Research Article

Nodal Solutions for Some Second-Order Semipositone Integral Boundary Value Problems

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Using bifurcation techniques, we first prove a global bifurcation theorem for nonlinear second-order semipositone integral boundary value problems. Then the existence and multiplicity of nodal solutions of the above problems are obtained. Finally, an example is worked out to illustrate our main results.

1. Introduction

In this paper, we consider the existence and multiplicity of nodal solutions for the following nonlinear second-order semipositone integral boundary value problems (BVP for short):

$$x''(t) + \lambda f(x(t)) = 0, \quad 0 < t < 1,$$
$$x(0) = 0, \qquad x(1) = \int_0^1 a(s) x(s) \, ds, \tag{1}$$

where $\lambda > 0$ is a parameter, $f \in C(R, R)$, and $a \in L[0,1]$ is nonnegative with $0 < \int_0^1 a^2(s) ds < 1$.

Boundary value problems with integral boundary conditions for ordinary differential equations arise in different areas of applied mathematics and physics. Moreover, they include two, three, multipoint, and nonlocal boundary value problems as special cases. For boundary value problems with integral boundary conditions and comments on their importance, we refer the reader to [1–12] and the references therein.

In [1], utilizing the fixed point index and Leray-Schauder degree theory, Zhang and Sun obtained some existence results for multiple solutions including sign-changing solutions under some technical hypotheses for the following integral boundary value problem:

$$x''(t) + f(x(t)) = 0, \quad 0 < t < 1$$
$$x(0) = 0, \qquad x(1) = \int_0^1 a(s) x(s) \, ds, \tag{2}$$

where $f \in C(R, R)$, $f(0) = 0$, $a \in L[0,1]$ is nonnegative with $\int_0^1 a^2(s) ds < 1$.

The purpose of this paper is to investigate the existence and multiplicity of sign-changing solutions of BVP (1), having a given number of zeros (so-called "nodal solution"). The existence of such solutions has been investigated for many types of nonlinear Sturm-Liouville problems with separated boundary conditions and multipoint boundary conditions in many recent papers; see [13–18]. Recently, Sun et al. [16] studied for the following m-point boundary value problems:

$$u''(t) + f(u(t)) = 0, \quad 0 < t < 1$$
$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{3}$$

where $\lambda > 0$ is a parameter, $f \in C(R, R)$, $f(0) = 0$, $\alpha_i \in (0,1)$, $i = 1, 2, \dots, m-2$, $0 < \sum_{i=1}^{m-2} \alpha_i < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$. By using Rabinowitz's

global bifurcation theorem, they obtained the existence and multiplicity of nodal solutions when $f_0 \in (0, +\infty)$, where

$$f_0 = \lim_{t \to 0} \frac{f(t)}{t}. \tag{4}$$

To the authors' knowledge, there are few papers that have considered the existence of nodal solutions for integral boundary value problems. In [1], Zhang and Sun have obtained sign-changing solutions of BVP (2), but no information is obtained regarding the number of zeros of the solution.

Motivated by [1, 15, 16], in this paper we investigate the existence and multiplicity of nodal solutions for BVP (1). The main features of this paper are as follows. First, the nonlinear term f is semipositone, and $f_0 \in (0, +\infty)$, where f_0 is defined as in (4). Next, the methods used here are Rabinowitz's global bifurcation theorem and some of the techniques used in [16], which are entirely different from [1, 7, 8]. Finally, the results we obtained are the existence of at least any given even number of nodal solutions.

