Hindawi Publishing Corporation
International Journal of Antennas and Propagation
Volume 2016, Article ID 3293418, 11 pages
http://dx.doi.org/10.1155/2016/3293418

Research Article

Geolocation of a Known Altitude Target Using TDOA and GROA in the Presence of Receiver Location Uncertainty

Bing Deng,[1,2] Le Yang,[3] Zheng Bo Sun,[2] and Hua Feng Peng[2]

[1] *Zhengzhou Institute of Information Science and Technology, Zhengzhou, Henan 450002, China*
[2] *National Key Laboratory of Science and Technology on Blind Signal Processing, Chengdu, Sichuan 610041, China*
[3] *School of Internet of Things (IoT) Engineering, Jiangnan University, Wuxi, Jiangsu 214122, China*

Correspondence should be addressed to Le Yang; le.yang.le@gmail.com

Received 2 June 2016; Accepted 17 August 2016

Academic Editor: Sotirios K. Goudos

This paper considers the problem of geolocating a target on the Earth surface using the target signal time difference of arrival (TDOA) and gain ratio of arrival (GROA) measurements when the receiver positions are subject to random errors. The geolocation Cramer-Rao lower bound (CRLB) is derived and the performance improvement due to the use of target altitude information is quantified. An algebraic geolocation solution is developed and its approximate efficiency under small Gaussian noise is established analytically. Its sensitivity to the target altitude error is also studied. Simulations justify the validity of the theoretical developments and illustrate the good performance of the proposed geolocation method.

1. Introduction

Passive target localization is a classical problem which has gained considerable attention in different application contexts, such as radar, sonar, navigation, tracking, and wireless communications [1–3]. Localization techniques have been extensively investigated for positioning parameters including the angle of arrival (AOA) [4], time difference of arrival (TDOA), and frequency difference of arrival (FDOA) of the target signal captured at spatially distributed receivers [2–4].

More recently, the use of the received signal strength (RSS) has been considered for target localization via, for example, microphone arrays [5]. Under the free-space propagation condition, the received signal energy is inversely proportional to the distance squared between the target and the receiver [5, 6]. This leads to the development of several received signal strength indicator- (RSSI-) based localization methods (see [5–10] and the references therein). But they require that the target transmit power is known, which renders them unsuitable for the passive localization of uncooperative targets. On the other hand, noting that the signal energies received at different receivers would be different, the utilization of the gain difference of arrival (GROA) measurement has been recognized to be useful for passive localization [11–13]. It needs the reciprocal of the received signal amplitude with respect to a reference receiver only to locate a target. The requirement for knowing the target signal transmit power is thus eliminated.

In the literature, several techniques have been proposed for target localization using GROA. Specifically, Cui et al. [14] considered using the signal TDOA and interaural level difference (ILD) obtained at two microphones for 2D sound source localization. Ho and Sun [11] utilized TDOA and GROA measurements jointly in 3D localization. They assumed the use of more than four sensors and proposed a closed-form two-step solution, which will be referred to as the two-step weighted least-squares (TSWLS) technique. The contribution of the GROA measurements to the improvement of target localization accuracy was studied. Different from the study in [11], Hao et al. [12, 13] considered the practical scenario where the known sensor positions have errors. They proposed in [12] a new closed-form algorithm that estimates both the unknown source and the sensor positions from TDOA and GROA measurements [12]. In [13], two bias mitigation methods, called BiasSub and BiasRed, were developed to reduce the estimation bias of the original TSWLS method [11].

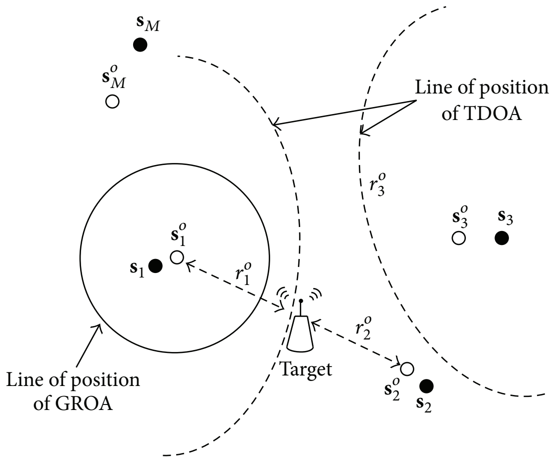

Figure 1: Target geolocation scenario.

In this work, we consider the passive geolocation of a target on the Earth surface using TDOA and GROA measurements in the presence of receiver position errors. The target altitude information can come from, for example, an altimeter [15, 16] or simply the prior information that the target is on the ground. The study begins with mathematically formulating the geolocation problem and deriving the geolocation Cramer-Rao lower bound (CRLB). The contribution of the target altitude information to improving the geolocation accuracy is investigated. The target geolocation problem is then cast into an equality-constrained optimization problem, where the equality constraint comes from the target altitude information and the cost function takes into account the presence of receiver position errors. An improved constrained weighted least-squares (ICWLS) solution is derived by following a similar approach as in [15]. Its approximate efficiency is established analytically. The sensitivity of the geolocation accuracy to the error in the target altitude information is quantitatively analyzed. Simulations corroborate the theoretical developments and show better performance of the proposed geolocation technique over a benchmark method.

The rest of this paper is organized as follows. Section 2 formulates the geolocation problem in consideration. Section 3 derives the geolocation CRLB. Section 4 presents the proposed ICWLS geolocation technique and the performance analysis with respect to the target position CRLB. Section 5 investigates the impact of the target altitude uncertainty on the geolocation accuracy. Section 6 gives the simulation results and Section 7 concludes the paper.

2. Problem Formulation

Consider the geolocation scenario shown in Figure 1. The target is located on the surface of Earth modeled as an oblate spheroid. The unknown target position vector in the geocentric coordinate system is denoted by $\mathbf{u}^o = [x^o, y^o, z^o]^T$, the elements of which are related to the geodetic coordinates of the target $[\alpha, \beta, h]^T$ via

$$\begin{aligned} x^o &= (N+h)\cos(\alpha)\cos(\beta), \\ y^o &= (N+h)\cos(\alpha)\sin(\beta), \\ z^o &= \left[N\left(1-e^2\right)+h\right]\sin(\alpha). \end{aligned} \tag{1}$$

Here, α and β denote the geodetic latitude and longitude of the target. $N = r/\sqrt{1-e^2\sin^2(\alpha)}$, where $r = 6378.137$ km is the equatorial radius of the spheroid Earth and $e = 0.081819190842$ is the eccentricity. h is the target altitude. This work assumes that h is known to the geolocation algorithm. Under this assumption, eliminating α and β in (1) yields an equality constraint on the target geocentric position $\mathbf{u}^o$, which is

$$\begin{gathered} f(\mathbf{u}^o) = \mathbf{u}^{oT}\mathbf{P}\mathbf{u}^o = (N+h)^2, \\ \mathbf{P} = \operatorname{diag}\left\{1, 1, \frac{(N+h)^2}{\left[N\left(1-e^2\right)+h\right]^2}\right\}. \end{gathered} \tag{2}$$

The signal emission of the target is captured by M receivers. Signal TDOAs and GROAs are estimated for target geolocation. The known geocentric coordinates of the receivers are corrupted by additive random noises and they are denoted by $\mathbf{s}_i = [x_i, y_i, z_i]^T = \mathbf{s}_i^o + \Delta\mathbf{s}_i$, where $i = 1, 2, \ldots, M$ and $\mathbf{s}_i^o = [x_i^o, y_i^o, z_i^o]^T$ are the unknown true receiver positions. $\Delta\mathbf{s}_i$ is the position error in $\mathbf{s}_i$. Collecting $\mathbf{s}_i$, we obtain $\mathbf{s} = [\mathbf{s}_1^T, \mathbf{s}_2^T, \ldots, \mathbf{s}_M^T]^T$, where the receiver position error vector is $\Delta\mathbf{s} = \mathbf{s} - \mathbf{s}^o = [\Delta\mathbf{s}_1^T, \Delta\mathbf{s}_2^T, \ldots, \Delta\mathbf{s}_M^T]^T$ and $\mathbf{s}^o = [\mathbf{s}_1^{oT}, \mathbf{s}_2^{oT}, \ldots, \mathbf{s}_M^{oT}]^T$ is the true receiver position vector. In this study, we assume $\Delta\mathbf{s}$ is a zero-mean Gaussian distributed random vector with covariance matrix $\mathbf{Q}_\mathbf{s}$.

Suppose the target signal captured at receiver 1 is $x_1(t) = s(t) + \xi_1(t)$, where $s(t)$ is the true target signal and $\xi_1(t)$ is the additive noise. The target signals captured at other receivers can then be expressed as [9, 10]

$$x_i(t) = \frac{1}{g_{i1}^o} s\left(t - \tau_{i1}^o\right) + \xi_i(t), \tag{3}$$

where $i = 2, 3, \ldots, M$. It is assumed that $s(t)$ and $\xi_i(t)$ are independent zero-mean Gaussian random signals [11–13]. τ_{i1}^o and g_{i1}^o are the true signal TDOA and GROA between receiver pair i and 1.

