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Research Article

Reproducing Kernel Method for Fractional Riccati Differential Equations

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This paper is devoted to a new numerical method for fractional Riccati differential equations. The method combines the reproducing kernel method and the quasilinearization technique. Its main advantage is that it can produce good approximations in a larger interval, rather than a local vicinity of the initial position. Numerical results are compared with some existing methods to show the accuracy and effectiveness of the present method.

1. Introduction

This paper deals with the numerical solution of the following fractional Riccati differential equation:

$$u^{\alpha}(x) = p(x) + q(x)u(x) + r(x)u^2(x), \quad 0 \le x \le T, \; 0 < \alpha \le 1, \quad u(0) = 0, \tag{1}$$

where $u^{\alpha}(x)$ denotes the Caputo fractional derivative of order α and

$$u^{\alpha}(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\tau)^{-\alpha} u'(\tau)\, d\tau. \tag{2}$$

Riccati differential equations arise in many fields [1]. The problem has attracted much attention and has been studied by many authors. However, deriving its analytical solution in an explicit form seems to be unlikely except for certain special situations. Recently, many numerical methods [2–9] have been proposed to solve integer order Riccati differential equations. However, the discussion on the numerical methods for fractional order Riccati differential equations is rare. Odibat and Momani [10] developed a modified homotopy perturbation method for fractional Riccati differential equations. Li [11] presented a numerical method for fractional differential equations based on Chebyshev wavelets. Hosseinnia et al. [12] applied an enhanced homotopy perturbation method for fractional Riccati differential equations. Yüzbaşı [13] introduced a numerical method for fractional Riccati differential equations using the Bernstein polynomial. Khader [14] developed the fractional Chebyshev finite difference method for fractional Riccati differential equations. Yang and Baleanu [15], Yang et al. [16], and Baleanu et al. [17] proposed local fractional variation iteration for fractional heat conduction and wave equations.

Recently, based on the reproducing kernel theory, Cui, Geng, and Lin presented the reproducing kernel method (RKM) for linear and nonlinear operator equations [18–21]. The method has been developed and applied to many problems [22–26].

The aim of this paper is to present a new method for fractional Riccati differential equations, based on the RKM and the quasilinearization technique.

The rest of the paper is organized as follows. In the next section, the quasilinearization technique is applied to fractional Riccati differential equation. The RKM for reduced linear fractional differential equations is introduced in Section 3. The numerical examples are presented in Section 4. Section 5 ends this paper with a brief conclusion.

2. Quasilinearization of Riccati Differential Equation (1)

In this section, the quasilinearization technique is applied to reduce (1) to a series of linear fractional problems. Define $f(x,u) = p(x) + r(x)u^2$. By choosing an appropriate initial approximation $u_0(x)$ for the function $u(x)$ in $f(x,u)$ and expanding $f(x,u)$ around $u_0(x)$, it follows that

$$f(x,u_1) = f(x,u_0) + (u_1 - u_0)\left.\frac{\partial f}{\partial u}\right|_{u=u_0} + \cdots. \quad (3)$$

Generally, one can write for $k = 1,2,\ldots$ (k = iteration index)

$$f(x,u_k) = f(x,u_{k-1}) + (u_k - u_{k-1})\left.\frac{\partial f}{\partial u}\right|_{u=u_{k-1}} + \cdots. \quad (4)$$

Therefore, the following iteration formula for (1) can be derived:

$$\begin{gathered} u_k^{\alpha}(x) + a_k(x)u_k(x) = f_k(x), \quad k = 1,2,\ldots, \\ u_k(0) = 0, \end{gathered} \quad (5)$$

where $a_k(x) = -[q(x) + (\partial f/\partial u)|_{u=u_{k-1}}] = -[q(x) + 2r(x)u_{k-1}(x)]$and $f_k(x) = f(x,u_{k-1}) - (\partial f/\partial u)|_{u=u_{k-1}} = p(x) - r(x)u_{k-1}^2(x)$ and $u_0(x)$ is the initial approximation.

Clearly, to solve (1), it suffices for us to solve the series of linear problem (5).

3. Method for Solving Linear Fractional Problem (5)

To illustrate how to solve (5) we consider the problem of solving

$$\begin{gathered} Lv(x) = v^{\alpha}(x) + a(x)v(x) = f(x), \quad 0 < x < T, \\ v(0) = 0, \end{gathered} \quad (6)$$

where $a(x)$ and $f(x)$ are continuous.

