Cirtoaje and Baiesu *Journal of Inequalities and Applications* 2011, **2011**:101
http://www.journalofinequalitiesandapplications.com/content/2011/1/101

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

An extension of Jensen's discrete inequality to half convex functions

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Abstract

We extend the right and left convex function theorems to weighted Jensen's type inequalities, and then combine the new theorems in a single one applicable to a half convex function *f*(*u*), defined on a real interval $\mathbb{I}$ and convex for $u \le s$ or $u \ge s$, where $s \in \mathbb{I}$. The obtained results are applied for proving some open relevant inequalities.

Keywords: weighted Jensen's discrete inequality, right convex function, left convex function, half convex function

1 Introduction

The right convex function theorem (RCF-Theorem) has the following statement (see [1-3]).

RCF-Theorem. *Let f(u) be a function defined on a real interval* $\mathbb{I}$ *and convex for* $u \ge s \in \mathbb{I}$. *The inequality*

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \tag{1}$$

holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ *satisfying* $x_1 + x_2 + \ldots + x_n \ge ns$ *if and only if*

$$f(x) + (n-1)f(y) \ge nf(s) \tag{2}$$

for all $x, y \in \mathbb{I}$ *which satisfy* $x \le s \le y$ *and* $x + (n - 1)y = ns$.

Replacing *f*(*u*) by *f*(-*u*), *s* by -*s*, *x* by -*x*, *y* by -*y*, and each x_i by $-x_i$ for $i = 1, 2, \ldots, n$, from RCF-Theorem we get the left convex function theorem (LCF-Theorem).

LCF-Theorem. *Let f(u) be a function defined on a real interval* $\mathbb{I}$ *and convex for* $u \le s \in \mathbb{I}$. *The inequality*

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \tag{3}$$

holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ *satisfying* $x_1 + x_2 + \ldots + x_n \le ns$ *if and only if*

$$f(x) + (n-1)f(y) \ge nf(s) \tag{4}$$

for all $x, y \in \mathbb{I}$ *which satisfy* $x \ge s \ge y$ *and* $\times + (n - 1)y = ns$.

Notice that from RCF- and LCF-Theorems, we get the following theorem, which we have called the half convex function theorem (HCF-Theorem).

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HCF-Theorem. *Let $f(u)$ be a function defined on a real interval $\mathbb{I}$ and convex for $u \le s$ or $u \ge s$, where $s \in \mathbb{I}$. The inequality*

$$f(x_1) + f(x_2) + \cdots + f(x_n) \ge nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \tag{5}$$

holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ satisfying $x_1 + x_2 + \ldots + x_n = ns$ if and only if

$$f(x) + (n-1)f(y) \ge nf(s) \tag{6}$$

for all $x, y \in \mathbb{I}$ which satisfy $x + (n - 1)y = ns$.

Applying RCF-, LCF-, and HCF-Theorems to the function $f(u) = g(e^u)$, and replacing s by $\ln r$, x by $\ln x$, y by $\ln y$, and each x_i by $\ln a_i$ for $i = 1, 2, \ldots, n$, we get the following corollaries, respectively.

RCF-Corollary. *Let g be a function defined on a positive interval $\mathbb{I}$ such that $f(u) = g(e^u)$ is convex for $e^u \ge r \in \mathbb{I}$. The inequality*

$$g(a_1) + g(a_2) + \cdots + g(a_n) \ge ng(\sqrt[n]{a_1 a_2 \ldots a_n}) \tag{7}$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1 a_2 \ldots a_n \ge r^n$ if and only if

$$g(a) + (n-1)g(b) \ge ng(r) \tag{8}$$

for all $a, b \in \mathbb{I}$ which satisfy $a \le r \le b$ and $ab^{n-1} = r^n$.

LCF-Corollary. *Let g be a function defined on a positive interval $\mathbb{I}$ such that $f(u) = g(e^u)$ is convex for $e^u \le r \in \mathbb{I}$. The inequality*

$$g(a_1) + g(a_2) + \cdots + g(a_n) \ge ng(\sqrt[n]{a_1 a_2 \ldots a_n}) \tag{9}$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1 a_2 \ldots a_n \le r^n$ if and only if

$$g(a) + (n-1)g(b) \ge ng(r) \tag{10}$$

for all $a, b \in \mathbb{I}$ which satisfy $a \ge r \ge b$ and $ab^{n-1} = r^n$.

HCF-Corollary. *Let g be a function defined on a positive interval $\mathbb{I}$ such that $f(u) = g(e^u)$ is convex for $e^u \le r$ or $e^u \ge r$, where $r \in \mathbb{I}$. The inequality*

$$g(a_1) + g(a_2) + \cdots + g(a_n) \ge ng(r) \tag{11}$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1 a_2 \ldots a_n = r^n$ if and only if

$$g(a) + (n-1)g(b) \ge ng(r) \tag{12}$$

for all $a, b \in \mathbb{I}$ which satisfy $ab^{n-1} = r^n$.