Now we give some notations and a global bifurcation theorem which will be used in Section 3. Let E be a real Banach space; Rabinowitz studied a nonlinear eigenvalue problem of the form

$$u = \lambda L u + H(\lambda, u), \tag{5}$$

where $\lambda > 0$ is a parameter, $L : E \to E$ is a compact linear map, $H : \varepsilon \equiv R \times E \to E$ is completely continuous, and $H(\lambda, u) = o(\|u\|)$ for u near 0 uniformly on bounded λ intervals. A solution of (5) is a pair $(\lambda, u) \in \varepsilon$ which satisfies (5). The closure of the set of nontrivial solutions of (5) is denoted by $\sum$. If there exist $\mu \in R^+ = [0, +\infty)$ and $0 \neq \nu \in E$ such that $\nu = \mu L \nu$, μ is said to be a positive eigenvalue of L and ν is said to be an eigenfunction corresponding to μ. The set of positive eigenvalues of L will be denoted by $r(L)$. The algebraic multiplicity of $\mu \in r(L)$ is dim $\cup_{j=1}^{\infty} N((I - \mu L)^j)$, where $N(A)$ denotes the null space of A. The following was shown in Theorem 1.3 and Theorem 1.25 of Rabinowitz [19] and Theorem 2 of Dancer [20].

Theorem A. *If $\mu \in r(L)$ is simple, then $\sum$ possess a maximal subcontinuum ζ_μ which can be decomposed into two subcontinua C_μ^+ and C_μ^- such that, for some neighborhood B of $(\mu, 0)$, $(\lambda, u) \in C_\mu^+(C_\mu^-) \cap B$, and $(\lambda, u) \neq (\mu, 0)$ imply $(\lambda, u) = (\lambda, \alpha \nu + w)$, where $\alpha > 0$ ($\alpha < 0$) and $|\lambda - \mu| = o(1)$, $\|w\| = o(|\alpha|)$ for α near 0. Moreover, either C_μ^+ and C_μ^- are both unbounded or $C_\mu^+ \cap C_\mu^- \neq \{(\mu, 0)\}$.*

This paper is arranged as the follows: some preliminaries and some lemmas are given including the study of the eigenvalues and eigenfunctions of the linearization of BVP (1) in Section 2. The main results are proved by using Theorem A in Section 3. A concrete example is given to illustrate the application of the main results in Section 4.

2. Some Preliminaries and Lemmas

Let $X = C[0, 1]$ with the norm $\|x\| = \max_{t \in [0,1]} |x(t)|$, $Y = \{x \in C^1[0, 1], x(0) = 0, x(1) = \int_0^1 a(s)x(s)ds\}$ with the norm

$\|x\|_1 = \max\{\|x\|, \|x'\|\}$, $Z = \{x \in C^2[0, 1], x(0) = 0, x(1) = \int_0^1 a(s)x(s)ds\}$ with the norm $\|x\|_2 = \max\{\|x\|, \|x'\|, \|x''\|\}$. Then X, Y, Z are Banach spaces.

For any C^1 function x, if $x(t_0) = 0$, then t_0 is said to be a simple zero of x if $x'(t_0) \neq 0$. For any integer $k \geq 1$ and any $\nu \in \{\pm\}$, as in [15], we define sets $T_k^\nu \subset Z$ consisting of the set of functions $x \in Z$ satisfying the following conditions:

   (i) $x(0) = 0$, $\nu x'(0) > 0$, and $x'(1) \neq 0$;

   (ii) x' has only simple zeros in $(0, 1)$ and has exactly k such zeros;

   (iii) x has a zero strictly between each two consecutive zeros of x'.

Note that $T_k^- = -T_k^+$ and let $T_k = T_k^+ \cup T_k^-$. It is easy to see that the sets T_k^+ and T_k^- are disjoint and open in Z. Let $E = R \times Y$ under the product topology, $\Phi_k^+ = R \times T_k^+$, $\Phi_k^- = R \times T_k^-$, and $\Phi_k = R \times T_k$.

In the following, we give some information on the spectrum structure of the linear integral boundary value problem corresponding to BVP (1):

$$x''(t) + \lambda x(t) = 0, \quad 0 < t < 1$$

$$x(0) = 0, \qquad x(1) = \int_0^1 a(s) x(s) ds. \tag{6}$$

Define the operators K on Y by

$$(Kx)(t) = \int_0^1 k(t, s) x(s) ds, \tag{7}$$

where

$$k(t, s) = G(t, s) + \frac{t \int_0^1 G(\tau, s) a(\tau) d\tau}{1 - \int_0^1 s a(s) ds},$$

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{8}$$

It is easy to prove the following lemma.

