolmogorov equation (1.1) has a global attractor $\mathcal{A}$ in H_α, and $\mathcal{A}$ attracts any bounded set of H_α in the H_α-norm.

Proof. From Theorem 3.1, we know that the solution of system (1.1) is a global weak solution for any $\phi \in H$. Hence, the solution $u(t, \phi)$ of system (1.1) can be written as

$$u(t, \varphi) = e^{tL}\varphi + \int_0^t e^{(t-\tau)L} G(u) d\tau. \tag{3.5}$$

Next, according to Lemma 2.2, we prove Theorem 3.2 in the following five steps.

Step 1. We prove that for any bounded set $U \subset H_{\frac{1}{2}}$ there is a constant $C > 0$ such that the solution $u(t, \phi)$ of system (1.1) is uniformly bounded by the constant C for any $\phi \in U$ and $t \ge 0$. To do that, we firstly check that system (1.1) has a global Lyapunov function as follows:

$$F(u) = \frac{1}{2}\int_{\Omega} (\beta |\Delta u|^2 + |\nabla u|^2 - u^2 + \frac{1}{2}u^4)dx, \tag{3.6}$$

In fact, if $u(t, \cdot)$ is a strong solution of system (1.1), we have

$$\frac{d}{dt}F(u(t,\varphi)) =< DF(u), \frac{du}{dt} >_H. \tag{3.7}$$

By (3.2) and (3.6), we get

$$\frac{du}{dt} = Lu + G(u) = -DF(u). \tag{3.8}$$

Hence, it follows from (3.7) and (3.8) that

$$\frac{dF(u)}{dt} =< DF(u), -DF(u) >_H = - \parallel DF(u) \parallel^2_H, \tag{3.9}$$

which implies that (3.6) is a Lyapunov function.

Integrating (3.9) from 0 to t gives

$$F(u(t,\varphi)) = -\int_0^t ||DF(u)||_H^2 dt + F(\varphi). \tag{3.10}$$

Using (3.6), we have

$$\begin{aligned} F(u) &= \frac{1}{2}\int_\Omega (\beta|\Delta u|^2 + |\nabla u|^2 - u^2 + \frac{1}{2}u^4)dx \\ &\geq \frac{1}{2}\int_\Omega (\beta|\Delta u|^2 - u^2 + \frac{1}{2}u^4)dx \\ &\geq \frac{1}{2}\int_\Omega (\beta|\Delta u|^2 - 1)dx \\ &\geq C_1\int_\Omega |\Delta u|^2 dx - C_2. \end{aligned}$$

Combining with (3.10) yields

$$\begin{aligned} C_1\int_\Omega |\Delta u|^2 dx - C_2 &\leq -\int_0^t ||DF(u)||_H^2 dt + F(\varphi), \\ C_1\int_\Omega |\Delta u|^2 dx + \int_0^t ||DF(u)||_H^2 dt &\leq F(\varphi) + C_2, \\ \int_\Omega |\Delta u|^2 dx &\leq C, \quad \forall t \geq 0, \varphi \in U, \end{aligned}$$

which implies

$$||u(t,\varphi)||_{H_{\frac{1}{2}}} \leq C. \quad \forall t \geq 0, \varphi \in U \subset H_{\frac{1}{2}}, \tag{3.11}$$

where C_1, C_2, and C are positive constants, and C only depends on ϕ.

Step 2. We prove that for any bounded set $U \subset H_\alpha(\frac{1}{2} \leq \alpha < 1)$ there exists $C > 0$ such that

$$||u(t,\varphi)||_{H_\alpha} \leq C, \quad \forall t \geq 0, \varphi \in U, \alpha < 1. \tag{3.12}$$

By $H_{\frac{1}{2}}(\Omega) \hookrightarrow L^6(\Omega)$, we have

$$\begin{aligned} ||G(u)||_H^2 &= \int_\Omega |G(u)|^2 dx = \int_\Omega |u - u^3|^2 dx = \int_\Omega |u^2 - 2u^4 + u^6| dx \\ &\leq \int_\Omega (|u|^2 + 2|u|^4 + |u|^6)dx \leq C\left(\int_\Omega |u|^6 dx + 1\right) \leq C\left(||u||_{H_{\frac{1}{2}}}^6 + 1\right). \end{aligned}$$

which implies that $G: H_{\frac{1}{2}} \rightarrow H$ is bounded.