2 Main results

In order to extend RCF-, LCF-, and HCF-Theorems to weighted Jensen's type inequalities, we need the following lemma.

Lemma 2.1 *Let q_1, q_2 and $r_1, r_2, \ldots, r_m$ be nonnegative real numbers such that*

$$r_1 + r_2 + \cdots + r_m = q_1 + q_2, \tag{13}$$

and let f be a convex function on $\mathbb{I}$. If $a, b \in \mathbb{I}$ $(a \le b)$ and $x_1, x_2, \ldots, x_m \in [a, b]$ such that

$$r_1x_1 + r_2x_2 + \cdots + r_mx_m = q_1a + q_2b, \tag{14}$$

then

$$r_1f(x_1) + r_2f(x_2) + \cdots + r_mf(x_m) \le q_1f(a) + q_2f(b). \tag{15}$$

The weighted right convex function theorem (WRCF-Theorem), weighted left convex function theorem (WLCF-Theorem), and weighted half convex function theorem (WHCF-Theorem) are the following.

WRCF-Theorem. *Let $f(u)$ be a function defined on a real interval $\mathbb{I}$ and convex for $u \ge s \in \mathbb{I}$, and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that*

$$p = \min\{p_1, p_2, \ldots, p_n\}, \quad p_1 + p_2 + \cdots + p_n = 1. \tag{16}$$

The inequality

$$p_1f(x_1) + p_2f(x_2) + \cdots + p_nf(x_n) \ge f(p_1x_1 + p_2x_2 + \cdots + p_nx_n) \tag{17}$$

holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ satisfying $p_1x_1 + p_2x_2 + \ldots + p_nx_n \ge s$ if and only if

$$pf(x) + (1 - p)f(y) \ge f(s) \tag{18}$$

for all $x, y \in \mathbb{I}$ such that $x \le s \le y$ and $px + (1 - p)y = s$.

WLCF-Theorem. *Let $f(u)$ be a function defined on a real interval $\mathbb{I}$ and convex for $u \le s \in \mathbb{I}$, and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that*

$$p = \min\{p_1, p_2, \ldots, p_n\}, \quad p_1 + p_2 + \cdots + p_n = 1. \tag{19}$$

The inequality

$$p_1f(x_1) + p_2f(x_2) + \cdots + p_nf(x_n) \ge f(p_1x_1 + p_2x_2 + \cdots + p_nx_n) \tag{20}$$

holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ satisfying $p_1x_1 + p_2x_2 + \ldots + p_nx_n \le s$ if and only if

$$pf(x) + (1 - p)f(y) \ge f(s) \tag{21}$$

for all $x, y \in \mathbb{I}$ such that $x \ge s \ge y$ and $px + (1 - p)y = s$.

WHCF-Theorem. *Let $f(u)$ be a function defined on a real interval $\mathbb{I}$ and convex for $u \le s$ or $u \ge s$, where $s \in \mathbb{I}$, and let $p_1, p_2, \ldots, p_n$ be positive real numbers such that*

$$p = \min\{p_1, p_2, \ldots, p_n\}, \quad p_1 + p_2 + \cdots + p_n = 1. \tag{22}$$

The inequality

$$p_1f(x_1) + p_2f(x_2) + \cdots + p_nf(x_n) \ge f(p_1x_1 + p_2x_2 + \cdots + p_nx_n) \tag{23}$$

holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ satisfying $p_1x_1 + p_2x_2 + \ldots + p_nx_n = s$ if and only if

$$pf(x) + (1 - p)f(y) \ge f(s) \tag{24}$$

for all $x, y \in \mathbb{I}$ such that $px + (1 - p)y = s$.

Notice that WLCF-Theorem can be obtained from WRCF-Theorem replacing $f(u)$ by $f(-u)$, s by $-s$, x by $-x$, y by $-y$, and each x_i by $-x_i$ for $i = 1,2, \ldots, n$.

On the other hand, applying WRCF-, WLCF-, and WHCF-Theorems to the function $f(u) = g(e^u)$ and replacing s by 1nr, x by 1nx, y by 1ny, and each x_i by In a_i for $i = 1, 2, \ldots n$, we get the following corollaries, respectively.

