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ON β-DUAL OF VECTOR-VALUED SEQUENCE SPACES OF MADDOX

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The β-dual of a vector-valued sequence space is defined and studied. We show that if an X-valued sequence space E is a BK-space having AK property, then the dual space of E and its β-dual are isometrically isomorphic. We also give characterizations of β-dual of vector-valued sequence spaces of Maddox $\ell(X,p)$, $\ell_\infty(X,p)$, $c_0(X,p)$, and $c(X,p)$.

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1. Introduction. Let $(X,\|\cdot\|)$ be a Banach space and $p=(p_k)$ a bounded sequence of positive real numbers. Let $\mathbb{N}$ be the set of all natural numbers, we write $x=(x_k)$ with x_k in X for all $k\in\mathbb{N}$. The X-valued sequence spaces of Maddox are defined as

$$\begin{aligned} c_0(X,p) &= \Big\{x=(x_k):\lim_{k\to\infty}\|x_k\|^{p_k}=0\Big\};\\ c(X,p) &= \Big\{x=(x_k):\lim_{k\to\infty}\|x_k-a\|^{p_k}=0 \text{ for some } a\in X\Big\};\\ \ell_\infty(X,p) &= \Big\{x=(x_k):\sup_k\|x_k\|^{p_k}<\infty\Big\};\\ \ell(X,p) &= \Big\{x=(x_k):\sum_{k=1}^{\infty}\|x_k\|^{p_k}<\infty\Big\}. \end{aligned} \tag{1.1}$$

When $X=\mathbb{K}$, the scalar field of X, the corresponding spaces are written as $c_0(p)$, $c(p)$, $\ell_\infty(p)$, and $\ell(p)$, respectively. All of these spaces are known as the sequence spaces of Maddox. These spaces were introduced and studied by Simons [7] and Maddox [3, 4, 5]. The space $\ell(p)$ was first defined by Nakano [6] and is known as the Nakano sequence space. Grosse-Erdmann [1] has investigated the structure of the spaces $c_0(p)$, $c(p)$, $\ell(p)$, and $\ell_\infty(p)$ and has given characterizations of β-dual of scalar-valued sequence spaces of Maddox.

In [8], Wu and Bu gave characterizations of Köthe dual of the vector-valued sequence space $\ell_p[X]$, where $\ell_p[X]$, $1<p<\infty$, is defined by

$$\ell_p[X]=\Big\{x=(x_k):\sum_{k=1}^{\infty}|f(x_k)|^p<\infty \text{ for each } f\in X'\Big\}. \tag{1.2}$$

In this paper, the β-dual of a vector-valued sequence space is defined and studied and we give characterizations of β-dual of vector-valued sequence spaces of Maddox

$\ell(X,p)$, $\ell_\infty(X,p)$, $c_0(X,p)$, and $c(X,p)$. Some results, obtained in this paper, are generalizations of some in [1, 3].

2. Notation and definitions. Let $(X,\|\cdot\|)$ be a Banach space. Let $W(X)$ and $\Phi(X)$ denote the space of all sequences in X and the space of all finite sequences in X, respectively. A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ and $k \in \mathbb{N}$ we write that x_k stand for the kth term of x. For $x \in X$ and $k \in \mathbb{N}$, we let $e^{(k)}(x)$ be the sequence $(0,0,0,\dots,0,x,0,\dots)$ with x in the kth position and let $e(x)$ be the sequence $(x,x,x,\dots)$. For a fixed scalar sequence $u=(u_k)$, the sequence space E_u is defined as

$$E_u = \{x=(x_k) \in W(X) : (u_k x_k) \in E\}. \tag{2.1}$$

An X-valued sequence space E is said to be *normal* if $(y_k) \in E$ whenever $\|y_k\| \le \|x_k\|$ for all $k \in \mathbb{N}$ and $(x_k) \in E$. Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a *K-space* if, for each $k \in \mathbb{N}$, the kth coordinate mapping $p_k : E \to X$, defined by $p_k(x) = x_k$, is continuous on E. In addition, if (E,τ) is a *Fréchet (Banach) space*, then E is called an FK-(BK)-space. Now, suppose that E contains $\Phi(X)$, then E is said to have *property AK* if $\sum_{k=1}^{n} e^{(k)}(x_k) \to x$ in E as $n \to \infty$ for every $x=(x_k) \in E$.

