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# THE LARGEST PROPER CONGRUENCE ON S(X)

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ABSTRACT. S(X) denotes the semigroup of all continuous selfmaps of the topological space X. In this paper, we find, for many spaces X, necessary and sufficient conditions for a certain type of congruence to be the largest proper congruence on S(X).

*KEY WORDS AND PHRASES. Semigroups of continuous selfmaps, largest proper congruence.*
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## 1. INTRODUCTION.

DEFINITION 1. A space X is <u>admissible</u> if it is compact, Hausdorff, contains more than one point and every continuous function from a closed subspace of X into X can be extended to a continuous selfmap of X .

REMARK: Any nondegenerate absolute retract is admissible as is any nondegenerate compact 0-dimensional metric space [1, p. 281, Corollary 3].

DEFINITION 2. A <u>unifying family</u> $\mathcal{G}$ is any nonempty family of nonempty subsets of X such that for any $A \in \mathcal{G}$ and any $f \in S(X)$, $f[A] \in \mathcal{G}$ whenever f is injective on A .

The collections of all subarcs, compact subsets, subcontinua, etc., are all unifying families. With every unifying family $\mathcal{G}$, we associate a congruence $\sigma(\mathcal{G})$ by defining $(f,g) \in \sigma(\mathcal{G})$ iff whenever either of the functions is injective on some $A \in \mathcal{G}$, then the two functions coincide on A . The verification that $\sigma(\mathcal{G})$ is a congruence is straightforward and will be omitted.

DEFINITION 3. A congruence of the form $\sigma(\mathcal{G})$ is referred to as a <u>unifying congruence</u>.

DEFINITION 4. A subspace Y of X is a <u>quasiretract</u> of X if there exists a continuous function from X into Y which is injective on Y .

It is immediate that every retract of X is a quasiretract. However, quasiretracts which are not retracts are abundant. For example, any subspace of X which contains a copy of X is a quasiretract so that quasiretracts of connected spaces need not even be connected and, similarly, quasiretracts of compact spaces need not be compact.

DEFINITION 5. A unifying family $\mathcal{G}$ is said to be normal if each $A \in \mathcal{G}$ is a quasiretract of $X$ and for each $A \in \mathcal{G}$ and each open subset of $G$ of $A$, there exists a $B \in \mathcal{G}$ such that $B \subset G$. The corresponding congruence $\sigma(A)$ will be referred to as a normal unifying congruence.

## 2. MAIN RESULTS.

We are now in a position to prove the main result of this paper.

THEOREM 6. *Let $X$ be admissible and let $\mathcal{G}$ be a normal unifying family. Then the normal unifying congruence $\sigma(\mathcal{G})$ is the largest proper congruence on $S(X)$ if and only if each nonempty open subset of each $A \in \mathcal{G}$ contains a copy of $X$.*

PROOF. (Sufficiency). It is immediate that all constant functions are related and that no constant function is related to the identity map so that $\sigma(\mathcal{G})$ is, indeed, a proper congruence. Let $\rho$ be any other congruence on $S(X)$ and suppose $\rho \not\subset \sigma(\mathcal{G})$. Then there exist two functions $f$ and $g$ in $S(X)$ such that $(f,g) \in \rho - \sigma(\mathcal{G})$ and this implies that there exists an $A \in \mathcal{G}$ such that one of the functions (say $f$) is injective on $A$ and $f(a) \neq g(a)$ for some $a \in A$. Then $G = \{x \in A: f(x) \neq g(x)\}$ is a nonempty open subset of $A$ and therefore contains a copy $Y$ of $X$. Let $h$ be any homeomorphism from $X$ onto $Y$ and define a mapping $k$ from $f[Y]$ into $X$ by $k(x) = (f \circ h)^{-1}(x)$. Now choose any point $p \in Y$. If $g(h(p)) \in f[Y]$, extend $k$ to a continuous selfmap $t$ of $X$. This can be done since $X$ is admissible. If $g(h(p)) \notin f[Y]$, choose any point $q \neq p$ and define a map $\hat{k}$ on $f[Y] \cup \{g(h(p))\}$ by $\hat{k}(x) = k(x)$ for $x \in f[Y]$ and $\hat{k}(g(h(p))) = q$. In this case also, $\hat{k}$ can be extended to a continuous selfmap $t$ of $X$ since $X$ is admissible. It is immediate that

$$(i,\ t \circ g \circ h) = (t \circ f \circ h,\ t \circ g \circ h) \in \rho \tag{2.1}$$