By the definition of Caputo fractional derivative, (6) is equivalent to the following equation:

$$\begin{gathered} \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}v'(\tau)\,d\tau + a(t)v(t) = f(t), \\ 0 \le t \le T, \\ v(0) = 0. \end{gathered} \quad (7)$$

If $u(t) \in C^1[0,T]$, then the improper integral $\int_0^t (t-\tau)^{-\alpha}v'(\tau)d\tau$ exists and

$$\begin{aligned} &\int_0^t (t-\tau)^{-\alpha}v'(\tau)\,d\tau \\ &= \int_0^t \frac{v'(\tau)-v'(t)}{(t-\tau)^{\alpha}}d\tau + v'(t)\int_0^t (t-\tau)^{-\alpha}d\tau \\ &= \frac{t^{1-\alpha}v'(t)}{1-\alpha} - \int_0^t \frac{v'(t)-v'(\tau)}{(t-\tau)^{\alpha}}d\tau \\ &= \frac{t^{1-\alpha}v'(t)}{1-\alpha} - \int_0^t g(\tau,v)\,d\tau. \end{aligned} \quad (8)$$

Table 1: Comparison of the numerical solutions with the other methods for $\alpha = 0.75$.

x	Ours	[10]	[11]
0.20	0.469516	0.428892	0.584307
0.40	0.933596	0.891404	1.024974
0.50	1.14488	1.132763	1.198621
0.60	1.33098	1.370240	1.349150
0.80	1.62153	1.794879	1.599235
1.00	1.81865	2.087384	1.801763

Define

$$g(\tau,v) = \begin{cases} \dfrac{v'(t)-v'(\tau)}{(t-\tau)^{\alpha}}, & \tau \ne t, \\ 0, & \tau = t. \end{cases} \quad (9)$$

This gives us a continuous function on $[0,t]$ and then integral $\int_0^t g(\tau,v)d\tau$ also exists and $\int_0^t ((v'(t)-v'(\tau))/(t-\tau)^{\alpha})d\tau = \int_0^t g(\tau,v)d\tau$.

Therefore, (6) can be converted into

$$\begin{gathered} \frac{1}{\Gamma(1-\alpha)}\left(\frac{t^{1-\alpha}v'(t)}{1-\alpha} - \int_0^t g(\tau,v)\,d\tau\right) + a(t)v(t) = f(t), \\ 0 \le t \le T, \\ v(0) = 0. \end{gathered} \quad (10)$$

Applying Hermite's quadrature formula to $\int_0^t ((v'(t) - v'(\tau))/(t-\tau)^{\alpha})d\tau$, one obtains

$$\int_0^t g(\tau,v)\,d\tau = \frac{\pi t}{2M}\sum_{k=1}^{M}\overline{g}(x_k,u(x_k))\sqrt{1-x_k^2}, \quad (11)$$

where $\overline{g}(x,u(x)) = g((t/2)(1+x),u((t/2)(1+x)))$, M is the number of nodes, and $x_k = \cos((2k-1)/2M)$, $k = 1,\ldots,M$.

Then (6) can be further equivalently approximated to

$$\begin{gathered} \frac{1}{\Gamma(1-\alpha)} \times\left(\frac{t^{1-\alpha}v'(t)}{1-\alpha} - \frac{\pi t}{2M}\sum_{k=1}^{M}\overline{g}(x_k,v(x_k))\sqrt{1-x_k^2}\right) \\ + a(t)v(t) = f(t), \quad 0 \le t \le T, \\ v(0) = 0. \end{gathered} \quad (12)$$

To apply the RKM to (12), it is necessary to construct the following reproducing kernel Hilbert space $W^3[0,T]$.