The spaces $c_0(p)$ and $c(p)$ are FK-spaces. In $c_0(X,p)$, we consider the function $g(x) = \sup_k \|x_k\|^{p_k/M}$, where $M = \max\{1, \sup_k p_k\}$, as a paranorm on $c_0(X,p)$, and it is known that $c_0(X,p)$ is an FK-space having property AK under the paranorm g defined as above. In $\ell(X,p)$, we consider it as a paranormed sequence space with the paranorm given by $\|(x_k)\| = (\sum_{k=1}^{\infty} \|x_k\|^{p_k})^{1/M}$. It is known that $\ell(X,p)$ is an FK-space under the paranorm defined as above.

For an X-valued sequence space E, define its Köthe dual with respect to the dual pair (X,X') (see [2]) as follows:

$$E^{\times}|_{(X,X')} = \left\{(f_k) \subset X' : \sum_{k=1}^{\infty} |f_k(x_k)| < \infty \ \forall x=(x_k) \in E\right\}. \tag{2.2}$$

In this paper, we denote $E^{\times}|_{(X,X')}$ by E^{α} and it is called the α-dual of E.

For a sequence space E, the β-dual of E is defined by

$$E^{\beta} = \left\{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x_k) \text{ converges } \forall\ (x_k) \in E\right\}. \tag{2.3}$$

It is easy to see that $E^{\alpha} \subseteq E^{\beta}$.

For the sake of completeness we introduce some further sequence spaces that will be considered as β-dual of the vector-valued sequence spaces of Maddox:

$$M_0(X,p) = \left\{x=(x_k) : \sum_{k=1}^{\infty} ||x_k|| M^{-1/p_k} < \infty \text{ for some } M \in \mathbb{N}\right\};$$

$$M_\infty(X,p) = \left\{x=(x_k) : \sum_{k=1}^{\infty} ||x_k|| n^{1/p_k} < \infty \ \forall n \in \mathbb{N}\right\};$$

$$\ell_0(X,p) = \left\{x = (x_k) : \sum_{k=1}^{\infty} ||x_k||^{p_k} M^{-p_k} < \infty \text{ for some } M \in \mathbb{N}\right\}, \quad p_k > 1 \ \forall k \in N;$$

$$cs[X'] = \left\{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x) \text{ converges } \forall\ x \in X\right\}. \tag{2.4}$$

When $X = \mathbb{K}$, the scalar field of X, the corresponding first two sequence spaces are written as $M_0(p)$ and $M_\infty(p)$, respectively. These two spaces were first introduced by Grosse-Erdmann [1].

3. Main results. We begin by giving some general properties of β-dual of vector-valued sequence spaces.

Proposition 3.1. *Let X be a Banach space and let E, E_1, and E_2 be X-valued sequence spaces. Then*
(i) $E^\alpha \subseteq E^\beta$.
(ii) *If* $E_1 \subseteq E_2$, *then* $E_2^\beta \subseteq E_1^\beta$.
(iii) *If* $E = E_1 + E_2$, *then* $E^\beta = E_1^\beta \cap E_2^\beta$.
(iv) *If E is normal, then* $E^\alpha = E^\beta$.

