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COMMON FIXED POINT THEOREMS AND APPLICATIONS

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ABSTRACT. The purpose of this paper is to discuss the existence of common fixed points for mappings in general quasi-metric spaces. As applications, some common fixed point theorems for mappings in probabilistic quasi-metric spaces are given. The results presented in this paper generalize some recent results.

KEY WORDS AND PHRASES. General quasi-metric space, probabilistic quasi-metric space, fixed point, periodic point, periodic index.
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1. INTRODUCTION

In this paper, we show the existence of common fixed points for commuting mappings in general quasi-metric spaces. As applications, we give some fixed point theorems for commuting mappings in probabilistic quasi-metric spaces. Our main results generalize and improve some recent results in [1], [4], [5] and [6].

Let $(G, \leq, <)$ be a partial order set satisfying the following conditions:

(G-1) 0 is the minimal element in G, i.e., $0 \leq u$ for all $u \in G$,

(G-2) for any $u, v \in G$, $\sup\{u, v\}$ exists and belongs to G,

(G-3) for any $u \in G$, $u \not< u$,

(G-4) for any $u, v, w \in G, u < w$ and $v < w \Rightarrow \sup\{u, v\} < w$, and $u < v$, $v \leq w \Rightarrow u < w$.

DEFINITION 1.1 Let X be a nonempty set. (X, r) is called a *general quasi-metric space* if $r : X \times X \to G = (G, \leq, <)$ satisfies the following conditions:

(QM-1) $r(x, y) = 0$ if and only if $x = y$,

(QM-2) $r(x, y) = r(y, x)$.

It follows from the definition that every general quasi-metric space includes a metric space as its special case.

DEFINITION 1.2 Let X be a nonempty set and let T be a self-mapping of X. A point $x \in X$ is called a *periodic point* of T if there exists a positive integer k such that $T^k x = x$. The least positive integer satisfying this condition is called the *periodic index* of x.

DEFINITION 1.3 A mapping $F : (-\infty, \infty) \to [0, \infty)$ is called a *distribution function* if it is nondecreasing and left-continuous with $\inf F(t) = 0$ and $\sup F(t) = 1$.

In what follows we always denote by $F(T)$ the set of all fixed points of T, $P(T)$ the set of all periodic points of T and $\mathcal{D}$ the set of all distribution functions, respectively, and let $\mathcal{D}^+ = \{F \in \mathcal{D} : F(t) = 0 \text{ for all } t < 0\}$.

DEFINITION 1.4 $(X, \mathcal{F})$ is called a *probabilistic quasi-metric space* if X is a nonempty set, $\mathcal{F}$ is a mapping from $X \times X$ into $\mathcal{D}^+$ (we shall denote the distribution function $\mathcal{F}(x, y)$ by $F_{x,y}(t)$ which represent the value of $F_{x,y}$ at $t \in (-\infty, \infty)$) satisfying the following conditions:

(PQM-1) $F_{x,y}(0) = 0$,

(PQM-2) $F_{x,y}(t) = 1$ for all $t > 0$ if and only if $x = y$,

(PQM-3) $F_{x,y}(t) = F_{y,x}(t)$ for all $t \in (-\infty, \infty)$.

DEFINITION 1.5 $(X, \mathcal{F})$ is called a *probabilistic metric space* if $(X, \mathcal{F})$ is a probabilistic quasi-metric space and the following condition is satisfied:

(PQM-4) if $F_{x,y}(t_1) = 1$ and $F_{y,z}(t_2) = 1$, then $F_{x,z}(t_1 + t_2) = 1$.

For more details on probabilistic metric spaces, refer to [3] and [7].

2. COMMON FIXED POINT THEOREMS

Now, we give our main theorems.

