Xu and Song *Boundary Value Problems* 2011, **2011**:15
http://www.boundaryvalueproblems.com/content/2011/1/15

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Critical parameter equations for degenerate parabolic equations coupled via nonlinear boundary flux

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Abstract

This paper deals with the critical parameter equations for a degenerate parabolic system coupled via nonlinear boundary flux. By constructing the self-similar supersolution and subsolution, we obtain the critical global existence parameter equation. The critical Fujita type is conjectured with the aid of some new results.
Mathematics Subject Classification (2000). 35K55; 35K57.

Keywords: degenerate parabolic system, global existence, blow-up

1 Introduction

In this paper, we consider the following degenerate parabolic equations

$$\frac{\partial u_i}{\partial t} = (u_i^{p_i})_{xx}, \quad (i = 1, 2, \ldots, k), \quad x > 0, \ 0 < t < T, \tag{1.1}$$

coupled via nonlinear boundary flux

$$-(u_i^{p_i})_x(0,t) = u_{i+1}^{q_{i+1}}(0,t), \quad (i = 1, 2, \ldots, k), \ u_{k+1} := u_1, \ q_{k+1} := q_1 \ \ 0 < t < T, \tag{1.2}$$

with continuous, nonnegative initial data

$$u_i(x,0) = u_{0i}(x), \quad (i = 1, 2, \ldots, k), \quad x > 0, \tag{1.3}$$

compactly supported in $\mathbb{R}_+$, where $p_i > 1$, $q_i > 0$, $(i = 1, 2, ..., k)$ are parameters.

Parabolic systems like (1.1)-(1.3) appear in several branches of applied mathematics. They have been used to models, for example, chemical reactions, heat transfer, or population dynamics (see [1] and the references therein).

As we shall see, under certain conditions the solutions of this problem can become unbounded in a finite time. This phenomenon is known as blow-up, and has been observed for several scalar equations since the pioneering work of Fujita [2]. For further references, see the review by Leivine [3]. Blow-up may also happen for systems (see [4-7]). Our main interest here will be to determine under which conditions there are solutions of (1.1)-(1.3) that blow up and, in the blow-up case, the speed at which blowup takes place, and the localization of blow-up points in terms of the parameters p_i, q_i, $(i = 1, 2, ..., k)$.

As a precedent, we have the work of Galaktionov and Levine [8], where they studied the single equation

$$\begin{aligned}
&u_{1t} = (u_1^{p_1})_{xx}, \quad x > 0, \ 0 < t < T, \\
&-(u_1^{p_1})_x(0,t) = u_1^{q_2}(0,t), \quad 0 < t < T, \\
&u_1(x,0) = u_{01}(x), \quad x > 0.
\end{aligned} \tag{1.4}$$

It was shown if $0 < q_2 \le q_0 = (p_1 + 1)/2$, then all nonnegative solutions of (1.4) are global in time, while for $q_2 > q_0$ there are solutions with finite time blow-up. That is, q_0 is the critical global existence exponent. Moreover, it was shown that $q_c := p_1 + 1$ is a critical exponent of Fujita type. Precisely, q_c has the following properties: if $q_0 < q_2 \le q_c$, the all nontrivial nonnegative solutions blow up in a finite time, while global nontrivial nonnegative solutions exist if $q_2 > q_c$.

We remark that there are some related works on the critical exponents for (1.1)-(1.3) in special cases.

In [9-11], the authors consider the case for $p_i = 1$, ($i = 1, 2, ..., k$).

In [12], the authors consider the case for $k = 2$.

For the system (1.1)-(1.3), instead of critical exponents there are critical parameter equations, one for global existence and another of Fujita type. This is the content of our first theorem.

To state our results, we introduce some useful symbols. Denote by

$$A = \begin{pmatrix} 1+p_1 & -2q_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1+p_2 & -2q_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1+p_{k-1} & -2q_k \\ -2q_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1+p_k \end{pmatrix}$$

A series of standard computations yield

$$\det A = \prod_{l=1}^{k}(1+p_l) - \prod_{l=1}^{k} 2q_l.$$

We shall see that det $A = 0$ is the critical global existence parameter equation. Let $(\alpha_1, \alpha_2, ..., \alpha_k)^T$ be the solution of the following linear algebraic system

$$A(\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \ \alpha_k)^T = (1, \ 1, \ldots, 1, \ 1)^T,$$

that is

$$\alpha_i = \frac{\prod_{l=1}^{k}(1+p_l)}{2q_i[\prod_{l=1}^{k}(1+p_l) - \prod_{l=1}^{k} 2q_l]} \sum_{m=i}^{k+i-1} \prod_{j=i}^{m} \frac{2q_j}{1+p_j}, \ q_{k+i} = q_i, \ p_{k+i} = p_i \ (i = 1, 2, \cdots, k). \tag{1.5}$$

We define

$$\beta_i = \frac{1 + (p_i - 1)\alpha_i}{2}, \quad (i = 1, 2, \cdots, k). \tag{1.6}$$

Theorem 1.1.