Hence, it follows from (2.2) and (3.5) that

$$\begin{aligned} ||u(t,\varphi)||_{H_\alpha} = ||e^{tL}\varphi + \int_0^t e^{(t-\tau)L} g(u)d\tau||_{H_\alpha} &\leq ||\varphi||_{H_\alpha} + \int_0^t ||(-L)^\alpha e^{(t-\tau)L} G(u)||_H d\tau \\ &\leq ||\varphi||_{H_\alpha} + \int_0^t ||(-L)^\alpha e^{(t-\tau)L}|| ||G(u)||_H d\tau \\ &\leq ||\varphi||_{H_\alpha} + C\int_0^t ||(-L)^\alpha e^{(t-\tau)L}||(||u||_{H_{\frac{1}{2}}}^6 + 1)d\tau \\ &\leq ||\varphi||_{H_\alpha} + C\int_0^t \tau^\beta e^{-\delta t} d\tau \leq C, \quad \forall t \geq 0, \varphi \in U \subset H_\alpha, \end{aligned}$$

where $\beta = \alpha (0 < \beta < 1)$. Hence, (3.12) holds.

Step 3. We prove that for any bounded set $U \subset H_\alpha (1 \le \alpha < \frac{3}{2})$ there exists $C > 0$ such that

$$||u(t,\varphi)||_{H_\alpha} \le C, \quad \forall t \ge 0, \varphi \in U \subset H_\alpha, \alpha < \frac{3}{2}. \tag{3.13}$$

In fact, by the embedding theorems of fractional order spaces [24]:

$$H^2(\Omega) \hookrightarrow W^{1,4}(\Omega), \quad H^2(\Omega) \hookrightarrow H^1(\Omega), \quad H_\alpha \hookrightarrow C^0(\Omega) \cap H^2(\Omega), \quad \alpha \ge \frac{1}{2},$$

we have

$$\begin{aligned}
||G(u)||^2_{H_{\frac{1}{2}}} &= \int_\Omega |(-L)^{\frac{1}{2}} G(u)|^2 dx =< (-L)^{\frac{1}{2}} G(u), (-L)^{\frac{1}{2}} G(u) >=< (-L)G(u), G(u) > \\
&= \int_\Omega [(\beta \Delta^2 G(u) - \Delta G(u)) G(u)] dx \le C \int_\Omega (|\Delta G(u)|^2 + |\nabla G(u)|^2) dx \\
&= C \int_\Omega (|(1-3u^2)\nabla u|^2 + |\Delta u - 6u(\nabla u)^2 - 3u^2 \Delta u|^2) dx \\
&\le C \int_\Omega (|u|^4 |\nabla u|^2 + |\nabla u|^2 + |\Delta u|^2 + |u|^2 |\nabla u|^4 + |u|^4 |\Delta u|^2) dx \\
&\le C \int_\Omega (sup_{x\in\Omega} |u|^4 |\nabla u|^2 + |\nabla u|^2 + |\Delta u|^2 + sup_{x\in\Omega} |u|^2 |\nabla u|^4 + sup_{x\in\Omega} |u|^4 |\Delta u|^2) dx \\
&\le C[sup_{x\in\Omega} |u|^4 \int_\Omega |\nabla u|^2 dx + \int_\Omega |\nabla u|^2 dx + \int_\Omega |\Delta u|^2 dx + sup_{x\in\Omega} |u|^2 \int_\Omega |\nabla u|^4 dx + sup_{x\in\Omega} |u|^4 \int_\Omega |\Delta u|^2 dx] \\
&\le C(||u||^4_{C^0} ||u||^2_{H^1} + ||u||^2_{H^1} + ||u||^2_{H^2} + ||u||^2_{C^0} ||u||^4_{W^{1,4}} + ||u||^4_{C^0} ||u||^2_{H^2}) \\
&\le C(||u||^4_{H_\alpha} ||u||^2_{H^1} + ||u||^2_{H^1} + ||u||^2_{H^2} + ||u||^2_{H_\alpha} ||u||^4_{W^{1,4}} + ||u||^4_{H_\alpha} ||u||^2_{H^2}) \\
&\le C(||u||^6_{H_\alpha} + ||u||^2_{H_\alpha}),
\end{aligned}$$

which implies

$$G: H_\alpha \to H_{\frac{1}{2}} \quad \text{is bounded for } \alpha \ge \frac{1}{2}. \tag{3.14}$$