Proof. Assertions (i), (ii), and (iii) are immediately obtained by the definitions. To prove (iv), by (i), it suffices to show only that $E^\beta \subseteq E^\alpha$. Let $(f_k) \in E^\beta$ and $x = (x_k) \in E$. Then $\sum_{k=1}^\infty f_k(x_k)$ converges. Choose a scalar sequence (t_k) with $|t_k| = 1$ and $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in \mathbb{N}$. Since E is normal, $(t_k x_k) \in E$. It follows that $\sum_{k=1}^\infty |f_k(x_k)|$ converges, hence $(f_k) \in E^\alpha$. □

If E is a BK-space, we define a norm on E^β by the formula

$$||(f_k)||_{E^\beta} = \sup_{\|(x_k)\| \le 1} \left| \sum_{k=1}^{\infty} f_k(x_k) \right|. \tag{3.1}$$

It is easy to show that $\|\cdot\|_{E^\beta}$ is a norm on E^β.

Next, we give a relationship between β-dual of a sequence space and its continuous dual. Indeed, we need a lemma.

Lemma 3.2. *Let E be an X-valued sequence space which is an FK-space containing $\Phi(X)$. Then for each $k \in \mathbb{N}$, the mapping $T_k : X \to E$, defined by $T_k x = e^k(x)$, is continuous.*

Proof. Let $V = \{e^k(x) : x \in X\}$. Then V is a closed subspace of E, so it is an FK-space because E is an FK-space. Since E is a K-space, the coordinate mapping $p_k : V \to X$ is continuous and bijective. It follows from the open mapping theorem that p_k is open, which implies that $p_k^{-1} : X \to V$ is continuous. But since $T_k = p_k^{-1}$, we thus obtain that T_k is continuous. □

Theorem 3.3. *If E is a BK-space having property AK, then E^β and E' are isometrically isomorphic.*

Proof. We first show that for $x=(x_k)\in E$ and $f\in E'$,

$$f(x)=\sum_{k=1}^{\infty} f(e^k(x_k)). \tag{3.2}$$

To show this, let $x=(x_k)\in E$ and $f\in E'$. Since E has property AK,

$$x=\lim_{n\to\infty}\sum_{k=1}^{n} e^{(k)}(x_k). \tag{3.3}$$

By the continuity of f, it follows that

$$f(x)=\lim_{n\to\infty}\sum_{k=1}^{n} f(e^{(k)}(x_k))=\sum_{k=1}^{\infty} f(e^{(k)}(x_k)), \tag{3.4}$$

so (3.2) is obtained. For each $k\in\mathbb{N}$, let $T_k: X\to E$ be defined as in Lemma 3.2. Since E is a BK-space, by Lemma 3.2, T_k is continuous. Hence $f\circ T_k\in X'$ for all $k\in\mathbb{N}$. It follows from (3.2) that

$$f(x)=\sum_{k=1}^{\infty}(f\circ T_k)(x_k)\quad \forall x=(x_k)\in E. \tag{3.5}$$

It implies, by (3.5), that $(f\circ T_k)_{k=1}^{\infty}\in E^{\beta}$. Define $\varphi: E'\to E^{\beta}$ by

$$\varphi(f)=(f\circ T_k)_{k=1}^{\infty}\quad \forall f\in E'. \tag{3.6}$$

It is easy to see that φ is linear. Now, we show that φ is onto. Let $(f_k)\in E^{\beta}$. Define $f: E\to K$, where K is the scalar field of X, by

$$f(x)=\sum_{k=1}^{\infty} f_k(x_k)\quad \forall x=(x_k)\in E. \tag{3.7}$$

For each $k\in\mathbb{N}$, let p_k be the kth coordinate mapping on E. Then we have

$$f(x)=\sum_{k=1}^{\infty}(f_k\circ p_k)(x)=\lim_{n\to\infty}\sum_{k=1}^{n}(f\circ p_k)(x). \tag{3.8}$$

Since f_k and p_k are continuous linear, so is also continuous $f\circ p_k$. It follows by Banach-Steinhaus theorem that $f\in E'$ and we have by (3.7) that; for each $k\in\mathbb{N}$ and each $z\in X$, $(f\circ T_k)(z)=f(e^{(k)}(z))=f_k(z)$. Thus $f\circ T_k=f_k$ for all $k\in\mathbb{N}$, which implies that $\varphi(f)=(f_k)$, hence φ is onto.