Multiplying the true TDOA τ_{i1}^o by the signal propagation speed c gives the true range difference of arrival (RDOA) r_{i1}^o, which is equal to

$$r_{i1}^o = c\tau_{i1}^o = r_i^o - r_1^o. \tag{4}$$

Here, $r_i^o = \|\mathbf{u}^o - \mathbf{s}_i^o\|$, $i = 1, 2, \ldots, M$, is the true distance between the target and receiver i. Under the condition that the signals $x_i(t)$ are received from line-of-sight (LOS) transmissions [17, 18], the true GROA g_{i1}^o in (3) is equal to [11–13]

$$g_{i1}^o = \frac{r_i^o}{r_1^o}. \tag{5}$$

It comes from the difference in the path loss from the target to receivers i and 1.

The target signal RDOA and GROA estimated from the received signals $x_i(t)$ are denoted by $r_{i1} = r_{i1}^o + \Delta r_{i1}$ and $g_{i1} = g_{i1}^o + \Delta g_{i1}$, where Δr_{i1} and Δg_{i1} are measurement noises. Collecting the obtained RDOA and GROA measurements gives

$$\mathbf{r} = [r_{21}, r_{31}, \ldots, r_{M1}]^T = \mathbf{r}^o + \Delta\mathbf{r}, \tag{6a}$$

$$\mathbf{g} = [g_{21}, g_{31}, \ldots, g_{M1}]^T = \mathbf{g}^o + \Delta\mathbf{g}, \tag{6b}$$

where $\mathbf{r}^o = [r_{21}^o, r_{31}^o, \ldots, r_{M1}^o]^T$ and $\mathbf{g}^o = [g_{21}^o, g_{31}^o, \ldots, g_{M1}^o]^T$. The RDOA and GROA measurement noise vectors, $\Delta\mathbf{r} = [\Delta r_{21}, \Delta r_{31}, \ldots, \Delta r_{M1}]^T$ and $\Delta\mathbf{g} = [\Delta g_{21}, \Delta g_{31}, \ldots, \Delta g_{M1}]^T$, are assumed to be independent zero-mean Gaussian random vectors [11–13]. Their covariance matrices are $\mathbf{Q}_\mathbf{r}$ and $\mathbf{Q}_\mathbf{g}$, and they are independent of the receiver position error vector $\Delta\mathbf{s}$ as well.

We are interested in identifying the target position $\mathbf{u}^o$ using the noisy TDOA and GROA measurements in $\mathbf{r}$ and $\mathbf{g}$, the erroneous receiver positions $\mathbf{s}_i$, and the equality constraint on $\mathbf{u}^o$ in (2).

3. Geolocation CRLB

The Cramer-Rao lower bound (CRLB) gives the lowest possible estimation covariance matrix for any unbiased estimator of deterministic parameters [19–21]. From the previous section, we have that the unknowns include the geocentric positions of the target and receivers. They can be collected in the composite unknown vector $\mathbf{\Phi}^o = [\mathbf{u}^{oT}, \mathbf{s}^{oT}]^T$. We are interested in deriving the CRLB of $\mathbf{u}^o$.

Note from (2) that $\mathbf{u}^o$ is equality-constrained and its CRLB is therefore a constrained one, which will thus be denoted by CCRLB($\mathbf{u}^o$). According to [15, 16], we have

$$\mathrm{CCRLB}(\mathbf{u}^o) = \mathbf{J}^{-1} - \mathbf{J}^{-1}\mathbf{F}\left(\mathbf{F}^T\mathbf{J}^{-1}\mathbf{F}\right)^{-1}\mathbf{F}^T\mathbf{J}^{-1}. \tag{7}$$

Here, $\mathbf{F}$ is the Jacobian of the constraint $f(\mathbf{u}^o)$ and, from (1), we have

$$\mathbf{F} = 2\mathbf{P}\mathbf{u}^o + \mathbf{u}^o\mathbf{P}_\mathbf{u}\mathbf{u}^{oT} - 2(N+h)\frac{\partial N}{\partial \mathbf{u}^o}, \tag{8}$$

where the expression of $\mathbf{P}_\mathbf{u}$ and $\partial N/\partial\mathbf{u}^o$ can be found in the Appendix.

$\mathbf{J}^{-1}$ is indeed the CRLB of the target position $\mathbf{u}^o$ when its altitude information is not available [11, 12]. Define $\mathbf{J}^{-1} = \mathrm{CRLB}(\mathbf{u}^o)$ for the sake of clarity. As a result, we can observe from (7) that the utilization of the target altitude information via (3) can in effect lead to improved performance in terms of reduced target geolocation CRLB.

We shall present the derivation of CRLB($\mathbf{u}^o$) to complete the CRLB analysis. For this purpose, let $\mathbf{m} = [\mathbf{r}^T, \mathbf{g}^T, \mathbf{s}^T]^T$ be the measurement vector containing the noisy TDOAs and GROAs as well as the erroneous receiver positions. Under the Gaussian noise model specified in Section 2, the logarithm of the probability density function (PDF) of $\mathbf{m}$ is [21, 22]

$$\begin{aligned}\ln f(\mathbf{m} \mid \mathbf{\Phi}^o) = k_1 &- \frac{1}{2}(\mathbf{r}-\mathbf{r}^o)^T\mathbf{Q}_\mathbf{r}^{-1}(\mathbf{r}-\mathbf{r}^o) + k_2 \\ &- \frac{1}{2}(\mathbf{g}-\mathbf{g}^o)^T\mathbf{Q}_\mathbf{g}^{-1}(\mathbf{g}-\mathbf{g}^o) + k_3 \\ &- \frac{1}{2}(\mathbf{s}-\mathbf{s}^o)^T\mathbf{Q}_\mathbf{s}^{-1}(\mathbf{s}-\mathbf{s}^o),\end{aligned} \tag{9}$$

where k_1, k_2, and k_3 are independent of the unknowns $\mathbf{\Phi}^o$. The Fisher information matrix (FIM) of $\mathbf{\Phi}^o$ is [20]

$$\mathrm{FIM}(\mathbf{\Phi}^o) = -E\left[\frac{\partial^2 \ln f(\mathbf{m} \mid \mathbf{\Phi}^o)}{\partial\mathbf{\Phi}^o\partial\mathbf{\Phi}^{oT}}\right]. \tag{10}$$

It can be expressed in the following block matrix form:

$$\mathrm{FIM}(\mathbf{\Phi}^o) = \begin{bmatrix}\mathbf{X} & \mathbf{Y} \\ \mathbf{Y}^T & \mathbf{Z}\end{bmatrix}, \tag{11}$$

where

$$\begin{aligned}\mathbf{X} &= -E\left[\frac{\partial^2 \ln f(\mathbf{m} \mid \mathbf{\Phi}^o)}{\partial\mathbf{u}^o\partial\mathbf{u}^{oT}}\right]^{-1} \\ &= \left(\frac{\partial\mathbf{r}^o}{\partial\mathbf{u}^o}\right)^T\mathbf{Q}_\mathbf{r}^{-1}\left(\frac{\partial\mathbf{r}^o}{\partial\mathbf{u}^o}\right) + \left(\frac{\partial\mathbf{g}^o}{\partial\mathbf{u}^o}\right)^T\mathbf{Q}_\mathbf{g}^{-1}\left(\frac{\partial\mathbf{g}^o}{\partial\mathbf{u}^o}\right), \\ \mathbf{Y} &= -E\left[\frac{\partial^2 \ln f(\mathbf{m} \mid \mathbf{\Phi}^o)}{\partial\mathbf{u}^o\partial\mathbf{s}^{oT}}\right]^{-1} \\ &= \left(\frac{\partial\mathbf{r}^o}{\partial\mathbf{u}^o}\right)^T\mathbf{Q}_\mathbf{r}^{-1}\left(\frac{\partial\mathbf{r}^o}{\partial\mathbf{s}^o}\right) + \left(\frac{\partial\mathbf{g}^o}{\partial\mathbf{u}^o}\right)^T\mathbf{Q}_\mathbf{g}^{-1}\left(\frac{\partial\mathbf{g}^o}{\partial\mathbf{s}^o}\right), \\ \mathbf{Z} &= -E\left[\frac{\partial^2 \ln f(\mathbf{m} \mid \mathbf{\Phi}^o)}{\partial\mathbf{s}^o\partial\mathbf{s}^{oT}}\right]^{-1} \\ &= \left(\frac{\partial\mathbf{r}^o}{\partial\mathbf{s}^o}\right)^T\mathbf{Q}_\mathbf{r}^{-1}\left(\frac{\partial\mathbf{r}^o}{\partial\mathbf{s}^o}\right) + \left(\frac{\partial\mathbf{g}^o}{\partial\mathbf{s}^o}\right)^T\mathbf{Q}_\mathbf{g}^{-1}\left(\frac{\partial\mathbf{g}^o}{\partial\mathbf{s}^o}\right) \\ &\quad + \mathbf{Q}_\mathbf{s}^{-1}.\end{aligned} \tag{12}$$

