Research Article

The Asymptotic Behaviours of a Class of Neutral Delay Fractional-Order Pantograph Differential Equations

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By using fractional calculus and the summation by parts formula in this paper, the asymptotic behaviours of solutions of nonlinear neutral fractional delay pantograph equations with continuous arguments are investigated. The asymptotic estimates of solutions for the equation are obtained, which may imply asymptotic stability of solutions. In the end, a particular case is provided to illustrate the main result and the speed of the convergence of the obtained solutions.

1. Introduction

Throughout the past decade, the study of the theory of fractional differential equations has gained a lot of great interests and attracted growing attention. Fractional differential equation has been used successfully in many applied fields such as physics, quantum mechanics, field theory, chemistry, optimal control, and engineering. Hence, the existence and uniqueness of fractional differential equations have been extensively researched by many experts and scholars [1–7]. It is well known that the pantograph equation arises in quite different fields of pure and applied mathematics and have been investigated extensively [8–10]. Recently, due to its importance in many applied fields and playing an extremely important role in explaining many different phenomena, it is also interesting to study the existence and asymptotic behaviours of the generalized and multipantograph equations. Yüzbasi et al. [8] investigated the numerical solution of generalized pantograph equation with a linear functional argument by virtue of introducing a collocation method based on the Bessel polynomials for the approximate solution of the pantograph equations. Rahimkhani et al. [11] obtained numerical solution of fractional pantograph differential equations by using generalized fractional-order Bernoulli wavelet. Yu [12] used variational iteration method to solve the multipantograph delay equation, and sufficient conditions were given to assure the convergence

of the method. Balachandran and Kiruthika [13] studied the existence of solutions of abstract fractional pantograph equations by using the fractional calculus and fixed point theorems. Peics [14] researched the asymptotic behaviour of solutions of difference equations with continuous argument and obtained asymptotic estimates. Doha et al. [15] using the Jacobi rational-Gauss method reduced solving the generalized pantograph equation to a system of algebraic equations and obtained reasonable numerical results. Motivated and inspired by thoughts and methods of the above recent papers, based on the difference pantograph equations [14], we establish the following neutral delay fractional pantograph equations with continuous arguments of the form

$$D^q \left[u(t) - A(t)u(t-1) \right] = B(t)u(p(t)),$$

$$t \in (0, \infty),$$

$$u(0) = u_0 = 0, \qquad (1)$$

$$u(t) = \varphi(t),$$

$$t \in [-1, 0],$$

where $0 < q < 1$, $A(t)$ and $B(t)$ are all continuous real function and are assumed to satisfy the following two hypotheses throughout this paper:

(H_1) $0 < A(t) < 1$ and $A(0) = 0$.

(H_2) There exists continuous real function $p(t)$ which satisfies the following: when $t > 0$, $0 < p(t) < t - \delta(t)$, where $0 < \delta'(t) < 1$ and $\lim_{t \to \infty} p(t) = \infty$; when $t = 0$, $\delta(0) = 0$, which implies $p(0) = 0$.

The aim of the paper is to research the asymptotic behaviour of solutions for fractional delay equations with continuous arguments (1), give asymptotic estimates for the speed of convergence of solutions, and apply our results to the particular cases. The paper is organized as follows. In Section 2, we present some notations and definitions of fractional calculus needed in the rest of the paper. Section 3 is devoted to the main results of the paper by using fractional calculus and the summation by parts formula. In Section 4, we obtain the speed of the convergence of the solutions for the initial value problem (1) by using Theorem 3 to deal with special case.

2. Preliminaries

For convenience of the reader, we need to introduce some notations and properties of fractional calculus which will be used in the proof of our results. Let $\mathbf{N}$ be the set of natural numbers, and let $\mathbf{R}$ be the set of real numbers and $\mathbf{R}_+ = (0, \infty)$.

