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Research Article

On the Approximate Controllability of Fractional Evolution Equations with Generalized Riemann-Liouville Fractional Derivative

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We discuss the approximate controllability of fractional evolution equations involving generalized Riemann-Liouville fractional derivative. The results are obtained with the help of the theory of fractional calculus, semigroup theory, and the Schauder fixed point theorem under the assumption that the corresponding linear system is approximately controllable. Finally, an example is provided to illustrate the abstract theory.

1. Introduction

Many social, physical, biological, and engineering problems can be described by fractional partial differential equations. In fact, fractional differential equations are considered as an alternative model to nonlinear differential equations. In the last two decades, fractional differential equations (see, e.g., [1–4] and the references therein) have attracted many scientists, and notable contributions have been made to both the theory and applications of fractional differential equations. Several researchers have studied the existence results of initial and boundary value problems involving fractional differential equations. The motivation for those works arises from both the development of the theory of fractional calculus itself and the applications of such constructions in various fields, including physics, chemistry, aerodynamics, and electrodynamics of complex medium. Recently, Zhou and Jiao [5] discussed the existence of mild solutions of fractional evolution and neutral evolution equations in an arbitrary Banach space in which the mild solution is defined using the probability density function and semigroup theory. Using the same method, Zhou et al. [6] gave a suitable definition of a mild solution for an evolution equation involving a Riemann-Liouville fractional derivative. Using sectorial operators, Shu and Wang [7] gave a definition of a mild solution for fractional differential equations with order $1 < \alpha < 2$ and established existence results. Agarwal et al. [8] studied the existence and dimension of the set of mild solutions of semilinear fractional differential equations inclusions. Hilfer [9] proposed a generalized Riemann-Liouville fractional derivative, for short, which includes Riemann-Liouville fractional derivative and Caputo fractional derivative. Very recently, Gu and Trujillo [10] investigated a class of evolution equations involving Hilfer fractional derivatives.

Recently, the approximate controllability of fractional semilinear evolution systems in abstract spaces has been studied by many researchers. In [11], Sakthivel et al. studied the approximate controllability of semilinear fractional differential systems. Kumar and Sukavanam [12, 13] obtained a new set of sufficient conditions for the approximate controllability of a class of semilinear delay control systems of fractional order by using the contraction principle and the Schauder fixed point theorem. Balasubramaniam et al. [14] derived sufficient conditions for the approximate controllability of impulsive fractional integrodifferential systems with nonlocal conditions in Hilbert space. Using the analytic resolvent method and the continuity of a resolvent in the uniform operator topology, Fan [15] derived existence and approximate controllability results of a fractional control system. Liu and Bin [16] studied existence of mild solutions and approximate controllability results for impulsive fractional abstract Cauchy problems involving Riemann-Liouville fractional derivatives. More recently, Mahmudov [17] formulated and proved a new set of sufficient conditions for the approximate controllability of fractional neutral type evolution equations in Banach spaces by using Schauder's fixed point theorem. However, the approximate controllability of fractional evolution equations with Hilfer fractional derivative has not yet been studied.

Motivated by the aforementioned papers, we study the approximate controllability of a class of fractional evolution equations:

$$\begin{gathered} D_{0^+}^{\nu,\mu} x(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in (0, b], \\ I_{0^+}^{(1-\nu)(1-\mu)} x(0) = x_0, \end{gathered} \tag{1}$$

where $D_{0^+}^{\nu,\mu}$ is the Hilfer fractional derivative, $0 \le \nu \le 1$, $0 < \mu < 1$, the state $x(\cdot)$ takes value in a Hilbert space X, and A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t),\ t > 0\}$ in X. The control function u takes values in a Hilbert space U, $u \in L^2([0, b], U)$, and $B : U \to X$ is a linear bounded operator. The function $f : [0, b] \times X \to X$ will be specified in later sections.

The focus of this paper is the study of the approximate controllability of fractional semilinear differential equations in Hilbert spaces. We will explore approximate controllability using techniques from [18]. The method is inspired by viewing the problem of approximate controllability as the limit of optimal control problems and replacing it via the convergence of resolvent operators (the resolvent condition, (R)).

2. Preliminaries

Define

$$\begin{aligned} &C^{\nu,\mu}([0, b], X) \\ &\quad = \Big\{x \in C((0, b], X) : \\ &\qquad \lim_{t\to 0^+} t^{(1-\lambda)(1-\mu)} x(t) \text{ exists and is finite}\Big\} \end{aligned} \tag{2}$$

with the norm $\|\cdot\|_{\nu,\mu}$ defined by

$$\|x\|_{\nu,\mu} = \sup_{0\le t\le b} \left| t^{(1-\lambda)(1-\mu)} x(t) \right|. \tag{3}$$

Obviously, $C^{\nu,\mu}([0, b], X)$ is a Banach space.

Let us recall the following definitions from fractional calculus.

Definition 1 (see [1]). The fractional integral of order $\alpha > 0$ with the lower limit a for a function $f : [a,\infty) \to \mathbb{R}$ is defined as

$$I_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,\ \alpha > 0, \tag{4}$$

provided that the right-hand side is pointwise defined on $[0,\infty)$, where Γ is the gamma function.

Definition 2 (see [9]). The Hilfer derivative of order $0 \le \nu \le 1$ and $0 < \mu < 1$ with lower limit a is defined as

$$D_{a^+}^{\nu,\mu} f(t) = I_{0^+}^{\nu(1-\mu)} \frac{d}{dt} I_{a^+}^{(1-\nu)(1-\mu)} f(t), \tag{5}$$

for functions such that the expression on the right-hand side exists.