Definition 1. $W^3[0,T] = \{u(x) \mid u''(x)$ is an absolutely continuous real value function, $u^{(3)}(x) \in L^2[0,T]$,

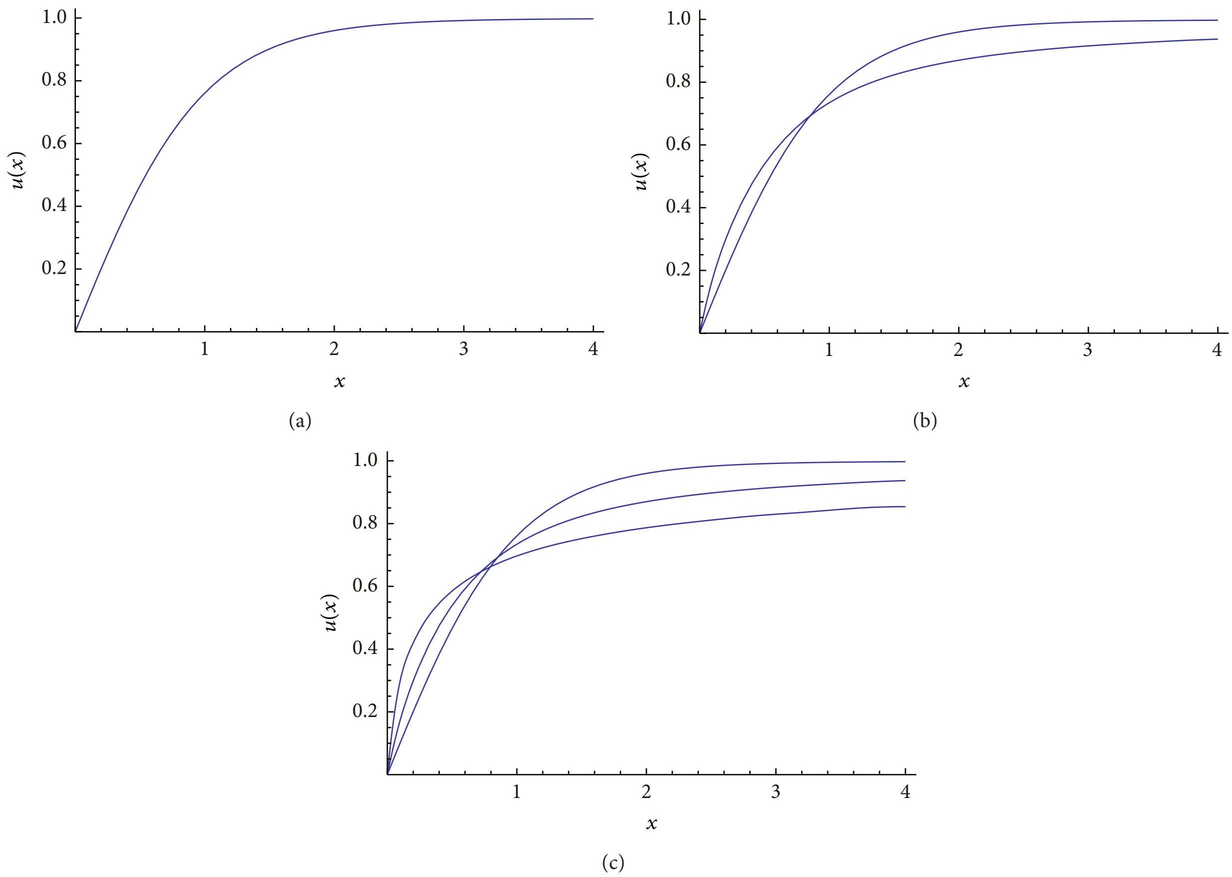

FIGURE 1: The behavior of approximate solution with different values of α ((a) $\alpha = 0.99$; (b) $\alpha = 0.75, 0.99$; (c) $\alpha = 0.5, 0.75, 0.99$).