Use the notations

$$\prod_{l=t}^{t-1} f(l) = 1,$$

$$\prod_{t=t-n}^{t} f(l) = f(t-n) f(t-n+1) \cdots f(t),$$

$$\sum_{\tau=t}^{t-1} f(\tau) = 0,$$

$$\sum_{\tau=t-n}^{t} f(\tau) = f(t-n) + f(t-n+1) + \cdots + f(t)$$

(2)

for $n \in \mathbf{N}$, $t \in \mathbf{R}_+$ and arbitrary real function f. The difference operator Δ is defined by $\Delta f(t) = f(t+1) - f(t)$, where the function $f(t)$ is defined for $t \in \mathbf{R}_+$. The difference operator Δ_t is defined by $\Delta_t g(t, a) = g(t+1, a) - g(t, a)$, where the function $g(t, a)$ is defined for $a, t \in \mathbf{R}_+$.

Let t_0 be a positive real number and set

$$t_{-1} = \min \{ \inf \{ p(s) : s \geq t_0 \}, t_0 - 1 \},$$

$$t_n = \inf \{ s : p(s) > t_{n-1} \}$$

(3)

for $n = 1, 2, \ldots$. Then $\{t_n\}$ is an increasing sequence such that

$$\lim_{n \to \infty} t_n = \infty,$$

$$\bigcup_{n=1}^{\infty} [t_{n-1}, t_n) = [t_0, \infty),$$

(4)

$$p(t) \in \bigcup_{i=0}^{n} [t_{i-1}, t_i),$$

$$t_n \leq t < t_{n+1}, \quad n = 0, 1, 2, \ldots.$$

Fix a point t such that $t_n \leq t < t_{n+1}$, define the natural number $k(t)$ such that $t - k(t) - 1 < t_n$ and $t - k(t) \geq t_n$, and the set $T(t)$ is defined by

$$T(t) = \{ t - k(t), t - k(t) + 1, \ldots, t - 1, t \}. \tag{5}$$

A solution of (1) is a function $u(t)$ which is defined for $t \geq t_{-1}$ and satisfies (1) for $t \geq t_0$. For a given real bounded continuous function ϕ on $t_{-1} \leq t < t_0$, (1) has a unique solution $u(t)$ satisfying the initial condition $u(t) = \phi(t)$.

Definition 1 (see [16]). The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in L_1(\mathbf{R}_+)$, is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \tag{6}$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 2 (see [16]). The Riemann-Liouville fractional derivative of order $\alpha > 0$, $n - 1 < \alpha < n$, $n \in N$, is defined by

$$D^\alpha = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) \, ds, \tag{7}$$

where the function $f(t)$ has absolutely continuous derivatives up to order $(n-1)$.

3. Main Results

In this section, let $M_0 = \sup_{t_{-1} \leq t < t_0} \rho(t) |u(t)|$, $N_0 = \inf_{t_{-1} \leq t < t_0} \rho(t) |u(t)|$.

Theorem 3. *Assume that the conditions (H_1) and (H_2) hold; there exists a real continuous function $\rho : [t_{-1}, \infty) \to (0, \infty)$ bounded on the initial interval $[t_{-1}, t_0]$ such that*

$$\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \frac{B(s)}{\rho(p(s))} ds \leq \frac{1 - A(t)}{\rho(t-1)}$$

$$\text{for } t \in [t_0, \infty) \tag{8}$$

and there exists a real number R such that

$$\prod_{n=0}^{j} (1 + R_n) \leq R, \quad \text{for all positive integers } j, \tag{9}$$

where

$$R_n = \sup_{t_n \leq t < t_{n+1}} \left\{ \rho(t) \sum_{\tau=t-k(t)}^{t} \frac{\Delta \rho(\tau-1)}{\rho(\tau) \rho(\tau-1)} \prod_{l=\tau+1}^{t} A(l) \right\}. \tag{10}$$

Assume that $u(t)$ is a solution of the initial value problem (1); then

$$|u(t)| \leq \frac{M_0 R}{\rho(t)} \quad \text{for } t \in [t_0, \infty). \tag{11}$$