The partial derivatives in (12) can be shown to be equal to

$$\frac{\partial\mathbf{r}^o}{\partial\mathbf{u}^o} = \begin{bmatrix}\boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_2^o}^T - \boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_1^o}^T \\ \boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_3^o}^T - \boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_1^o}^T \\ \vdots \\ \boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_M^o}^T - \boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_1^o}^T\end{bmatrix},$$

$$\frac{\partial \mathbf{r}^o}{\partial \mathbf{s}^o} = \begin{bmatrix} \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_1} & -\boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_2} & \mathbf{0}^T & \cdots & \mathbf{0}^T \\ \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_1} & \mathbf{0}^T & -\boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_3} & \cdots & \mathbf{0}^T \\ & & \cdots & & \\ \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_1} & \mathbf{0}^T & \mathbf{0}^T & \cdots & -\boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_M} \end{bmatrix},$$

$$\frac{\partial \mathbf{g}^o}{\partial \mathbf{u}^o} = \frac{1}{r_1^{o2}} \begin{bmatrix} r_1^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_2} - r_2^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_1} \\ r_1^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_3} - r_3^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_1} \\ \vdots \\ r_1^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_M} - r_M^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_1} \end{bmatrix},$$

$$\frac{\partial \mathbf{g}^o}{\partial \mathbf{s}^o} = \frac{1}{r_1^{o2}} \cdot \begin{bmatrix} r_2^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_1} & -r_1^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_2} & \mathbf{0}^T & \cdots & \mathbf{0}^T \\ r_3^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_1} & \mathbf{0}^T & -r_1^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_3} & \cdots & \mathbf{0}^T \\ & & \cdots & & \\ r_3^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_1} & \mathbf{0}^T & \mathbf{0}^T & \cdots & -r_1^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}^o_M} \end{bmatrix}, \tag{13}$$

where $\mathbf{0}$ is a 3×1 vector of zeros and $\boldsymbol{\rho}_{\mathbf{a},\mathbf{b}} = (\mathbf{a} - \mathbf{b})/\|\mathbf{a} - \mathbf{b}\|$ denotes a unit vector from $\mathbf{b}$ to $\mathbf{a}$.

From (11) and the definition of $\boldsymbol{\Phi}^o = [\mathbf{u}^{oT}, \mathbf{s}^{oT}]^T$, we have that $\mathbf{J}^{-1} = \mathrm{CRLB}(\mathbf{u}^o) = (\mathbf{X} - \mathbf{Y}\mathbf{Z}^{-1}\mathbf{Y}^T)^{-1}$. This completes the geolocation CRLB derivation.

4. Algorithm

Geolocating the target with known altitude h using TDOA and GROA measurements, r_{i1} and g_{i1}, is nontrivial, mainly because the unknown target position $\mathbf{u}^o$ is nonlinearly related to the measurements (see (4) and (5)). The problem is further complicated by the presence of receiver position errors and the equality constraint on $\mathbf{u}^o$ (see (2)). In [11, 23], with the availability of accurate receiver positions and without geometric constraints on the target position, closed-form solutions for TDOA- and GROA-based localization were developed by introducing extra variables to transform the nonlinear equations into pseudolinear ones and invoke the application of linear estimation techniques. They were shown to outperform the iterative Taylor-Series based methods [16, 21, 23] that require good initial solution guesses to avoid local convergence and may even have a divergence problem. Similar ideas have been applied to tackle, for example, the problem of target localization using TDOA and FDOA measurements [24]. Nevertheless, the aforementioned techniques either did not take into account receiver position errors or cannot cope with the equality constraint on the target position.

In this section, we shall propose a new solution for the TDOA- and GROA-based target geolocation problem described in Section 2. The algorithm development first follows the approach employed in [25, 26] to cast the geolocation problem into a constrained weighted least-squares (CWLS) optimization problem. It takes the presence of receiver position errors into consideration via modifying the weighting matrix appropriately and the target altitude information is included as an additional equality constraint.

The obtained CWLS minimization problem is solved using a technique developed on the basis of the method originally proposed in [15] for geolocation of a known altitude target using TDOA and FDOA measurements. The geolocation method, also referred to as the improved CWLS (ICWLS) solution, will be shown to have an estimation covariance matrix approximately equal to the geolocation CRLB in (7) when the receiver position errors and the measurement noise are small.

In the following, Section 4.1 gives the CWLS formulation of the TDOA- and GROA-based geolocation problem in consideration. Section 4.2 presents the solution to the CWLS optimization problem. Section 4.3 carries out the performance analysis. To facilitate the algorithm development, we convert the equality constraint on the target position $\mathbf{u}^o$ given in (2) into its equivalent form [16]

$$\mathbf{u}^{oT}\mathbf{u}^o = (N+h)^2 + \left\{1 - \frac{(N+h)^2}{\left[N\left(1-e^2\right)+h\right]^2}\right\} z^{o2}. \tag{14}$$

4.1. CWLS Formulation. Rearranging (4), we have

$$r_i^o = r_{i1}^o + r_1^o. \tag{15}$$

Squaring both sides and replacing r_i^{o2} with $\|\mathbf{u}^o - \mathbf{s}_i^o\|^2$ yield

$$r_{i1}^{o2} + 2r_{i1}^o r_1^o = \mathbf{s}_i^{oT}\mathbf{s}_i^o - \mathbf{s}_1^{oT}\mathbf{s}_1^o - 2\left(\mathbf{s}_i^o - \mathbf{s}_1^o\right)^T \mathbf{u}^o. \tag{16}$$

Expressing the true values in terms of their noisy quantities $r_{i1}^o = r_{i1} - \Delta r_{i1}$ and $\mathbf{s}_i^o = \mathbf{s}_i - \Delta\mathbf{s}_i$ and substituting the first-order approximation [20]

$$r_1^o = \left\|\mathbf{u}^o - \mathbf{s}_1^o\right\| \approx \left\|\mathbf{u}^o - \mathbf{s}_1\right\| + \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}_1}\Delta\mathbf{s}_1, \tag{17}$$

we arrive at the TDOA equation, after ignoring second-order error terms,

$$\begin{aligned} \varepsilon_{t,i} &\doteq 2r_i^o \Delta r_{i1} + 2\left(\mathbf{u}^o - \mathbf{s}_i\right)^T \Delta\mathbf{s}_i - 2r_i^o \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}_1}\Delta\mathbf{s}_1 \\ &= r_{i1}^2 - \mathbf{s}_i^T\mathbf{s}_i + \mathbf{s}_1^T\mathbf{s}_1 + 2\left(\mathbf{s}_i - \mathbf{s}_1\right)^T \mathbf{u}^o + 2\hat{r}_1^o r_{i1}, \end{aligned} \tag{18}$$

where $i = 2, 3, \ldots, M$ and $\hat{r}_1^o = \|\mathbf{u}^o - \mathbf{s}_1\|$. Similarly, rearranging (5) gives

$$g_{i1}^o r_1^o - r_1^o = r_i^o - r_1^o. \tag{19}$$

Putting $r_{i1}^o = r_i^o - r_1^o$ yields

$$\left(g_{i1}^o - 1\right) r_1^o = r_{i1}^o. \tag{20}$$

Substituting $g_{i1}^o = g_{i1} - \Delta g_{i1}$ and $\mathbf{s}_i^o = \mathbf{s}_i - \Delta\mathbf{s}_i$ and using (17), we have that the GROA equation is

$$\begin{aligned} \varepsilon_{g,i} &\doteq \Delta r_{i1} - \hat{r}_1^o \Delta g_{i1} + \left(g_{i1} - 1\right) \boldsymbol{\rho}^T_{\mathbf{u}^o,\mathbf{s}_1}\Delta\mathbf{s}_1 \\ &= r_{i1} - \left(g_{i1} - 1\right)\hat{r}_1^o, \end{aligned} \tag{21}$$

where, again, the second-order error terms have been neglected.