WRCF-Corollary *Let g be a function defined on a positive interval* $\mathbb{I}$ *such that* $f(u) = g(e^u)$ *is convex for* $\ln u \geq r \in \mathbb{I}$, *and let* $p_1, p_2, ..., p_n$ *be positive real numbers such that*

$$p = \min\{p_1, p_2, \ldots, p_n\}, \quad p_1 + p_2 + \cdots + p_n = 1. \tag{25}$$

The inequality

$$p_1 g(a_1) + p_2 g(a_2) + \cdots + p_n g(a_n) \geq g(a_1^{p_1} a_2^{p_2} \ldots a_n^{p_n}) \tag{26}$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *satisfying* $a_1^{p_1} a_2^{p_2} \ldots a_n^{p_n} \geq r$ *if and only if*

$$pg(a) + (1-p)g(b) \geq g(r) \tag{27}$$

for all $a, b \in \mathbb{I}$ *such that* $a \leq r \leq b$ *and* $a^p b^{1-p} = r$.

WLCF-Corollary. *Let g be a function defined on a positive interval* $\mathbb{I}$ *such that* $f(u) = g(e^u)$ *is convex for* $e^u \leq r \in \mathbb{I}$, *and let* $p_1, p_2, ..., p_n$ *be positive real numbers such that*

$$p = \min\{p_1, p_2, \ldots, p_n\}, \quad p_1 + p_2 + \cdots + p_n = 1. \tag{28}$$

The inequality

$$p_1 g(a_1) + p_2 g(a_2) + \cdots + p_n g(a_n) \geq g(a_1^{p_1} a_2^{p_2} \ldots a_n^{p_n}) \tag{29}$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *satisfying* $a_1^{p_1} a_2^{p_2} \ldots a_n^{p_n} \leq r$ *if and only if*

$$pg(a) + (1-p)g(b) \geq g(r) \tag{30}$$

for all $a, b \in \mathbb{I}$ *such that* $a \geq r \geq b$ *and* $a^p b^{1-p} = r$.

WHCF-Corollary. *Let g be a function defined on a positive interval* $\mathbb{I}$ *such that* $f(u) = g(e^u)$ *is convex for* $e^u \leq r$ *or* $e^u \geq r$, *where* $r \in \mathbb{I}$, *and let* $p_1, p_2, ..., p_n$ *be positive real numbers such that*

$$p = \min\{p_1, p_2, \ldots, p_n\}, \quad p_1 + p_2 + \cdots + p_n = 1. \tag{31}$$

The inequality

$$p_1 g(a_1) + p_2 g(a_2) + \cdots + p_n g(a_n) \geq g(a_1^{p_1} a_2^{p_2} \ldots a_n^{p_n}) \tag{32}$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *satisfying* $a_1^{p_1} a_2^{p_2} \ldots a_n^{p_n} = r$ *if and only if*

$$pg(a) + (1-p)g(b) \geq g(r) \tag{33}$$

for all $a, b \in \mathbb{I}$ *such that* $a^p b^{1-p} = r$.

***Remark* 2.2**. Let us denote

$$g(u) = \frac{f(u) - f(s)}{u - s}, \quad h(x, y) = \frac{g(x) - g(y)}{x - y}. \tag{34}$$

In some applications, it is useful to replace the hypothesis

$$pf(x) + (1-p)f(y) \geq f(s) \tag{35}$$

in WRCF-, WLCF-, and WHCF-Theorems by the equivalent condition:

$h(x, y) \geq 0$ for all $x, y \in \mathbb{I}$ such that $px + (1 - p)y = s$.

This equivalence is true since

$$\begin{aligned} pf(x) + (1-p)f(y) - f(s) &= p[f(x) - f(s)] + (1-p)[f(y) - f(s)] \\ &= p(x-s)g(x) + (1-p)(y-s)g(y) \\ &= p(1-p)(x-y)[g(x) - g(y)] \\ &= p(1-p)(x-y)^2 h(x,y). \end{aligned}$$

***Remark* 2.3**. The required inequalities in WRCF-, WLCF-, and WHCF-Theorems turn into equalities for $x_1 = x_2 = \ldots = x_n$. In addition, on the assumption that $p_1 = \min\{p_1, p_2, \ldots, p_n\}$, equality holds for $x_1 = x$ and $x_2 = \ldots = x_n = y$ if there exist $x, y \in \mathbb{I}$, $x \neq y$ such that $px + (1-p)y = s$ and $pf(x) + (1-p)f(y) = f(s)$.