Lemma 1. *The linear operator $K : Y \to Y$ is completely continuous. Moreover, $(\lambda, x) \in (0, \infty) \times C^2[0, 1]$ is a solution of (6) if and only if $(\lambda, x) \in E$ is a solution of the operator equation $x = \lambda K x$.*

We now define a function $G : (0, +\infty) \to R$ by

$$G(\eta) = \sin \eta - \int_0^1 a(s) \sin(\eta s) ds, \quad \eta \in (0, +\infty). \tag{9}$$

Lemma 2. *All the zeros of $G(\eta)$ are simple.*

Proof. Suppose that η is a double zero of $G(\eta)$; that is,

$$\sin \eta = \int_0^1 a(s) \sin(\eta s) ds, \qquad \cos \eta = \int_0^1 s a(s) \cos(\eta s) ds. \tag{10}$$

Hence,

$$
1 = \left(\int_0^1 a(s) \sin(\eta s) \, ds \right)^2 + \left(\int_0^1 sa(s) \cos(\eta s) \, ds \right)^2
$$

$$
\leq \int_0^1 a^2(s) \, ds \int_0^1 \sin^2(\eta s) \, ds
$$

$$
+ \int_0^1 s^2 a^2(s) \, ds \int_0^1 \cos^2(\eta s) \, ds \tag{11}
$$

$$
\leq \int_0^1 a^2(s) \, ds < 1,
$$

which shows that (10) cannot hold, and so $G(\eta)$ has only simple zeros. $\qquad\square$

Lemma 3. *Suppose that $a(s)$ is symmetrical in $[0,1]$. Then $G(\eta)$ has no zero on $[0, \pi/2]$ and, for each $i \geq 1$, $G(\eta)$ has exactly one zero $\eta_i = 2i\pi$ on $I_{2i} := ((2i-(1/2))\pi, (2i+(1/2))\pi)$.*

Proof. Since $0 < \int_0^1 a^2(s)ds < 1$ and $\int_0^1 a(s)ds \leq (\int_0^1 a^2(s)ds)^{1/2}$, we have

$$
0 < \int_0^1 a(s) \, ds < 1. \tag{12}
$$

Now, (12) implies that $G(\eta) > 0$ on $(0, \pi/2]$; that is, $G(\eta)$ has no zero in this interval, and also

$$
G\left(\left(2i - \frac{1}{2}\right)\pi\right) < 0, \quad G\left(\left(2i + \frac{1}{2}\right)\pi\right) > 0, \quad i \geq 1. \tag{13}
$$

For each integer $i \geq 1$, by the symmetry of $a(s)$ in $[0,1]$, we have

$$
\int_0^1 a(s) \sin(2i\pi s) \, ds
$$

$$
= \int_0^1 a(1-t) \sin[2i\pi(1-t)] \, dt
$$

$$
= -\int_0^1 a(1-t) \sin 2i\pi t \, dt \tag{14}
$$

$$
= -\int_0^1 a(t) \sin 2i\pi t \, dt.
$$

So, $2 \int_0^1 a(s) \sin(2i\pi s)ds = 0$; that is, $\int_0^1 a(s) \sin(2i\pi s)ds = 0$.

Thus, $G(2i\pi) = 0$. That is, $G(\eta)$ has one zero $\eta_i = 2i\pi$ on each interval $I_{2i} := ((2i-(1/2))\pi, (2i+(1/2))\pi)$.

For any fixed integer $i \geq 1$, suppose that $G(\eta)$ has another zero $\bar{\eta}_i$ on I_{2i}. In view of the continuity of $G(\eta)$ and (13), then $G(\eta)$ has the third zero $\tilde{\eta}_i$ on I_{2i}. Without loss of generality, we may assume that $\bar{\eta}_i < \tilde{\eta}_i$. We have the following three cases to consider.