Collect $\varepsilon_{t,i}$ into $\boldsymbol{\varepsilon}_{\mathbf{t}} = [\varepsilon_{t,2}, \varepsilon_{t,3}, \dots, \varepsilon_{t,M}]^T$. Similarly, define $\boldsymbol{\varepsilon}_{\mathbf{g}} = [\varepsilon_{g,2}, \varepsilon_{g,3}, \dots, \varepsilon_{g,M}]^T$. Stacking (18) and (21) for $i = 2, 3, \dots, M$, respectively, and combining the results yield

$$\boldsymbol{\varepsilon}_{\mathbf{tg}} = \mathbf{h}_{\mathbf{tg}} - \mathbf{G}_{\mathbf{tg}}\mathbf{u}^o + \mathbf{g}_{\mathbf{tg}}\widehat{r}_1^o, \tag{22}$$

where

$$\mathbf{h}_{\mathbf{tg}} = \left[\mathbf{h}_{\mathbf{t}}^T, \mathbf{h}_{\mathbf{g}}^T\right]^T,$$

$$\mathbf{h}_{\mathbf{t}} = \begin{bmatrix} r_{21}^2 - \mathbf{s}_2^T\mathbf{s}_2 + \mathbf{s}_1^T\mathbf{s}_1 \\ r_{31}^2 - \mathbf{s}_3^T\mathbf{s}_3 + \mathbf{s}_1^T\mathbf{s}_1 \\ \vdots \\ r_{M1}^2 - \mathbf{s}_M^T\mathbf{s}_M + \mathbf{s}_1^T\mathbf{s}_1 \end{bmatrix},$$

$$\mathbf{h}_{\mathbf{g}} = \begin{bmatrix} r_{21} \\ r_{31} \\ \vdots \\ r_{M1} \end{bmatrix},$$

$$\mathbf{G}_{\mathbf{tg}} = -2\begin{bmatrix} (\mathbf{s}_2 - \mathbf{s}_1)^T \\ (\mathbf{s}_3 - \mathbf{s}_1)^T \\ \vdots \\ (\mathbf{s}_M - \mathbf{s}_1)^T \\ \mathbf{0}^T \\ \mathbf{0}^T \\ \vdots \\ \mathbf{0}^T \end{bmatrix}, \tag{23}$$

$$\mathbf{g}_{\mathbf{tg}} = -\begin{bmatrix} -2r_{21} \\ -2r_{31} \\ \vdots \\ -2r_{M1} \\ g_{21} - 1 \\ g_{31} - 1 \\ \vdots \\ g_{M1} - 1 \end{bmatrix}.$$

From (18) and (21), the equation error vector $\boldsymbol{\varepsilon}_{\mathbf{tg}}$ is

$$\boldsymbol{\varepsilon}_{\mathbf{tg}} = \mathbf{B}_{\mathbf{tg}}\Delta\mathbf{r} + \mathbf{C}_{\mathbf{tg}}\Delta\mathbf{g} + \mathbf{D}_{\mathbf{tg}}\Delta\mathbf{s},$$

$$\mathbf{B}_{\mathbf{tg}} = \begin{bmatrix} \mathbf{B}_{11} \\ \mathbf{B}_{12} \end{bmatrix},$$

$$\mathbf{C}_{\mathbf{tg}} = \begin{bmatrix} \mathbf{C}_{11} \\ \mathbf{C}_{12} \end{bmatrix},$$

$$\mathbf{D}_{\mathbf{tg}} = \begin{bmatrix} \mathbf{D}_{11} \\ \mathbf{D}_{12} \end{bmatrix}, \tag{24}$$

where $\mathbf{B}_{11}$, $\mathbf{B}_{12}$, $\mathbf{C}_{11}$, $\mathbf{C}_{12}$, $\mathbf{D}_{11}$, and $\mathbf{D}_{12}$ are equal to

$$\begin{aligned}
\mathbf{B}_{11} &= 2\begin{bmatrix} r_2^o & 0 & \cdots & 0 \\ 0 & r_3^o & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & r_M^o \end{bmatrix}, \\
\mathbf{C}_{11} &= \mathbf{O}_{(M-1)\times(M-1)}, \\
\mathbf{D}_{11} &= 2\begin{bmatrix} -r_2^o\boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_1}^T & (\mathbf{u}^o - \mathbf{s}_2)^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\ -r_3^o\boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_1}^T & \mathbf{0}^T & (\mathbf{u}^o - \mathbf{s}_3)^T & \cdots & \mathbf{0}^T \\ & & \cdots\cdots & & \\ -r_M^o\boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_1}^T & \mathbf{0}^T & \mathbf{0}^T & \cdots & (\mathbf{u}^o - \mathbf{s}_M)^T \end{bmatrix}, \\
\mathbf{B}_{12} &= \mathbf{I}_{(M-1)\times(M-1)}, \\
\mathbf{C}_{12} &= -\widehat{r}_1^o\mathbf{I}_{(M-1)\times(M-1)}, \\
\mathbf{D}_{12} &= \begin{bmatrix} (g_{21} - 1)\,\boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_1}^T & \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\ (g_{31} - 1)\,\boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_1}^T & \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (g_{M1} - 1)\,\boldsymbol{\rho}_{\mathbf{u}^o,\mathbf{s}_1}^T & \mathbf{0}^T & \mathbf{0}^T & \cdots & \mathbf{0}^T \end{bmatrix},
\end{aligned} \tag{25}$$

where $\mathbf{0}$ denotes a 3×1 zero vector and $\mathbf{I}_{(M-1)\times(M-1)}$ represents an $(M-1)\times(M-1)$ identity matrix.

The solution equation in (22) is nonlinear with respect to the unknown target position $\mathbf{u}^o$, because $\widehat{r}_1^o = \|\mathbf{u}^o - \mathbf{s}_1\|$ is also dependent on $\mathbf{u}^o$. Moreover, recall from Section 2 that the TDOA noise $\Delta\mathbf{r}$, the GROA noise, and the receiver position error $\Delta\mathbf{s}$ are all zero-mean Gaussian distributed. As a result, the equation error vector $\boldsymbol{\varepsilon}_{\mathbf{tg}}$ is a zero-mean Gaussian random vector. Therefore, the CWLS estimator for $\mathbf{u}^o$ needs to minimize the following cost function:

$$\zeta = \left(\mathbf{h}_{\mathbf{tg}} - \mathbf{G}_{\mathbf{tg}}\mathbf{u}^o + \mathbf{g}_{\mathbf{tg}}\widehat{r}_1^o\right)^T \mathbf{W}\left(\mathbf{h}_{\mathbf{tg}} - \mathbf{G}_{\mathbf{tg}}\mathbf{u}^o + \mathbf{g}_{\mathbf{tg}}\widehat{r}_1^o\right), \tag{26}$$

where $\mathbf{W}$ is the weighting matrix equal to [15]

$$\begin{aligned}
\mathbf{W} &= E\left[\boldsymbol{\varepsilon}_{\mathbf{tg}}\boldsymbol{\varepsilon}_{\mathbf{tg}}^T\right]^{-1} \\
&= \left(\mathbf{B}_{\mathbf{tg}}\mathbf{Q}_{\mathbf{r}}\mathbf{B}_{\mathbf{tg}}^T + \mathbf{C}_{\mathbf{tg}}\mathbf{Q}_{\mathbf{g}}\mathbf{C}_{\mathbf{tg}}^T + \mathbf{D}_{\mathbf{tg}}\mathbf{Q}_{\mathbf{s}}\mathbf{D}_{\mathbf{tg}}^T\right)^{-1}.
\end{aligned} \tag{27}$$

The constraints come from the target altitude information (see (14)) as well as the functional relationship $\widehat{r}_1^o = \|\mathbf{u}^o - \mathbf{s}_1\|$. In particular, we have

$$\mathbf{s}_1^T\mathbf{s}_1 - 2\mathbf{s}_1^T\mathbf{u}^o + \mathbf{u}^{oT}\mathbf{u}^o - \widehat{r}_1^{o2} = 0. \tag{28}$$

In summary, the CWLS optimization problem for the considered TDOA- and GROA-based target geolocation is

$$\min_{\mathbf{u}^o, \hat{r}_1^o} \quad \zeta$$
$$\mathbf{s}_1^T\mathbf{s}_1 - 2\mathbf{s}_1^T\mathbf{u}^o + \mathbf{u}^{oT}\mathbf{u}^o - \hat{r}_1^{o2} = 0$$
$$\mathbf{u}^{oT}\mathbf{u}^o = (N+h)^2 + \left\{1 - \frac{(N+h)^2}{\left[N\left(1-e^2\right)+h\right]^2}\right\} z^{o2}. \tag{29}$$