Therefore, it follows from (3.12) and (3.14) that

$$||G(u)||_{H_{\frac{1}{2}}} < C, \quad \forall t \ge 0, \varphi \in U \subset H_\alpha, \frac{1}{2} \le \alpha < 1. \tag{3.15}$$

Then, using same method as that in Step 2, we get from (3.15) that

$$\begin{aligned}
||u(t,\varphi)||_{H_\alpha} &= ||e^{tL}\varphi + \int_0^t e^{(t-\tau)L} G(u) d\tau ||_{H_\alpha} \le ||\varphi||_{H_\alpha} + \int_0^t ||(-L)^\alpha e^{(t-\tau)L} G(u)||_H d\tau \\
&\le ||\varphi||_{H_\alpha} + C \int_0^t ||(-L)^{\alpha - \frac{1}{2}} e^{(t-\tau)L}|| \, ||G(u)||_{H_{\frac{1}{2}}} d\tau \\
&\le ||\varphi||_{H_\alpha} + C \int_0^t \tau^\beta e^{-\delta t} d\tau \le C, \quad \forall t \ge 0, \varphi \in U \subset H_\alpha,
\end{aligned}$$

where $\beta = \alpha - \frac{1}{2} (0 < \beta < 1)$. Hence, (3.13) holds.

Step 4. We prove that for any bounded set $U \subset H_\alpha (\alpha \ge 0)$ there exists $C > 0$ such that

$$||u(t,\varphi)||_{H_\alpha} \le C, \quad \forall t \ge 0, \varphi \in U \subset H_\alpha, \alpha \ge 0. \tag{3.16}$$

In fact, by the embedding theorems of fractional order spaces [24]:

$$\begin{aligned}
&H^4(\Omega) \hookrightarrow H^3(\Omega) \hookrightarrow H^2(\Omega), \quad H^4(\Omega) \hookrightarrow W^{2,4}(\Omega), \\
&\qquad H_\alpha \hookrightarrow C^1(\Omega) \cap H^4(\Omega), \quad \alpha \ge 1.
\end{aligned}$$

we have

$$
\begin{aligned}
||G(u)||_{H_1}^2 &= ||(-L)G(u)||^2 \leq C\int_\Omega (|\Delta^2 G(u)|^2 + |\Delta G(u)|^2)dx \\
&\leq C\int_\Omega [(|\Delta^2 u| + 30|\nabla u|^2|\Delta u| + 12|u||\Delta u|^2 + 18|u||\nabla u||\nabla\Delta u| + 3|u|^2|\Delta^2 u|)^2 \\
&\quad +(|\Delta u| + 6|u||\nabla u|^2 + 3|u|^2|\Delta u|)^2]dx \\
&\leq C\int_\Omega (|\Delta^2 u|^2 + |\nabla u|^4|\Delta u|^2 + |u|^2|\Delta u|^4 + |u|^2|\nabla u|^2|\nabla\Delta u|^2 + |u|^4|\Delta^2 u|^2 \\
&\quad +|\Delta u|^2 + |u|^2|\nabla u|^4 + |u|^4|\Delta u|^2)dx \\
&\leq C\int_\Omega (|\Delta^2 u|^2 + sup_{x\in\Omega}|\nabla u|^4|\Delta u|^2 + sup_{x\in\Omega}|u|^2|\Delta u|^4 + sup_{x\in\Omega}|u|^2 sup_{x\in\Omega}|\nabla u|^2|\nabla\Delta u|^2 \\
&\quad +sup_{x\in\Omega}|u|^4|\Delta^2 u|^2 + |\Delta u|^2 + sup_{x\in\Omega}|u|^2 sup_{x\in\Omega}|\nabla u|^4 + sup_{x\in\Omega}|u|^4|\Delta u|^2)dx \\
&\leq C[\int_\Omega |\Delta^2 u|^2 dx + sup_{x\in\Omega}|\nabla u|^4\int_\Omega |\Delta u|^2 dx + sup_{x\in\Omega}|u|^2\int_\Omega |\Delta u|^4 dx + sup_{x\in\Omega}|u|^2 sup_{x\in\Omega}|\nabla u|^2\int_\Omega |\nabla\Delta u|^2 dx \\
&\quad +sup_{x\in\Omega}|u|^4\int_\Omega |\Delta^2 u|^2 dx + \int_\Omega |\Delta u|^2 dx + sup_{x\in\Omega}|u|^2 sup_{x\in\Omega}|\nabla u|^4\int_\Omega dx + sup_{x\in\Omega}|u|^4\int_\Omega |\Delta u|^2 dx] \\
&\leq C(||u||_{H^4}^2 + ||u||_{C^1}^4||u||_{H^2}^2 + ||u||_{C^0}^2||u||_{W^{2,4}}^4 + ||u||_{C^0}^2||u||_{C^1}^2||u||_{H^3}^2 \\
&\quad +||u||_{C^0}^4||u||_{H^4}^2 + ||u||_{H^2}^2 + ||u||_{C^0}^2||u||_{C^1}^4 + ||u||_{C^0}^4||u||_{H^2}^2) \\
&\leq C(||u||_{H^4}^2 + ||u||_{H_\alpha}^4||u||_{H^2}^2 + ||u||_{H_\alpha}^2||u||_{W^{2,4}}^4 + ||u||_{H_\alpha}^4||u||_{H^3}^2 \\
&\quad +||u||_{H_\alpha}^4||u||_{H^4}^2 + ||u||_{H^2}^2 + ||u||_{H_\alpha}^6 + ||u||_{H_\alpha}^4||u||_{H^2}^2) \\
&\leq C(||u||_{H_\alpha}^6 + ||u||_{H_\alpha}^2)
\end{aligned}
$$