Finally, we show that φ is linear isometry. For $f\in E'$, we have

$$\begin{aligned}\|f\| &= \sup_{\|(x_k)\|\le 1}\left|f((x_k))\right| \\ &= \sup_{\|(x_k)\|\le 1}\left|\sum_{k=1}^{\infty} f(e^{(k)}(x_k))\right| \quad \text{(by (3.2))} \\ &= \sup_{\|(x_k)\|\le 1}\left|\sum_{k=1}^{\infty}(f\circ T_k)(x_k)\right| \\ &= \left\|(f\circ T_k)_{k=1}^{\infty}\right\|_{E^{\beta}} \\ &= \left\|\varphi(f)\right\|_{E^{\beta}}.\end{aligned} \tag{3.9}$$

Hence φ is isometry. Therefore, $\varphi : E' \to E^\beta$ is an isometrically isomorphism from E' onto E^β. This completes the proof. □

We next give characterizations of β-dual of the sequence space $\ell(X,p)$ when $p_k > 1$ for all $k \in \mathbb{N}$.

Theorem 3.4. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k > 1$ for all $k \in \mathbb{N}$. Then $\ell(X,p)^\beta = \ell_0(X',q)$, where $q = (q_k)$ is a sequence of positive real numbers such that $1/p_k + 1/q_k = 1$ for all $k \in \mathbb{N}$.*

Proof. Suppose that $(f_k) \in \ell_0(X',q)$. Then $\sum_{k=1}^\infty \|f_k\|^{q_k} M^{-q_k} < \infty$ for some $M \in \mathbb{N}$. Then for each $x = (x_k) \in \ell(X,p)$, we have

$$\begin{aligned}
\sum_{k=1}^\infty |f_k(x_k)| &\le \sum_{k=1}^\infty \|f_k\| M^{-1/p_k} M^{1/p_k} \|x_k\| \\
&\le \sum_{k=1}^\infty \left(\|f_k\|^{q_k} M^{-q_k/p_k} + M\|x_k\|^{p_k}\right) \\
&= \sum_{k=1}^\infty \|f_k\|^{q_k} M^{-(q_k-1)} + M\sum_{k=1}^\infty \|x_k\|^{p_k} \\
&= M\sum_{k=1}^\infty \|f_k\|^{q_k} M^{-q_k} + M\sum_{k=1}^\infty \|x_k\|^{p_k} \\
&< \infty,
\end{aligned} \tag{3.10}$$

which implies that $\sum_{k=1}^\infty f_k(x_k)$ converges, so $(f_k) \in \ell(X,p)^\beta$.

On the other hand, assume that $(f_k) \in \ell(X,p)^\beta$, then $\sum_{k=1}^\infty f_k(x_k)$ converges for all $x = (x_k) \in \ell(X,p)$. For each $x = (x_k) \in \ell(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in \mathbb{N}$. Since $(t_k x_k) \in \ell(X,p)$, by our assumption, we have $\sum_{k=1}^\infty f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^\infty |f_k(x_k)| < \infty \quad \forall x \in \ell(X,p). \tag{3.11}$$

We want to show that $(f_k) \in \ell_0(X',q)$, that is, $\sum_{k=1}^\infty \|f_k\|^{q_k} M^{-q_k} < \infty$ for some $M \in \mathbb{N}$. If it is not true, then

$$\sum_{k=1}^\infty \|f_k\|^{q_k} m^{-q_k} = \infty \quad \forall m \in \mathbb{N}. \tag{3.12}$$

It implies by (3.12) that for each $k \in \mathbb{N}$,

$$\sum_{i>k} \|f_i\|^{q_i} m^{-q_i} = \infty \quad \forall m \in \mathbb{N}. \tag{3.13}$$