4.2. ICWLS Solution. To find the solution to (29), we first approximate the second constraint using $\mathbf{u}^{oT}\mathbf{u}^o = (r+h)^2$ to produce an initial geolocation result, where $r = 6378.137$ km is the equatorial radius of the spheroid Earth. The associated Lagrangian is

$$\begin{aligned} L\left(\mathbf{u}, \hat{r}_1^o, \lambda_1, \lambda_2\right) &= \left(\mathbf{h}_{\mathbf{tg}} - \mathbf{G}_{\mathbf{tg}}\mathbf{u}^o + \mathbf{g}_{\mathbf{tg}}\hat{r}_1^o\right)^T \mathbf{W}\left(\mathbf{h}_{\mathbf{tg}} - \mathbf{G}_{\mathbf{tg}}\mathbf{u}^o + \mathbf{g}_{\mathbf{tg}}\hat{r}_1^o\right) \\ &+ \lambda_1\left[\hat{r}_1^{o2} - \mathbf{s}_1^T\mathbf{s}_1 + 2\mathbf{s}_1^T\mathbf{u}^o - (r+h)^2\right] \\ &+ \lambda_2\left[\mathbf{u}^{oT}\mathbf{u}^o - (r+h)^2\right], \end{aligned} \tag{30}$$

where λ_1 and λ_2 are the Lagrange multipliers. Differentiating $L(\mathbf{u}, \hat{r}_1^o, \lambda_1, \lambda_2)$ with respect to $\mathbf{u}^o$ and $\hat{r}_1^o$ and setting the results to zeros, we obtain that the initial geolocation result $\mathbf{u}$ is equal to

$$\mathbf{u} = \mathbf{G}_1\left(\mathbf{G}_{\mathbf{tg}}^T\mathbf{W}\mathbf{G}_2\mathbf{r}_1 - \lambda_1\mathbf{s}_1\right), \tag{31}$$

$$\mathbf{G}_1 = \left(\mathbf{G}_{\mathbf{tg}}^T\mathbf{W}\mathbf{G}_{\mathbf{tg}} + \lambda_2\mathbf{I}\right)^{-1}, \tag{32}$$

$$\mathbf{r}_1 = \left[1, \hat{r}_1^o, \hat{r}_1^{o2}\right]^T, \tag{33}$$

$$\mathbf{G}_2 = \left[\mathbf{h}_{\mathbf{tg}}, \mathbf{g}_{\mathbf{tg}}, \mathbf{0}\right], \tag{34}$$

$$\mathbf{g}_{\mathbf{tg}}^T\mathbf{W}\left(\mathbf{G}_2\mathbf{r}_1 - \mathbf{G}_{\mathbf{tg}}\mathbf{u}\right) + \lambda_1\hat{r}_1^o = 0. \tag{35}$$

Define $\mathbf{g}_1 = [\mathbf{s}_1^T\mathbf{s}_1 + (r+h)^2, 0, -1]^T$. The first equality constraint in (29) now becomes

$$2\mathbf{s}_1^T\mathbf{u}^o = \mathbf{g}_1^T\mathbf{r}_1. \tag{36}$$

Putting (36) into (31) gives

$$\begin{aligned} \lambda_1 &= \mathbf{g}_2^T\mathbf{r}_1, \\ \mathbf{g}_2^T &= \frac{2\mathbf{s}_1^T\mathbf{G}_1\mathbf{G}_{\mathbf{tg}}^T\mathbf{W}\mathbf{G}_2 - \mathbf{g}_1^T}{2\mathbf{s}_1^T\mathbf{G}_1\mathbf{s}_1}. \end{aligned} \tag{37}$$

Substituting (37) into (31), we have

$$\begin{aligned} \mathbf{u} &= \mathbf{G}_3\mathbf{r}_1, \\ \mathbf{G}_3 &= \mathbf{G}_1\left(\mathbf{G}_{\mathbf{tg}}^T\mathbf{W}\mathbf{G}_2 - \mathbf{s}_1\mathbf{g}_2^T\right). \end{aligned} \tag{38}$$

Using (37) and (38) to simplify (35) yields

$$\begin{aligned} \mathbf{g}_3\mathbf{r}_1 &= 0, \\ \mathbf{g}_3 &= \mathbf{g}_2^T + 2\mathbf{g}_{\mathbf{tg}}^T\mathbf{W}\left(\mathbf{G}_2 - \mathbf{G}_{\mathbf{tg}}\mathbf{G}_3\right), \end{aligned} \tag{39}$$

which is a polynomial in terms of $\hat{r}_1^o$. For a given λ_2, one can find two roots for $\hat{r}_1^o$, and there is only one positive solution for $\hat{r}_1^o$ in most cases. Putting the result back into (38), we can obtain an initial estimate of the target position that is dependent on the value of λ_2. In other words, the initial target position estimate from the optimization problem (29) can in fact be expressed as $\mathbf{u}(\lambda_2)$. Applying it into the equality constraint from the target altitude information $\mathbf{u}^{oT}\mathbf{u}^o = (r+h)^2$ produces an equation for λ_2, which may be solved using Newton's method with an initial solution guess $\lambda_2 = 0$, as in [15].

We can improve the geolocation result by first utilizing $\mathbf{u}$ to find an estimate of the target geodetic latitude α and $N = r/\sqrt{1 - e^2\sin^2(\alpha)}$ via

$$\alpha = \tan^{-1}\left[\frac{\mathbf{u}(3)}{\sqrt{\mathbf{u}(1)^2 + \mathbf{u}(2)^2}\left(1-e^2\right)}\right]. \tag{40}$$

Putting the estimated N into (29) and repeating the procedure that finds $\mathbf{u}$ yield an improved target position estimate. The above process can be iterated several times until convergence.

Another aspect that needs to be addressed is the evaluation of the weighting matrix $\mathbf{W}$ that involves the unknown true target position $\mathbf{u}^o$. To bypass this difficulty, we can first set $\mathbf{W}$ to $(\mathrm{diag}[\mathbf{Q}_\mathbf{r}, \mathbf{Q}_\mathbf{g}])^{-1}$ and use (31) to (39) to obtain an initial estimate of $\mathbf{u}^o$. A better $\mathbf{W}$ can then be produced so that a more precise estimation of $\mathbf{u}^o$ can be found. These steps are interleaved with the iterations that refine the altitude constraint in (29).

4.3. Performance Analysis. We shall derive the estimation covariance matrix of the ICWLS solution and contrast it with the target geolocation CRLB in (7) to establish the approximate efficiency of the proposed TDOA- and GROA-based geolocation technique. For this purpose, first express the ICWLS solutions in terms of their true values and estimation errors as $\mathbf{u} = \mathbf{u}^o + \Delta\mathbf{u}$ and $\hat{r}_1 = \hat{r}_1^o + \Delta\hat{r}_1$. Similarly, we may write the regressand and regressor in (22) as $\mathbf{h}_{\mathbf{tg}} = \mathbf{h}_{\mathbf{tg}}^o + \Delta\mathbf{h}_{\mathbf{tg}}$ and $\mathbf{g}_{\mathbf{tg}} = \mathbf{g}_{\mathbf{tg}}^o + \Delta\mathbf{g}_{\mathbf{tg}}$. We have

$$\boldsymbol{\varepsilon}_{\mathbf{tg}} = \mathbf{G}_{\mathbf{tg}}\Delta\mathbf{u} - \mathbf{g}_{\mathbf{tg}}^o\Delta\hat{r}_1^o. \tag{41}$$

Applying (41), we may rewrite (30) as, after neglecting the second-order error terms,

$$\begin{aligned} L\left(\Delta\mathbf{u}, \Delta\hat{r}_1, \lambda_1, \lambda_2\right) &= \left(\boldsymbol{\varepsilon}_{\mathbf{tg}} - \mathbf{G}_{\mathbf{tg}}\Delta\mathbf{u} + \mathbf{g}_{\mathbf{tg}}^o\Delta\hat{r}_1\right)^T \\ &\cdot \mathbf{W}\left(\boldsymbol{\varepsilon}_{\mathbf{tg}} - \mathbf{G}_{\mathbf{tg}}\Delta\mathbf{u} + \mathbf{g}_{\mathbf{tg}}^o\Delta\hat{r}\right) \\ &+ \lambda_1\left[2\left(\mathbf{s}_1 - \mathbf{u}^o\right)^T\Delta\mathbf{u} + 2\hat{r}_1^o\Delta\hat{r}_1\right] + \lambda_2\left(2\mathbf{u}^{oT}\Delta\mathbf{u}\right). \end{aligned} \tag{42}$$

Setting the partial derivatives of $L(\Delta\mathbf{u}, \Delta\hat{r}_1, \lambda_1, \lambda_2)$ with respect to $\Delta\mathbf{u}$, $\Delta\hat{r}_1$, λ_1, and λ_2 to zeros yields

$$-\mathbf{G}_{\mathbf{tg}}^{T}\mathbf{W}\left(\boldsymbol{\varepsilon}_{\mathbf{tg}} - \mathbf{G}_{\mathbf{tg}}\Delta\mathbf{u} + \mathbf{g}_{\mathbf{tg}}^{o}\Delta\hat{r}_1\right) + \lambda_1\left(\mathbf{s}_1 - \mathbf{u}^o\right) + \lambda_2\mathbf{u}^o = 0, \tag{43}$$

$$\mathbf{g}_{\mathbf{tg}}^{oT}\mathbf{W}\left(\boldsymbol{\varepsilon}_{\mathbf{tg}} - \mathbf{G}_{\mathbf{tg}}\Delta\mathbf{u} + \mathbf{g}_{\mathbf{tg}}^{o}\Delta\hat{r}\right) + \lambda_1\hat{r}_1^o = 0, \tag{44}$$

$$\Delta\hat{r}_1 = -\frac{1}{\hat{r}_1^o}\left(\mathbf{s}_1 - \mathbf{u}^o\right)^T\Delta\mathbf{u}, \tag{45}$$

$$\mathbf{u}^{oT}\Delta\mathbf{u} = 0. \tag{46}$$

Substitution of (45) into (44) leads to

$$\lambda_1 = -\frac{1}{\hat{r}_1^o}\mathbf{g}_{\mathbf{tg}}^{oT}\mathbf{W}\left(\boldsymbol{\varepsilon}_{\mathbf{tg}} - \mathbf{G}_{\mathbf{s}}^{oT}\Delta\mathbf{u}\right), \qquad \mathbf{G}_{\mathbf{s}}^{oT} = \mathbf{G}_{\mathbf{tg}} + \frac{1}{\hat{r}_1^o}\mathbf{g}_{\mathbf{tg}}^{o}\left(\mathbf{s}_1 - \mathbf{u}^o\right)^T. \tag{47}$$