Remark 3. When $\nu = 0$, $0 < \mu < 1$, the Hilfer fractional derivative coincides with the classical Riemann-Liouville fractional derivative:

$$D_{a^+}^{0,\mu} f(t) = \frac{d}{dt} I_{a^+}^{1-\mu} f(t) = {}^{L}D_{a^+}^{\mu} f(t). \tag{6}$$

When $\nu = 1$, $0 < \mu < 1$, the Hilfer fractional derivative coincides with the classical Caputo fractional derivative:

$$D_{a^+}^{1,\mu} f(t) = I_{a^+}^{1-\mu} \frac{d}{dt} f(t) = {}^{C}D_{a^+}^{\mu} f(t). \tag{7}$$

For $x \in X$, we define two families of operators $\{\mathcal{S}_{\nu,\mu}(t) : t \ge 0\}$ and $\{\mathcal{P}_{\mu}(t) : t \ge 0\}$ by

$$\begin{gathered} \mathcal{S}_{\nu,\mu}(t) = I_{0^+}^{\nu(1-\mu)} \mathcal{P}_{\mu}(t), \qquad \mathcal{P}_{\mu}(t) = t^{\mu-1} \mathcal{T}_{\mu}(t), \\ \mathcal{T}_{\mu}(t) = \int_0^\infty \mu\theta \Psi_{\mu}(\theta) S(t^{\mu}\theta)\, d\theta, \end{gathered} \tag{8}$$

where

$$\Psi_{\mu}(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-n\mu)} \sin(n\pi\alpha), \quad \theta \in (0,\infty), \tag{9}$$

is a function of Wright-type defined on $(0,\infty)$ which satisfies

$$\begin{gathered} \Psi_{\alpha}(\theta) \ge 0, \qquad \int_0^\infty \Psi_{\alpha}(\theta)\, d\theta = 1, \\ \int_0^\infty \theta^{\zeta} \Psi_{\mu}(\theta)\, d\theta = \frac{\Gamma(1+\zeta)}{\Gamma(1+\mu\zeta)}, \quad \zeta \in (-1,\infty). \end{gathered} \tag{10}$$

Lemma 4 (see [10]). *The operators $\mathcal{S}_{\nu,\mu}$ and $\mathcal{P}_{\mu}$ have the following properties.*

(i) *For any fixed $t > 0$, $\mathcal{S}_{\nu,\mu}(t)$ and $\mathcal{P}_{\mu}(t)$ are linear and bounded operators, and*

$$\begin{gathered} \left\|\mathcal{P}_{\mu}(t) x\right\| \le \frac{M t^{\mu-1}}{\Gamma(\mu)} \|x\|, \\ \left\|\mathcal{S}_{\nu,\mu}(t) x\right\| \le \frac{M t^{(\nu-1)(1-\mu)}}{\Gamma(\nu(1-\mu)+\mu)} \|x\|. \end{gathered} \tag{11}$$

(ii) *$\{\mathcal{P}_{\mu}(t) : t > 0\}$ is compact, if $\{S(t) : t > 0\}$ is compact.*

In this paper we adopt the following definition of mild solution of the initial-value problem (1); see [10].

Definition 5. A solution $x(\cdot;u) \in C([0,b],X)$ is said to be a mild solution of (1) if for any $u \in L^2([0,b],U)$ the integral equation

$$x(t) = \mathcal{S}_{\nu,\mu}(t)\,x_0 + \int_0^t \mathcal{P}_\mu(t-s)\left[Bu(s) + f(s,x(s))\right]ds \tag{12}$$

is satisfied, for all $0 \le t \le b$.

Let $x(b;u)$ be the state value of (12) at the terminal time b corresponding to the control u. We introduce the set $\mathfrak{R}(b) = \{x(b;u) : u \in L_2([0,b],U)\}$, which is called the reachable set of system (12) at terminal time T, and denote its closure in X by $\overline{\mathfrak{R}(b)}$.

Definition 6. The system (1) is said to be approximately controllable on $[0,b]$ if $\overline{\mathfrak{R}(b)} = X$; that is, given an arbitrary $\varepsilon > 0$ it is possible to steer from the point x_0 to within a distance ε from all points in the state space X at time b.

Remark 7. (i) When $\nu = 0$, the fractional equation (12) simplifies to the classical Riemann-Liouville fractional equation which has been studied by Zhou et al. in [6]. In this case

$$\mathcal{S}_{0,\mu}(t) = \mathcal{P}_\mu(t) = t^{\mu-1}\mathcal{T}_\mu(t), \quad 0 < t \le b. \tag{13}$$

(ii) When $\nu = 1$, the fractional equation (12) simplifies to the classical Caputo fractional equation which has been studied by Zhou and Jiao in [5]. In this case

$$\mathcal{S}_{1,\mu}(t) = \mathcal{S}_\mu(t), \quad 0 \le t \le b, \tag{14}$$

where $\mathcal{S}_\mu(t)$ is defined in [5].

3. Main Results

To investigate the approximate controllability of system (12), we impose the following conditions:

(H1) $S(t)$, $t > 0$, is compact;

(H2) the function $f : [0,b]\times X \to X$ satisfies the following:

- (a) $f(t,\cdot) : X \to X$ is continuous for each $t \in (0,b]$,
- (b) for each $x \in X$, $f(\cdot,x) : (0,b] \to X$ is strongly measurable;

(H3) there is a constant $\mu_1 \in (0,\mu)$ and $n \in L^{1/\mu_1}([0,b],\mathbb{R}^+)$ such that, for every $x \in X$ and almost all $t \in [0,b]$, we have

$$\|f(t,x)\| \le n(t). \tag{15}$$

Consider the following linear fractional differential system:

$$\begin{gathered} D_{0^+}^{\nu,\mu}x(t) = Ax(t) + Bu(t), \quad t \in (0,b], \\ I_{0^+}^{(1-\nu)(1-\mu)}x(0) = x_0. \end{gathered} \tag{16}$$

The approximate controllability for the linear fractional system (16) is a natural generalization of approximate controllability of linear first order control system. It is convenient at this point to introduce the following controllability and resolvent operators associated with (16):

$$\begin{gathered} L_0^b = \int_0^b \mathcal{P}_\mu(b-s)\,Bu(s)\,ds, \\ \Gamma_0^b = \int_0^b \mathcal{P}_\mu(b-s)\,BB^*\mathcal{P}_\mu^*(b-s)\,ds, \\ R\left(\varepsilon,\Gamma_0^b\right) = \varepsilon\left(\varepsilon I + \Gamma_0^b\right)^{-1}, \end{gathered} \tag{17}$$

respectively, where B^* denotes the adjoint of B and $\mathcal{P}_\mu^*(t)$ is the adjoint of $\mathcal{P}_\mu(t)$. It is straightforward to show that the operator L_0^b is a linear bounded operator for $1/2 < \mu \le 1$.

We also impose the following resolvent condition:

(R) for every $h \in X$, $\varepsilon(\varepsilon I + \Gamma_0^b)^{-1}(h)$ converges to zero as $\varepsilon \to 0^+$ in strong topology.