(i) Consider $\bar{\eta}_i < \eta_i < \tilde{\eta}_i$. By (9) and (12), we have

$$
G'(\eta_i) = G'(2i\pi) = \cos(2i\pi) - \int_0^1 sa(s) \cos(2i\pi s) \, ds
$$

$$
= 1 - \int_0^1 sa(s) \cos(2i\pi s) \, ds \tag{15}
$$

$$
> 0.
$$

From (13) and Lemma 2, it is easy to see that $G'(\eta_i) < 0$, which contradicts to (15).

(ii) Consider $\bar{\eta}_i < \tilde{\eta}_i < \eta_i$. From (13) and Lemma 2, it is easy to see that $G'(\tilde{\eta}_i) < 0$. So, we have

$$
\sin(\tilde{\eta}_i) = \int_0^1 a(s) \sin(\tilde{\eta}_i s) \, ds,
$$

$$
0 < \cos(\tilde{\eta}_i) < \int_0^1 sa(s) \cos(\tilde{\eta}_i s) \, ds. \tag{16}
$$

Hence,

$$
1 < \left(\int_0^1 a(s) \sin(\tilde{\eta}_i s) \, ds \right)^2
$$

$$
+ \left(\int_0^1 sa(s) \cos(\tilde{\eta}_i s) \, ds \right)^2
$$

$$
\leq \int_0^1 a^2(s) \, ds \int_0^1 \sin^2(\tilde{\eta}_i s) \, ds \tag{17}
$$

$$
+ \int_0^1 s^2 a^2(s) \, ds \int_0^1 \cos^2(\tilde{\eta}_i s) \, ds
$$

$$
\leq \int_0^1 a^2(s) \, ds < 1,
$$

which is a contradiction.

(iii) Consider $\eta_i < \bar{\eta}_i < \tilde{\eta}_i$. Similar to the proof of Case (ii), we can also lead to a contradiction.

Therefore, $G(\eta)$ has exactly one zero $\eta_i = 2i\pi$ on $I_{2i} := ((2i-(1/2))\pi, (2i+(1/2))\pi)$ for each $i \geq 1$. $\qquad\square$

As the proof of Lemma 4 in [1], it is easy to obtain the following lemma.

Lemma 4. *(1) For each $k \geq 1$, $\eta_k \in I_k := ((k-(1/2))\pi, (k+(1/2))\pi)$ is one zero of $G(\eta)$ if and only if $\lambda_k = \eta_k^2$ is an eigenvalue of K. In addition, $\varphi_k(t) = \sin(\eta_k t)$ is an eigenfunction corresponding to λ_k and $\varphi_k(t) \in T_k^+$.*

(2) The algebraic multiplicity of each positive eigenvalue λ_n $(n = 1, 2, \ldots)$ of K is 1.

Lemma 5. *Suppose that $a(s)$ is symmetrical in $[0,1]$. Then*

(1) *there exists a subsequence $\{\lambda_{n_i}\}$ of the eigenvalue sequence $\{\lambda_n\}$ of K such that $\lambda_{n_i} = (2i\pi)^2$ $(i = 1, 2, \ldots)$, and the eigenfunction φ_{n_i} corresponding to λ_{n_i} is $\varphi_{n_i}(t) = \sin(2i\pi t)$;*

(2) *$\varphi_{n_i}(t) \in T_{2i}^+$ for $i = 1, 2, \ldots$.*

Proof. From Lemmas 3 and 4, conclusion (1) can be obtained immediately. Noticing that $\varphi_{n_i}(t) = \sin(2i\pi t)$, $i = 1, 2, \ldots$, it is easy to check that $\varphi_{n_i}(t) \in T_{2i}^+$ for $i = 1, 2, \ldots$. $\square$

Define the operators F and A on Y by
$(Fx)(t) = f(x(t))$ and $Ax(t) = (KFu)(t)$ for $t \in [0, 1]$, respectively, where the operator K is defined as in (7).

It is easy to see that $A : Y \to Y$ is completely continuous. By direct computation, we can easily get the following lemma.