Putting (45) and (47) into (43), we obtain

$$\Delta\mathbf{u} = \left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\boldsymbol{\varepsilon}_{\mathbf{tg}} - \lambda_2\mathbf{u}^o\right). \tag{48}$$

Moreover, from (46) and (48), we have

$$\lambda_2 = \frac{\mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\boldsymbol{\varepsilon}_{\mathbf{tg}}}{\mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o}. \tag{49}$$

By putting (49) into (48), the geolocation error of the ICWLS solution can be shown to be equal to

$$\Delta\mathbf{u} = \left[\mathbf{I} - \frac{\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o\mathbf{u}^{oT}}{\mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o}\right]\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1} \cdot \mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\boldsymbol{\varepsilon}_{\mathbf{tg}}. \tag{50}$$

With the assumption that the TDOA and GROA measurement errors are zero-mean Gaussian distributed, we have $E(\boldsymbol{\varepsilon}_{\mathbf{tg}}) = 0$ and

$$E\left(\Delta\mathbf{u}\right) = 0. \tag{51}$$

This indicates that, under small measurement and receiver position errors, the proposed ICWLS geolocation solution is unbiased.

The covariance matrix of $\Delta\mathbf{u}$, from (50), is

$$\operatorname{cov}\left(\mathbf{u}\right) = \left[\mathbf{I} - \frac{\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o\mathbf{u}^{oT}}{\mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o}\right]\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}. \tag{52}$$

It can be expressed in the following equivalent form:

$$\operatorname{cov}\left(\mathbf{u}\right) = \left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1} - \left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1} \cdot \mathbf{u}^o\left[\mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o\right]^{-1} \cdot \mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}. \tag{53}$$

Again, under small measurement and receiver position errors, we can show that

$$\operatorname{cov}\left(\mathbf{u}\right) \approx \operatorname{CCRLB}\left(\mathbf{u}^o\right). \tag{54}$$

This verifies the approximate efficiency of the proposed ICWLS geolocation solution.

5. Effect of Altitude Error

The development of the ICWLS solution assumes the availability of the precise knowledge on the target altitude h. In practice, this is rarely the case. We shall investigate the impact of the uncertainty in the target altitude information on the geolocation accuracy. Different from the errors in the TDOA and GROA measurements, which are assumed to be random, the altitude error, denoted by Δh, is generally unknown but deterministic.

The analysis starts with replacing h in (30) with $h + \Delta h$ and defining $r_h = N + h$. Following the same approach that finds the ICWLS geolocation error in (50), we obtain that the geolocation error now becomes

$$\Delta\mathbf{u} = \left[\mathbf{I} - \frac{\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o\mathbf{u}^{oT}}{\mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o}\right]\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1} \cdot \mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\boldsymbol{\varepsilon}_{\mathbf{tg}} + \frac{\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o}{\mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o}r_h\Delta h. \tag{55}$$

Because the altitude error Δh is deterministic, we have

$$E\left(\Delta\mathbf{u}\right) = \frac{\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o}{\mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o}r_h\Delta h \neq 0. \tag{56}$$

This implies that the presence of target altitude error would make the ICWLS geolocation result biased, as expected. Moreover, the second moment of $\Delta\mathbf{u}$ is

$$E\left(\Delta\mathbf{u}\Delta\mathbf{u}^T\right) = \left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1} - \frac{\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o\mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}}{\mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o} \times \left[1 - \frac{\left(r_h\Delta h\right)^2}{\mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o}\right]. \tag{57}$$

Comparing with (53), we can notice that if the following condition is fulfilled:

$$\Delta h \leq \frac{1}{r_h}\sqrt{\mathbf{u}^{oT}\left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}\mathbf{u}^o}, \tag{58}$$

we have

$$E\left(\Delta\mathbf{u}\Delta\mathbf{u}^T\right) \leq \left(\mathbf{G}_{\mathbf{s}}^{oT}\mathbf{W}\mathbf{G}_{\mathbf{s}}^{o}\right)^{-1}. \tag{59}$$

Table 1: True positions of the receivers.

Receiver number i	Longitude (°)	Latitude (°)	Altitude (m)
1	104.0214E	30.6535N	525
2	104.0250E	30.6800N	565
3	104.0486E	30.6796N	575
4	104.0555E	30.6476N	542
5	104.0213E	30.6690N	534
6	104.0350E	30.6807N	557
7	104.0555E	30.6618N	552
8	104.0381E	30.6507N	511

This means that when the target altitude is known imprecisely but its error satisfies (58), exploring it can still improve the geolocation performance over the case where only TDOA and GROA measurements are utilized. However, the altitude error may significantly degrade the geolocation accuracy, if condition (58) is violated.

6. Simulations

Consider $M = 8$ receivers whose true positions are summarized in Table 1. The target is located at (104.0381E°, 30.6650N°) with an altitude of 500 m. The covariance matrices of the TDOA and GROA measurement errors are set to be $\mathbf{Q_r} = c^2\sigma_t^2\mathbf{R}$ and $\mathbf{Q_g} = \sigma_g^2\mathbf{R}$, and that of the receiver position error is $\mathbf{Q_s} = \sigma_s^2\mathbf{I}$. $\mathbf{R}$ is an $(M-1)\times(M-1)$ matrix with the diagonal elements being equal to 1 and the off-diagonal elements all equal to 0.5.

The geolocation accuracy of the proposed ICWLS solution is quantified by the root mean square error (RMSE), defined as $\mathrm{RMSE}(\mathbf{u}) = \sqrt{\sum_{l=1}^{L}\|\mathbf{u}_l - \mathbf{u}^o\|^2/L}$. $\mathbf{u}_l$ is the target position estimate at the lth ensemble run, and $L = 5000$ is the total number of ensemble runs. In each ensemble run, the TDOA and GROA measurements and the erroneous receiver positions are generated by adding to the true values independent zero-mean Gaussian noise with covariance matrices $\mathbf{Q_r}$, $\mathbf{Q_g}$, and $\mathbf{Q_s}$.

For the purpose of comparison, we simulate a benchmark technique, referred to as the improved two-step weighted least-squares (ITSWLS) algorithm. The improved two-step method is developed on the basis of the two-step TDOA- and GROA-based localization algorithm originally proposed in [11]. We follow the approach in [4] to generalize the solution from [11] to take into consideration the presence of receiver position errors. Equation (28) is also included as an additional solution equation in the first-step processing of the benchmark technique to account for the target altitude information. All simulations were performed using MATLAB R2009b on a desktop PC with an Intel i5-3470 3.2 GHz CPU and 2.0 GB RAM (the code for implementing the proposed ICWLS method can be provided upon request).

Figure 2 compares the geolocation accuracies of the ICWLS and ITSWLS solutions as a function of the standard

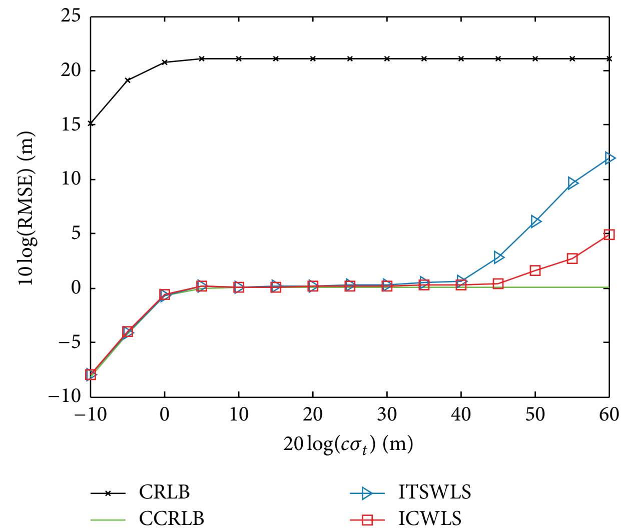

Figure 2: Comparison of target geolocation RMSEs of ITSWLS and ICWLS as a function of $c\sigma_t$.