Remark 8. The assumption (R) is equivalent to the approximate controllability of the linear system (16); see [19, 20].

In order to formulate the controllability problem in the form in which the fixed point theorem is readily applicable, it is assumed that the corresponding linear system is approximately controllable. It will be shown that system (1) is approximately controllable provided that we can show for all $\varepsilon > 0$ there exists a continuous function $x \in C([0,b],X)$ such that

$$\begin{gathered} u_\varepsilon(t,x) = B^*\mathcal{P}_\mu^*(b-t)\left(\varepsilon I + \Gamma_0^b\right)^{-1}p(x), \\ x(t) = \mathcal{S}_{\nu,\mu}(t)\,x_0 + \int_0^t \mathcal{P}_\mu(t-s)\left[Bu(s) + f(s,x(s))\right]ds, \end{gathered} \tag{18}$$

where

$$p(x) = h - \mathcal{S}_{\nu,\mu}(b)\,x_0 - \int_0^b \mathcal{P}_\mu(b-s)\,f(s,x(s))\,ds. \tag{19}$$

Based on this observation, our goal is to find conditions for the solvability of (18). Note also that it will be shown that the control in (18) drives the system (1) from x_0 to

$$h - \varepsilon\left(\varepsilon I + \Gamma_0^b\right)^{-1}p(x) \tag{20}$$

provided that the system (18) has a solution.

For all $\varepsilon > 0$, consider the operator $\Phi_\varepsilon : C^{\nu,\mu}([0,b],X) \to C^{\nu,\mu}([0,b],X)$ defined as follows:

$$\begin{aligned} \left(\Phi_\varepsilon x\right)(t) :=& \ \mathcal{S}_{\nu,\mu}(t)\,x_0 \\ &+ \int_0^t \mathcal{P}_\mu(t-s)\left[Bu_\varepsilon(s,x) + f(s,x(s))\right]ds. \end{aligned} \tag{21}$$

Let $x \in C^{\nu,\mu}([0,b],X)$. Observe that

$$\begin{aligned}
&\lim_{t\to 0^+} t^{(1-\nu)(1-\mu)} \mathcal{S}_{\nu,\mu}(t)\, x_0 \\
&= \lim_{t\to 0^+} \frac{t^{(1-\nu)(1-\mu)}}{\Gamma(\nu(1-\mu))} \int_0^t (t-s)^{\nu(1-\mu)-1} s^{\mu-1} \mathcal{P}_\mu(s)\, x_0 ds \\
&= \lim_{t\to 0^+} \frac{1}{\Gamma(\nu(1-\mu))} \int_0^1 (1-s)^{\nu(1-\mu)-1} s^{\mu-1} \mathcal{P}_\mu(ts)\, x_0 ds \\
&= \frac{x_0}{\Gamma(\nu(1-\mu)+\mu)}.
\end{aligned} \tag{22}$$

Define $t^{(1-\nu)(1-\mu)}(\Phi_\varepsilon x)(t)$ as follows:

$$t^{(1-\nu)(1-\mu)}(\Phi_\varepsilon x)(t) := \begin{cases} t^{(1-\nu)(1-\mu)} \mathcal{S}_{\nu,\mu}(t)\, x_0 \\ \quad + t^{(1-\nu)(1-\mu)} \int_0^t \mathcal{P}_\mu(t-s) \\ \qquad \times [Bu_\varepsilon(s,x) + f(s,x(s))]\, ds, \\ \qquad 0 < t \le b, \\ \dfrac{x_0}{\Gamma(\nu(1-\mu)+\mu)}, \\ \qquad t = 0. \end{cases} \tag{23}$$

It will be shown that for all $\varepsilon > 0$ the operator $\Phi_\varepsilon : C^{\nu,\mu}([0,b],X) \to C^{\nu,\mu}([0,b],X)$ has a fixed point. To prove this we will employ the Schauder fixed point theorem.

Lemma 9. *Let $0 \le \nu \le 1$ and $1/2 < \mu \le 1$. If assumptions (H1)–(H3) hold, then for any $\varepsilon > 0$ the control function $u_\varepsilon(t,x)$ has the following properties:*

(i) $\|u_\varepsilon(t,x)\| \le (M_B M (b-t)^{\mu-1}/\varepsilon\Gamma(\mu))(\|h\| + (Mb^{(\nu-1)(1-\mu)}/\Gamma(\nu(1-\mu)+\mu))\|x_0\| + (M/\Gamma(\mu))((1-\mu_1)^{1-\mu_1} b^{\mu-\mu_1}/(\mu-\mu_1)^{1-\mu_1})\|n\|_{1/\mu_1})$,

(ii) *for any $t \in [0,b]$ we have $\lim_{n\to\infty}\|u_\varepsilon(t,x_n) - u_\varepsilon(t,x)\| = 0$, where $M_B = \|B\|$, $\|n\|_{1/\mu_1}$ is L^{1/μ_1} norm of n.*

Proof. (i) By the definition of $u_\varepsilon(t,x)$ we have

$$\begin{aligned}
&\|u_\varepsilon(t,x)\| \\
&\le \left\| B^* \mathcal{P}_\mu^*(b-t) \left(\varepsilon I + \Gamma_0^T\right)^{-1} p(x) \right\| \\
&\le \frac{M_B M (b-t)^{\mu-1}}{\Gamma(\mu)} \left\| \left(\varepsilon I + \Gamma_0^T\right)^{-1} p(x) \right\| \\
&\le \frac{M_B M (b-t)^{\mu-1}}{\varepsilon\Gamma(\mu)} \|p(x)\| \\
&\le \frac{M_B M (b-t)^{\mu-1}}{\varepsilon\Gamma(\mu)} \\
&\quad \cdot \left(\|h\| + \left\|\mathcal{S}_{\nu,\mu}(b)\, x_0\right\| + \left\| \int_0^b \mathcal{P}_\mu(b-s) f(s,x(s))\, ds \right\| \right).
\end{aligned} \tag{24}$$