Proof. It is easy to see that (1) is equivalent to the following integral equation:

$$u(t) - A(t)u(t-1)$$

$$= u_0 - A(0)u(-1)$$

$$+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B(s)u(p(s))ds; \tag{12}$$

that is,

$$u(t) = A(t)u(t-1)$$

$$+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B(s)u(p(s))ds. \tag{13}$$

Let $y(t) = u(t)\rho(t)$; then the function $y(t)$ will satisfy the equation

$$\frac{y(t)}{\rho(t)} = A(t)\frac{y(t-1)}{\rho(t-1)}$$

$$+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} B(s)\frac{y(p(s))}{\rho(p(s))}ds. \tag{14}$$

Let $t \in [t_n, t_{n+1})$ and $\tau \in T(t)$; then (12) is equivalent to

$$\Delta_\tau \left\{ \frac{y(\tau-1)}{\rho(\tau-1)} \prod_{l=t-k(t)}^{\tau-1} \frac{1}{A(l)} \right\}$$

$$= \left(\frac{y(\tau)}{\rho(\tau)} - A(\tau)\frac{y(\tau-1)}{\rho(\tau-1)} \right) \prod_{l=t-k(t)}^{\tau} \frac{1}{A(l)} \tag{15}$$

$$= \frac{1}{\Gamma(q)} \int_0^\tau (\tau-s)^{q-1} B(s)\frac{y(p(s))}{\rho(p(s))}ds \prod_{l=t-k(t)}^{\tau} \frac{1}{A(l)}.$$

Summing up both sides of the above equality from $t - k(t)$ to t, we get

$$y(t) = \frac{\rho(t)}{\rho(t-k(t)-1)}y(t-k(t)-1) \prod_{l=t-k(t)}^{t} A(l)$$

$$+ \rho(t)$$

$$\cdot \sum_{\tau=t-k(t)}^{t} \left\{ \frac{1}{\Gamma(q)} \int_0^\tau (\tau-s)^{q-1} B(s)\frac{y(p(s))}{\rho(p(s))}ds \right\} \tag{16}$$

$$\cdot \prod_{l=\tau+1}^{t} A(l).$$

So from (16), we obtain

$$\left| y(t) \right| \leq \frac{\rho(t)}{\rho(t-k(t)-1)} \left| y(t-k(t)-1) \right| \prod_{l=t-k(t)}^{t} A(l)$$

$$+ \frac{\rho(t)}{\Gamma(q)} \sum_{\tau=t-k(t)}^{t} \int_0^\tau (\tau-s)^{q-1} \frac{B(s)}{\rho(p(s))} \left| y(p(s)) \right| ds \tag{17}$$

$$\cdot \prod_{l=\tau+1}^{t} A(l).$$

Let

$$\mu_n = \sup_{t_{n-1} \leq t < t_n} \left| y(t) \right|,$$

$$M_n = \max \{ \mu_0, \mu_1, \ldots, \mu_n \}. \tag{18}$$

Since $|y(p(t))| \leq M_n$ for $\tau \in T(t)$ and $t_n \leq t < t_{n+1}$, by utilizing the summation by parts formula, we can get

$$\left| y(t) \right| \leq M_n \left(\frac{\rho(t)}{\rho(t-k(t)-1)} \prod_{l=t-k(t)}^{t} A(l) + \frac{\rho(t)}{\Gamma(q)} \right.$$

$$\cdot \sum_{\tau=t-k(t)}^{t} \int_t^\tau (\tau-s)^{q-1} \frac{B(s)}{\rho(p(s))} \prod_{l=\tau+1}^{t} A(l) \right)$$

$$\leq M_n \left(\frac{\rho(t)}{\rho(t-k(t)-1)} \prod_{l=t-k(t)}^{t} A(l) + \rho(t) \right.$$