Lemma 6. $(\lambda, x) \in (0, \infty) \times C^2[0, 1]$ *is a solution of BVP* (1) *if and only if* $(\lambda, x) \in E$ *is a solution of equation*

$$x = \lambda Ax. \tag{18}$$

For $y \in Y$, by the mean-value theorem for the integral, there exists a point $\xi \in (0, 1)$ such that

$$y(1) = \int_0^1 a(s) y(s)\, ds = y(\xi) \int_0^1 a(s)\, ds. \tag{19}$$

Let $x_0(t) = 1$, $\omega_0(t) = K(x_0(t))$ for each $t \in [0, 1]$ and $e(t) = \xi(1 - \xi)t \int_0^1 a(s)ds/(1 - \xi \int_0^1 a(s)ds)$. Let $\omega(t) = M_0\omega_0(t)$ for each $t \in [0, 1]$, where $M_0 > 0$ is a constant to be defined later. The set W is defined by

$$W = \{x \in X \mid x(t) + \omega(t) \tag{20}$$
$$\geq \|x(t) + \omega(t)\| e(t), t \in [0, 1]\}.$$

Obviously $W \in X$ is a closed convex set, and, for each $\tau > 0$,

$$\tau W = \{y = \tau x \mid x \in W\}$$
$$= \left\{y \in X \mid \frac{1}{\tau} y(t) + \omega(t)\right.$$
$$\left. \geq \left\|\frac{1}{\tau} y(t) + \omega(t)\right\| e(t), t \in [0, 1]\right\} \tag{21}$$
$$= \{y \in X \mid y(t) + \tau\omega(t)$$
$$\geq \|y(t) + \tau\omega(t)\| e(t), t \in [0, 1]\}.$$

Lemma 7. *Let* M_0 *be a positive number such that* $f(x) \geq -M_0$, *for each* $x \in R$. *Then*

(1) $A : Y \to W$;

(2) *for each* $0 < \tau_1 < \tau_2 < \infty$, $\tau_1 W \subset \tau_2 W$.

Proof. (1) For each $x \in Y$, let $y = Ax + \omega = L(Fx + M_0x_0)$. By direct computation we have

$$y''(t) + (f(x(t)) + M_0) = 0, \quad 0 < t < 1$$
$$y(0) = 0, \qquad y(1) = \int_0^1 a(s) y(s)\, ds. \tag{22}$$

Since $f(x(t)) + M_0 \geq 0$, then $y''(t) \leq 0$, and so y is a concave function on $[0, 1]$. From (7), it is easy to see that

$$y(1) = \frac{1}{1 - \int_0^1 sa(s)\, ds}$$
$$\times \iint_0^1 G(\tau, s) a(\tau) [f(x(s)) + M_0]\, d\tau\, ds \geq 0. \tag{23}$$

Using the concavity of y and the boundary condition $y(0) = 0$, $y(1) \geq 0$, we can see that $y(t) \geq 0$ for each $t \in [0, 1]$ and $\|y\| = \max_{t \in [0, 1]} y(t)$; we have from the concavity of y that

$$y(t) \leq \frac{y(1) - y(\xi)}{1 - \xi} (t - 1) + y(1), \quad t \in [0, \xi]. \tag{24}$$

By (19), we have $y(\xi) = y(1)/\int_0^1 a(s)ds$. Hence,

$$y(t)$$
$$\leq y(1) \frac{\int_0^1 a(s)(1 - \xi)\, ds + (1 - t)\left(1 - \int_0^1 a(s)\, ds\right)}{\int_0^1 a(s)(1 - \xi)\, ds}$$
$$\leq y(1) \frac{\int_0^1 a(s)(1 - \xi)\, ds + \left(1 - \int_0^1 a(s)\, ds\right)}{\int_0^1 a(s)(1 - \xi)\, ds}$$
$$= y(1) \frac{1 - \xi \int_0^1 a(s)\, ds}{\int_0^1 a(s)(1 - \xi)\, ds}, \quad t \in [0, \xi]. \tag{25}$$