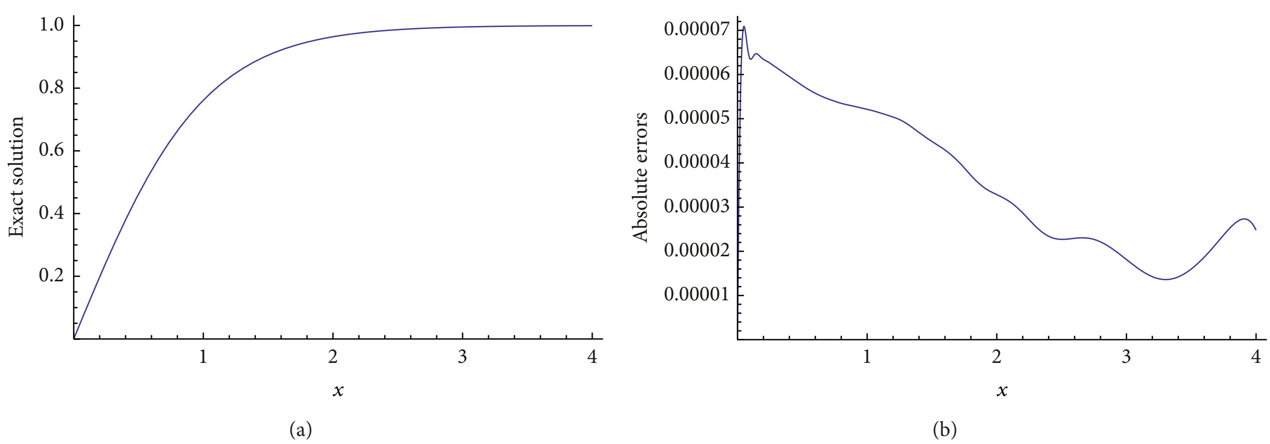

FIGURE 2: Comparison of approximate solutions with the exact solutions for $\alpha = 1$ ((a) exact solution; (b) absolute errors).

$u(0) = 0\}$. The inner product and norm in $W^3[0, T]$ are given, respectively, by

$$\begin{aligned}&(u(y), v(y))_3\\&\quad= u(0)v(0) + u'(0)v'(0) + u''(0)v''(0) + \int_0^T u^{(3)}v^{(3)}dy,\\&\|u\|_3 = \sqrt{(u,u)_3}, \qquad u, v \in W^3[0, T].\end{aligned} \tag{13}$$

Theorem 2. *$W^3[0, T]$ is a reproducing kernel space and its reproducing kernel is*

$$k(x, y) = \begin{cases} k_1(x, y), & y \le x, \\ k_1(y, x), & y > x, \end{cases} \tag{14}$$

where $k_1(x, y) = y^2(7\,x\,y^4 - y^5 + 35\,x^3 y\,(4 + y) - 21\,x^2(-60 + y^3))/5040$.

Table 2: Comparison of the numerical solutions with the other methods for $\alpha = 0.90$.

x	Ours	[13]	IABMM [12]	MHPM [10]
0.20	0.312985	0.314869	—	—
0.40	0.695357	0.697544	—	—
0.50	0.901484	0.903695	0.8621	0.9010
0.60	1.10576	1.107866	—	—
0.80	1.47606	1.477707	—	—
1.00	1.76417	1.764520	1.7356	1.8720

Table 3: Numerical results for $\alpha = 0.99,\ 1$ on $[0, 4]$.

x	Ours ($\alpha = 0.99$)	Ours ($\alpha = 1$)	Exact solution ($\alpha = 1$)
0.5	0.769552	0.756446	0.756014
1.0	1.69828	1.68978	1.689500
1.5	2.19159	2.19599	2.195630
2.0	2.35063	2.35838	2.357770
2.5	2.39373	2.40046	2.40028
3.0	2.40539	2.41058	2.41081
3.5	2.40899	2.41302	2.41339
4.0	2.41047	2.41374	2.41401

Table 4: Comparison of the numerical solutions with the other methods for $\alpha = 0.75$.

x	Ours	[13]	IABMM [12]	MHPM [10]
0.20	0.307359	0.3099755	0.3117	0.3138
0.40	0.480346	0.4816318	0.4855	0.4929
0.60	0.597542	0.5977827	0.6045	0.5974
0.80	0.679657	0.6788495	0.6880	0.6604
1.00	0.738213	0.7368368	0.7478	0.7183

Table 5: Comparison of the numerical solutions with the other methods for $\alpha = 0.90$.

x	Ours	[13]	IABMM [12]	MHPM [10]
0.20	0.237652	0.2387891	0.2393	0.2391
0.40	0.421766	0.4225830	0.4234	0.4229
0.60	0.565673	0.5661715	0.5679	0.5653
0.80	0.674464	0.6746270	0.6774	0.6740
1.00	0.754632	0.7545890	0.7584	0.7569

Definition 3. $W^1[0,T] = \{u(x) \mid u(x)$ is an absolutely continuous real value function, $u'(x) \in L^2[0,T]\}$. The inner product and norm in $W^1[0,T]$ are given, respectively, by

$$(u(y), v(y))_1 = u(0)\,v(0) + \int_0^T u'v'dy,$$
$$\|u\|_1 = \sqrt{(u,u)_1}, \quad u, v \in W^1[0,T]. \tag{15}$$

