Research Article

Extremal Inverse Eigenvalue Problem for a Special Kind of Matrices

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We consider the following inverse eigenvalue problem: to construct a special kind of matrix (real symmetric doubly arrow matrix) from the minimal and maximal eigenvalues of all its leading principal submatrices. The necessary and sufficient condition for the solvability of the problem is derived. Our results are constructive and they generate algorithmic procedures to construct such matrices.

1. Introduction

Peng et al. in [1] solved two inverse eigenvalue problems for symmetric arrow matrices and, in the other article [2], a correction, for one of the problems stated in [1], has been presented as well. In recent paper [3], Nazari and Beiranvand introduced an algorithm to construct symmetric quasi-antibidiagonal matrices that having its given eigenvalues. Pickmann et al. in [4] introduced an algorithm for inverse eigenvalue problem on symmetric tridiagonal matrices. In this paper we introduced symmetric doubly arrow matrix as follows:

$$
A = \begin{pmatrix}
a_1 & b_1 & \cdots & b_{s-1} & 0 & \cdots & 0 \\
cb_1 & a_2 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 & 0 & \cdots & 0 \\
b_{s-1} & 0 & \cdots & a_s & b_s & \cdots & b_{n-1} \\
0 & 0 & \cdots & b_s & a_{s+1} & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & b_{n-1} & 0 & \cdots & a_n
\end{pmatrix}, \quad a_j, b_j \in \mathbb{R},
\tag{1}
$$

where $b_j \geq 0$, $1 \leq s \leq n$. If $s = 1$ or $s = n$; then the matrix A of the form (1) is a symmetric arrow matrix as follows:

$$
B = \begin{pmatrix}
a_1 & b_1 & \cdots & b_{n-1} \\
b_1 & a_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
b_{n-1} & 0 & \cdots & a_n
\end{pmatrix}, \quad a_j, b_j \in \mathbb{R}.
\tag{2}
$$

This family of matrices appears in certain symmetric inverse eigenvalue and inverse Sturm-Liouville problems [5, 6], which arise in many applications [7–12], including modern control theory and vibration analysis [7, 8]. In this paper, we construct matrix A of the form (1), from a special kind of spectral information, which only recently is being considered. Since this type of matrix structure generalizes the well-known arrow matrices, we think that it will also become of interest in applications.

We will denote I_j as the identity matrix of order j; A_j as the $j \times j$ leading principal submatrix of A; $P_j(\lambda)$ as the characteristic polynomial of A_j; and $\lambda_1^{(j)} \leq \lambda_2^{(j)} \leq \cdots \leq \lambda_j^{(j)}$ as the eigenvalues of A_j.

We want to solve the following problem.

Problem 1. Given the $2n - 1$ real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, find an $n \times n$ matrix A of the form (1) such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and maximal eigenvalues of A_j, $j = 1, 2, \ldots, n$.

Our work is motivated by the results in [2]. There, the authors solved this kind of inverse eigenvalue problem for symmetric arrow matrix B of the form (2).

Theorem 2 (see [2]). *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, be given. Then there exists an $n \times n$ matrix B of the form (2), such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and maximal eigenvalues of its leading principal submatrix B_j, $j = 1, 2, \ldots, n$, if and only if*

$$\lambda_1^{(n)} \le \cdots \le \lambda_1^{(3)} \le \lambda_1^{(2)} \le \lambda_1^{(1)} \le \lambda_2^{(2)} \le \lambda_3^{(3)} \le \cdots \le \lambda_n^{(n)}. \tag{3}$$

Theorem 3 (see [2]). *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, be given. Then there exists a unique $n \times n$ matrix B of the form (2), with $a_j \in \mathbb{R}$ and $b_j > 0$, such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and the maximal eigenvalues of its leading principal submatrix B_j, $j = 1, 2, \ldots, n$, if and only if*

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_n^{(n)}. \tag{4}$$

In this paper, we will show that Theorems 2 and 3 are also right for symmetric doubly arrow matrix A in (1) by a similar method.