From the concavity of y, we have for each $t \in [\xi, 1]$ that

$$y(t) \leq \frac{y(\xi)}{\xi} t \leq \frac{y(\xi)}{\xi}$$
$$\leq y(1) \frac{1 - \xi \int_0^1 a(s)\, ds}{\int_0^1 a(s)\xi(1 - \xi)\, ds}. \tag{26}$$

It follows from (25) and (26) that

$$y(1) \geq \frac{\xi(1 - \xi) \int_0^1 a(s)\, ds}{1 - \xi \int_0^1 a(s)\, ds} \|y\|. \tag{27}$$

Then we have from the concavity of y that

$$y(t) \geq (y(1) - y(0)) t = y(1) t$$
$$\geq \frac{\xi(1 - \xi) \int_0^1 a(s)\, ds}{1 - \xi \int_0^1 a(s)\, ds} \|y\| t = \|y\| e(t); \tag{28}$$

that is,

$$Ax(t) + \omega(t) \geq \|Ax(t) + \omega(t)\| e(t), \quad t \in [0, 1]. \tag{29}$$

This implies that $A : Y \to W$, and, therefore, conclusion (1) holds.

(2) Since $A\theta = \theta$, from (1), we see that

$$\omega(t) = A\theta(t) + \omega(t) \geq \|A\theta(t) + \omega(t)\| e(t)$$
$$= \|\omega(t)\| e(t), \quad t \in [0, 1]. \tag{30}$$

For each $x \in \tau_1 W$, we have

$$x(t) + \tau_1\omega(t) \geq \|x(t) + \tau_1\omega(t)\| e(t), \quad t \in [0, 1]. \tag{31}$$

Then by (30) and (31), we have

$$
\begin{aligned}
x(t) + \tau_2 \omega(t) = x(t) &+ \tau_1 \omega(t) + (\tau_2 - \tau_1)\omega(t) \\
&\geq \big[\|x(t) + \tau_1 \omega(t)\| \\
&\quad + (\tau_2 - \tau_1)\|\omega(t)\|\big] e(t) \\
&\geq \|x(t) + \tau_2 \omega(t)\| e(t), \quad t \in [0,1].
\end{aligned}
\tag{32}
$$

This implies that $x \in \tau_2 W$. Thus, $\tau_1 W \subset \tau_2 W$. $\qquad\square$

3. Main Results

Theorem 8. *Suppose that $a(s)$ is symmetrical in $[0,1]$, $f_0 \in (0,\infty)$, and there exists $M_0 > 0$ such that $f(x) \geq -M_0$ for each $x \in R$. Then for each integer $i > 0$ and each $\nu = +$, or $-$, there exists an unbounded maximal subcontinuum $C_{n_i}^\nu$ of solutions of BVP (1) in $\Phi_{2i}^\nu \cup \{((2i\pi)^2/f_0, 0)\}$, which meets $\{((2i\pi)^2/f_0, 0)\}$ in Σ and satisfies*

(1) $C_{n_i}^+ \cap (\{\lambda\} \times Y) \neq \emptyset$ for each $i \geq 1$, $\lambda \geq (2i\pi)^2/f_0$;

(2) $C_{n_i}^- \cap (\{\lambda\} \times Y) \neq \emptyset$ for each $i \geq 1$, $\lambda \geq (2i\pi)^2/f_0$.

Proof. Since $f_0 \in (0,\infty)$, the operator equation (18) can be rewritten as

$$
x = \lambda f_0 K x + H(\lambda, x).
\tag{33}
$$

Here $H(\lambda, x) = \lambda A x - \lambda f_0 K x$ and K is defined as in (7). Obviously, it is easy to see that $H(\lambda, u) = o(\|u\|_1)$ for u near 0 uniformly on bounded λ intervals. Notice that K is a compact linear map on Y. A solution of BVP (1) is a pair $(\lambda, x) \in E$. By $f_0 \in (0,\infty)$, the known curve of solutions $\{(\lambda, 0) \mid \lambda \in R^+\}$ will henceforth be referred to as the trivial solutions. The closure of the set on nontrivial solutions of BVP (1) will be denoted by Σ as in Theorem A.