Table 6: Comparison of the numerical solutions with the other methods for $\alpha = 1.0$.

x	Exact	Ours	[13]	MHPM [10]
0.20	0.197375	0.19738	0.197375	0.197375
0.40	0.379949	0.379956	0.379948	0.379944
0.60	0.53705	0.537061	0.537049	0.536857
0.80	0.664037	0.664053	0.664036	0.661706
1.00	0.761594	0.761618	0.761594	0.746032

Theorem 4. *$W^1[0,T]$ is a reproducing kernel space and its reproducing kernel is*

$$\overline{k}(x,y) = \begin{cases} 1+y, & y \le x, \\ 1+x, & y > x. \end{cases} \tag{16}$$

Put

$$Lv(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{t^{1-\alpha}v'(t)}{1-\alpha} - \frac{\pi t}{2M} \times \sum_{k=1}^{M} \overline{g}(x_k, v(x_k))\sqrt{1-x_k^2}\right) + a(t)\,v(t). \tag{17}$$

Clearly, $L : W^3[0,T] \to W^1[0,T]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{k}(x, x_i)$ and $\psi_i(x) = L^*\varphi_i(x)$, where L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W^3[0,T]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i}\beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0, i = 1, 2, \ldots). \tag{18}$$

Theorem 5. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,T]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W^3[0,T]$.*

Theorem 6. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,T]$ and the solution of* (12) *is unique, then the solution of* (12) *is*

$$v(t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik} f(x_k)\,\overline{\psi}_i(t). \tag{19}$$

Now, an approximate solution $V_N(x)$ of (6) can be obtained by the N-term intercept of the exact solution $v(x)$ and

$$v_N(x) = \sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik} f(x_k)\,\overline{\psi}_i(t). \tag{20}$$

Similarly, the approximate solutions $u_k(x)$ can be obtained:

$$u_{k,N}(x) = \sum_{j=1}^{N} A_j\overline{\psi}_j(x), \tag{21}$$

where $A_j = \sum_{l=1}^{j}\beta_{jl} f_k(x_l)$.

4. Numerical Examples

Example 1. Consider the following fractional Riccati differential equation [10–13]:

$$u^{\alpha}(x) = 1 + 2u(x) - u^{2}(x),$$
$$0 \le x \le T, \ 0 < \alpha \le 1, \quad (22)$$
$$u(0) = 0.$$

The exact solution for $\alpha = 1$ can be easily determined to be

$$u(x) = 1 + \sqrt{2}\ \tanh\left(\sqrt{2}\,x + \frac{\log\left(\left(-1+\sqrt{2}\right)/\left(1+\sqrt{2}\right)\right)}{2}\right). \quad (23)$$

Applying the proposed method, taking $T = 1,\ k = 3,\ M = 30,\ N = 50$, the numerical results compared with other methods are listed in Tables 1 and 2. Taking $T = 4,\ k = 3,\ M = 30,\ N = 50$, the numerical results on $[0, 4]$ are listed in Table 3. From Table 3, it is easily found that the present approximations are effective for a larger interval, rather than a local vicinity of the initial position.

Example 2. Consider the following fractional Riccati differential equation [10–14]:

$$u^{\alpha}(x) = 1 + u^{2}(x), \quad 0 \le x \le T, \ 0 < \alpha \le 1$$
$$u(0) = 0. \quad (24)$$

The exact solution for $\alpha = 1$ can be easily determined to be

$$u(x) = \frac{e^{2x} - 1}{e^{2x} + 1}. \quad (25)$$

According to the present method, taking $T = 1,\ k = 3,\ M = 50,\ N = 50$, the numerical results compared with other methods are given in Tables 4, 5, and 6. Taking $T = 4,\ k = 5,\ M = 50,\ N = 80$, the numerical results on $[0, 4]$ are shown in Figures 1 and 2. From these figures we can conclude that the obtained numerical solutions are in excellent agreement with the exact solution for a larger interval.

5. Conclusion

In this paper, combining the RKM, the numerical integral, and quasilinearization techniques, a new numerical method is proposed for fractional Riccati differential equations. The main advantage of this method is that it can provide accurate numerical approximations on a larger interval. Numerical results compared with the existing methods show that the present method is a powerful method for solving fractional Riccati differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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