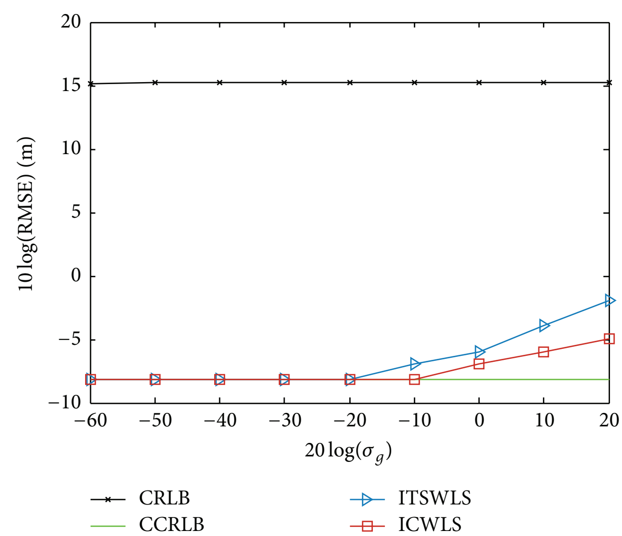

Figure 3: Comparison of target geolocation RMSEs of ITSWLS and ICWLS as a function of σ_g.

deviation of the TDOA measurement noise σ_t. The standard deviations of the GROA measurement and receiver position errors are $\sigma_g = 10^{-3}$ and $\sigma_s = 10^{-6}$ m. An altitude error of 10 m is assumed.

Figure 3 plots the estimation RMSEs of the two simulated geolocation algorithms as a function of the standard deviation of the GROA measurement noise σ_g. In this simulation, we set $\sigma_t = 10^{-9}$ s and $\sigma_s = 10^{-6}$ m. The altitude error remains to be 10 m.

Figure 4 shows as a function of the standard deviation of the receiver position error σ_s the geolocation RMSE of the two considered geolocation techniques. We set $\sigma_t = 10^{-9}$ s and $\sigma_g = 10^{-3}$ while keeping the altitude error at 10 m.

Also included in Figures 2–4 are the associated target geolocation CRLBs in (7) (CCRLB($\mathbf{u}^o$)) and the CRLBs of

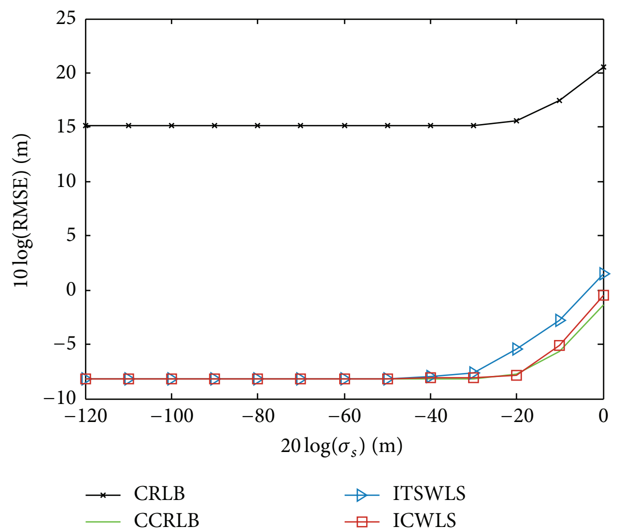

FIGURE 4: Comparison of target geolocation RMSEs of ITSWLS and ICWLS as a function of σ_s.

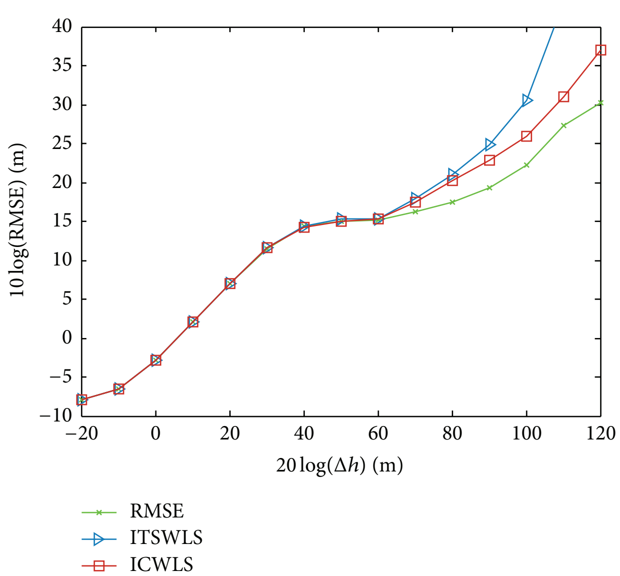

FIGURE 5: Comparison of target geolocation RMSEs of ITSWLS and ICWLS as a function of Δh.

the target position (CRLB($\mathbf{u}^o$)) when the altitude information is absent.

In the last experiment, we investigate the effect of target altitude error. The results are summarized in Figure 5, where, as a function of the target altitude error Δh, the geolocation RMSEs of the two algorithms under consideration are contrasted with respect to the theoretical results given in (57). We set $\sigma_t = 10^{-9}$ s, $\sigma_g = 10^{-3}$, and $\sigma_s = 10^{-6}$ m.

We obtain the following observations from Figures 2–5:

(1) Comparing CCRLB($\mathbf{u}^o$) and CRLB($\mathbf{u}^o$) reveals that exploring the target altitude information can significantly improve the target geolocation accuracy.

(2) Both the ICWLS and ITSWLS methods are able to attain the CRLB accuracy under small noise conditions. But ICWLS appears to be more robust to larger noise levels. This might be explained by noting that, within ICWLS, the functional relationship between the unknown target position $\mathbf{u}^o$ and the nuisance parameter $\hat{r}_1^o$ (i.e., $\hat{r}_1^o = \|\mathbf{u}^o - \mathbf{s}_1\|$) is explored as an equality constraint on $\mathbf{u}^o$. On the other hand, ITSWLS first ignores them being dependent and utilizes their functional relationship in a separate processing stage.

(3) In this simulation, the geolocation RMSE from simulations matches the theoretical value well. This justifies the validity of the analysis in Section 5.

7. Conclusion

This work investigated geolocating a target on the Earth surface from TDOA and GROA measurements. The practical scenario where the known receiver positions have errors was also taken into consideration. CRLB analysis showed that the use of target altitude information can improve the target geolocation accuracy. An algebraic closed-form geolocation solution, based on formulating the geolocation task as an equality-constrained optimization problem, was developed. It can reach the CRLB accuracy under small Gaussian noise and it was shown via simulations to be able to outperform a benchmark technique at relatively large noise levels.

Appendix

This appendix derives the matrix $\mathbf{F}$ in (8). By the matrix derivative lemma [27] and from (2), we have that the Jacobian $\mathbf{F}$ of the constraint $f(\mathbf{u}^o)$ is

$$\begin{aligned}\mathbf{F} &= \frac{\partial\left[\mathbf{u}^{oT}\mathbf{P}\mathbf{u}^o - (N+h)^2\right]}{\partial\mathbf{u}^o} \\ &= \frac{\left(\partial\mathbf{u}^{oT}\right)\mathbf{P}\mathbf{u}^o}{\partial\mathbf{u}^o} + \frac{\mathbf{u}^{oT}(\partial\mathbf{P})\mathbf{u}^o}{\partial\mathbf{u}^o} + \frac{\mathbf{u}^{oT}\mathbf{P}\left(\partial\mathbf{u}^o\right)}{\partial\mathbf{u}^o} \\ &\quad - 2(N+h)\frac{\partial N}{\partial\mathbf{u}^o} \\ &= 2\mathbf{P}\mathbf{u}^o + \mathbf{u}^o\mathbf{P}_\mathbf{u}\mathbf{u}^{oT} - 2(N+h)\frac{\partial N}{\partial\mathbf{u}^o},\end{aligned} \tag{A.1}$$

where

$$\mathbf{P}_\mathbf{u} = \frac{\partial\mathbf{P}}{\partial\mathbf{u}^o} = \frac{\partial\mathbf{P}}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial\mathbf{u}^o}. \tag{A.2}$$

By the chain rule of the partial derivative, we have

$$\frac{\partial\mathbf{P}}{\partial\alpha} = \frac{\partial\mathbf{P}}{\partial N}\cdot\frac{\partial N}{\partial\alpha}, \tag{A.3}$$

where

$$\frac{\partial \mathbf{P}}{\partial N} = \operatorname{diag}\left\{0, 0, \frac{2e^2 h (N+h)}{\left[N\left(1-e^2\right)+h\right]^3}\right\},$$
$$\frac{\partial N}{\partial \alpha} = \frac{re^2 \sin(\alpha)\cos(\alpha)}{\left[1-e^2\sin^2(\alpha)\right]^{3/2}}. \tag{A.4}$$