THEOREM 2.1 Let (X, r) be a general quasi-metric space and let S and T be two commuting self-mappings of X. If for any $x \in X$ and any two positive integers $n, q \geq 2$ with

$$\begin{aligned} &T^i x \neq T^j x, \quad 0 \leq i < j \leq n-1, \\ &S^{i'} x \neq S^{j'} x, \quad 0 \leq i' < j' \leq q-1, \end{aligned} \tag{2.1}$$

$$r(T^n S^i x, S^q T^i x) < \max\left\{ \sup_{1 \leq j \leq n, 1 \leq j' \leq q} r(T^j x, S^{j'} x), \sup_{1 \leq j \leq n} r(T^j x, x), \sup_{1 \leq j' \leq q} r(S^{j'} x, x) \right\} \tag{2.2}$$

for $i = 1, 2, ..., n-1$ and $j = 1, 2, ..., q-1$, then S and T have a common fixed point in X if and only if there exist integers m, n, p, q, $m > n \geq 0$, $p > q \geq 0$, and a point $x \in X$ such that

$$T^m x = S^p x = T^n x = S^q x \tag{2.3}$$

and either $F(S) \neq \emptyset$ or $F(T) \neq \emptyset$. If this condition is satisfied, then either $T^n x$ or $S^q x$ is a common fixed point of S and T.

PROOF. Let $x^* \in X$ be a common fixed point of T, i.e., $x^* = Sx^* = Tx^*$. Then (2.3) is true in case $m = p = 1$ and $n = q = 0$.

Conversely, suppose that there exist a point $x \in X$ and four integers m, n, p, q, $m > n \geq 0$, $p > q \geq 0$, such that (2.3) is satisfied. Without loss of generality, we can assume that $x \in F(S)$ and m is the minimal integer satisfying $T^k x = T^n x$, $k > n$. Putting $y = T^n x$ and $p_1 = m - n$, we have $T^{p_1} y = y$ and p_1 is the minimal integer satisfying $T^k y = y$, $k \geq 1$.

By (2.2), it follows that

$$r(T^n x, T^i x) < \max\left\{ \sup_{1 \leq j \leq n} \{r(T^j x, x)\}, \sup_{1 \leq j \leq n} \{r(T^j x, x\}, 0 \right\},$$

i.e.,

$$r(T^n x, T^i x) < \sup_{1 \leq j \leq n} \{r(T^j x, x)\}. \tag{2 4}$$

Next, we prove that y is a common fixed point of S and T. Suppose the contrary. Then y is not a fixed point of T. Also, $p \geq 2$ and

$$T^i y \neq T^j y, \quad 0 \leq i < j \leq p_1 - 1.$$

By (2.4), it follows that, for $i = 1, 2, ..., p_1 - 1$,

$$r(y, T^i y) = r(T^{p_1} y, T^i y) < \sup_{1 \leq j \leq p_1} \{r(T^j y, y)\} \leq \sup_{1 \leq j \leq p_1 - 1} \{r(T^j y, y)\}.$$

It follows from (G-4) that

$$\sup_{1 \leq j \leq p-1} \{r(y, T^j y)\} < \sup_{1 \leq j \leq p-1} \{r(T^j y, y)\},$$

which is a contradiction. Therefore, $y = T^m x$ is a fixed point of T. Further, since S and T are commuting, we have

$$y = T^m x = T^m Sx = ST^m x = Sy,$$

i.e., $y = T^m x$ is a common fixed point of S and T. In this case, when $x \in F(T)$, we have, by interchanging the role of S and T, that $y = S^q x$ is a common fixed point of S and T. This completes the proof.