3 Proof of Lemma 2.1

Consider only the nontrivial case $a < b$. Since $x_1, x_2, \ldots, x_m \in [a, b]$ there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \in [0, 1]$ such that

$$x_i = \lambda_i a + (1-\lambda_i)b, \quad i = 1, 2, \ldots, m.$$

From

$$\lambda_i = \frac{x_i - b}{a - b}, \quad i = 1, 2, \ldots, m,$$

we have

$$\sum_{i=1}^m r_i\lambda_i = \frac{1}{a-b}\Big(\sum_{i=1}^m r_i x_i - b\sum_{i=1}^m r_i\Big) = \frac{1}{a-b}[q_1 a + q_2 b - b(q_1 + q_2)] = q_1.$$

Thus, according to Jensen's inequality, we get

$$\begin{aligned} \sum_{i=1}^m r_i f(x_i) &\leq \sum_{i=1}^m r_i[\lambda_i f(a) + (1-\lambda_i)f(b)] \\ &= [f(a) - f(b)]\sum_{i=1}^m r_i\lambda_i + f(b)\sum_{i=1}^m r_i \\ &= [f(a) - f(b)]q_1 + f(b)(q_1 + q_2) \\ &= q_1 f(a) + q_2 f(b). \end{aligned}$$

4 Proof of WRCF-Theorem

Since the necessity is obvious, we prove further the sufficiency. Without loss of generality, assume that $x_1 \leq x_2 \leq \ldots \leq x_n$. If $x_1 \geq s$, then the required inequality follows by Jensen's inequality for convex functions. Otherwise, since

$$p_1x_1 + p_2x_2 + \cdots + p_nx_n \geq (p_1 + p_2 + \cdots + p_n)s,$$

there exists $k \in \{1, 2, \ldots, n-1\}$ such that

$$x_1 \leq \cdots \leq x_k < s \leq x_{k+1} \leq \cdots \leq x_n.$$

Let us denote

$$q = p_1 + \cdots + p_k,$$

By Jensen's inequality, we have

$$\sum_{i=k+1}^{n} p_i f(x_i)) \geq (p_{k+1} + \cdots + p_n) f(z) = (1-q) f(z),$$

where

$$z = \frac{p_{k+1}x_{k+1} + \cdots + p_n x_n}{p_{k+1} + \cdots + p_n}, \quad z \geq s, \ z \in \mathbb{I}.$$

Thus, it suffices to prove that

$$\sum_{i=1}^{k} p_i f(x_i) + (1-q) f(z) \geq f(S), \tag{36}$$

where

$$S = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = \frac{p_1 x_1 + p_2 x_2 + \cdots + p_n x_n}{p_1 + p_2 + \cdots + p_n}, \quad s \leq S \leq z.$$

Let y_i, $i = 1, 2, ..., k$, defined by

$$p x_i + (1-p) y_i = s.$$

We will show that

$$z \geq y_1 \geq y_2 \geq \cdots \geq y_k > s.$$

We have

$$y_1 \geq y_2 \geq \cdots \geq y_k,$$
$$y_k - s = \frac{p(s - x_k)}{1-p} > 0,$$
$$y_1 = \frac{s - p x_1}{1-p} \leq \frac{S - p x_1}{1-p} = \frac{(p_1 - p) x_1 + p_2 x_2 + \cdots + p_n x_n}{(p_1 - p) + p_2 + \cdots + p_n}.$$

Since $p_1 - p = p_1 - \min\{p_1, p_2, ..., p_n\} \geq 0$, we get

$$\frac{(p_1 - p) x_1 + p_2 x_2 + \cdots + p_n x_n}{(p_1 - p) + p_2 + \cdots + p_n} \leq \frac{p_2 x_2 + \cdots + p_n x_n}{p_2 + \cdots + p_n} \leq z,$$

and hence $y_1 \leq z$. Now, from $z \geq y_1 \geq y_2 \geq ... \geq yk > s$, it follows that $y_1, y_2, \ldots, y_k \in \mathbb{I}$. Then, by hypothesis, we have

$$p f(x_i) + (1-p) f(y_i) \geq f(s)$$

for $i = 1, 2, ..., k$. Summing all these inequalities multiplied by p_i/p, respectively, we get

$$\sum_{i=1}^{k} p_i f(x_i) + \frac{1-p}{p} \sum_{i=1}^{k} p_i f(y_i) \geq \frac{q}{p} f(s).$$

Therefore, to prove (36), it suffices to show that

$$\frac{q}{p} f(s) + (1-q) f(z) \geq \frac{1-p}{p} \sum_{i=1}^{k} p_i f(y_i) + f(S). \tag{37}$$

Since $S \in [s, z]$, $y_i \in (s, z]$ for $i = 1, 2, ..., k$,

$$\frac{q}{p} + 1 - q = \frac{1-p}{p}\sum_{i=1}^{k} p_i + 1,$$

$$\frac{q}{p}s + (1-q)z = \frac{1-p}{p}\sum_{i=1}^{k} p_i y_i + S,$$

(37) is a consequence of Lemma 2.1, where $m = k + 1$, $a = s$, $b = z$, $q_1 = q/p$, $q_2 = 1 - q$, $r_m = 1$, $x_m = S$, $r_i = (1 - p)p_i/p$ and $x_i = y_i$ for $i = 1, 2, ..., k$.