(I) *If* $\prod_{l=1}^{k}(1+p_l) \ge \prod_{l=1}^{k} 2q_l$(*i.e.* det $A \ge 0$), *every nonnegative solution of (1.1)-(1.3) is global in time.*

(II) *If* $\prod_{l=1}^{k}(1+p_l) < \prod_{l=1}^{k} 2q_l$ (*i.e.* det $A < 0$) *and there exists* j ($1 \le j \le k$) *such that* $\alpha_j + \beta_j \le 0$, *then every nonnegative, nontrivial solution blows up in finite time.*

(III) *If* $\prod_{l=1}^{k}(1+p_l) < \prod_{l=1}^{k} 2q_l$ *(i.e.* $\det A < 0$*), with* $\alpha_i + \beta_i > 0$ $(i = 1, 2, ..., k)$*, there exist nonnegative solutions with blow-up and nonnegative solutions that are global.*

Therefore, the critical global existence parameter equation is

$$\prod_{l=1}^{k}(1+p_l) = \prod_{l=1}^{k} 2q_l \ (i.e.\ \det A = 0)$$

and the critical Fujita type parameter equation is

$$\min\{\alpha_1+\beta_1,\ \alpha_2+\beta_2, \ldots, \alpha_k+\beta_k\} = 0.$$

The values of α_i, β_i $(i = 1, 2, ..., k)$ are the exponents of self-similar solutions to problem (1.1)-(1.2). Such self-similar solutions are studied in Section 2, and play an important role in the proof of Theorem 1.1.

Let us observe that if we take $k = 2$, the critical parameter equations coincide with those found in [12].

The rest of this paper is organized as follows. In the next section, we study the existence of self-similar solutions of different type. In Section 3 we give some results concerning existence, comparison, monotonicity and uniqueness. In Section 4 we find the critical parameter equations (Theorem 1.1).

2 Self-similar solutions

In this section, we consider different kinds of self-similar solutions of problem (1.1)-(1.2). We have the following results.

Theorem 2.1. *Let*

$$u_i(x,t) = (T-t)^{\alpha_i} f_i(\xi_i), \quad \xi_i = x(T-t)^{-\beta_i},\ i = 1, 2, \ldots, k. \tag{2.1}$$

If

$$\prod_{l=1}^{k}(1+p_l) < \prod_{l=1}^{k} 2q_l, \tag{2.2}$$

there is a self-similar solution of problem (1.1)-(1.2) blowing up in a finite time $T > 0$*, of form (2.1). Moreover, the support of* f_i *is* $\mathbb{R}_+$ *if* $\beta_i > 0$*, and a compact set if* $\beta_i \le 0$ $(i = 1, 2, ..., k)$*.*

Theorem 2.2. *Let*

$$u_i(x,t) = t^{\alpha_i} f_i(\xi_i),\ \xi_i = xt^{-\beta_i},\ i = 1, 2, \ldots, k. \tag{2.3}$$

(a) *If*

$$\prod_{l=1}^{k}(1+p_l) > \prod_{l=1}^{k} 2q_l, \tag{2.4}$$

then there exist functions f_i *positive in* $\mathbb{R}_+$*, such that* u_i *given in (2.3) is a self-similar solution of problem (1.1)-(1.2) global in time. These solutions have* $\alpha_i > 0$ *and thus their initial data are identically zero. Then* $\beta_i < 0$ $(i = 1, 2, ..., k)$*.*

(b)*If*

$$\alpha_i + \beta_i > 0,\ \ i = 1,\ 2, \ldots,\ k,$$

then there exist functions f_i, compactly supported in $\mathbb{R}_+$, such that u_i given in (2.3) is a self-similar solution of problem (1.1)-(1.2) global in time. These solutions have $\alpha_i < 0$ and thus they decay to zero as $t \to \infty$. Then $\beta_i > 0$, and hence their supports expand as time increases.

Remark 2.2. If there exists j ($1 \leq j \leq k$) such that $\alpha_j + \beta_j \leq 0$, there are no profiles $f_i \in L^1(\mathbb{R}_+)$ such that u_i ($i = 1, 2, ..., k,$) given by (2.3) is a solution. Indeed

$$\int_0^\infty u_j(x,t)dx = t^{\alpha_j+\beta_j}\int_0^\infty f_j(\xi_j)d\xi_j.$$

Then, if $\alpha_j + \beta_j \leq 0$, the mass of u_j would not increase, a contradiction.

Theorem 2.3. *Let*

$$u_i(x,t) = e^{\alpha_i t} f_i(\xi_i), \quad \xi_i = xe^{-\beta_i t}, \qquad i = 1, 2, \ldots, k. \tag{2.5}$$

If

$$\prod_{l=1}^{k}(1+p_l) = \prod_{l=1}^{k} 2q_l, \tag{2.6}$$

for any $\alpha_1 > 0$, there is a self-similar solution of problem (1.1)-(1.2) global in time of form (2.5) where

$$\alpha_i = \alpha_1 \prod_{j=2}^{i} \frac{1+p_{j-1}}{2q_j} \; (i = 2, \ldots, k), \quad \beta_i = \frac{(p_i - 1)\alpha_i}{2} \; (i = 1, 2, \ldots, k). \tag{2.7}$$

Moreover, the supports of f_i ($i = 1, 2, ..., k$) are compact.

Remark 2.3. The solutions are in principle weak. However, if they are positive everywhere, they are also classical.

In order to prove these theorems, we will use the following results of Gilding and Peletier (see [13-15]):

Theorem 2.4. *Let a, b, $V \in \mathbb{R}$ and $U \geq 0$. For fixed a and b, let S_A denote the set of values of (U, V) such that there exists a weak, nonnegative, compactly supported solution f_1 of*

$$(f_1^{p_1})''(\eta) + a\eta f_1'(\eta) = bf_1(\eta), \quad 0 < \eta < \infty, \tag{2.8}$$

$$f_1(0) = U, \tag{2.9}$$

$$(f_1^{p_1})'(0) = V, \tag{2.10}$$

and let S_B denote the set of values (U, V) for which there exists a bounded, positive, classical solution f_1 of (2.8)-(2.10).