If $H(\lambda, x) \equiv 0$, then (33) becomes a linear system

$$
x = \lambda f_0 K x.
\tag{34}
$$

By Lemmas 3, 4, and 5, (34) possesses an increasing subsequence $\{\lambda_{n_i}/f_0\} = \{(2i\pi)^2/f_0\}$ of simple eigenvalues sequence $\{\lambda_n/f_0\}$ and $(2i\pi)^2/f_0 \to +\infty$ as $i \to +\infty$. Any eigenfunction $\varphi_k(t) = \sin(\eta_k t)$ corresponding to λ_k/f_0 is in T_k^+. Moreover, $\varphi_{n_i}(t) \in T_{2i}^+$ for $i = 1, 2, \ldots$ and $\varphi_k(t) \notin T_{2i}^+$ for $k \neq n_i$.

Consider (33) as a bifurcation problem from the trivial solution. From Theorem A and $f_0 \in (0,\infty)$, it follows that, for each integer $i \geq 1$, $\sum$ possess a maximal subcontinuum $C_{n_i} \subseteq E$ which can be decomposed into two subcontinua $C_{n_i}^+$, $C_{n_i}^-$ such that, for some neighborhood B of $((2i\pi)^2/f_0, 0)$,

$$
(\lambda, x) \in C_{n_i}^+\left(C_{n_i}^-\right) \cap B, \quad (\lambda, x) \neq \left(\frac{(2i\pi)^2}{f_0}, 0\right),
\tag{35}
$$

implying $(\lambda, x) = (\lambda, \alpha\varphi_{n_i} + w)$, where $\alpha > 0$ ($\alpha < 0$) and $|\lambda - ((2i\pi)^2/f_0)| = o(1)$, $\|w\|_1 = o(|\alpha|)$ for α near 0.

By (18) and the continuity of the operator $A : Y \to Z$, the set $C_{n_i}^\nu$ lies in $R \times Z$ and the injection $C_{n_i}^\nu \to R \times Z$ is continuous. Moreover, note that $C_{n_i}^\nu \cap (\{0\} \times Z) = \emptyset$. So, $C_{n_i}^\nu$ is also a continuum in $R \times Z$, and the above properties hold in $R \times Z$.

Since T_{2i} is open in Z and $\varphi_{n_i}(t) \in T_{2i}^+$, we know that

$$
\frac{x}{\alpha} = \varphi_{n_i} + \frac{w}{\alpha} \in T_{2i}^+
\tag{36}
$$

for $0 \neq \alpha$ sufficiently small. Then there exists $\epsilon_0 > 0$ such that, for $\epsilon \in (0, \epsilon_0)$, we have

$$
(\lambda, x) \in \Phi_{2i}, \quad \left(C_{n_i} \setminus \left\{\left(\frac{(2i\pi)^2}{f_0}, 0\right)\right\}\right) \cap B_\epsilon \subset \Phi_{2i},
\tag{37}
$$

where B_ϵ is an open ball in $R \times Z$ of radius ϵ centered at $((2i\pi)^2/f_0, 0)$. Since T_{2i}^ν is open in Z, it can follow, similar to the proof of Proposition 4.1 in [15], that

$$
(\lambda, x) \in C_{n_i} \cap (R \times \partial T_{2i}) \implies x = 0,
\tag{38}
$$

which means $C_{n_i} \setminus \{((2i\pi)^2/f_0, 0)\} \cap \partial\Phi_{2i} = \emptyset$. Consequently, C_{n_i} lies in $\Phi_{2i} \cup \{((2i\pi)^2/f_0, 0)\}$.

Similarly we can obtain that $C_{n_i}^\nu$ lies in $\Phi_{2i}^\nu \cup \{((2i\pi)^2/f_0, 0)\}$ ($\nu = +$ or $-$). Noticing that $T_k^+ \cap T_k^- = \emptyset$, it can be obtained that $C_{n_i}^+ \cap C_{n_i}^- = ((2i\pi)^2/f_0, 0)$. From Theorem A, we know that $C_{n_i}^+$ and $C_{n_i}^-$ are unbounded in $R \times Z$.