5 Applications

Proposition 5.1. *Let $a_1, a_2, ..., a_n$ ($n \geq 3$) be positive real numbers such that $a_1a_2 \ldots a_n = 1$. If p and q are nonnegative real numbers such that $p + q \geq n - 1$, then* [4]

$$\frac{1}{1+pa_1+qa_1^2} + \frac{1}{1+pa_2+qa_2^2} + \cdots + \frac{1}{1+pa_n+qa_n^2} \geq \frac{n}{1+p+q}.$$

Proof. Write the desired inequality as

$$g(a_1) + g(a_2) + \cdots + g(a_n) \geq ng(1),$$

where

$$g(t) = \frac{1}{1+pt+qt^2}, \quad t > 0.$$

To prove this inequality, we apply HCF-Corollary for $r = 1$. Let

$$f(u) = g(e^u) = \frac{1}{1+pe^u+qe^{2u}}, \quad u \in \mathbb{R}.$$

Using the second derivative,

$$f''(u) = \frac{e^u[4q^2e^{3u} + 3pqe^{2u} + (p^2-4q)e^u - p]}{(1+pe^u+qe^{2u})^3},$$

we will show that $f(u)$ is convex for $e^u \geq r = 1$. We need to show that

$$4q^2t^3 + 3pqt^2 + (p^2 - 4q)t - p \geq 0$$

for $t \geq 1$. Indeed,

$$\begin{aligned} 4q^2t^3 + 3pqt^2 + (p^2-4q)t - p &\geq (4q^2 + 3pq + p^2 - 4q - p)t \\ &= [(p+2q)(p+q-2) + 2q^2 + p]t > 0, \end{aligned}$$

because $p + q \geq n - 1 \geq 2$.

By HCF-Corollary, it suffices to prove that $g(a) + (n - 1)g(b) \geq ng(1)$ for all $a, b > 0$ such that $ab^{n-1} = 1$. We write this inequality as

$$\frac{b^{2n-2}}{b^{2n-2}+pb^{n-1}+q} + \frac{n-1}{1+pb+qb^2} \geq \frac{n}{1+p+q}.$$

Applying the Cauchy-Schwarz inequality, it suffices to prove that

$$\frac{(b^{n-1}+n-1)^2}{(b^{2n-2}+pb^{n-1}+q)+(n-1)(1+pb+qb^2)} \geq \frac{n}{1+p+q},$$

which is equivalent to

$$pB + qC \geq A,$$

where

$$\begin{aligned} A &= (n-1)(b^{n-1}-1)^2 \geq 0, \\ B &= (b^{n-1}-1)^2 + nE = \frac{A}{n-1} + nE, \\ C &= (b^{n-1}-1)^2 + nF = \frac{A}{n-1} + nF, \end{aligned}$$

with

$$E = b^{n-1} + n - 2 - (n-1)b, \quad F = 2b^{n-1} + n - 3 - (n-1)b^2.$$

By the AM-GM inequality applied to $n - 1$ positive numbers, we have $E \geq 0$ and $F \geq 0$ for $n \geq 3$. Since $A \geq 0$ and $p + q \geq n - 1$, we have

$$pB + qC - A \geq pB + qC - \frac{(p+q)A}{n-1} = n(pE + qF) \geq 0.$$

Equality holds for $a_1 = a_2 = \dots = a_n = 1$.

***Remark* 5.2**. For $p + q = n - 1$ and $n \geq 3$, by Proposition 5.1 we get the following beautiful inequality

$$\frac{1}{1 + pa_1 + qa_1^2} + \frac{1}{1 + pa_2 + qa_2^2} + \cdots + \frac{1}{1 + pa_n + qa_n^2} \geq 1.$$

If $p = n - 1$ and $q = 0$, then we get the well-known inequality

$$\frac{1}{1 + (n-1)a_1} + \frac{1}{1 + (n-1)a_2} + \cdots + \frac{1}{1 + (n-1)a_n} \geq 1.$$

***Remark* 5.3**. For $p = q = \frac{1}{r}$, $0 < r \leq \frac{2}{n-1}$ and $n \geq 3$, by Proposition 5.1 we obtain the inequality

$$\frac{1}{r + a_1 + a_1^2} + \frac{1}{r + a_2 + a_2^2} + \cdots + \frac{1}{r + a_n + a_n^2} \geq \frac{n}{r+2}.$$