Using the Hölder inequality and (H3) yields

$$\begin{aligned}
&\|u_\varepsilon(t,x)\| \\
&\le \frac{M_B M (b-t)^{\mu-1}}{\varepsilon\Gamma(\mu)} \cdot \left(\|h\| + \frac{Mb^{(\nu-1)(1-\mu)}}{\Gamma(\nu(1-\mu)+\mu)} \|x_0\| + \frac{M}{\Gamma(\mu)} \int_0^b (b-s)^{\mu-1} n(s)\, ds \right) \\
&\le \frac{M_B M (b-t)^{\mu-1}}{\varepsilon\Gamma(\mu)} \times \left(\|h\| + \frac{Mb^{(\nu-1)(1-\mu)}}{\Gamma(\nu(1-\mu)+\mu)} \|x_0\| + \frac{M}{\Gamma(\mu)} \left(\int_0^t (t-s)^{(\mu-1)/(1-\mu_1)} ds \right)^{1-\mu_1} \cdot \left(\int_0^t n^{1/\mu_1}(s)\, ds \right)^{\mu_1} \right) \\
&\le \frac{M_B M (b-t)^{\mu-1}}{\varepsilon\Gamma(\mu)} \cdot \left(\|h\| + \frac{Mb^{(\nu-1)(1-\mu)}}{\Gamma(\nu(1-\mu)+\mu)} \|x_0\| + \frac{M}{\Gamma(\mu)} \frac{(1-\mu_1)^{1-\mu_1} b^{\mu-\mu_1}}{(\mu-\mu_1)^{1-\mu_1}} \|n\|_{1/\mu_1} \right).
\end{aligned} \tag{25}$$

(ii) Assume that $\lim_{n\to\infty}\|x_n - x\|_{\nu,\mu} = 0$. Then we have

$$\lim_{n\to\infty} x_n(s) = x(s), \quad 0 < s \le b. \tag{26}$$

From (H2), it follows that

$$\lim_{n\to\infty} f(s, x_n(s)) = f(s, x(s)) \quad \text{a.e. in } [0,b]. \tag{27}$$

Using (H3), we get

$$(b-s)^{\mu-1} \left\| f(s,x_n(s)) - f(s,x(s)) \right\| \le 2(b-s)^{\mu-1} n(s), \quad \text{a.e. in } [0,b]. \tag{28}$$

Since $s \longrightarrow 2(b-s)^{\mu-1}n(s)$ is integrable on $[0,b]$, by the Lebesgue dominated convergence theorem, we have

$$\int_0^b (b-s)^{\mu-1}\left\|f(s,x_n(s)) - f(s,x(s))\right\| ds \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{29}$$

Further, it follows that

$$\begin{aligned} &\left\|p(x_n) - p(x)\right\| \\ &\quad \le \frac{M}{\Gamma(\mu)} \int_0^b (b-s)^{\mu-1}\left\|f(s,x_n(s)) - f(s,x(s))\right\| ds \longrightarrow 0, \\ &\qquad \text{as } n \longrightarrow \infty. \end{aligned} \tag{30}$$

Thus

$$\begin{aligned} &\left\|u_\varepsilon(t,x_n) - u_\varepsilon(t,x)\right\| \\ &\quad = \left\|B^*\mathscr{P}_\mu^*(b-t)\left(\varepsilon I + \Gamma_0^b\right)^{-1}\left(p(x_n) - p(x)\right)\right\| \\ &\quad \le \frac{M_B M}{\varepsilon\Gamma(\mu)}\left\|p(x_n) - p(x)\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{31}$$

□

Lemma 10. *Let $0 \le \nu \le 1$ and $1/2 < \mu \le 1$. Under assumptions (H1)–(H3), for any $\varepsilon > 0$ there exists a positive number $r := r(\varepsilon)$ such that $\Phi_\varepsilon(B_r) \subset B_r$, where*

$$B_r := \left\{x \in C^{\nu,\mu}([0,b],X) : \|x\|_{\nu,\mu} \le r\right\}. \tag{32}$$

Proof. Let $\varepsilon > 0$ be fixed and $x \in B_r$. Since $x(t)$ is continuous, it follows from (H2) that $f(t,x(t))$ is a measurable function on $[0,b]$. Using the Hölder inequality and (H3) yields

$$\begin{aligned} &\left\|t^{(1-\nu)(1-\mu)}(\Phi_\varepsilon x)(t)\right\| \\ &\quad \le \left\|t^{(1-\nu)(1-\mu)}\mathcal{S}_{\nu,\mu}(t)x_0\right\| \\ &\qquad + \left\|t^{(1-\nu)(1-\mu)}\int_0^t \mathscr{P}_\mu(t-s) f(s,x(s))\, ds\right\| \\ &\qquad + \left\|t^{(1-\nu)(1-\mu)}\int_0^t \mathscr{P}_\mu(t-s) B u_\varepsilon(s,x)\, ds\right\| \\ &\quad =: I_1 + I_2 + I_3. \end{aligned} \tag{33}$$

We estimate each of I_i, $i = 1,2,3$, separately. By the assumption (H3), we have

$$\begin{aligned} I_1 &\le \left\|t^{(1-\nu)(1-\mu)}\mathcal{S}_{\nu,\mu}(t)x_0\right\| \le \frac{M}{\Gamma(\nu(1-\mu)+\mu)}\left\|x_0\right\|, \\ I_2 &\le t^{(1-\nu)(1-\mu)}\int_0^t \left\|\mathscr{P}_\mu(t-s) f(s,x(s))\right\| ds \\ &\le \frac{Mt^{(1-\nu)(1-\mu)}}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\left\|f(s,x(s))\right\| ds \\ &\le \frac{Mt^{(1-\nu)(1-\mu)}}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1} n(s)\, ds \\ &\le \frac{Mt^{(1-\nu)(1-\mu)}}{\Gamma(\mu)}\left(\int_0^t (t-s)^{(\mu-1)/(1-\mu_1)} ds\right)^{1-\mu_1} \\ &\quad \cdot \left(\int_0^t n^{1/\mu_1}(s)\, ds\right)^{\mu_1} \\ &\le \frac{Mb^{(1-\nu)(1-\mu)}}{\Gamma(\mu)}\frac{\left(1-\mu_1\right)^{1-\mu_1} b^{\mu-\mu_1}}{\left(\mu-\mu_1\right)^{1-\mu_1}}\|n\|_{1/\mu_1}. \end{aligned} \tag{34}$$