which implies

$$G : H_\alpha \to H_1 \quad \text{is bounded for } \alpha \geq 1. \tag{3.17}$$

Therefore, it follows from (3.13) and (3.17) that

$$||G(u)||_{H_1} < C, \quad \forall t \geq 0, \varphi \in U \subset H_\alpha, 1 \leq \alpha < \frac{3}{2}. \tag{3.18}$$

Then, we get from (3.18) that

$$
\begin{aligned}
||u(t,\varphi)||_{H_\alpha} = ||e^{tL}\varphi + \int_0^t e^{(t-\tau)L}G(u)d\tau||_{H_\alpha} &\leq ||\varphi||_{H_\alpha} + \int_0^t ||(-L)^\alpha e^{(t-\tau)L}G(u)||_H d\tau \\
&\leq ||\varphi||_{H_\alpha} + \int_0^t ||(-L)^{\alpha-1}e^{(t-\tau)L}||||G(u)||_{H_1}d\tau \\
&\leq ||\varphi||_{H_\alpha} + C\int_0^t \tau^\beta e^{-\delta t}d\tau \leq C, \quad \forall t \geq 0, \varphi \in U \subset H_\alpha,
\end{aligned}
$$

where $\beta = \alpha - 1 (0 < \beta < 1)$. Hence, (3.16) holds.

By doing the same procedures as Steps 1-4, we can prove that (3.16) holds for all $\alpha \geq 0$.

Step 5. We show that for any $\alpha \geq 0$, system (1.1) has a bounded absorbing set in H_α. We first consider the case of $\alpha = \frac{1}{2}$.

From Theorem 3.1 we have known that the extended Fisher-Kolmogorov equation possesses a global attractor in H space, and the global attractor of this equation consists of equilibria with their stable and unstable manifolds. Thus, each trajectory has to converge to a critical point. From (3.9) and (3.16), we deduce that for any $\varphi \in H_{\frac{1}{2}}$ the solution $u(t, \phi)$ of system (1.1) converges to a critical point of F. Hence, we only need to prove the following two properties:

(1) $F(u) \to \infty \Leftrightarrow ||u||_{H_{\frac{1}{2}}} \to \infty$,

(2) the set $S = \{u \in H_{\frac{1}{2}} | DF(u) = 0\}$ is bounded.

Property (1) is obviously true, we now prove (2) in the following. It is easy to check if $DF(u) = 0$, u is a solution of the following equation

$$\begin{cases} \beta\Delta^2 u - \Delta u - u + u^3 = 0, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0. \end{cases} \tag{3.19}$$

Taking the scalar product of (3.19) with u, then we derive that

$$\int_\Omega (\beta|\Delta u|^2 + |\nabla u|^2 - |u|^2 + |u|^4)dx = 0.$$

Using Hölder inequality and the above inequality, we have

$$\int_\Omega (|\Delta u|^2 + |\nabla u|^2 + |u|^4)dx \le C,$$

where $C > 0$ is a constant. Thus, property (2) is proved.