In addition, for $r = \frac{2}{n-1}$, $n \geq 3$, we get

$$\sum_{i=1}^{n} \frac{1}{2 + (n-1)(a_i + a_i^2)} \geq \frac{1}{2}.$$

***Remark* 5.4**. For $p = 2r$, $q = r^2$, $r \geq \sqrt{n} - 1$ and $n \geq 3$, by Proposition 5.1 we obtain

$$\frac{1}{(1 + ra_1)^2} + \frac{1}{(1 + ra_2)^2} + \cdots + \frac{1}{(1 + ra_n)^2} \geq \frac{n}{(1+r)^2}.$$

Proposition 5.5. *Let* $a_1, a_2, \dots, a_n$ *(*$n \geq 4$*) be positive real numbers such that* $a_1 a_2 \dots a_n = 1$. *If* p, q, r *are nonnegative real numbers such that* $p + q + r \geq n - 1$, *then* [4]

$$\sum_{i=1}^{n} \frac{1}{1 + pa_i + qa_i^2 + ra_i^3} \geq \frac{n}{1 + p + q + r}.$$

Proof. Write the required inequality as

$$g(a_1) + g(a_2) + \cdots + g(a_n) \geq ng(1),$$

where

$$g(t) = \frac{1}{1 + pt + qt^2 + rt^3}, \quad t > 0,$$

and apply HCF-Corollary to $g(t)$ for $r = 1$. Let

$$f(u) = g(e^u) = \frac{1}{1 + pe^u + qe^{2u} + re^{3u}},$$

defined on $\mathbb{R}$. For $n \geq 4$, which implies $p + q + r \geq 3$, we claim that f is convex for $e^u \geq 1$. Since

$$f''(u) = \frac{t[9r^2t^5 + 11qrt^4 + (2pr + 4q^2)t^3 + (3pq - 9r)t^2 + (p^2 - 4q)t - p]}{(1 + pt + qt^2 + rt^3)^3},$$

where $t = e^u \geq 1$, we need to show that

$$9r^2t^5 + 11qrt^4 + (2pr + 4q^2)t^3 + (3pq - 9r)t^2 + (p^2 - 4q)t - p \geq 0$$

Since

$$9r^2t^5 + 11qrt^4 + (2pr + 4q^2)t^3 - p \geq (9r^2 + 11qr + 2pr + 4q^2)t^3 - pt,$$

it suffices to show that

$$(9r^2 + 11qr + 2pr + 4q^2)t^2 + (3pq - 9r)t + p^2 - p - 4q \geq 0.$$

Using the inequality $t^2 \geq 2t - 1$, we still have to prove that $At + B \geq 0$, where

$$A = 18r^2 + 22qr + 4pr + 8q^2 + 3pq - 9r,$$
$$B = -9r^2 - 11qr - 2pr - 4q^2 + p^2 - p - 4q.$$

Since $p + q + r \geq 3$, we have

$$A \geq 18r^2 + 22qr + 4pr + 8q^2 + 3pq - 3r(p + q + r)$$
$$= 15r^2 + 19qr + pr + 8q^2 + 3pq \geq 0.$$

Therefore,

$$At + B \geq A + B = p^2 + 4q^2 + 9r^2 + 3pq + 11qr + 2pr - (p + 4q + 9r)$$
$$\geq p^2 + 4q^2 + 9r^2 + 3pq + 11qr + 2pr - \frac{(p + 4q + 9r)(p + q + r)}{3}$$
$$= \frac{2(p - r)^2 + 9q^2 + 16r^2 + 4pq + 20qr}{3} > 0.$$

According to HCF-Corollary, it suffices to prove that $g(a) + (n - 1)\, g(b) \geq ng(1)$ for all $a, b > 0$ such that $ab^{n-1} = 1$. We write this inequality as

$$\frac{b^{3n-3}}{b^{3n-3}+pb^{2n-2}+qb^{n-1}+r}+\frac{n-1}{1+pb+qb^2+rb^3}\geq\frac{n}{1+p+q+r},$$

or

$$p^2A_{11}+q^2A_{22}+r^2A_{33}+pqA_{12}+qrA_{23}+rpA_{31}\geq Ap+Bq+Cr,$$

where

$$A_{11}=b^{2n-2}(b^n-nb+n-1),$$
$$A_{22}=b^{n-1}(b^{2n}-nb^2+n-1),$$
$$A_{33}=b^{3n}-nb^3+n-1,$$

$$A_{12}=b^{3n-1}+b^{3n-2}+(n-1)(b^{2n-2}+b^{n-1})-n(b^{2n}+b^n),$$
$$A_{23}=b^{3n}+b^{3n-1}+(n-1)(b^{n-1}+1)-n(b^{n+2}+b^2),$$
$$A_{31}=b^{3n}+b^{3n-2}+(n-1)(b^{2n-2}+1)-n(b^{2n+1}+b),$$

$$A=b^{2n-2}[(n-1)b^n-nb^{n-1}+1],$$
$$B=b^{n-1}[(n-1)b^{2n}-nb^{2n-2}+1],$$
$$C=(n-1)b^{3n}-nb^{3n-3}+1.$$