(a) *If $b < 0$ and $2a + b < 0$, then $S_A = \{(0, 0)\}$ and $S_B = \emptyset$.*

(b) *If $b < 0$ and $2a + b = 0$, then $S_A = \{(0, V): 0 \leq V < \infty\}$ and $S_B = \emptyset$.*

(c) *If $b \leq 0$ and $2a + b > 0$, then there exists a unique V_* such that $S_A = \{(U, U^{(p_1+1)/2}V_*) : 0 \leq U < 1\}$ and $S_B = \{(U, V): 0 \leq U < \infty, U^{(p_1+1)/2}V_* < V < \infty\}$, where $V_* > 0$ if $a + b < 0$, $V_* = 0$ if $a + b = 0$, and $V_* < 0$ if $a + b > 0$.*

(d) *If* $b > 0$ *and* $a \geq 0$, *then there exists a unique* $V_* < 0$ *such that* $S_A = \{(U, U^{(p_1+1)/2}V_*) \ : \ 0 \leq U < 1\}$*and* $S_B = \emptyset$.

(e) *If* $b > 0$ *and* $a < 0$, *or* $b = 0$ *and* $a \leq 0$, *then* $S_A = \{(0, 0)\}$ *and there exists a unique* V_* *such that* $S_B = \{(U, U^{(p1+1)/2}V_*)\colon 0 \leq U < \infty\}$, *where* $V_* < 0$ *if* $b > 0$ *and* $V_* = 0$ *if* $b = 0$.

Moreover, for each $(U, V) \in S_A \cup S_B$ *there exists at most one weak solution of (2.8)-(2.10).*

Remark 2.4. In the case where $a = ((p_1 - 1)/2)b > 0$, we have $V_* = -1$. This is a consequence of the existence for a self-similar solution of exponential form for the scalar problem (1.4) with $q_2 = (p_1 + 1)/2$ (see [8]).

Proof of Theorem 2.1. We consider solutions of form (2.1). Imposing that the porous equations (1.1) are fulfilled, we get the following relations for the parameters:

$$\alpha_i - 1 = \alpha_i p_i - 2\beta_i, \quad i = 1, 2, \ldots, k. \tag{2.11}$$

On the other hand, the boundary conditions (1.2) imply that

$$\alpha_i p_i - \beta_i = \alpha_{i+1} q_{i+1}, \quad i = 1, 2, \ldots, k, \ \ \alpha_{k+1} = \alpha_1, \ \ q_{k+1} = q_1. \tag{2.12}$$

Solving the linear systems (2.11)-(2.12), we get that α_i, β_i ($i = 1, 2, ..., k$) are given by (1.5) and (1.6). Therefore, $\alpha_i < 0$ ($i = 1, 2, ..., k$) if and only if $\prod_{l=1}^{k}(1 + p_l) < \prod_{l=1}^{k} 2q_l$. On the other hand, the profiles must satisfy

$$(f_i^{p_i})''(\xi_i) - \beta_i \xi_i f_i'(\xi_i) = -\alpha_i f_i(\xi_i), \quad i = 1, 2, \ldots, k, \tag{2.13}$$

plus the boundary conditions

$$-(f_i^{p_i})'(0) = f_{i+1}^{q_{i+1}}(0), \quad i = 1, 2, \ldots, k, \ \ q_{k+1} = q_1, \ \ f_{k+1} = f_1. \tag{2.14}$$

Then f_i satisfy (2.8) with coefficients $a_i = -\beta_I$, $b_i = -\alpha_i$ ($i = 1, 2, ..., k$). Thus, Theorem 2.4 parts (d) and (e) says that there is an one-parameter family (parameter U_i) of (2.8) satisfying

$$f_i(0) = U_i, \ \ (f_i^{p_i})'(0) = U_i^{(p_i+1)/2} V_{*i},$$

where $V_{*i} < 0$ ($i = 1, 2, ..., k$) are constants. The profile f_i has compact support if $\beta_i \leq 0$ and is positive in $\mathbb{R}_+$ if $\beta_i > 0$. We choose U_i such that the boundary conditions (2.14) are fulfilled, that is

$$-U_i^{(p_i+1)/2} V_{*i} = U_{i+1}^{q_{i+1}}, \quad i = 1, 2, \ldots, k, \ \ U_{k+1} = U_1, q_{k+1} = q_1.$$

Taking logarithms, this is equivalent to

$$A(\ln U_1, \ \ln U_2, \ldots, \ln U_{k-1}, \ \ln U_k)^T = -2(\ln|V_{*1}|, \ \ln|V_{*2}|, \ldots, \ln|V_{*k-1}|, \ \ln|V_{*k}|)^T. \tag{2.15}$$

As $\prod_{l=1}^{k}(1 + p_l) \neq \prod_{l=1}^{k} 2q_l$ (*i.e.* det $A \neq 0$), the above system has a unique solution. □

Proof of Theorem 2.2. We are considering solutions of the form (2.3). Imposing that the equations (1.1) and that boundary conditions (1.2) are fulfilled, we get that the exponents should satisfy the relations (2.11)-(2.12). Hence they are given by (1.5)-(1.6). Moreover, the boundary conditions for the profiles are given by (2.14). However, the equations for the profiles are now different:

$$(f_i^{p_i})''(\xi_i) + \beta_i \xi_i f_i'(\xi_i) = \alpha_i f_i(\xi_i), \quad i = 1,\ 2, \dots, k. \tag{2.16}$$

Thus, f_i satisfy (2.8) with coefficients $a_i = \beta_i$, $b_i = \alpha_i$ (i = 1, 2, ..., k).

(I) If $\alpha_i > 0$, that is, if (2.4) holds, then $\beta_i < 0$ (i = 1, 2, ..., k). Therefore, applying Theorem 2.4 part (d) as in the proof of Theorem 2.1, and taking the solutions of (2.15) as values for parameters, we obtain that there exist positive profiles f_i (i = 1, 2, ..., k) solving (2.16) and satisfying (2.14).