By (3.12), let $m_1 = 1$, then there is a $k_1 \in \mathbb{N}$ such that

$$\sum_{k \le k_1} \|f_k\|^{q_k} m_1^{-q_k} > 1. \tag{3.14}$$

By (3.13), we can choose $m_2 > m_1$ and $k_2 > k_1$ with $m_2 > 2^2$ such that

$$\sum_{k_1<k\le k_2} \|f_k\|^{q_k} m_2^{-q_k} > 1. \tag{3.15}$$

Proceeding in this way, we can choose sequences of positive integers (k_i) and (m_i) with $1 = k_0 < k_1 < k_2 < \cdots$ and $m_1 < m_2 < \cdots$, such that $m_i > 2^i$ and

$$\sum_{k_{i-1}<k\le k_i} \|f_k\|^{q_k} m_i^{-q_k} > 1. \tag{3.16}$$

For each $i \in \mathbb{N}$, choose x_k in X with $\|x_k\| = 1$ for all $k \in \mathbb{N}$, $k_{i-1} < k \le k_i$ such that

$$\sum_{k_{i-1}<k\le k_i} |f_k(x_k)|^{q_k} m_i^{-q_k} > 1 \quad \forall i \in \mathbb{N}. \tag{3.17}$$

Let $a_i = \sum_{k_{i-1}<k\le k_i} |f_k(x_k)|^{q_k} m_i^{-q_k}$. Put $y = (y_k)$, $y_k = a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k-1} x_k$ for all $k \in \mathbb{N}$ with $k_{i-1} < k \le k_i$. By using the fact that $p_k q_k = p_k + q_k$ and $p_k(q_k - 1) = q_k$ for all $k \in \mathbb{N}$, we have that for each $i \in \mathbb{N}$,

$$\begin{aligned}
\sum_{k_{i-1}<k\le k_i} \|y_k\|^{p_k} &= \sum_{k_{i-1}<k\le k_i} \left\| a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k-1} x_k \right\|^{p_k} \\
&= \sum_{k_{i-1}<k\le k_i} a_i^{-p_k} m_i^{-p_k q_k} |f_k(x_k)|^{q_k} \\
&= \sum_{k_{i-1}<k\le k_i} a_i^{-p_k} m_i^{-p_k} m_i^{-q_k} |f_k(x_k)|^{q_k} \\
&\le a_i^{-1} m_i^{-1} \sum_{k_{i-1}<k\le k_i} m_i^{-q_k} |f_k(x_k)|^{q_k} \\
&\le a_i^{-1} m_i^{-1} a_i \\
&= m_i^{-1} \\
&< \frac{1}{2^i},
\end{aligned} \tag{3.18}$$

so we have that $\sum_{k=1}^{\infty} \|y_k\|^{p_k} \le \sum_{i=1}^{\infty} 1/2^i < \infty$. Hence, $y = (y_k) \in \ell(X,p)$. For each $i \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{k_{i-1}<k\le k_i} |f_k(y_k)| &= \sum_{k_{i-1}<k\le k_i} \left| f_k\left(a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k-1} x_k\right)\right| \\
&= \sum_{k_{i-1}<k\le k_i} a_i^{-1} m_i^{-q_k} |f_k(x_k)|^{q_k} \\
&= a_i^{-1} \sum_{k_{i-1}<k\le k_i} m_i^{-q_k} |f_k(x_k)|^{q_k} \\
&= 1,
\end{aligned} \tag{3.19}$$

so that $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts (3.11). Hence $(f_k) \in \ell_0(X', q)$. The proof is now complete. ◻

The following theorem gives a characterization of β-dual of $\ell(X,p)$ when $p_k \le 1$ for all $k \in \mathbb{N}$. To do this, the following lemma is needed.