Now, we show that system (1.1) has a bounded absorbing set in H_α for any $\alpha \ge \frac{1}{2}$, i.e., for any bounded set $U \subset H_\alpha$ there are $T > 0$ and a constant $C > 0$ independent of ϕ such that

$$||u(t,\varphi)||_{H_\alpha} \le C, \quad \forall t \ge T, \varphi \in U. \tag{3.20}$$

From the above discussion, we know that (3.20) holds as $\alpha = \frac{1}{2}$. By (3.5) we have

$$u(t,\varphi) = e^{(t-T)L}u(T,\varphi) + \int_0^t e^{(t-\tau)L}G(u)d\tau. \tag{3.21}$$

Let $B \subset H_{\frac{1}{2}}$ be the bounded absorbing set of system (1.1), and $T_0 > 0$ such that

$$u(t,\varphi) \in B, \quad \forall t \ge T_0, \varphi \in U \subset H_\alpha\left(\alpha \ge \frac{1}{2}\right). \tag{3.22}$$

It is well known that

$$||e^{tL}|| \le Ce^{-t\lambda_1^2},$$

where $\lambda_1 > 0$ is the first eigenvalue of the equation

$$\begin{cases} \beta\Delta^2 u - \Delta u = \lambda u, \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0. \end{cases}$$

Hence, for any given $T > 0$ and $\varphi \in U \subset H_\alpha(\alpha \ge \frac{1}{2})$. We have

$$||e^{(t-\tau)L}u(t,\varphi)||_{H_\alpha} = ||(-L)^\alpha e^{(t-\tau)L}u(t,\varphi)||_H \to 0, \quad as \quad t \to \infty. \tag{3.23}$$

From (3.21),(3.22) and Lemma 2.4, for any $\frac{1}{2} \le \alpha < 1$ we get that

$$\begin{aligned} ||u(t,\varphi)||_{H_\alpha} &\le ||e^{(t-T_0)L}u(T_0,\varphi)||_{H_\alpha} + \int_{T_0}^t ||(-L)^\alpha e^{(t-\tau)L}G(u)||d\tau \\ &\le ||e^{(t-T_0)L}u(T_0,\varphi)||_{H_\alpha} + C\int_0^{t-T_0} \tau^{-\alpha}e^{-\lambda_1\tau}d\tau, \end{aligned} \tag{3.24}$$

where $C > 0$ is a constant independent of ϕ.

Then, we infer from (3.23) and (3.24) that (3.20) holds for all $\frac{1}{2} \le \alpha < 1$. By the iteration method, we have that (3.20) holds for all $\alpha \ge \frac{1}{2}$.

Finally, this theorem follows from (3.16), (3.20) and Lemma 2.2. The proof is completed.

Acknowledgements
The author is very grateful to the anonymous referees whose careful reading of the manuscript and valuable comments enhanced presentation of the manuscript. Foundation item: the National Natural Science Foundation of China (No. 11071177).

Author details
[1]College of Mathematics, Sichuan University, Chengdu, Sichuan 610041, PR China [2]College of Mathematics and Software Science, Sichuan Normal University, Chengdu, Sichuan 610066, PR China

Competing interests
The author declares that they have no competing interests.