Let $\lambda \geq (2i\pi)^2/f_0$ be fixed. For each $0 < \tau < \lambda$ and $x \in Z$, (τ, x) is a solution of (18), and by Lemma 7, $x = \tau A x \in \tau W \subset \lambda W$. Thus,

$$
x(t) + \lambda \omega(t) \geq \|x(t) + \lambda \omega(t)\| e(t), \quad t \in [0,1],
\tag{39}
$$

since

$$
\begin{aligned}
\omega(t) = M_0 \omega_0(t) &\leq M_0 \frac{t}{1 - \xi \int_0^1 a(s)\, ds} \int_0^1 (1-s)\, ds \\
&< \frac{(M_0 + 1)t}{2\left(1 - \xi \int_0^1 a(s)\, ds\right)} \leq c_1 e(t), \quad t \in [0,1],
\end{aligned}
\tag{40}
$$

where $c_1 \geq (M_0 + 1)/2\xi \int_0^1 a(s)(1-\xi)\, ds$, and so

$$
x(t) \geq (\|x\| - \lambda\|\omega\| - \lambda c_1) e(t), \quad t \in [0,1].
\tag{41}
$$

Let $R(\lambda) = 2(\lambda\|\omega\| + \lambda c_1)$. Then for each $(\tau, x) \in \Sigma$, $\tau \neq 0$, $\|x\| \geq R(\lambda)$, we have $x(t) \geq 0$ for $t \in [0,1]$. This implies that

$$
\begin{aligned}
C_{n_i}^+ \cap ([0,\lambda] \times \{x \in Y \mid \|x\| = R(\lambda)\}) = \emptyset \quad (i \geq 1), \\
C_{n_i}^- \cap ([0,\lambda] \times \{x \in Y \mid \|x\| = R(\lambda)\}) = \emptyset \quad (i \geq 1).
\end{aligned}
\tag{42}
$$

Thus, the conclusion holds and the proof is complete. $\qquad\square$

Immediately, from Theorem 8, we have the following result.

Theorem 9. *Suppose that all the conditions of Theorem 8 hold. Then, for each $\lambda > (2i\pi)^2/f_0$, BVP (1) has at least $2i$ nodal solutions $u_{n_1}^+, u_{n_1}^-, u_{n_2}^+, u_{n_2}^-, \ldots, u_{n_i}^+, u_{n_i}^-$ in Y such that $u_{n_i}^+$ has $(2i-1)$ zeros in $(0,1)$ and is positive near $t = 0$ and $u_{n_i}^-$ has $(2i-1)$ zeros in $(0,1)$ and is negative near $t = 0$.*

4. An Example

Consider the following nonlinear second-order integral boundary value problem:

$$x''(t) + f(x(t)) = 0, \quad 0 < t < 1$$
$$x(0) = 0, \qquad x(1) = \int_0^1 (s - s^2) x(s) \, ds, \tag{43}$$

where $f(x) = 104\pi^2 \sin(x)$.

By direct computation, it is easy to see that $f_0 = \lim_{x\to 0}(104\pi^2 \sin(x)/\sin(x)) = 104\pi^2$, so, $(2i\pi)^2/f_0 = 4i^2\pi^2/104\,\pi^2 = i^2/26$.

Next, we check that all the conditions of Theorem 9 hold. Take $a(s) = s - s^2$. It is clear that $0 < \int_0^1 a^2(s)ds = 1/30 < 1$, and $a(s)$ is symmetrical in $[0,1]$, and $a(s) \in L[0,1]$ is nonnegative. Since $f(x) = 104\pi^2 \sin(x) \geq 104\pi^2$, $f_0 = 104\pi^2 \in (0,\infty)$, and $\lambda = 1$. It follows from Theorem 9, when $i^2/26 < 1$, we have $i = 1, 2, 3, 4, 5$, so the boundary value problem (43) has at least 10 nodal solutions in $C^1[0,1]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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