On the other hand, by using Theorem 3 of [6], we have the following:

THEOREM 2.2 Let (X, r) be a general quasi-metric space and let S and T be two commuting self-mappings of X. Assume that for any $x, y \in X$, $x \neq y$, there exists a positive integer $p(x, y)$ such that

$$\begin{aligned} r((ST)^{p(x,y)}x, (ST)^{p(x,y)}y) < \sup\{& r(x,y), r(x, (ST)^{p(x,y)}x), r(y, (ST)^{p(x,y)}y), \\ & r(x, (ST)^{p(x,y)}y), r(y, (ST)^{p(x,y)}x)\}. \end{aligned}$$

Then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST with periodic index k such that for any $u, v \in A = \{x, STx, ..., (ST)^{k-1}x\}$, $u \neq v$, there exist $x', y' \in A$, $x' \neq y'$, satisfying the following conditions:

$$(ST)^{p(x',y')}x' = u, \quad (ST)^{p(x',y')}y' = v$$

and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(St) \neq \emptyset$. If these conditions are satisfied, then the point x is the unique common fixed point of S and T.

PROOF. The necessity condition is obvious.

The sufficiency condition follows from Theorem 3 of [6] as follows: In Theorem 3 of [6], if we replace T by ST, we can conclude that ST has a unique fixed point x in X. Since S and T are commuting,

$$Sx = S(STx) = ST(Sx)$$

and so Sx is also a fixed point of ST. Uniqueness gives $Sx = x$. Similarly, $Tx = x$. This completes the proof.

The following is a special case of Theorem 2.2 by setting $p(x, y) = p(x)$:

COROLLARY 2.3 Let (X, r) be a general quasi-metric space and let S and T be two commuting self-mappings of X. Assume that for any $x \in X$, there exists a positive integer $p(x)$ such that every $y \in X$, $x \neq y$,

$$\begin{aligned} r((ST)^{p(x)}x, (ST)^{p(x)}y) < \sup\{& r(x,y), r(x, (ST)^{p(x)}x), r(y, (ST)^{p(x)}x), r(y, (ST)^{p(x)}y), \\ & r(x, (ST)^{p(x)}y), r(y, (ST)^{p(x)}x)\}. \end{aligned}$$

Then S and T has a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST with periodic index k such that for any $u, v \in A = \{x, STx, ..., (ST)^{k-1}x\}$, $u \neq v$, there exist $x', y' \in A$, $x' \neq y'$, satisfying the following conditions:

$$(ST)^{p(x')}x' = u, \quad (ST)^{p(x')}y' = v$$

and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If these conditions are satisfied, then the point x is the unique common fixed point of S and T.

The following is obtained from Corollary 2.3 by setting $p(x) = p$:

COROLLARY 2.4 Let (X, r) be a general quasi-metric space and let S and T be two commuting self-mappings of X. Assume that there exists a positive integer p such that for any $x, y \in X$, $x \neq y$,

$$r((ST)^p x, (ST)^p y) < \sup\{r(x,y), r(x, (ST)^p x), r(y, (ST)^p y), r(x, (ST)^p y), r(y, (ST)^p x)\}.$$

Then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If this condition is satisfied, then the point x is the unique common fixed point of S and T.

By setting $p = 1$ in Corollary 2.4, we have the following:

COROLLARY 2.5 Let (X, r) be a general quasi-metric space and let S and T be two commuting self-mappings of X. Assume that for any $x, y \in X$, $x \neq y$,

$$r(STx, STy) < \sup\{r(x,y), r(x, STx), r(y, STy), r(x, STy), r(y, STx)\}.$$

Then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If this condition is satisfied, then the point x is the unique common fixed point of S and T.

By using Theorem 5 of [6], we have the following:

THEOREM 2.6 Let (X, r) be a general quasi-metric space and let S and T be two commuting self-mappings of X. Assume that there exist positive integers p, q such that for any x, y, $\in X$, $x \neq y$,

$$r((ST)^p x, (ST)^q y) < \sup\{r(x,y), r(x, (ST)^p x), r(y, (ST)^q y), r(x, (ST)^q y), r(y, (ST)^p x)\}.$$

Then S and T have a fixed point in X if and only if there exists a periodic point $x \in X$ of ST with periodic index k which satisfies the following condition:

$$k \neq 2|p_2 - q_2|,$$

where $p = p_1 k + p_2$, $q = q_1 k + q_2$, $0 \leq p_2, q_2 < k$ and p_1, q_1 are non-negative integers and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If this condition is satisfied, then the point x is the unique common fixed point of S and T.