Lemma 3.5. *Let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $\ell_\infty(X,p) = \bigcup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$.*

Proof. Let $x \in \ell_\infty(X,p)$, then there is some $n \in \mathbb{N}$ with $\|x_k\|^{p_k} \le n$ for all $k \in \mathbb{N}$. Hence $\|x_k\| n^{-1/p_k} \le 1$ for all $k \in \mathbb{N}$, so that $x \in \ell_\infty(X)_{(n^{-1/p_k})}$. On the other hand, if $x \in \bigcup_{n=1}^{\infty} \ell_\infty(X)_{(n^{-1/p_k})}$, then there are some $n \in \mathbb{N}$ and $M > 1$ such that $\|x_k\| n^{-1/p_k} \le M$ for every $k \in \mathbb{N}$. Then we have $\|x_k\|^{p_k} \le nM^{p_k} \le nM^{\alpha}$ for all $k \in \mathbb{N}$, where $\alpha = \sup_k p_k$. Hence $x \in \ell_\infty(X,p)$. □

Theorem 3.6. *Let $p = (p_k)$ be a bounded sequence of positive real numbers with $p_k \le 1$ for all $k \in \mathbb{N}$. Then $\ell(X,p)^\beta = \ell_\infty(X',p)$.*

Proof. If $(f_k) \in \ell(X,p)^\beta$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for every $x = (x_k) \in \ell(X,p)$, using the same proof as in Theorem 3.4, we have

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \forall x = (x_k) \in \ell(X,p). \tag{3.20}$$

If $(f_k) \notin \ell_\infty(X',p)$, it follows by Lemma 3.5 that $\sup_k \|f_k\| m^{-1/p_k} = \infty$ for all $m \in \mathbb{N}$. For each $i \in \mathbb{N}$, choose sequences (m_i) and (k_i) of positive integers with $m_1 < m_2 < \cdots$ and $k_1 < k_2 < \cdots$ such that $m_i > 2^i$ and $\|f_{k_i}\| m_i^{-1/p_{k_i}} > 1$. Choose $x_{k_i} \in X$ with $\|x_{k_i}\| = 1$ such that

$$|f_{k_i}(x_{k_i})| m_i^{-1/p_{k_i}} > 1. \tag{3.21}$$

Let $y = (y_k)$, $y_k = m_i^{-1/p_{k_i}} x_{k_i}$ if $k = k_i$ for some i, and 0 otherwise. Then $\sum_{k=1}^{\infty} \|y_k\|^{p_k} = \sum_{i=1}^{\infty} 1/m_i < \sum_{i=1}^{\infty} 1/2^i = 1$, so that $(y_k) \in \ell(X,p)$ and

$$\begin{aligned} \sum_{k=1}^{\infty} |f_k(y_k)| &= \sum_{i=1}^{\infty} \left| f_{k_i}\left(m_i^{-1/p_{k_i}} x_{k_i} \right) \right| \\ &= \sum_{i=1}^{\infty} m_i^{-1/p_{k_i}} |f_{k_i}(x_{k_i})| \\ &= \infty \quad (\text{by } (3.21)), \end{aligned} \tag{3.22}$$

and this is contradictory to (3.20), hence $(f_k) \in \ell_\infty(X',p)$.

Conversely, assume that $(f_k) \in \ell_\infty(X',p)$. By Lemma 3.5, there exists $M \in \mathbb{N}$ such that $\sup_k \|f_k\| M^{-1/p_k} < \infty$, so there is a $K > 0$ such that

$$\|f_k\| \le KM^{1/p_k} \quad \forall k \in \mathbb{N}. \tag{3.23}$$

Let $x = (x_k) \in \ell(X,p)$. Then there is a $k_0 \in \mathbb{N}$ such that $M^{1/p_k}\|x_k\| \le 1$ for all $k \ge k_0$. By $p_k \le 1$ for all $k \in \mathbb{N}$, we have that, for all $k \ge k_0$,