We proceed to evaluate $\partial\alpha/\partial\mathbf{u}^o$. Let $\boldsymbol{\varphi} = [\alpha, \beta, h]^T$ be the geodetic coordinates of the target. From (1), we have that $\partial\mathbf{u}^o/\partial\boldsymbol{\varphi}$ is equal to

$$\mathbf{T}_1 = \begin{bmatrix} \frac{\partial x^o}{\partial \alpha} & \frac{\partial x^o}{\partial \beta} & \frac{\partial x^o}{\partial h} \\ \frac{\partial y^o}{\partial \alpha} & \frac{\partial y^o}{\partial \beta} & \frac{\partial y^o}{\partial h} \\ \frac{\partial z^o}{\partial \alpha} & \frac{\partial z^o}{\partial \beta} & \frac{\partial z^o}{\partial h} \end{bmatrix}$$
$$= \begin{bmatrix} -(N+h)\sin\alpha\cos\beta, -(N+h)\cos\alpha\sin\beta, \cos\alpha\cos\beta \\ -(N+h)\sin\alpha\sin\beta, (N+h)\cos\alpha\cos\beta, \cos\alpha\sin\beta \\ \left(N\left(1-e^2\right)+h\right)\cos\alpha, 0, \sin\alpha \end{bmatrix}. \tag{A.5}$$

Then, it is easy to show that

$$\begin{bmatrix} \frac{\partial \alpha}{\partial x^o} & \frac{\partial \alpha}{\partial y^o} & \frac{\partial \alpha}{\partial z^o} \\ \frac{\partial \beta}{\partial x^o} & \frac{\partial \beta}{\partial y^o} & \frac{\partial \beta}{\partial z^o} \\ \frac{\partial h}{\partial x^o} & \frac{\partial h}{\partial y^o} & \frac{\partial h}{\partial z^o} \end{bmatrix} = \mathbf{T}_1^{-1} = \mathbf{T}_2, \tag{A.6}$$

$$\frac{\partial \alpha}{\partial \mathbf{u}^o} = \mathbf{T}_2(1,:), \tag{A.7}$$

where $\mathbf{T}_2(1,:)$ denotes the first row of $\mathbf{T}_2$. Then

$$\frac{\partial N}{\partial \mathbf{u}^o} = \frac{\partial N}{\partial \alpha}\frac{\partial \alpha}{\partial \mathbf{u}^o} = \frac{re^2\sin(\alpha)\cos(\alpha)}{\left[1-e^2\sin^2(\alpha)\right]^{3/2}}\mathbf{T}_2(1,:)^T. \tag{A.8}$$

Putting (A.3), (A.7), and (A.8) into (A.2) yields the desired Jacobian $\mathbf{F}$.

Competing Interests

The authors declare that they have no competing interests regarding the publication of this paper.

Acknowledgments

This study was supported by the National Natural Science Foundation of China (no. 61172140 and no. 61304264).

References

[1] A. Beck, P. Stoica, and J. Li, "Exact and approximate solutions of source localization problems," *IEEE Transactions on Signal Processing*, vol. 56, no. 5, pp. 1770–1778, 2008.

[2] G. H. Zhu and D. Z. Feng, "Bi-iterative method for moving source localisation using TDOA and FDOA measurements," *Electronics Letters*, vol. 51, no. 1, pp. 8–10, 2015.

[3] K. C. Ho, "Bias reduction for an explicit solution of source localization using TDOA," *IEEE Transactions on Signal Processing*, vol. 60, no. 5, pp. 2101–2114, 2012.

[4] K. C. Ho, X. Lu, and L. Kovavisaruch, "Source localization using TDOA and FDOA measurements in the presence of receiver location errors: analysis and solution," *IEEE Transactions on Signal Processing*, vol. 55, no. 2, pp. 684–696, 2007.

[5] M. R. Gholami, R. M. Vaghefi, and E. G. Ström, "RSS-based sensor localization in the presence of unknown channel parameters," *IEEE Transactions on Signal Processing*, vol. 61, no. 15, pp. 3752–3759, 2013.

[6] R. M. Vaghefi, M. R. Gholami, R. M. Buehrer, and E. G. Strom, "Cooperative received signal strength-based sensor localization with unknown transmit powers," *IEEE Transactions on Signal Processing*, vol. 61, no. 6, pp. 1389–1403, 2013.

[7] A. Coluccia, "Reduced-bias ML-based estimators with low complexity for self-calibrating RSS ranging," *IEEE Transactions on Wireless Communications*, vol. 12, no. 3, pp. 1220–1230, 2013.

[8] D. Li and Y. H. Hu, "Least square solution of energy based acoustic source localization problems," in *Proceedings of the International Conference on Parallel Processing Workshops (ICPPW '04)*, pp. 443–446, Montreal, Canada, August 2004.

[9] X. Sheng and Y.-H. Hu, "Maximum likelihood multiple-source localization using acoustic energy measurements with wireless sensor networks," *IEEE Transactions on Signal Processing*, vol. 53, no. 1, pp. 44–53, 2005.

[10] D. Blatt and A. O. Hero III, "Energy-based sensor network source localization via projection onto convex sets," *IEEE Transactions on Signal Processing*, vol. 54, no. 9, pp. 3614–3619, 2006.

[11] K. C. Ho and M. Sun, "Passive source localization using time difference of arrival and gain ratios of arrival," *IEEE Transactions on Signal Processing*, vol. 56, no. 2, pp. 464–477, 2008.

[12] B. J. Hao, Z. Li, and Y. M. Ren, "Passive multiple disjoint sources localization using TDOAs and GROAs," *Acta Ectronica Sinica*, vol. 39, no. 3, pp. 2374–2381, 2012 (Chinese).

[13] B.-J. Hao, Z. Li, P.-W. Wan, J.-B. Si, P.-H. Qi, and L. Guan, "Bias reduction for passive source localization based on TDOA and GROA," *Acta Electronica Sinica*, vol. 42, no. 3, pp. 477–484, 2014 (Chinese).

[14] W. Cui, Z. Cao, and J. Wei, "Dual-microphone source location method in 2-D space," in *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP '06)*, pp. 845–848, Toulouse, France, May 2006.

[15] K. C. Ho and Y. T. Chan, "Geolocation of a known altitude object from TDOA and FDOA measurements," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 33, no. 3, pp. 770–783, 1997.

[16] Y. L. Cao, L. Peng, J. Li, L. Yang, and F. Guo, "A new iterative algorithm for geolocating a known altitude target using TDOA and FDOA measurements in the presence of satellite location uncertainty," *Chinese Journal of Aeronautics*, vol. 28, no. 5, pp. 1510–1518, 2015.

[17] L. E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders, *Fundamentals of Acoustics*, John Wiley & Sons, New York, NY, USA, 1982.

[18] T. K. Sarkar, M. C. Wicks, and M. Salazar-Palma, *Smart Antennas* , Wiley-IEEE Press, New York, NY, USA, 2003.

[19] L. Yang and K. C. Ho, "An approximately efficient TDOA localization algorithm in closed-form for locating multiple disjoint

sources with erroneous sensor positions," *IEEE Transactions on Signal Processing*, vol. 57, no. 12, pp. 4598–4615, 2009.

[20] L. Yang and K. C. Ho, "Alleviating sensor position error in source localization using calibration emitters at inaccurate locations," *IEEE Transactions on Signal Processing*, vol. 58, no. 1, pp. 67–83, 2010.

[21] W. H. Foy, "Position-location solutions by Taylor-series estimation," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 12, no. 2, pp. 187–194, 1976.

[22] B. Hao and Z. Li, "BFGS quasi-Newton location algorithm using TDOAs and GROAs," *Journal of Systems Engineering and Electronics*, vol. 24, no. 3, pp. 341–348, 2013.

[23] K. C. Ho and W. W. Xu, "An accurate algebraic solution for moving source location using TDOA and FDOA measurements," *IEEE Transactions on Signal Processing*, vol. 52, no. 9, pp. 2453–2463, 2004.

[24] M. Sun and K. C. Ho, "An asymptotically efficient estimator for TDOA and FDOA positioning of multiple disjoint sources in the presence of sensor location uncertainties," *IEEE Transactions on Signal Processing*, vol. 59, no. 7, pp. 3434–3440, 2011.

[25] H. G. Yu, G. M. Huang, J. Gao, and B. Liu, "An efficient constrained weighted least squares algorithm for moving source location using TDOA and FDOA measurements," *IEEE Transactions on Wireless Communications*, vol. 11, no. 1, pp. 44–47, 2012.

[26] F. Y. Qu and X. W. Meng, "Source localization using TDOA and FDOA measurements based on constrained total least squares algorithm," *Journal of Electronics & Information Technology*, vol. 11, no. 1, pp. 44–48, 2012 (Chinese).

[27] S. Haykin, *Adaptive Filter Theory*, Prentice-Hall, Englewood Cliffs, NJ, USA, 2002.