PROOF. The necessity condition is obvious.

To prove converse, if we use Theorem 5 of [6] by replacing T with ST, then ST has a unique fixed point x in X. Therefore, employing the same argument as in the proof of Theorem 2.2, it follows that the point x is the unique fixed point of S and T. This completes the proof.

REMARK. Theorems 2.1 ~ 2.6 generalize some main results in [1], [2] and [4].

3. APPLICATIONS TO PROBABILISTIC QUASI-METRIC SPACES

First of all, we define partial orders " $\leq$ " and " $<$ " on $\mathcal{D}^+$ as follows, respectively: For any F_1, $F_2 \in \mathcal{D}^+$ and $t > 0$,

$$F_1 \leq F_2 \Rightarrow F_1(t) \geq F_2(t),$$
$$F_1 < F_2 \Rightarrow F_1(t) > F_2(t).$$

In the sequel, we denote $G = (\mathcal{D}^+, \leq, <)$. It is obvious that G satisfies the following conditions:

(G-1) there exists a minimal element $0 \overset{\text{def}}{=} H \in G$, where

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0, \end{cases}$$

(G-2) for any $F_1,\ F_2 \in G$,

$$\sup\{F_1, F_2\}(t) \overset{\text{def}}{=} \min\{F_1(t), F_2(t)\},$$

(G-3) for any $F \in G,\ F \not< F$,

(G-4) for any $F_1,\ F_2,\ F_3 \in G$,

$$F_1 < F_3,\ F_2 < F_3 \Rightarrow \sup\{F_1, F_2\} < F_3\,,$$
$$F_1 < F_2,\ F_2 \le F_3 \Rightarrow F_1 < F_3\,.$$

THEOREM 3.1 (Embedding Theorem) Let $(X, \mathcal{F})$ be a probabilistic quasi-metric space. Then $(X, \mathcal{F})$ is a general quasi-metric space, where $G = (D^+, \le, <)$ is the partial order set induced by the way as above.

PROOF. Let $r(x, y) = F_{x,y}$ for all $x,\ y \in X$. It is easy to see that r satisfies the conditions (QM-2) and (QM-2) of Definition 1.1.

The following results are obtained from Theorems 2.1 ~ 2.6 and Theorem 3.1 immediately:

THEOREM 3.2 Let $(X, \mathcal{F})$ be a probabilistic quasi-metric space and let S and T be two commuting self-mappings of X. If for any $x \in X$ and positive integer $n,\ q \ge 2$ with

$$T^i x \ne T^j x\,, \quad 0 \le i < j \le n-1\,,$$
$$S^i x \ne S^j x\,, \quad 0 \le i < j \le q-1\,,$$
$$F_{T^n S^i x, S^q T^i x}(T) > \min\left\{ \min_{1 \le j \le n, 1 \le j' \le q} F_{T^j x, S^{j'} x}(t), \min_{1 \le j \le n} F_{T^j x, x}(t), \min_{1 \le j' \le q} F_{S^{j'} x, x}(t) \right\}$$

for $i = 1, 2, ..., n-1$ and $j = 1, 2, ..., q-1$, then S and T have a common fixed point in X if and only if there exist integers $m,\ p,\ q,\ m > n \ge 0,\ p,\ q \ge 0$, and a point $x \in X$ such that

$$T^m x = S^p x = T^n x = S^q x$$

and either $F(S) \ne \emptyset$ or $F(T) \ne \emptyset$. If this condition is satisfied, then either $T^m x$ or $S^q x$ is a common fixed point of S and T.