Combining the estimates (33)-(34) yields

$$\begin{aligned} I_1 + I_2 &< \frac{M}{\Gamma(\nu(1-\mu)+\mu)}\left\|x_0\right\| \\ &\quad + \frac{Mb^{(1-\nu)(1-\mu)}}{\Gamma(\mu)}\frac{\left(1-\mu_1\right)^{1-\mu_1} b^{\mu-\mu_1}}{\left(\mu-\mu_1\right)^{1-\mu_1}}\|n\|_{1/\mu_1} := \Delta. \end{aligned} \tag{35}$$

Next, observe that

$$\begin{aligned} I_3 &\le t^{(1-\nu)(1-\mu)}\int_0^t \left\|\mathscr{P}_\mu(t-s) B u_\varepsilon(s,x)\right\| ds \\ &= t^{(1-\nu)(1-\mu)}\int_0^t \Big\|\mathscr{P}_\mu(t-s) BB^*\mathscr{P}_\mu^*(b-s) \\ &\qquad \cdot \left(\varepsilon I + \Gamma_0^T\right)^{-1} p(x)\Big\| ds \\ &\le t^{(1-\nu)(1-\mu)}\int_0^t \left\|\mathscr{P}_\mu(t-s) BB^*\mathscr{P}_\mu^*(b-s)\right\| ds \\ &\quad \cdot \left\|\left(\varepsilon I + \Gamma_0^T\right)^{-1} p(x)\right\| \\ &\le \frac{M_B^2 M^2 t^{(1-\nu)(1-\mu)}}{\Gamma^2(\mu)} \\ &\quad \cdot \int_0^t (t-s)^{\mu-1}(b-s)^{\mu-1} ds \left\|\left(\varepsilon I + \Gamma_0^T\right)^{-1} p(x)\right\| \\ &= \frac{M_B^2 M^2 t^{(1-\nu)(1-\mu)}}{\Gamma^2(\mu)}\frac{b^{2\mu-1}}{\mu}\left\|\left(\varepsilon I + \Gamma_0^T\right)^{-1} p(x)\right\| \end{aligned}$$

$$\leq \frac{1}{\varepsilon}\frac{M_B^2 M^2 t^{(1-\nu)(1-\mu)}}{\Gamma^2(\mu)}\frac{b^{2\mu-1}}{\mu}\|p(x)\|$$
$$\leq \frac{1}{\varepsilon}\frac{M_B^2 M^2}{\Gamma^2(\mu)}\frac{b^{2\mu-1}}{\mu}\left(b^{(1-\nu)(1-\mu)}\|h\|+\Delta\right). \tag{36}$$

Thus,

$$\left\|t^{(1-\nu)(1-\mu)}(\Phi_\varepsilon x)(t)\right\| \leq \Delta + \frac{1}{\varepsilon}\frac{M_B^2 M^2}{\Gamma^2(\mu)}\frac{b^{2\mu-1}}{\mu}\left(b^{(1-\nu)(1-\mu)}\|h\|+\Delta\right). \tag{37}$$

The last two inequalities imply that for large enough $r > 0$ the following inequality holds:

$$\left\|t^{(1-\nu)(1-\mu)}(\Phi_\varepsilon z_r)(t)\right\| \leq r. \tag{38}$$

Therefore, Φ_ε maps B_r into itself. □

Lemma 11. *Let $0 \leq \nu \leq 1$ and $1/2 < \mu \leq 1$. If assumptions (H1)–(H3) hold, then the set $\{\Phi_\varepsilon x : x \in B_r\}$ is an equicontinuous family of functions on $[0,b]$.*

Proof. For $0 < t < t+h \leq b$, we have

$$\begin{aligned}
&\left\|(t+h)^{(1-\nu)(1-\mu)}(\Phi_\varepsilon z)(t+h) - t^{(1-\nu)(1-\mu)}(\Phi_\varepsilon z)(t)\right\| \\
&\leq \left\|(t+h)^{(1-\nu)(1-\mu)}\mathcal{S}_{\nu,\mu}(t+h)x_0 - t^{(1-\nu)(1-\mu)}\mathcal{S}_{\nu,\mu}(t)x_0\right\| \\
&+ \Big\|\int_t^{t+h}(t+h-s)^{(1-\nu)(1-\mu)}(t+h-s)^{\mu-1} \cdot \mathcal{T}_\mu(t+h-s)\left[Bu_\varepsilon(s,z)+f(s,z(s))\right]ds\Big\| \\
&+ \Big\|\int_0^t\left((t+h-s)^{(1-\nu)(1-\mu)}(t+h-s)^{\mu-1} - (t-s)^{(1-\nu)(1-\mu)}(t-s)^{\mu-1}\right) \times \mathcal{T}_\mu(t+h-s)\left[Bu_\varepsilon(s,z)+f(s,z(s))\right]ds\Big\| \\
&+ \Big\|\int_0^t(t-s)^{(1-\nu)(1-\mu)}(t-s)^{\mu-1} \cdot \left(\mathcal{T}_\mu(t+h-s)-\mathcal{T}_\mu(t-s)\right) \cdot \left[Bu_\varepsilon(s,z)+f(s,z(s))\right]ds\Big\| \\
&\leq I_4+I_5+I_6+I_7.
\end{aligned} \tag{39}$$

For $0 < t < t+h \leq b$, we have

$$I_4 \leq \left\|(t+h-s)^{(1-\nu)(1-\mu)}\mathcal{S}_{\nu,\mu}(t+h) - (t-s)^{(1-\nu)(1-\mu)}\mathcal{S}_{\nu,\mu}(t)\right\|\|x_0\|. \tag{40}$$

By Lemma 4, we know that $t^{(1-\nu)(1-\mu)}\mathcal{S}_{\nu,\mu}(t)$ is uniformly continuous on $[0,b]$, which enables us to deduce that $\lim_{h\to 0^+} I_4 = 0$.

By condition (H3), we deduce that $\lim_{h\to 0^+} I_5 = 0$.

Noting that

$$\begin{aligned}
&\Big|(t+h-s)^{(1-\nu)(1-\mu)}(t+h-s)^{\mu-1} - (t-s)^{(1-\nu)(1-\mu)}(t-s)^{\mu-1}\Big| m(s) \\
&\leq (t-s)^{(1-\nu)(1-\mu)}(t-s)^{\mu-1}m(s),
\end{aligned} \tag{41}$$

and $\int_0^t (t-s)^{(1-\nu)(1-\mu)}(t-s)^{\mu-1}m(s)ds$ exists, it follows from the Lebesgue dominated convergence theorem that

$$\int_0^t \Big|(t+h-s)^{(1-\nu)(1-\mu)}(t+h-s)^{\mu-1} - (t-s)^{(1-\nu)(1-\mu)}(t-s)^{\mu-1}\Big| m(s) \longrightarrow 0, \tag{42}$$

as $h \to 0^+$. It follows that $\lim_{h\to 0^+} I_6 = 0$.