$$M^{1/p_k}\|x_k\| \le (M^{1/p_k}\|x_k\|)^{p_k} = M\|x_k\|^{p_k}. \tag{3.24}$$

Then

$$\begin{aligned}\sum_{k=1}^{\infty}|f_k(x_k)| &\le \sum_{k=1}^{k_0}\|f_k\|\|x_k\|+\sum_{k=k_0+1}^{\infty}\|f_k\|\|x_k\| \\ &\le \sum_{k=1}^{k_0}\|f_k\|\|x_k\|+K\sum_{k=k_0+1}^{\infty}M^{1/p_k}\|x_k\| \quad \text{(by (3.23))} \\ &\le \sum_{k=1}^{k_0}\|f_k\|\|x_k\|+KM\sum_{k=k_0+1}^{\infty}\|x_k\|^{p_k} \quad \text{(by (3.24))} \\ &< \infty.\end{aligned} \tag{3.25}$$

This implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, hence $(f_k)\in\ell(X,p)^{\beta}$. □

Theorem 3.7. *Let $p=(p_k)$ be a bounded sequence of positive real numbers. Then $\ell_\infty(X,p)^{\beta}=M_\infty(X',p)$.*

Proof. If $(f_k)\in M_\infty(X',p)$, then $\sum_{k=1}^{\infty}\|f_k\|m^{1/p_k}<\infty$ for all $m\in\mathbb{N}$, we have that for each $x=(x_k)\in\ell_\infty(X,p)$, there is $m_0\in\mathbb{N}$ such that $\|x_k\|\le m_0^{1/p_k}$ for all $k\in\mathbb{N}$, hence $\sum_{k=1}^{\infty}|f_k(x_k)|\le\sum_{k=1}^{\infty}\|f_k\|\|x_k\|\le\sum_{k=1}^{\infty}\|f_k\|m_0^{1/p_k}<\infty$, which implies that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so that $(f_k)\in\ell_\infty(X,p)^{\beta}$.

Conversely, assume that $(f_k)\in\ell_\infty(X,p)^{\beta}$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x=(x_k)\in\ell_\infty(X,p)$, by using the same proof as in Theorem 3.4, we have

$$\sum_{k=1}^{\infty}|f_k(x_k)|<\infty \quad \forall x=(x_k)\in\ell_\infty(X,p). \tag{3.26}$$

If $(f_k)\notin M_\infty(X',p)$, then $\sum_{k=1}^{\infty}\|f_k\|M^{1/p_k}=\infty$ for some $M\in\mathbb{N}$. Then we can choose a sequence (k_i) of positive integers with $0=k_0<k_1<k_2<\cdots$ such that

$$\sum_{k_{i-1}<k\le k_i}\|f_k\|M^{1/p_k}>i \quad \forall i\in\mathbb{N}. \tag{3.27}$$

And we choose x_k in X with $\|x_k\|=1$ such that for all $i\in\mathbb{N}$,

$$\sum_{k_{i-1}<k\le k_i}|f_k(x_k)|M^{1/p_k}>i. \tag{3.28}$$

Put $y=(y_k)$, $y_k=M^{1/p_k}x_k$. Clearly, $y\in\ell_\infty(X,p)$ and

$$\sum_{k=1}^{\infty}|f_k(y_k)|\ge\sum_{k_{i-1}<k\le k_i}|f_k(x_k)|M^{1/p_k}>i \quad \forall i\in\mathbb{N}. \tag{3.29}$$

Hence $\sum_{k=1}^{\infty}|f_k(y_k)|=\infty$, which contradicts (3.26). Hence $(f_k)\in M_\infty(X',p)$. The proof is now complete. □

Theorem 3.8. *Let $p=(p_k)$ be a bounded sequence of positive real numbers. Then $c_0(X,p)^{\beta}=M_0(X',p)$.*