THEOREM 3.3 Let $(X, \mathcal{F})$ be a probabilistic quasi-metric space and let S and T be two commuting self-mappings of X. If there exist positive integers $p,\ q$ such that for any $x,\ y \in X,\ x \ne y$, and for all $t > 0$,

$$F_{(ST)^p x, (ST)^q y}(t) > \min\left\{F_{x,y}(t), F_{x,(ST)^p x}(t), F_{y,(ST)^q y}(t), F_{x,(ST)^q y}(t), F_{y,(ST)^p x}(t)\right\},$$

then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST with periodic index k which satisfies the condition (2.5) in Theorem 2.6 and either $F(S) \cap P(ST) \ne \emptyset$ or $F(T) \cap P(ST) \ne \emptyset$. If this condition is satisfied, then the point x is the unique common fixed point of S and T.

The following is a special case of Theorem 3.3 obtained by setting $p = q = 1$:

COROLLARY 3.4 Let $(X, \mathcal{F})$ be a probabilistic quasi-metric space and let S and T be two commuting self-mappings of X. If for any $x,\ y \in X,\ x \ne y$, and $t > 0$,

$$F_{STx, STy}(t) > \min\{F_{x,y}(t), F_{x,STx}(t), F_{y,STy}(t), F_{x,STy}(t), F_{y,STx}(t)\}\,,$$

then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST and either $F(S) \cap P(ST) \ne \emptyset$ or $F(T) \cap P(ST) \ne \emptyset$. If this condition is satisfied, then the point x is the unique common fixed point of S and T.

THEOREM 3.5 Let $(X, \mathcal{F})$ be a probabilistic quasi-metric space and let S and T be two commuting self-mappings of X. If for any $x \in X$, $x \neq y$, there exists a positive integer $p(x, y)$ such that for all $t > 0$,

$$F_{(ST)^{p(x,y)}x,(ST)^{p(x,y)}y}(t) > \min\{F_{x,y}(t), F_{x,(ST)^{p(x,y)}x}(t), F_{y,(ST)^{p(x,y)}y}(t), F_{x,(ST)^{p(x,y)}y}(t), F_{y,(ST)^{p(x,y)}x}(t)\},$$

then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST with periodic index k such that for any $u, v \in A = \{x, (ST)x, ..., (ST)^{k-1}\}$, $u \neq v$, there exist x', $y' \in A$, $x' \neq y'$, satisfying the following conditions:

$$(St)^{p(x',y')}x' = u, \quad (ST)^{p(x',y')}y' = v$$

and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If these conditions are satisfied, then the point x is the unique common fixed point of S and T.

By setting $p(x, y) = p(x)$ in Theorem 3.5, we have the following:

COROLLARY 3.6 Let $(X, \mathcal{F})$ be a probabilistic quasi-metric space and let S and T be two commuting self-mappings of X. If there exists a positive integer $p(x)$ such that for every $y \in X$, $x \neq y$, and for all $t > 0$,

$$F_{(ST)^{p(x)}x,(ST)^{p(x)}y}(t) > \min\{F_{x,y}(t), F_{x,(ST)^{p(x)}x}(t), F_{y,(ST)^{p(x)}y}(t), F_{x,(ST)^{p(x)}y}(t), F_{y,(ST)^{p(x)}x}(t)\},$$

then S and T have a common fixed point in X if and only if there exists a periodic point $x \in X$ of ST with periodic index k such that for any u, $v \in A = \{x, (ST)x, ..., (ST)^{k-1}x\}, u \neq v$, there exist x', $y' \in A$, $x' \neq y'$, satisfying the following conditions:

$$(ST)^{p(x')}x' = u, \quad (ST)^{p(x')}y' = v$$

and either $F(S) \cap P(ST) \neq \emptyset$ or $F(T) \cap P(ST) \neq \emptyset$. If these conditions are satisfied, then the point x is the unique common fixed point of T.

REMARK. Theorems 3.3 ~ 3.6 include Theorems 1 ~ 5 in [4] and Theorems 3.3 ~ 3.6 in [5] as special cases.

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