Since $A,B,C\geq 0$ (from the AM-GM inequality applied to n positive numbers) and $p+q+r\geq n-1$, it suffices to show that

$$(n-1)(p^2A_{11}+q^2A_{22}+r^2A_{33}+pqA_{12}+qrA_{23}+rpA_{31})\geq$$
$$\geq(p+q+r)(Ap+Bq+Cr),$$

which is equivalent to

$$p^2B_{11}+q^2B_{22}+r^2B_{33}+pqB_{12}+qrB_{23}+rpB_{31}\geq 0, \tag{38}$$

where

$$B_{11}=(n-1)A_{11}-A=nb^{2n-2}[b^{n-1}-(n-1)b+n-2],$$
$$B_{22}=(n-1)A_{22}-B=nb^{n-1}[b^{2n-2}-(n-1)b^2+n-2],$$
$$B_{33}=(n-1)A_{33}-C=n[b^{3n-3}-(n-1)b^3+n-2],$$

$$B_{12}=(n-1)A_{12}-A-B$$
$$=nb^{n-1}[2b^{2n-2}-(n-1)b^{n+1}+(n-2)b^{n-1}-(n-1)b+n-2]$$
$$=nb^{2n-2}[2b^{n-1}-(n-1)b^2+n-3]+nb^{n-1}[b^{n-1}-(n-1)b+n-2],$$

$$B_{23}=(n-1)A_{23}-B-C$$
$$=n[2b^{3n-3}-(n-1)b^{n+2}+(n-2)b^{n-1}-(n-1)b^2+n-2],$$

$$B_{31}=(n-1)A_{31}-C-A$$
$$=n[2b^{3n-3}-(n-1)b^{2n+1}+(n-2)b^{2n-2}-(n-1)b+n-2].$$

We see that B_{11}, B_{22}, B_{33}, $B_{12} \geq 0$ (by the AM-GM inequality applied to n - 1 positive numbers). Also, we have

$$\begin{aligned}\frac{B_{23}}{n} =& b^{n-1}[3b^{n-1} - (n-1)b^3 + n - 4] + 2b^{n-1}(b^{n-1} - 1)^2 \\ &+ b^{2n-2} - (n-1)b^2 + n - 2 \geq 0,\end{aligned}$$

since

$$3b^{n-1} - (n-1)b^3 + n - 4 \geq 0, \quad b^{2n-2} - (n-1)b^2 + n - 2 \geq 0$$

(by the AM-GM inequality applied to n - 1 positive numbers). Using the inequality $b^{n-1} - (n - 1)b + n - 2 \geq 0$,we get $B_{31} \geq D$, where

$$D = nb^{n-1}[2b^{2n-2} - (n-1)b^{n+2} + (n-2)b^{n-1} - 1].$$

To prove (38), it suffices to show that $p^2 B_{11} + r^2 B_{33} + prD \geq 0$. This is true if $4B_{11}B_{33} \geq D^2$; that is,

$$\begin{aligned}&4[b^{n-1} - (n-1)b + n - 2][b^{3n-3} - (n-1)b^3 + n - 2] \geq \\ &\geq [2b^{2n-2} - (n-1)b^{n+2} + (n-2)b^{n-1} - 1]^2.\end{aligned} \tag{39}$$

In the case $n = 4$, (39) becomes in succession

$$\begin{aligned}4(b^3 - 3b + 2)(b^9 - 3b^3 + 2) &\geq (b^6 - 2b^3 + 1)^2, \\ 4(b-1)^2(b+2)(b^3-1)^2(b^3+2) &\geq (b^3-1)^4, \\ (b-1)^2(b^3-1)^2(3b^4 + 5b^3 - 3b^2 + 6b + 15) &\geq 0.\end{aligned}$$

Clearly, the last inequality is true. The inequality (39) also holds for $n \geq 5$, but we leave this to the reader to prove. Equality occurs for $a_1 = a_2 = \ldots = a_n = 1$.