(II) If $\alpha_i < 0$ and $\alpha_i + \beta_i > 0$ (i = 1, 2, ..., k), we can apply Theorem 2.4 part (c) as in the proof of Theorem 2.1 and taking the solutions of (2.15) as the parameters, we obtain that there exist compactly supports profiles f_i (i = 1, 2, ..., k) solving (2.16) and satisfying the boundary conditions (2.14).

Proof of Theorem 2.3. We are considering solutions of the form (2.5). Though the boundary conditions (1.2) impose (2.12) again, now equations (1.1) impose different relations for the exponents. Namely

$$\alpha_i = \alpha_i p_i - 2\beta_i, \quad i = 1,\ 2, \dots, k. \tag{2.17}$$

Thus,

$$A(\alpha_1, \alpha_2, \dots, \alpha_{k-1},\ \alpha_k)^T = (0,\ 0, \dots, 0,\ 0)^T. \tag{2.18}$$

There are nontrivial solutions of (2.18) if and only if $\prod_{l=1}^k (1 + p_l) = \prod_{l=1}^k 2q_l$ (*i.e.* det $A = 0$). In this case, β_1, α_I, β_i (i = 2, ...,k) are related to α_1 by (2.7).

The boundary conditions for the profiles are again given by (2.14), while the equations for the profiles are given by (2.16). If $\alpha_1 > 0$, then $\beta_1, \alpha_i, \beta_i > 0$ (i = 2, ..., k) and $\beta_i = ((p_i - 1)/2)\alpha_i$ (i = 1, ..., k). Hence, using Remark 2.4, we have solutions of (2.16) with $V_{*i} = -1$ (i = 1, 2, ...,k). Choosing one of the solutions of (2.15) with right-hand side zero (again we are using $\prod_{l=1}^k (1 + p_l) = \prod_{l=1}^k 2q_l$ (*i.e.* det A = 0)), we obtain that there exist compactly supported profiles f_i (i = 1, 2, ..., k) solving (2.16) and satisfying (2.14).

3 Existence and uniqueness

First, we state a theorem that guarantees the existence of a solution. It can be obtained using a standard monotonicity argument following ideas from [16].

Theorem 3.1. *Given continuous, compactly supported initial data $u_{0i}(x)$ (i = 2, ..., k), there exists a local in time continuous weak solution of (1.1)-(1.3). Moreover, if the initial data are smooth and compatible in sense that*

$$-(u_{0i}^{p_i})_x(0) = u_{0i+1}^{q_{i+1}}(0), \quad i = 2, \dots, k, \quad u_{0k+1}(x) = u_{01}(x),$$

then the solution has continuous time derivatives down to t = 0.

Proof. Let us consider the Neumann problem

$$\begin{aligned} &w_t = (w^r)_{xx}, \quad x > 0,\ 0 < t < \tau, \\ &-(w^r)_x(0,t) = h(t), \quad 0 < t < \tau, \\ &w(x,0) = w_0(x), \quad x > 0, \end{aligned} \tag{3.1}$$

with $r > 1$. We define the operator $M_{q_{i+1}} : C([0,\tau]) \to C([0,\tau])$ as $M_{q_{i+1}}(h)(t) = w^{q_{i+1}}(0,t)$, where $w(x, t)$ is the unique solution of (3.1) with $r = p_i$ and initial condition $w_0(x) = u_{0i}(x)$ $\left(i = 1, 2, \dots, k, M_{q_{k+1}} = M_{q_1}, w^{q_{k+1}} = w^{q_1}\right)$.

It has been proved in [17] that $M_{q_i}(i = 1, 2, ..., k)$ is continuous and compact. Moreover, they are order preserving.

Now let $A(h) = M_{q_k} \circ M_{q_{k-1}} \circ \cdots \circ M_{q_2} \circ M_{q_1}(h)$. Using the method of monotone iterations, one can prove that there exist $\tau > 0$ such that A has a fixed point in $C([0, \tau])$. This fixed point provides us with a continuous weak solution of (1.1)-(1.3) up to time τ.

In order to obtain the regularity of the solution with compatible initial data, we only have to observe that the solution of (3.1) is regular if $-(w_0^r)_x = h(0)$ (see [18]).

Remark 3.1. If the initial data are compactly support, the solution u_i ($i = 1, 2, ..., k$) also has compact support as long as it exists.

Remark 3.2. If the initial data are nontrivial, we can assume that they satisfy $u_{0i}(x) > 0$ ($i = 1, 2, ..., k$). If not, $u_i(0, t)$ ($i = 1, 2, ..., k$) eventually become positive (compare with a Barenblatt solution of the corresponding equation).

Next, we define what called a subsolution and a supersolution for (1.1)-(1.2).

Definition 3.1. $(\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{k-1}, \underline{u}_k)$*is a subsolution of (1.1)-(1.2) if it satisfies*

$$\frac{\partial \underline{u}_i}{\partial t} \le (\underline{u}_i^{p_i})_{xx}\ x > 0,\ 0 < t < T, \quad i = 1,\ 2, \ldots, k, \tag{3.2}$$

$$-(\underline{u}_i^{p_i})_x(0, t) \le \underline{u}_{i+1}^{q_{i+1}}(0, t),\ \underline{u}_{k+1}^{q_{k+1}} = \underline{u}_1^{q_1},\ 0 < t < T, \quad i = 1,\ 2, \ldots, k. \tag{3.3}$$

Definition 3.2. *We call* $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{k-1}, \bar{u}_k)$ *a supersolution of (1.1)-(1.2) of it satisfies (3.2)-(3.3) with the opposite inequalities.*

With these definitions of super and subsolutions, we can state a comparison lemma.