where $i$ denotes the identity map. Furthermore we assert that

$$t \circ g \circ h(p) \neq p\ . \tag{2.2}$$

This is immediate in the case where $g(h(p)) \notin f[Y]$ for then, $t(g(h(p))) = q$. As for the case where $g(h(p)) \in f[Y]$, suppose $t \circ g \circ h(p) = p$. Then $(f \circ h)^{-1} \circ g \circ h(p) = p$ which implies $g(h(p)) = f(h(p))$. But this is a contradiction since $h(p) \in Y$ and $f$ and $g$ differ at each point of $Y$. Thus, (2.2) has been verified. Now let $r = t \circ g \circ h$. Since $p \in A$ and $r(p) \neq p$, there exists an open subset $H$ of $A$ containing $p$ such that $c\ell\, r[H] \cap c\ell\, H = \emptyset$. Then $H$ contains a copy $Z$ of $X$ and we let $\alpha$ be any homeomorphism from $X$ onto $Z$. Define a mapping $\beta$ on $Z \cup c\ell\, r[H]$ by $\beta(x) = \alpha^{-1}(x)$ for $x \in Z$ and $\beta(x) = p$ for $x \in c\ell\, r[H]$. Since $X$ is admissible, $\beta$ has an extension to a continuous selfmap $\gamma$ of $X$. Now $(i,r) \in \rho$ from (2.1) and this implies that

$$(i, \langle p \rangle) = (\gamma \circ \alpha,\ \gamma \circ r \circ \alpha) \in \rho \tag{2.3}$$

where $\langle p \rangle$ denotes the constant function which maps everything into the point $p$. Thus, for any two functions $v,w \in S(X)$ we have

$$(v,\langle p\rangle) = (i \circ v,\ \langle p\rangle \circ v) \in \rho \tag{2.4}$$

and similarly

$$(w,\langle p\rangle) = (i \circ w,\ \langle p\rangle \circ w) \in \rho\ . \tag{2.5}$$

Statements (2.4) and (2.5) together imply that $(v,w) \in \rho$. That is, $\rho$ is the universal congruence. This completes the sufficiency portion of the proof.

(Necessity). Suppose now that $\sigma(\mathcal{G})$ is the largest proper congruence on $S(X)$ and let $\mathcal{J}$ be the family of all subspaces of $X$ which are homeomorphic to $X$. Then $\mathcal{J}$ is a unifying family and the unifying congruence $\sigma(\mathcal{J})$ is a proper congruence on $X$. To see this, observe that for any point $p \in X$, $(i,\langle p\rangle) \notin \sigma(\mathcal{J})$ where, as before, $i$ denotes the identity map and $\langle p\rangle$ is the function which maps everything into the point $p$. Thus, we have

$$\sigma(\mathcal{J}) \subset \sigma(\mathcal{G})\ . \tag{2.6}$$

Now take any $A \in \mathcal{G}$ and let $G$ be any open subset of $A$. Since the unifying family $\mathcal{G}$ is normal, there exists a $B \in \mathcal{G}$ such that $B \subset G$. Furthermore, $B$ is a quasi-retract of $X$ so there exists a continuous function $f$ from $X$ into $B$ which is injective on $B$. Choose any point $p \in X$ and note that $(f,\langle p\rangle) \notin \sigma(\mathcal{G})$. It then follows from this and (2.6) that $(f,\langle p\rangle) \notin \sigma(\mathcal{J})$. This means that there is some $Y$ in $\mathcal{J}$ on which $f$ is injective. Thus, $f[Y] \subset B \subset G$ and the proof is complete since $f[Y]$ is homeomorphic to $X$.

COROLLARY 6. *Let $X$ be any N-dimensional admissible space which is a subspace of the Euclidean N-cell $I^N$ and let $\mathcal{G}$ consist of all subspaces of $X$ which are homeomorphic to $I^N$. Then $\sigma(\mathcal{G})$ is the largest proper congruence on $S(X)$.*

PROOF. It follows from Theorem IV3 [2, p. 44] that $\mathcal{G}$ is a nonempty collection and it is immediate that $\mathcal{G}$ is a normal unifying family. The conclusion now follows from Theorem 6.

EXAMPLE: Let $D = \{(x,y) \in R^2 : x^2 + y^2 \le 1\}$, let $J = \{(x,y) \in R^2 : x = 0$ and $1 \le y \le 2\}$ and let $X = D \cup J$ where the topology is that induced by the Euclidean plane. Let $\mathcal{G}_1$ consist of all subspaces of $X$ which are homeomorphic to $D$, let $\mathcal{G}_2$ consist of all subspaces of $X$ which are homeomorphic to the disjoint union of two copies of $D$ and let $\mathcal{G}_3$ consist of all subspaces of $X$ which are homeomorphic to either $X$ or $J$. The space $S$ is an absolute retract so it is certainly admissible. Moreover, one easily verifies that $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ are all normal unifying families. Theorem 6 therefore applies and it follows that $\sigma(\mathcal{G}_1) = \sigma(\mathcal{G}_2)$ is the largest proper congruence on $S(X)$ while $\sigma(\mathcal{G}_3)$ is not the largest proper congruence.

In closing, we remark that the results in this paper extend and supplement some of the results in Chapter 6 of [3].

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