$$\cdot \sum_{\tau=t-k(t)}^{t} \frac{1-A(\tau)}{\rho(\tau-1)} \prod_{l=\tau+1}^{t} A(l) \right)$$

$$= M_n \left(\frac{\rho(t)}{\rho(t-k(t)-1)} \prod_{l=t-k(t)}^{t} A(l) + \rho(t) \right.$$

$$\cdot \sum_{\tau=t-k(t)}^{t} \frac{1}{\rho(\tau-1)} \Delta_\tau \left(\prod_{l=\tau+1}^{t} A(l) \right) \right) \tag{19}$$

$$= M_n \left(\frac{\rho(t)}{\rho(t-k(t)-1)} \prod_{l=t-k(t)}^{t} A(l) + \rho(t) \right.$$

$$\cdot \frac{1}{\rho(\tau-1)} \left(\prod_{l=\tau}^{t} A(l) \Big|_{t-k(t)}^{t+1} \right.$$

$$+ \rho(t) \sum_{\tau=t-k(t)}^{t} \frac{\Delta\rho(\tau-1)}{\rho(\tau)\rho(\tau-1)} \prod_{l=\tau+1}^{t} A(l) \right) = M_n \left(1 \right.$$

$$+ \rho(t) \sum_{\tau=t-k(t)}^{t} \frac{\Delta\rho(\tau-1)}{\rho(\tau)\rho(\tau-1)} \prod_{l=\tau+1}^{t} A(l) \right) \leq M_n (1$$

$$+ R_n).$$

From the above inequality, we get

$$1 + R_n \geq 0,$$

$$M_{n+1} \leq M_n(1+R_n) \quad \forall n = 0, 1, 2, \ldots. \tag{20}$$

Further, we can obtain

$$M_{n+1} \leq M_n(1+R_n) \leq M_{n-1}(1+R_n)(1+R_{n-1})$$

$$\leq \cdots \leq M_0 \prod_{j=0}^{n} (1+R_j). \tag{21}$$

Consequently,

$$\left| y(t) \right| \leq M_0 R; \tag{22}$$

that is, $|u(t)| \leq M_0 R/\rho(t)$. □

Remark 4. In Theorem 3, suppose that the function $\rho(t)$ is monotone increasing; then the sequence $\{R_n\}$ has only positive members and the assumption

$$\prod_{n=0}^{\infty} (1 + R_n) < \infty \qquad (23)$$

implies the existence of a real number R that satisfies condition (9). If the function $\rho(t)$ is monotone decreasing, then condition (9) is satisfied with $R = 1$.

Remark 5. In Theorem 3, if $B(t)$ is positive continuous real function, then the solution $u(t)$ of the initial value problem (1) is nonnegative for $t \in [t_0, \infty)$.

Theorem 6. *If the conditions (H_1) and (H_2) hold, there exists a real continuous function $\rho : [t_{-1}, \infty) \to (0, \infty)$ bounded on the initial interval $[t_{-1}, t_0]$ such that*

$$\frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \frac{B(s)}{\rho(p(s))} ds \geq \frac{1 - A(t)}{\rho(t - 1)} \qquad (24)$$

$$for\ t \in [t_0, \infty)$$

and there exists a real number r such that

$$\prod_{n=0}^{j} (1 + r_n) \geq r, \quad for\ all\ positive\ integers\ j, \qquad (25)$$

where

$$r_n = \inf_{t_n \leq t < t_{n+1}} \left\{ \rho(t) \sum_{\tau=t-k(t)}^{t} \frac{\Delta\rho(\tau - 1)}{\rho(\tau)\rho(\tau - 1)} \prod_{l=\tau+1}^{t} A(l) \right\}. \qquad (26)$$