Lemma 3.1 *Let* $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{k-1}, \bar{u}_k)$*be a supersolution and* $(\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_{k-1}, \underline{u}_k)$*be a subsolution. If*

$$\underline{u}_i(x, 0) \le \bar{u}_i(x, 0), \quad i = 1,\ 2, \ldots, k,$$

with

$$\underline{u}_i(0, 0) \le \bar{u}_i(0, 0), \quad i = 1,\ 2, \ldots, k,$$

then

$$\underline{u}_i(x, t) \le \bar{u}_i(x, t), \quad i = 1,\ 2, \ldots, k,$$

as long as both super and subsolutions exist.

Proof. It is standard, therefore we omit the details. Assume that the result is false. Let t_0 be the maximum time such that

$$\underline{u}_i(x, t) \le \bar{u}_i(x, t),\ i = 1,\ 2, \ldots, k,$$

up to t_0. This time t_0 must be positive, by continuity. At that time, we must have $\underline{u}_j(0, t_0) = \bar{u}_j(0, t_0)$ for some j ($1 \le j \le k$). Let us assume that $\underline{u}_1(0, t_0) = \bar{u}_1(0, t_0)$. Now the result follows by an application of Hopf's lemma. Indeed, $\bar{u}_1 - \underline{u}_1$ satisfies a uniformly parabolic equation in a neighborhood of $x = 0$, attains a minimum at $(0, t_0)$, and the corresponding flux is greater or equal than zero, a contradiction.

Now we state a lemma that guarantees that, for certain initial data, the solution of (1.1)-(1.3) increases in time.

Lemma 3.2 *Let $u_{0i}(x)$ be the initial data for (1.1) -(1.3) such that $u_{0i}(x)$ are smooth, satisfy the compatibility condition at the boundary and $(u_{0i}^{p_i})_{xx} \geq 0$. Then $u_i(x, t)$ increases in time, i.e., $u_{it}(x, t) \geq 0$ (i = 1, 2, ...,k).*

Proof. Let $w_i = u_{it}$. Then, as the solutions are smooth (Theorem 3.1), we can differentiate to obtain the $(w_1, ..., w_k)$ is a solution of

$$w_{it} = (p_i u_i^{p_i-1} w_i)_{xx}, \quad i = 1, \quad 2, \dots k, \tag{3.4}$$

$$-(p_i u_i^{p_i-1} w_i)_x(0, t) = q_{i+1} u_{i+1}^{q_{i+1}-1} w_{i+1}(0, t), \quad q_{k+1} = q_1, \quad u_{k+1} = u_1, \quad w_{k+1} = w_1, \tag{3.5}$$

with initial data satisfying

$$w_i(x, 0) \geq 0, \quad i = 1, 2, \dots, k.$$

To conclude the proof we apply the maximum principle. Due to the degeneration of the equations this cannot be done directly. A standard regularization procedure is needed (see [8] for details).

Next, we deal with the problem of uniqueness versus non-uniqueness for (1.1)-(1.3) on the case of vanishing initial data ($u_{0i}(x)$ = 0, i = 1, 2, ..., k).

Theorem 3.2

(a) *Let $\prod_{l=1}^{k}(1 + p_l) > \prod_{l=1}^{k} 2q_l$. Then there exists a nontrivial solution with zero initial data that becomes positive at × = 0 instantaneously. Then there is no uniqueness for problem (1.1)-(1.3) with zero initial data.*

(b) *Let $\prod_{l=1}^{k}(1 + p_l) \leq \prod_{l=1}^{k} 2q_l$. Then the solution of (1.1)-(1.3) with zero initial data is unique.*

Proof.

(a) The self-similar solutions constructed in Theorem 2.2 become positive at x = 0 instantaneously.

(b) We can construct small supersolution with the aid of the self-similar ones of exponential form that we found in Theorem 2.3. First, choose $\widetilde{q}_1 \leq q_1$ such that $\bar{u}_i(x, t) = e^{\alpha_i(t+\tau)} f_i(xe^{-\beta_i(t+\tau)}), \quad i = 1, 2, \dots, k,$.

$$\bar{u}_i(x, t) = e^{\alpha_i(t+\tau)} f_i(xe^{-\beta_i(t+\tau)}), \quad i = 1, 2, \dots, k,$$

where $\alpha_1 > 0$ is arbitrary and β_1, α_i, β_i, (i = 2, ..., k) are given by (2.7). Now we observe that $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{k-1}, \bar{u}_k)$ be a supersolution is a supersolution of (1.1)-(1.3) as long as $\overline{u}_1(0, t) \leq 1$. By the comparison Lemma 3.1, we obtain that every solution has initial data identically zero satisfies

$$\bar{u}_i(x, t) \geq u_i(x, t), \quad i = 1, 2, \dots, k.$$

As $\bar{u}_i$ can be chosen as small as we want (using τ negative and large enough) we conclude that $\bar{u}_i \equiv 0$ $(i = 1, 2, \dots, k)$.

4 Blow-up versus global existence

We devote this section to prove Theorem 1.1. We borrow ideas from [8]. However, the fact that we are dealing with a system instead of a single equation forces us to develop a significantly different proof. We will organize the proof in several lemmas.

Our first lemma proves part (I) of Theorem 1.1.

Lemma 4.1. *If* $\prod_{l=1}^{k}(1+p_l) \geq \prod_{l=1}^{k} 2q_l$(*i.e.* det $A \geq 0$), *every nonnegative solution of (1.1)-(1.3) is global in time.*

Proof. It is enough to construct global supersolutions with initial data as large as needed. We achieve this with the aid of the self-similar solutions of exponential form that we found in Theorem 2.3.