The paper is organized as follows. In Section 2, we discuss some properties of A. In Section 3, we solve Problem 1 by giving a necessary and sufficient condition for the existence of the matrix A in (1) and also solve the case in which the matrix A, in Problem 1, is required to have all its entries b_i positive. Finally, In Section 4 we show some examples to illustrate the results.

2. Properties of the Matrix A

Lemma 4. *Let A be a matrix of the form (1). Then the sequence of characteristic polynomials $\{P_j(\lambda)\}_{j=1}^n$ satisfies the recurrence relation:*

$$p_1(\lambda) = (\lambda - a_1),$$

$$p_j(\lambda) = (\lambda - a_j) p_{j-1}(\lambda) - b_{j-1}^2 \prod_{i=2}^{j-1} (\lambda - a_i),$$

$$j = 2, 3, \ldots, s, \tag{5}$$

$$p_j(\lambda) = (\lambda - a_j) p_{j-1}(\lambda) - b_{j-1}^2 \prod_{i=s+1}^{j-1} (\lambda - a_i) p_{s-1}(\lambda),$$

$$j = s + 1, \ldots, n.$$

Proof. It is easy to verify by expanding the determinant. $\quad\square$

Lemma 5 (see [2]). *Let $p(\lambda)$ be a monic polynomial of degree n with all real zeroes. If λ_1 and λ_n are, respectively, the minimal and maximal zeroes of $p(\lambda)$, then*

(1) *if $\mu < \lambda_1$, we have that $(-1)^n p(\mu) > 0$;*

(2) *if $\mu > \lambda_n$, we have that $p(\mu) > 0$.*

Observe that, from the Cauchy interlacing property, the minimal and the maximal eigenvalue, $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, respectively, of each leading principal submatrix A_j, $j = 1, 2, \ldots, n$, of the matrix A in (1) satisfy the relations

$$\lambda_1^{(n)} \le \cdots \le \lambda_1^{(3)} \le \lambda_1^{(2)} \le \lambda_1^{(1)} \le \lambda_2^{(2)} \le \lambda_3^{(3)} \le \cdots \le \lambda_n^{(n)}, \tag{6}$$

$$\lambda_1^{(j)} \le a_i \le \lambda_j^{(j)}, \quad i = 1, 2, \ldots, j; \; j = 1, 2, \ldots, n. \tag{7}$$

Lemma 6. *Let $\{P_j(\lambda)\}_{j=1}^n$ be the polynomials defined in (5), whose minimal and maximal zeroes, $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, respectively, satisfy the relations (6) and (7), and*

$$h_j$$

$$=
\begin{cases}
p_{j-1}\left(\lambda_1^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - p_{j-1}\left(\lambda_j^{(j)}\right) \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right), \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = 2, 3, \ldots, s, \\[6pt]
p_{j-1}\left(\lambda_1^{(j)}\right) p_{s-1}\left(\lambda_j^{(j)}\right) \\
\quad \times \prod_{i=s+1}^{j-1}\left(\lambda_j^{(j)} - a_i\right) - p_{j-1}\left(\lambda_j^{(j)}\right) p_{s-1}\left(\lambda_1^{(j)}\right) \\
\quad \times \prod_{i=s+1}^{j-1}\left(\lambda_1^{(j)} - a_i\right), \quad j = s+1, \ldots, n.
\end{cases}
\tag{8}$$