Assume that $u(t)$ is a solution of the positive initial value problem (1); then

$$u(t) \geq \frac{N_0 r}{\rho(t)} \quad for\ t \in [t_0, \infty). \qquad (27)$$

Proof. It is similar to the proof of Theorem 3. It is seen that (1) is equivalent to the following integral equation:

$$y(t) = \frac{\rho(t)}{\rho(t - k(t) - 1)} y(t - k(t) - 1) \prod_{l=t-k(t)}^{t} A(l)$$

$$+ \rho(t)$$

$$\cdot \sum_{\tau=t-k(t)}^{t} \left\{ \frac{1}{\Gamma(q)} \int_0^{\tau} (\tau - s)^{q-1} B(s) \frac{y(p(s))}{\rho(p(s))} ds \right\} \qquad (28)$$

$$\cdot \prod_{l=\tau+1}^{t} A(l).$$

By utilizing the hypothesis (24), it is easy to see that the function $B(t)$ is positive, by the virtue of Remark 5, and it

implies that the function $y(t)$ is also positive. So we can get the following estimate:

$$y(t) \geq \frac{\rho(t)}{\rho(t - k(t) - 1)} y(t - k(t) - 1) \prod_{l=t-k(t)}^{t} A(l)$$

$$+ \frac{\rho(t)}{\Gamma(q)} \sum_{\tau=t-k(t)}^{t} \int_0^{\tau} (\tau - s)^{q-1} \frac{B(s)}{\rho(p(s))} y(p(s)) ds \qquad (29)$$

$$\cdot \prod_{l=\tau+1}^{t} A(l).$$

Define for $n = 0, 1, 2, \ldots,$

$$v_n = \inf_{t_{n-1} \leq t < t_n} |y(t)|,$$

$$N_n = \min \{v_0, v_1, \ldots, v_n\}. \qquad (30)$$

Because of $y(p(\tau)) \geq N_n$ for $\tau \in T(t)$ and $t_n \leq t < t_{n+1}$, by means of the summation by parts formula, we can draw the conclusion

$$y(t) \geq N_n \left(1 + \rho(t) \sum_{\tau=t-k(t)}^{t} \frac{\Delta\rho(\tau - 1)}{\rho(\tau)\rho(\tau - 1)} \prod_{l=\tau+1}^{t} A(l) \right) \qquad (31)$$

$$\geq N_n (1 + r_n).$$

From the above inequality, it follows that

$$N_{n+1} \geq N_n (1 + r_n) \geq N_0 \prod_{j=0}^{n} (1 + r_j). \qquad (32)$$

Hence, $y(t) \geq N_0 r$; that is, $u(t) \geq N_0 r / \rho(t)$. $\qquad \square$

Remark 7. In Theorem 6, if the function $\rho(t)$ is monotone decreasing, then the sequence $\{r_n\}$ defined by (26) is all negative and the assumption $\prod_{n=0}^{\infty} (1 + r_n) > 0$ implies the existence of a real number $r > 0$ which satisfies condition (25). If the function $\rho(t)$ is monotone increasing, then condition (25) is satisfied with $r = 1$.

Theorem 8. *Assume that conditions (H_1) and (H_2) hold; there exists a real continuous function $\rho : [t_{-1}, \infty) \to (0, \infty)$ bounded on the initial interval $[t_{-1}, t_0]$ such that*

$$\frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} \frac{B(s)}{\rho(p(s))} ds = \frac{1 - A(t)}{\rho(t - 1)} \qquad (33)$$

$$for\ t \in [t_0, \infty)$$

and there exist real numbers $r > 0$ and R such that conditions (9) and (25) hold for all positive integer j. Suppose that $u(t)$ is a solution of the positive initial value problem (1); then

$$0 < \frac{N_0 r}{\rho(t)} \leq u(t) \leq \frac{M_0 R}{\rho(t)} \quad for\ t \in [t_0, \infty). \qquad (34)$$

Proof. According to the consequence of Theorems 3 and 6, we can easily obtain the proof. $\qquad \square$

4. Particular Case

In this section, we use Theorem 3 to deal with special case. By seeking an appropriate function $\rho(t)$, we can also obtain the speed of the convergence of the solutions for the initial value problem (1).