First we choose $\widetilde{q}_1 \geq q_1$ such that $2\widetilde{q}_1 \prod_{l=2}^{k} 2q_l = \prod_{l=1}^{k}(1+p_l)$ and we let

$$\bar{u}_i(x,t) = e^{\alpha_i(t+\tau)} f_i(xe^{-\beta_i(t+\tau)}), \quad i = 1,\ 2, \ldots, k,$$

where $\alpha_1 > 0$ is arbitrary and β_1, α_i, β_i, (i = 2, ..., k) are given by (2.7). Now we observe that $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_{k-1}, \bar{u}_k)$ is a supersolution of (1.1)-(1.3) as long as $\bar{u}_1(0,t) \geq 1$. This can be done by choosing τ large enough. This also allows to assume $\bar{u}_i(x,0) \geq u_{0i}(x)(i = 1,\ 2, \ldots, k)$. Then, by the comparison Lemma 3.1, we obtain that every solution is global.

Now we construct subsolutions with finite time blow-up.

Lemma 4.2. *Let* $\prod_{l=1}^{k}(1+p_l) < \prod_{l=1}^{k} 2q_l$ (*i.e.* det $A < 0$), *then there exist compactly supported functions* g_i *(i = 1, 2, ..., k), such that*

$$\underline{u}_i(x,t) = (T-t)^{\alpha_i} g_i(\xi_i),\ \xi_i = x(T-t)^{-\beta_i}, \quad i = 1,\ 2, \ldots, k,$$

is a subsolution of (1.1)-(1.2).

Proof. To satisfy (3.2) and (3.3), we need that

$$\begin{aligned} &(g_i^{p_i})''(\xi_i) \geq -\alpha_i g_i(\xi_i) + \beta_i \xi_i g'_i(\xi_i), \quad i = 1,\ 2, \ldots, k, \\ -&(g_i^{p_i})'(0) \leq g_{i+1}^{q_{i+1}}(0), \quad i = 1,\ 2, \ldots, k,\ \ q_{k+1} = q_1,\ \ g_{k+1} = g_1. \end{aligned}$$

We choose

$$g_i(\xi_i) = A_i(a_i - \xi_i)_+^{1/(p_i-1)}, \quad i = 1,\ 2, \ldots, k.$$

Inserting this in the equation, we get

$$\frac{p_i}{(p_i-1)^2} A_i^{p_i-1} \geq -\alpha_i(a_i - \xi_i)_+ - \frac{\beta_i}{p_i - 1}\xi_i \textit{ for } 0 \leq \xi_i \leq a_i, \quad i = 1, 2, \ldots, k.$$

Hence, it is enough to impose

$$\frac{p_i}{(p_i-1)^2} A_i^{p_i-1} \geq -\alpha_i a_i + \frac{|\beta_i|}{p_i-1} a_i, \quad i = 1,\ 2, \ldots, k,$$

that is

$$C_i A_i^{p_i-1} \geq a_i, \quad i = 1,\ 2, \ldots, k. \tag{4.1}$$

The boundary conditions impose

$$\frac{p_i}{p_i-1} A_i^{p_i} a_i^{1/(p_i-1)} \geq A_{i+1}^{q_{i+1}} a_{i+1}^{q_{i+1}/(p_{i+1}-1)}, \quad i = 1, 2, \ldots, k,\ A_{k+1} = A_1,\ \ q_{k+1} = q_1,\ \ a_{k+1} = a_1. \tag{4.2}$$

Let

$$b_i = A_i a_i^{1/(p_i-1)}, \quad i = 1,\ 2, \ldots, k.$$

Then conditions (4.2) become

$$\frac{p_i}{p_i - 1} A_i^{p_i - 1} b_i \geq b_{i+1}^{q_{i+1}}, \quad i = 1,\ 2, \ldots, k,\ b_{k+1}^{q_{k+1}} = b_1^{q_1}. \tag{4.3}$$

We fix $b_i = 1$ ($i = 1, 2, ..., k$) and then A_i large enough (and thus a_i small) to satisfy (4.1) and (4.3).

Corollary 4.1 *Let* $\prod_{l=1}^{k}(1 + p_l) < \prod_{l=1}^{k} 2q_l$ *(i.e.* det $A < 0$). *Then there exist solutions of (1.1)-(1.3) that blow up in a finite time.*

Proof. We only have to apply Lemma 3.1, to obtain that every solution (u_1, ..., u_k) that begins above the subsolutions provided by Lemma 4.2 has finite time blow-up.

Lemma 4.3 *Let* $\prod_{l=1}^{k}(1 + p_l) < \prod_{l=1}^{k} 2q_l$*(i.e.* det $A < 0$). *If there exists j ($1 \leq j \leq k$) such that $\alpha_j + \beta_j \leq 0$, then every nontrivial solution of (1.1)-(1.3) blows up in finite time.*

Proof. Without loss of generality, we consider the case $\alpha_1 + \beta_1 \leq 0$.