Received: 31 May 2011 Accepted: 25 October 2011 Published: 25 October 2011

References

1. Dee, GT, Saarloos, W: Bistable systems with propagating fronts leading to pattern formation. Phys Rev Lett. **60**(25), 2641–2644 (1988). doi:10.1103/PhysRevLett.60.2641
2. Saarloos, W: Dynamical velocity selection: marginal stability. Phys Rev Lett. **58**(24), 2571–2574 (1987). doi:10.1103/PhysRevLett.58.2571
3. Saarloos, W: Front propagation into unstable states: marginal stability as a dynamical mechanism for velocity selection. Phys Rev. **37A**(1), 211–229 (1988)
4. Peletier, LA, Troy, WC: Spatial patterns described by the extended Fisher-Kolmogorov equation: Kinks. Diff Integral Eqn. **8**, 1279–1304 (1995)
5. Peletier, LA, Troy, WC: Chaotic spatial patterns described by the extended Fisher-Kolmogorov equation. J Diff Eqn. **129**, 458–508 (1996). doi:10.1006/jdeq.1996.0124
6. Peletier, LA, Troy, WC: A topological shooting method and the existence of kinds of the extended Fisher-Kolmogorov equation. Topol Methods Nonlinear Anal. **6**, 331–355 (1997)
7. Peletier, LA, Troy, WC, VanderVorst, RCAM: Stationary solutions of a fourth-order nonlinear diffusion equation. Diff Uravneniya. **31**, 327–337 (1995)
8. Van der Berg, JB: Uniqueness of solutions for the extended Fisher-Kolmogorov equation. C R Acad Sci Paris Ser I. **326**, 417–431 (1998)
9. Kwapisz, J: Uniqueness of the stationary wave for the extended Fisher-Kolmogorov equation. J Diff Eqn. **165**, 235–253 (2000). doi:10.1006/jdeq.1999.3750
10. Tersian, S, Chaparova, J: Periodic and homoclinic solutions of extended Fisher-Kolmogorov equations. J Math Anal Appl. **260**, 490–506 (2001). doi:10.1006/jmaa.2001.7470
11. Smets, D, Van den Berg, JB: Homoclinic solutions for Swift-Hohenberg and suspension bridge type equations. J Differ Eqn. **184**(1), 78–96 (2002). doi:10.1006/jdeq.2001.4135
12. Li, CY: Homoclinic orbits of two classes of fourth order semilinear differential equations with periodic nonlinearity. J Appl Math Comput. **27**, 107–116 (2008). doi:10.1007/s12190-008-0045-4
13. Hale, JK: Asymptotic Behaviour of Dissipative Systems. American Mathematical Society, Providence. (1988)
14. Lu, S, Wu, H, Zhong, CK: Attractors for nonautonomous 2D Navier-Stokes equations with normal external forces. Discrete Contin Dyn Syst. **13**(3), 701–719 (2005)
15. Ma, QF, Wang, SH, Zhong, CK: Necessary and sufficient conditions for the existence of global attractors for semigroups and applications. Indiana Univ Math J. **51**(6), 1541–1559 (2002). doi:10.1512/iumj.2002.51.2255
16. Zhong, CK, Sun, C, Niu, M: On the existence of global attractor for a class of infinite dimensional nonlinear dissipative dynamical systems. Chin Ann Math B. **26**(3), 1–8 (2005)
17. Temam, R: Infinite-Dimensional Dynamical Systems in Mechanics and Physics. In Appl Math Sci, vol. 68,Springer, New York (1997)
18. Zhong, CK, Yang, M, Sun, C: The existence of global attractors for the norm-to-weak continuous semigroup and application to the nonlinear reaction-diffusion equation. J Differ Eqn. **223**(2), 367–399 (2006). doi:10.1016/j.jde.2005.06.008
19. Nicolaenko, B, Scheurer, B, Temam, R: Some global dynamical properties of a class of pattern formation equations. Commun Part Diff Eqn. **14**, 245–297 (1989). doi:10.1080/03605308908820597
20. Ma, T, Wang, SH: Bifurcation Theory and Applications. World Scietific Series. Nonlinear Sci Ser A Monogr Treatises. World Scientific, Singapore. **153** (2005)
21. Ma, T, Wang, SH: Stability and Bifurcation of Nonlinear Evolution Equations. Science Press, China (in Chinese) (2007)
22. Ma, T, Wang, SH: Phase Transition Dynamics in Nonlinear Sciences. Springer, New York (2011, in press)
23. Song, LY, Zhang, YD, Ma, T: Global attractor of the Cahn-Hilliard equation in H^k spaces. J Math Anal Appl. **355**, 53–62 (2009). doi:10.1016/j.jmaa.2009.01.035
24. Pazy, A: Semigroups of Linear Operators and Applications to Partial Differential Equations. In Appl Math Sci, vol. 44, Springer (2006)

doi:10.1186/1687-2770-2011-39
Cite this article as: Luo: Global attractor of the extended Fisher-Kolmogorov equation in H^k spaces. *Boundary Value Problems* 2011 **2011**:39.