Proof. Suppose $(f_k) \in M_0(X',p)$, then $\sum_{k=1}^{\infty} \|f_k\| M^{-1/p_k} < \infty$ for some $M \in \mathbb{N}$. Let $x = (x_k) \in c_0(X,p)$. Then there is a positive integer K_0 such that $\|x_k\|^{p_k} < 1/M$ for all $k \geq K_0$, hence $\|x_k\| < M^{-1/p_k}$ for all $k \geq K_0$. Then we have

$$\sum_{k=K_0}^{\infty} |f_k(x_k)| \leq \sum_{k=K_0}^{\infty} ||f_k|| ||x_k|| \leq \sum_{k=K_0}^{\infty} ||f_k|| M^{-1/p_k} < \infty. \tag{3.30}$$

It follows that $\sum_{k=1}^{\infty} f_k(x_k)$ converges, so that $(f_k) \in c_0(X,p)^{\beta}$.

On the other hand, assume that $(f_k) \in c_0(X,p)^{\beta}$, then $\sum_{k=1}^{\infty} f_k(x_k)$ converges for all $x = (x_k) \in c_0(X,p)$. For each $x = (x_k) \in c_0(X,p)$, choose scalar sequence (t_k) with $|t_k| = 1$ such that $f_k(t_k x_k) = |f_k(x_k)|$ for all $k \in \mathbb{N}$. Since $(t_k x_k) \in c_0(X,p)$, by our assumption, we have $\sum_{k=1}^{\infty} f_k(t_k x_k)$ converges, so that

$$\sum_{k=1}^{\infty} |f_k(x_k)| < \infty \quad \forall x \in c_0(X,p). \tag{3.31}$$

Now, suppose that $(f_k) \notin M_0(X',p)$. Then $\sum_{k=1}^{\infty} \|f_k\| m^{-1/p_k} = \infty$ for all $m \in \mathbb{N}$. Choose $m_1, k_1 \in \mathbb{N}$ such that

$$\sum_{k \leq k_1} ||f_k|| m_1^{-1/p_k} > 1 \tag{3.32}$$

and choose $m_2 > m_1$ and $k_2 > k_1$ such that

$$\sum_{k_1 < k \leq k_2} ||f_k|| m_2^{-1/p_k} > 2. \tag{3.33}$$

Proceeding in this way, we can choose $m_1 < m_2 < \cdots$, and $0 = k_1 < k_2 < \cdots$ such that

$$\sum_{k_{i-1} < k \leq k_i} ||f_k|| m_i^{-1/p_k} > i. \tag{3.34}$$

Take x_k in X with $\|x_k\| = 1$ for all $k, k_{i-1} < k \leq k_i$ such that

$$\sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \forall i \in \mathbb{N}. \tag{3.35}$$

Put $y = (y_k)$, $y_k = m_i^{-1/p_k} x_k$ for $k_{i-1} < k \leq k_i$, then $y \in c_0(X,p)$ and

$$\sum_{k=1}^{\infty} |f_k(y_k)| \geq \sum_{k_{i-1} < k \leq k_i} |f_k(x_k)| m_i^{-1/p_k} > i \quad \forall i \in \mathbb{N}. \tag{3.36}$$

Hence we have $\sum_{k=1}^{\infty} |f_k(y_k)| = \infty$, which contradicts (3.31), therefore $(f_k) \in M_0(X',p)$. This completes the proof. □

Theorem 3.9. *Let $p = (p_k)$ be a bounded sequence of positive real numbers. Then $c(X,p)^{\beta} = M_0(X',p) \cap cs[X']$.*

Proof. Since $c(X,p) = c_0(X,p) + E$, where $E = \{e(x) : x \in X\}$, it follows by Proposition 3.1(iii) and Theorem 3.8 that $c(X,p)^{\beta} = M_0(X',p) \cap E^{\beta}$. It is obvious by definition that $E^{\beta} = \{(f_k) \subset X' : \sum_{k=1}^{\infty} f_k(x)$ converges for all $x \in X\} = cs[X']$. Hence we have the theorem. □

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