For $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned}
I_7 &\leq \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)(t-s)^{(1-\nu)(1-\mu)}(t-s)^{\mu-1} \\
&\cdot \left|\mathcal{T}_\mu(t+h-s)-\mathcal{T}_\mu(t-s)\right| \cdot \left|Bu_\varepsilon(s,z)+f(s,z(s))\right| ds.
\end{aligned} \tag{43}$$

Since the compactness of $\mathcal{T}_\mu(t)$ $(t > 0)$ implies the continuity of $\mathcal{T}_\mu(t)$ $(t > 0)$ in the uniform operator topology, it can be easily seen that $\lim_{h\to 0^+} I_7 = 0$.

The case $t = 0$ and $0 < h \leq b$ follows from (23).

Thus, the set $\{\Phi_\varepsilon x : x \in B_r\}$ is an equicontinuous family of functions in $C^{\nu,\mu}([0,b],X)$. □

Lemma 12. *Let $0 \leq \nu \leq 1$ and $1/2 < \mu \leq 1$. Let assumptions (H1)–(H3) hold. For any $t \in [0,b]$ the set $V(t) := \{(\Phi_\varepsilon x)(t) : x \in B_r\}$ is relatively compact in X.*

Proof. Let $0 < t \leq b$ be fixed and let λ be a real number satisfying $0 < \lambda < t$. For $\delta > 0$ define the operator $\Phi_\varepsilon^{\lambda,\delta}$ on B_r by

$$\begin{aligned}
\left(\Phi_\varepsilon^{\lambda,\delta}x\right)(t) &:= \frac{1}{\Gamma(\nu(1-\mu))}S(\lambda^\mu\delta) \cdot \int_\lambda^t \frac{s^{\mu-1}}{(t-s)^{1-\nu(1-\mu)}} \cdot \int_\delta^\infty \mu\theta\Psi_\mu(\theta)S(s^\mu\theta-\lambda^\mu\delta)\,d\theta\,ds x_0 \\
&+ \mu S(\lambda^\mu\delta)\int_0^{t-\lambda}\int_\delta^\infty \theta(t-s)^{\mu-1}\Psi_\mu(\theta) \cdot S\left((t-s)^\mu\theta-\lambda^\mu\delta\right)d\theta \cdot \left[Bu_\varepsilon(s,z)+f(s,z(s))\right]ds.
\end{aligned} \tag{44}$$

Since $S(t)$ is a compact operator, the set $\{(\Phi_{\varepsilon}^{\lambda,\delta}x)(t) : x \in B_r\}$ is relatively compact in X. Moreover, for each $x \in B_r$, we have

$$\begin{aligned}
&\left\| (\Phi_{\varepsilon}x)(t) - \left(\Phi_{\varepsilon}^{\lambda,\delta}x\right)(t) \right\| \\
&\quad \leq \frac{1}{\Gamma(\nu(1-\mu))} \\
&\qquad \cdot \left\| \int_0^t \frac{s^{\mu-1}}{(t-s)^{1-\nu(1-\mu)}} \int_0^{\delta} \mu\theta\Psi_{\mu}(\theta) S(s^{\mu}\theta)\, d\theta\, ds x_0 \right\| \\
&\qquad + \frac{1}{\Gamma(\nu(1-\mu))} \\
&\qquad \cdot \left\| \int_0^{\lambda} \frac{s^{\mu-1}}{(t-s)^{1-\nu(1-\mu)}} \int_{\delta}^{\infty} \mu\theta\Psi_{\mu}(\theta) S(s^{\mu}\theta)\, d\theta\, ds x_0 \right\| \\
&\qquad + \mu \left\| \int_0^t \int_0^{\delta} \theta (t-s)^{\mu-1} \Psi_{\mu}(\theta) S((t-s)^{\mu}\theta) \right. \\
&\qquad\qquad \left. \cdot \left[Bu_{\varepsilon}(s,x) + f(s,x(s)) \right] d\theta\, ds \right\| \\
&\qquad + \mu \left\| \int_{t-\lambda}^t \int_{\delta}^{\infty} \theta (t-s)^{\mu-1} \Psi_{\mu}(\theta) S((t-s)^{\mu}\theta) \right. \\
&\qquad\qquad \left. \cdot \left[Bu_{\varepsilon}(s,x) + f(s,x(s)) \right] d\theta\, ds \right\| \\
&\quad =: I_8 + I_9 + I_{10} + I_{11}.
\end{aligned} \tag{45}$$

A similar argument as before yields

$$\begin{aligned}
I_8 &\leq \frac{\mu M}{\Gamma(\nu(1-\mu))} \int_0^t \frac{s^{\mu-1}}{(t-s)^{1-\nu(1-\mu)}} ds \left(\int_0^{\delta} \theta\Psi_{\mu}(\theta)\, d\theta \right) \|x_0\| \\
&\leq \frac{\mu M}{\Gamma(\nu(1-\mu))} \frac{1}{t^{(1-\nu)(1-\mu)}} \\
&\quad \cdot \int_0^1 (1-s)^{\nu(1-\mu)-1} s^{\mu-1} ds \left(\int_0^{\delta} \theta\Psi_{\mu}(\theta)\, d\theta \right) \|x_0\| \\
&\leq \frac{\mu M}{\Gamma(\nu(1-\mu))} \frac{1}{t^{(1-\nu)(1-\mu)}} \\
&\quad \cdot B(\nu(1-\mu), \mu) \left(\int_0^{\delta} \theta\Psi_{\mu}(\theta)\, d\theta \right) \|x_0\|, \\
I_9 &\leq \frac{\mu M}{\Gamma(\nu(1-\mu))} \\
&\quad \cdot \int_0^{\lambda} \frac{s^{\mu-1}}{(t-s)^{1-\nu(1-\mu)}} ds \left(\int_{\delta}^{\infty} \theta\Psi_{\mu}(\theta)\, d\theta \right) \|x_0\| \\
&\leq \frac{\mu M b^{\nu(1-\mu)-1}}{\Gamma(\nu(1-\mu))\Gamma(1+\mu)} \frac{\lambda^{\mu}}{\mu} \left(\int_{\delta}^{\infty} \theta\Psi_{\mu}(\theta)\, d\theta \right) \|x_0\|,
\end{aligned} \tag{46}$$

where we have used the equality

$$\int_0^{\infty} \theta_{\mu}^{\beta} \Psi_{\mu}(\theta)\, d\theta = \frac{\Gamma(1+\beta)}{\Gamma(1+\mu\beta)}. \tag{47}$$