Assume that there exists a global nonnegative nontrivial solution of (1.1)-(1.3), we make the following change of variables

$$\varphi_i(\xi_i, \tau) = (1 + t)^{-\alpha_i} u_i(\xi_i(1 + t)^{\beta_i}, t),\ \tau = \log(1 + t), \quad i = 1,\ 2, \ldots, k. \tag{4.4}$$

These functions satisfy

$$\varphi_{i\tau} = (\varphi_i^{p_i})_{\xi_i\xi_i} + \beta_i\xi_i\varphi_{i\xi_i} - \alpha_i\varphi_i, \quad i = 1, 2, \ldots, k, \tag{4.5}$$

$$-(\varphi_i^{p_i})_{\xi_i}(0, \tau) = \varphi_{i+1}^{q_{i+1}}(0, \tau), \quad i = 1,\ 2, \ldots, k,\ \varphi_{k+1}^{q_{k+1}} = \varphi_1^{q_1}. \tag{4.6}$$

As $u_i(x, t)$ ($i = 1, 2, ..., k$) are by hypothesis global, the same is true for ϕ_i ($i = 1, 2, ..., k$,). We will construct a solution $(\widehat{\varphi}_1, \ldots, \widehat{\varphi}_k)$ to system (4.5)-(4.6) increasing with time, with initial data $(\widehat{\varphi}_{01}, \ldots, \widehat{\varphi}_{0k})$ such that $\widehat{\varphi}_{0i}(\xi_i) \leq u_i(\xi_i, 0)$ $(i = 1, 2, \ldots, k)$. We will prove that $(\widehat{\varphi}_1, \ldots, \widehat{\varphi}_k)$ cannot exists globally, thus contradicting the global existence of (u_1, ..., u_k). In order to achieve our goal, we use an adaptation for systems of the general monotonicity for single quasilinear equation described in [19].

We take initial data $(\widehat{\varphi}_{01}, \ldots, \widehat{\varphi}_{0k})$ satisfying

$$(\widehat{\varphi}_{0i}^{p_i})_{\xi_i\xi_i} + \beta_i\xi_i(\widehat{\varphi}_{0i})_{\xi_i} - \alpha_i\widehat{\varphi}_{0i} \geq 0, \quad i = 1,\ 2, \ldots, k,$$

and the compatibility conditions

$$-(\widehat{\varphi}_{0i}^{p_i})_{\xi_i}(0) = \widehat{\varphi}_{0i+1}^{q_{i+1}}(0), \quad i = 1,\ 2, \ldots, k,\ \widehat{\varphi}_{0k+1}^{q_{k+1}} = \widehat{\varphi}_{01}^{q_1}.$$

Hence, arguing as in Lemma 3.2, we have that $\widehat{\varphi}_{i\tau} \geq 0$ $(i = 1,\ 2, \ldots, k)$.

Following an idea for scalar equation from [8], we set

$$\widehat{\varphi}_{01}(\xi_1) = h(\xi_1 + b),$$

where h is the Barenblatt profile

$$h(\xi_1) = a_{p_1}(c - \xi_1^2)_+^{1/(p_1 - 1)}.$$

Then we have

$$(\widehat{\varphi}_{01}^{p_1})_{\xi_1\xi_1} + \beta_1\xi_1(\widehat{\varphi}_{01})_{\xi_1} - \alpha_1\widehat{\varphi}_{01} = -\frac{1}{p_1+1}bh_{\xi_1}(\xi_1+b)$$
$$+(\beta_1 - \frac{1}{p_1+1})\xi_1 h_{\xi_1}(\xi_1+b) + (-\alpha_1 - \frac{1}{p_1+1})h(\xi_1+b).$$

The last expression is nonnegative if $\beta_1 - 1/(p_1 + 1) \le 0$ and $-\alpha_1 - 1/(p_1 + 1) \ge 0$. But these two conditions are equivalent $\alpha_1 + \beta_1 \le 0$.

Now we take $\widetilde{\alpha}_i,\ \widetilde{\beta}_i > 0$ such that $\widetilde{\alpha}_i \ge \alpha_i,\ \widetilde{\beta}_i \ge \beta_i (i = 1,\ 2, \ldots, k)$. We take as $\widehat{\varphi}_{0i}$ a solution to

$$(\widehat{\varphi}_{0i}^{p_i})'' = -\widetilde{\beta}_i\xi_i\widehat{\varphi}'_{0i} + \widetilde{\alpha}_i\widehat{\varphi}_{0i}, \quad i = 1,\ 2, \ldots, k.$$

There is one-parameter family of solution to this equation (see Theorem 2.4), with $\widehat{\varphi}_{0i} \ge 0,\ \widehat{\varphi'}_{0i} \le 0 (i = 2, \cdots, k)$. Hence,

$$(\widehat{\varphi}_{0i}^{p_i})'' \ge -\beta_i\xi_i\widehat{\varphi}'_{0i} + \alpha_i\widehat{\varphi}_{0i}, \quad i = 2, \ldots, k.$$

Moreover,

$$\widehat{\varphi}_{0i}(0) = U_i,\ (\widehat{\varphi}_{0i}^{p_i})'(0) = U_i^{(p_i+1)/2}V_{*i}, \quad i = 2, \ldots, k,$$

where $V_{*i} < 0$ is a constant and U_i is the free parameter.

We still have to control the boundary conditions. In order to do this, we choose the constants c, b and U_i (i = 2, ...,k) conveniently. They have to satisfy

$$\frac{2p_1 a_{p_1}}{p_1 - 1}b(c-b^2)^{1/(p_1-1)} = U_2^{q_2},\ b \in (0, c^{1/2}),$$
$$-V_{*i}U_i^{(p_i+1)/2} = a_{p_1}^{q_{i+1}}(c-b^2)^{q_{i+1}/(p_1-1)},\ i = 2, \ldots, k.$$

Thus, we choose

$$U_2 = c_2 b^{2q_3/(2q_2q_3-p_2-1)},\ \ U_i = c_i b^{2(p_2-1)q_{i+1}}/((p_i+1)(2q_2q_3-p_2-1)), (i = 3, \ldots, k,\ q_{k+1} = q_1),$$
$$c = b^2 + \gamma b^{(p_1-1)(p_2-1)/(2q_2q_3-p_2-1)},$$

where c_i (i = 2, ..., k) and γ are positive constants. Taking b small enough, the initial data $(\widehat{\varphi}_{01}, \ldots, \widehat{\varphi}_{0k})$ is below ($u_1(\xi_1,0)$, ...,$u_k(\xi_k, 0)$). This can be done as u_{0i} (i = 1, 2, ... k) can be assumed to be positive at the origin.