Corollary 9. *Let $p(t)$ satisfy $p_1 t \leq p(t) \leq p_2 t$ for real numbers $0 < p_1 \leq p_2 < 1$. Suppose that conditions (H_1) and (H_2) hold; there exist real numbers S and β such that $0 < S \leq 1$, $0 < \beta < 1$,*

$$\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \frac{|B(s)|}{\rho(p(s))} ds \leq \frac{S(1-A(t))}{\rho(p(t))},$$

$$\beta \leq (1 - A(t))$$

$$for \ t \geq t_0 > \frac{p_1}{1-p_1}, \quad (35)$$

$$\log \frac{1}{S} \left(\log \frac{1}{p_1} - \log \frac{1}{p_2} \right) < \log \frac{1}{p_1} \log \frac{1}{p_2}.$$

Let $u(t)$ be a solution of the initial value problem (1); then

$$|u(t)| \leq \frac{M_0 R}{\left(t - p_1 / (1 - p_1)\right)^k} \quad for \ t \leq t_0, \quad (36)$$

where $k = \log S / \log p_1$, $M_0 = \sup_{t_{-1} \leq t < t_0} \{(t - p_1/(1 - p_1))^k |u(t)|\}$, and

$$R = \prod_{n=0}^{\infty} \left(1 + \frac{k t_0^k (1-p_1)^{k+1}}{\beta p_1^k (t_0 (1-p_1) - p_2^n)^{k+1}} \left(\frac{p_2^{k+1}}{p_1^k} \right)^n \right). \quad (37)$$

Proof. By using relations $p(t_{n+1}) = t_n$ and $p_1 t \leq p(t) \leq p_2 t$, we can obtain

$$\frac{t_0}{p_2^n} \leq t_n \leq \frac{t_0}{p_1^n} \quad for \ n = 1, 2, \dots . \quad (38)$$

Let $\rho(t) = (t - p_1/(1 - p_1))^k$. Because of $S = p_1^k$, we have

$$\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \frac{|B(s)|}{\rho(p(s))} ds \leq \frac{S(1-A(t))}{\rho(pt)}$$

$$\leq \frac{S(1-A(t))}{\rho(p_1 t)} = \frac{S(1-A(t))}{p_1^k (t - 1/(1-p_1))^k}$$

$$= \frac{S(1-A(t))}{p_1^k (t - 1 - p_1/(1-p_1))^k} = \frac{1-A(t)}{\rho(t-1)}. \quad (39)$$

Consequently, condition (8) is effective. Furthermore, we obtain

$$R_n = \sup_{t_n \leq t < t_{n+1}} \left\{ \left(t - \frac{p_1}{1-p_1} \right)^k \right.$$

$$\times \sum_{\tau=t-k(t)}^{t} \frac{(\tau - p_1/(1-p_1))^k - (\tau - 1/(1-p_1))^k}{(\tau - p_1/(1-p_1))^k (\tau - 1/(1-p_1))^k}$$

$$\left. \cdot \prod_{l=\tau+1}^{t} A(l) \right\} \leq \sup_{t_n \leq t < t_{n+1}} \left\{ t^k (1-\beta)^t \right.$$

$$\left. \cdot \sum_{\tau=t-k(t)}^{t} \frac{k}{(\tau - 1/(1-p_1))^{k+1}} \left(\frac{1}{1-\beta} \right)^r \right\}$$

$$\leq \sup_{t_n \leq t < t_{n+1}} \left\{ \frac{k t^k}{\beta (t - k(t) - 1/(1-p_1))^{k+1}} \right\}$$

$$\leq \frac{k t_0^k (1-p_1)^{k+1}}{\beta p_1^k (t_0 (1-p_1) - p_2^n)^{k+1}} \left(\frac{p_2^{k+1}}{p_1^k} \right)^n. \quad (40)$$

It follows that Theorem 3 and Remark 4 imply the conclusion. □

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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