From (46), it follows that

$$I_8 \longrightarrow 0, \quad I_9 \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0^+,\ \delta \longrightarrow 0^+. \tag{48}$$

Similarly,

$$I_{10} \longrightarrow 0, \quad I_{11} \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0^+,\ \delta \longrightarrow 0^+. \tag{49}$$

Consequently, for each $x \in B_r$,

$$\left\| (\Phi_{\varepsilon}x)(t) - \left(\Phi_{\varepsilon}^{\lambda,\delta}x\right)(t) \right\| \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0^+,\ \delta \longrightarrow 0^+. \tag{50}$$

Therefore, there exist relatively compact sets arbitrarily close to the set $\{(\Phi_{\varepsilon}x)(t) : x \in B_r\}$. Hence, the set $\{(\Phi_{\varepsilon}x)(t) : x \in B_r\}$ is relatively compact in X. □

Lemma 13. *Let $0 \leq \nu \leq 1$ and $1/2 < \mu \leq 1$. If assumptions (H1)–(H3) hold, then the operator $\Phi_{\varepsilon} : C^{\nu,\mu}([0,b],X) \to C^{\nu,\mu}([0,b],X)$ is continuous on B_r.*

Proof. Observe that, for all $t \in [0,b]$, $x_n, x \in B_r$, we have

$$\begin{aligned}
&\left\| t^{(1-\nu)(1-\mu)} (\Phi_{\varepsilon}x_n)(t) - t^{(1-\nu)(1-\mu)} (\Phi_{\varepsilon}x)(t) \right\| \\
&\quad \leq \frac{M t^{(1-\nu)(1-\mu)}}{\Gamma(\mu)} \\
&\qquad \cdot \int_0^t (t-s)^{\mu-1} \\
&\qquad\quad \cdot \left(\left\| f(s,x_n(s)) - f(s,x(s)) \right\| \right. \\
&\qquad\qquad \left. + M_b \left\| u_{\varepsilon}(s,x_n) - u_{\varepsilon}(s,x) \right\| \right) ds.
\end{aligned} \tag{51}$$

The rest of the proof is similar to the proof of Lemma 9. □

Theorem 14. *If assumptions (H1)–(H3) hold and $1/2 < \mu \leq 1$, then there exists a solution to (18).*

Proof. According to infinite-dimensional version of the Ascoli-Arzela theorem if (i) for $t \in [0,b]$, the set $V(t) := \{(\Phi_{\varepsilon}x)(t) : x \in B_r\}$ is relatively compact in X; (ii) family $\{\Phi_{\varepsilon}x : x \in B_r\}$ is uniformly bounded and equicontinuous, and then $\{\Phi_{\varepsilon}x : x \in B_r\}$ is relatively compact family in $C^{\nu,\mu}([0,b],X)$. Properties (i) and (ii) follow from Lemmas 10–12. By Lemma 13, for any $\varepsilon > 0$, the operator Φ_{ε} is continuous. Thus from the Schauder fixed point theorem Φ_{ε} has a fixed point. Therefore, the fractional control system (18) has a solution on $[0,b]$. The proof is complete. □

We are now in a position to state and prove the main result of the paper.

Theorem 15. *Let $0 \leq \nu \leq 1$ and $1/2 < \mu \leq 1$. Suppose that conditions (H1)–(H3) (R) are satisfied. Then system (1) is approximately controllable on $[0,b]$.*

Proof. Let $\varepsilon > 0$ and let x^ε be a fixed point of Φ_ε in $B_{r(\varepsilon)}$. Then x^ε is a mild solution of (1) on $[0, b]$ under the control

$$u_\varepsilon(t, x^\varepsilon) = B^* \mathcal{S}^*_{\nu,\mu}(b-t)\left(\varepsilon I + \Gamma_0^T\right)^{-1} p(x^\varepsilon),$$
$$p(x^\varepsilon) = h - \mathcal{S}_{\nu,\mu}(b) x_0 - \int_0^b \mathcal{P}_\mu(b-s) f(s, x^\varepsilon(s))\, ds \tag{52}$$

and satisfies the following equality:

$$\begin{aligned} x^\varepsilon(b) &= \mathcal{S}_{\nu,\mu}(b) x_0 \\ &\quad + \int_0^b (b-s)^{\alpha-1} \mathcal{P}_\mu(b-s) \\ &\quad \cdot [Bu_\varepsilon(s, x^\varepsilon) + f(s, x^\varepsilon(s))]\, ds \\ &= \mathcal{S}_{\nu,\mu}(b) x_0 + \left(-\varepsilon I + \varepsilon I + \Gamma_0^b\right)\left(\varepsilon I + \Gamma_0^b\right)^{-1} p(x^\varepsilon) \\ &\quad + \int_0^b \mathcal{P}_\mu(b-s) f(s, x^\varepsilon(s))\, ds \\ &= h - \varepsilon\left(\varepsilon I + \Gamma_0^b\right)^{-1} p(x^\varepsilon). \end{aligned} \tag{53}$$

It follows from (H3) that for all $\varepsilon > 0$

$$\int_0^b \|f(s, x^\varepsilon(s))\|^{1/\mu_1} ds \le \int_0^T n^{1/\mu_1}(s)\, ds. \tag{54}$$