To conclude the proof, we will show that $(\widehat{\varphi}_1, \ldots, \widehat{\varphi}_k)$ converge to a self-similar profile that does not exist in this range of parameters.

Lemma 4.4. *There exists j (*$1 \le j \le k$*) such that*

$$\lim_{\tau\to\infty}\widehat{\varphi}_j(\xi_j, \tau) = \widetilde{\varphi}_j(\xi_j) < \infty,\ \forall\xi_j > 0. \tag{4.7}$$

Proof. It is clear that $\widehat{\varphi}_{i\xi_i} \le 0\ (i = 1,\ 2, \ldots, k)$. Let us suppose that

$$\widehat{\varphi}_i(\xi_i, \tau) \to \infty \textit{ uniformly in } (0, \xi_{i0}), \quad i = 1,\ 2, \ldots, k.$$

In the original variables $(\widehat{u}_1, \ldots, \widehat{u}_k)$, we have that for any $M > 0$ there is a value such that

$$(1+t_0)^{\alpha_i}M \le \widehat{u}_i(x, t_0) \textit{ for } 0 < x(1+t_0)^{-\beta_i} < \xi_{i0},\ i = 1,\ 2, \cdots, k. \tag{4.8}$$

Now we will check that, under these conditions, we can put one of the blowing up subsolutions constructed in Lemma 4.2 below these data. This would lead to a contradiction, as $(\widehat{u}_1, \dots, \widehat{u}_k)$ is global. In order to do this, we need

$$\begin{aligned} &(1+t_0)^{a_1} M \geq A_1 a_1^{1/(p_1-1)} T^{\alpha_1}, \\ &\xi_{10}(1+t_0)^{\beta_1} \geq a_1 T^{\beta_1}. \end{aligned} \tag{4.9}$$

The first equation says that the height at $x = 0$ of $\widehat{u}_1$ is bigger than that of $\underline{u}_1$, and the second says that the support of $\widehat{u}_1$ is bigger than the support of $\underline{u}_1$. Imposing analogous conditions for $\widehat{u}_i$ and $\underline{u}_i (i = 2, \dots, k)$ we get

$$\begin{aligned} &(1+t_0)^{a_i} M \geq A_i a_i^{1/(p_i-1)} T^{\alpha_i}, \\ &\xi_{i0}(1+t_0)^{\beta_i} \geq a_i T^{\beta_i}. \end{aligned} \tag{4.10}$$

Taking $T = 1 + t_0$, then a_i small enough and A_i large enough (i = 1, 2, ..., k), and then M large, then the $2k$ conditions (4.9)-(4.10) are fulfilled.

Let us remark this parametric evolution comparison method to prove global nonexistence for arbitrary data first introduced in [20], for scalar quasilinear heat equation.

End of the proof of Lemma 4.3. Let us assume that (4.7) holds. Using standard arguments, see [8], we may pass to the limit to obtain that

$$(\widetilde{\varphi}_1^{p_1})_{\xi_1\xi_1} + \beta_1 \xi_1 \widetilde{\varphi}_{1\xi_1} - \alpha_1 \widetilde{\varphi}_1 = 0. \tag{4.11}$$

Let $z = \widetilde{\varphi}_1^{p_1}$, then

$$z_{\xi_1\xi_1} + \frac{\beta_1}{p_1} \xi_1 z^{(1-p_1)/p_1} z_{\xi_1} \leq 0.$$

Hence, in (0, ξ_{10}), $z \geq c > 0$,

$$z_{\xi_1\xi_1} \leq C z_{\xi_1}.$$

We conclude that z and therefore $\widetilde{\varphi}_1$ cannot be unbounded at $\xi_1 = 0$. In particular, $0 < \widetilde{\varphi}_1(0) \leq C$. Then, considering the regularity of $\widetilde{\varphi}_1$ in the region where $\widetilde{\varphi}_1 > 0$, we can pass to the limit in the boundary condition for $(\widehat{\varphi}_1^{p_1})_{\xi_1}$ to obtain that

$$-(\widetilde{\varphi}_1^{p_1})_{\xi_1}(0) = \widetilde{\varphi}_2^{q_2}(0). \tag{4.12}$$

However, as $\alpha_1 + \beta_1 \leq 0$, problem (4.11)-(4.12) does not have a nontrivial solution, see Theorem 2.4.

If (4.7) holds for some $j > 1$, we can proceed as before to obtain that $\widetilde{\varphi}_j(0) < \infty$. Thus, we can pass to the limit in the boundary condition for $\widehat{\varphi}_j$, obtaining

$$-(\widetilde{\varphi}_j^{p_j})_{\xi_j}(0) = \widetilde{\varphi}_{j+1}^{q_{j+1}}(0).$$

As $\widetilde{\varphi}_{j+1}(0) \geq \widetilde{\varphi}_{j+1}(\xi_{j+1})$, this implies that $\widetilde{\varphi}_{j+1}$ is finite for every $\xi_{j+1} \geq 0$. We get the same contradiction as before.

Acknowledgements
We would like to thank Professor Dimitru Motreanu, Christopher Rualizo and the referees for their valuable comments and suggestions.

Authors' contributions
The authors declare that the work was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 1 May 2011 Accepted: 19 August 2011 Published: 19 August 2011

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doi:10.1186/1687-2770-2011-15
Cite this article as: Xu and Song: **Critical parameter equations for degenerate parabolic equations coupled via nonlinear boundary flux.** *Boundary Value Problems* 2011 **2011**:15.