***Remark* 5.6**. For $n = 4$ and $p + q + r = 3$, by Proposition 5.5 we get the following beautiful inequality

$$\sum_{i=1}^{4} \frac{1}{1 + pa_i + qa_i^2 + ra_i^3} \geq 1.$$

In addition, for $p = q = r = 1$, we get the known inequality ([2])

$$\sum_{i=1}^{4} \frac{1}{1 + a_i + a_i^2 + a_i^3} \geq 1.$$

Conjecture 5.7. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 a_2 \ldots a_n = 1$, and let $k_1, k_2, \ldots, k_m$ be nonnegative real numbers such that $k_1 + k_2 + \ldots + k_m \geq n - 1$. If $m \leq n - 1$, then*

$$\sum_{i=1}^{n} \frac{1}{1 + k_1 a_i + k_2 a_i^2 + \cdots + k_m a_i^m} \geq \frac{n}{1 + k_1 + k_2 + \cdots + k_m}. \tag{40}$$

***Remark* 5.8**. For $m = n - 1$ and $k_1 = k_2 = \ldots = k_m = 1$, (40) turns into the known beautiful inequality ([2])

$$\sum_{i=1}^{n} \frac{1}{1 + a_i + a_i^2 + \cdots + a_i^{n-1}} \geq 1.$$

***Remark* 5.9**. For $k_1 = \binom{m}{1} r, k_2 = \binom{m}{2} r^2, \ldots, k_m = \binom{m}{m} r^m$, (40) turns into the known inequality [1,2]

$$\sum_{i=1}^{n} \frac{1}{(1 + ra_i)^m} \geq \frac{n}{(1 + r)^m},$$

which holds for $1 \leq m \leq n - 1$ and $r \geq \sqrt[m]{n} - 1$. □

Proposition 5.10. *If $x_1, x_2, ..., x_n$ are nonnegative real numbers such that*

$$x_1 + 2x_2 + \cdots + nx_n = \frac{n(n+1)}{2},$$

then

$$(n-1)(n+2)\left[x_1^3 + 2x_2^3 + \cdots + nx_n^3 - \frac{n(n+1)}{2}\right] \geq$$
$$\geq 2(n^2 + n - 1)\left[x_1^2 + 2x_2^2 + \cdots + nx_n^2 - \frac{n(n+1)}{2}\right].$$

Proof. Since the inequality is trivial for $n = 1$, consider further that $n \geq 2$. Write the inequality as

$$p_1 f(x_1) + p_2 f(x_2) + \cdots + p_n f(x_n) \geq f(p_1 x_1 + p_2 x_2 + \cdots + p_n x_n),$$

where

$$p_i = \frac{2i}{n(n+1)}, \quad i = 1, 2, \ldots, n,$$
$$f(u) = (n-1)(n+2)u^3 - 2(n^2 + n - 1)u^2, \quad u \geq 0.$$

The function $f(u)$ is convex for $u \geq s = 1$, since

$$f''(u) = 6(n-1)(n+2)u - 4(n^2 + n - 1)$$
$$\geq 6(n-1)(n+2) - 4(n^2 + n - 1) = 2(n^2 + n - 4) > 0$$

for $u \geq 1$. According to WHCF-Theorem and Remark 2.2, it suffices to prove that $h(x, y) \geq 0$ for all $x, y \geq 0$ such that $px + (1 - p)y = 1$, where

$$p = \min\{p_1, p_2, \ldots, p_n\} = \frac{2}{n(n+1)}.$$

We have

$$g(u) = \frac{f(u) - f(1)}{u - 1} = (n-1)(n+2)(u^2 + u + 1) - 2(n^2 + n - 1)(u + 1),$$
$$h(x, y) = \frac{g(x) - g(y)}{x - y} = (n-1)(n+2)(x + y + 1) - 2(n^2 + n - 1).$$

From $px + (1 - p)y = 1$, we get

$$x + y = x + \frac{1 - px}{1 - p} = \frac{1 + (1 - 2p)x}{1 - p}$$
$$= \frac{n(n+1) + (n^2 + n - 4)x}{(n-1)(n+2)} \geq \frac{n(n+1)}{(n-1)(n+2)},$$

and hence

$$h(x, y) \geq (n-1)(n+2)\left[\frac{n(n+1)}{(n-1)(n+2)} + 1\right] - 2(n^2 + n - 1) = 0.$$

This completes the proof. Equality holds for $a_1 = a_2 = \ldots a_n = 1$, and also for $a_1 = 0$ and $a_2 = \cdots = a_n = \frac{n(n+1)}{(n-1)(n+2)}$.

Acknowledgements
The authors are grateful to the referees for their useful comments.

Authors' contributions
VC conceived and proved the main results and their applications. AB performed numerical verification for all applications and prepared the first manuscript. Both authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

Received: 17 April 2011 Accepted: 31 October 2011 Published: 31 October 2011

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doi:10.1186/1029-242X-2011-101
Cite this article as: Cirtoaje and Baiesu: An extension of Jensen's discrete inequality to half convex functions. *Journal of Inequalities and Applications* 2011 **2011**:101.