Consequently, the sequence $\{f(\cdot, x^\varepsilon(\cdot))\}$ is bounded. As such, there is a subsequence, still denoted by $\{f(\cdot, x^\varepsilon(\cdot))\}$, that converges weakly to, say, $f(\cdot)$ in $L^{1/\mu_1}([0, b], X)$. Then

$$\begin{aligned} &\|p(x^\varepsilon) - p\| \\ &\quad = \left\| \int_0^b \mathcal{P}_\mu(b-s)[f(s, x^\varepsilon(s)) - f(s)]\, ds \right\| \\ &\quad \le \sup_{0\le t\le b} \left\| \int_0^t \mathcal{P}_\mu(t-s)[f(s, x^\varepsilon(s)) - f(s)]\, ds \right\| \longrightarrow 0, \end{aligned} \tag{55}$$

as $\varepsilon \to 0^+$, because of the compactness of an operator

$$f(\cdot) \longrightarrow \int_0^\cdot \mathcal{P}_\mu(\cdot - s) f(s)\, ds : L^{1/\mu_1}([0, b], X) \longrightarrow C([0, b], X), \tag{56}$$

where

$$p = h - \mathcal{S}_{\nu,\mu} x_0 - \int_0^b \mathcal{P}_\mu(b-s) f(s)\, ds. \tag{57}$$

Then, by (53), the assumption (R), and $\|\varepsilon(\varepsilon I + \Gamma_0^b)^{-1}\| \le 1$, it follows that

$$\begin{aligned} &\|x^\varepsilon(b) - h\| \\ &\quad = \left\|\varepsilon\left(\varepsilon I + \Gamma_0^b\right)^{-1}(p(x^\varepsilon) - p) + \varepsilon\left(\varepsilon I + \Gamma_0^b\right)^{-1}(p)\right\| \\ &\quad \le \|p(x^\varepsilon) - p\| + \left\|\varepsilon\left(\varepsilon I + \Gamma_0^b\right)^{-1}(p)\right\| \longrightarrow 0 \end{aligned} \tag{58}$$

as $\varepsilon \to 0^+$. This gives the approximate controllability. The theorem is proved. □

Remark 16. Theorem 15 is a generalization of the existing results on the approximate controllability of fractional differential equations. When $\nu = 0$, the fractional control system (12) simplifies to the classical Riemann-Liouville fractional control equation which has been studied by Liu and Bin [16]. When $\nu = 1$, the fractional equation (12) simplifies to the classical Caputo fractional control system which has been studied by Sakthivel et al. [11].

4. Applications

The partial differential system arises in the mathematical modeling of heat transfer

$$\begin{aligned} &D_{0^+}^{\nu,3/4} x(t,\theta) = x_{\theta\theta}(t,\theta) + b(\theta) u(t) + f(t, x(t,\theta)), \\ &x(t, 0) = x(t, \pi) = 0, \quad t > 0, \\ &I_{0^+}^{(1/4)(1-\nu)} x(0) = x_0, \quad 0 < \theta < \pi,\ 0 \le t \le b, \end{aligned} \tag{59}$$

where $u \in L_2[0, b]$, $X = L_2[0, \pi]$, $h \in X$, $0 \le \nu \le 1$, $\mu = 3/4$, and $f : R \times R \to R$ is continuous and uniformly bounded. Let $B \in L(R, X)$ be defined as

$$\begin{aligned} (Bu)(\theta) &= b(\theta) u, \\ B^* \nu &= \sum_{n=1}^\infty \langle b, e_n\rangle \langle \nu, e_n\rangle, \end{aligned} \tag{60}$$

where $0 \le \theta \le \pi$, $u \in R$, and $b(\theta) \in L_2[0, \pi]$, and let $A : X \to X$ be operator defined by $Az = z''$ with domain

$$\begin{aligned} D(A) = \{&z \in X \mid z, z' \text{ are absolutely continuous}, \\ &z'' \in X,\ z(0) = z(\pi) = 0\}. \end{aligned} \tag{61}$$

Then

$$Az = \sum_{n=1}^\infty -n^2 \langle z, e_n\rangle e_n, \quad z \in D(A), \tag{62}$$

where $e_n(\theta) = \sqrt{2/\pi}\sin n\theta$, $0 \le x \le \pi$, $n = 1, 2, \ldots$. It is known that A generates a compact semigroup $S(t)$, $t > 0$, in X and is given by

$$S(t) x = \sum_{n=1}^\infty e^{-n^2 t} \langle x, e_n\rangle e_n, \quad x \in X. \tag{63}$$

Moreover, for any $x \in X$ we have

$$\begin{aligned} \mathcal{T}_{3/4}(t) &= \frac{3}{4}\int_0^\infty \theta \Psi_{3/4}(\theta) S\left(t^{3/4}\theta\right) d\theta, \\ \mathcal{T}_{3/4}(t) x &= \frac{3}{4}\sum_{n=1}^\infty \int_0^\infty \theta \Psi_{3/4}(\theta) \exp\left(-n^2 t^{3/4}\theta\right) d\theta \langle x, e_n\rangle e_n. \end{aligned} \tag{64}$$

In order to show that the associated linear system is approximately controllable on $[0, b]$, we need to show that $(b-s)^{\alpha-1} B^* \mathcal{T}_\mu (b-s) x = 0 \Rightarrow x = 0$. Indeed, observe that

$$\begin{aligned}
&(b-s)^{\mu-1} B^* \mathcal{T}_\mu (b-s)\, x \\
&\quad = (b-s)^{\mu-1} \\
&\qquad \cdot \sum_{n=1}^{\infty} (b, e_n) \frac{3}{4} \int_0^\infty \theta \Psi_{3/4}(\theta) \exp\left(-n^2 t^{3/4} \theta\right) d\theta \, \langle x, e_n \rangle \\
&\quad = (b-s)^{\mu-1} \frac{3}{4} \\
&\qquad \cdot \sum_{n=1}^{\infty} \int_0^\infty \theta \Psi_{3/4}(\theta) \exp\left(-n^2 t^{3/4} \theta\right) d\theta \, \langle b, e_n \rangle \langle x, e_n \rangle = 0.
\end{aligned} \tag{65}$$

So, $\langle x, e_n \rangle = 0 \Rightarrow x = 0$ provided that $\langle b, e_n \rangle = \int_0^\pi b(\theta) e_n(\theta) d\theta \neq 0$ for $n = 1, 2, 3, \ldots$. Therefore, the associated linear system is approximately controllable provided that $\int_0^\pi b(\theta) e_n(\theta) d\theta \neq 0$ for $n = 1, 2, 3, \ldots$. Because of the compactness of the semigroup $S(t)$ (and consequently $\mathcal{T}_{3/4}$) generated by A, the associated linear system of (59) is not exactly controllable but it is approximately controllable. Hence, according to Theorem 15, system (59) will be approximately controllable on $[0, b]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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