Then

$$\widetilde{h}_j = (-1)^{j-1} h_j \ge 0, \quad j = 2, 3, \ldots, n. \tag{9}$$

Proof. From Lemma 5, we have

$$(-1)^{j-1} p_{j-1}\left(\lambda_1^{(j)}\right) \ge 0, \qquad p_{j-1}\left(\lambda_j^{(j)}\right) \ge 0,$$

$$j = 2, 3, \ldots, n,$$

$$(-1)^{s-1} p_{s-1}\left(\lambda_1^{(j)}\right) \ge 0, \qquad p_{s-1}\left(\lambda_j^{(j)}\right) \ge 0,$$

$$j = s+1, \ldots, n. \tag{10}$$

Moreover, from (7)

$$\prod_{i=2}^{j-1}\left(\lambda_j^{(j)} - a_i\right) \ge 0, \qquad (-1)^j \prod_{i=2}^{j-1}\left(\lambda_1^{(j)} - a_i\right) \ge 0,$$

$$j = 2, 3, \ldots, s,$$

$$\prod_{i=s+1}^{j-1}\left(\lambda_j^{(j)} - a_i\right) \ge 0, \qquad (-1)^{j-s+1} \prod_{i=s+1}^{j-1}\left(\lambda_1^{(j)} - a_i\right) \ge 0,$$

$$j = s+1, \ldots, n. \tag{11}$$

Clearly $\widetilde{h}_j \ge 0$ from (10) and (11). $\quad\square$

Lemma 7 (see [2]). *Let A be a matrix of the form* (2) *with $b_i \neq 0$, $i = 1, 2, \ldots, n-1$. Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, respectively, be the minimal and maximal eigenvalues of the leading principal submatrix A_j, $j = 1, 2, \ldots, n$, of A. Then*

$$\lambda_1^{(j)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_j^{(j)},$$

$$\lambda_1^{(j)} < a_i < \lambda_j^{(j)}, \quad i = 2, 3, \ldots, j \tag{12}$$

for each $j = 2, 3, \ldots, n$.

3. Solution of Problem 1

The following theorem solves Problem 1. In particular, the theorem shows that condition (6) is necessary and sufficient for the existence of the matrix A in (1).

Theorem 8. *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, be given. Then there exists an $n \times n$ matrix A of the form* (1)*, such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and maximal eigenvalues of its leading principal submatrix A_j, $j = 1, 2, \ldots, n$, if and only if*

$$\lambda_1^{(n)} \leq \cdots \leq \lambda_1^{(3)} \leq \lambda_1^{(2)} \leq \lambda_1^{(1)} \leq \lambda_2^{(2)} \leq \lambda_3^{(3)} \leq \cdots \leq \lambda_n^{(n)}. \tag{13}$$

Proof. Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, satisfy (13). Observe that

$$A_1 = (a_1) = \left(\lambda_1^{(1)} \right) \tag{14}$$

and $p_1(\lambda) = \lambda - a_1$. From Theorem 2, there exists A_j, $j = 2, \ldots, s$ with $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ as its minimal and maximal eigenvalues, respectively. To show the existence of A_j, $j = s+1, \ldots, n$ with $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ as its minimal and maximal eigenvalues, respectively, is equivalent to showing that the system of equations

$$p_j\left(\lambda_1^{(j)}\right) = \left(\lambda_1^{(j)} - a_j\right) p_{j-1}\left(\lambda_1^{(j)}\right)$$

$$- b_{j-1}^2 \prod_{i=s+1}^{j-1} \left(\lambda_1^{(j)} - a_i\right) p_{s-1}\left(\lambda_1^{(j)}\right) = 0,$$

$$p_j\left(\lambda_j^{(j)}\right) = \left(\lambda_j^{(j)} - a_j\right) p_{j-1}\left(\lambda_j^{(j)}\right) \tag{15}$$

$$- b_{j-1}^2 \prod_{i=s+1}^{j-1} \left(\lambda_j^{(j)} - a_i\right) p_{s-1}\left(\lambda_j^{(j)}\right) = 0$$

has real solution a_j and b_{j-1}, $j = s+1, \ldots, n$. If the determinant

$$h_j = p_{j-1}\left(\lambda_1^{(j)}\right) p_{s-1}\left(\lambda_j^{(j)}\right)$$

$$\times \prod_{i=s+1}^{j-1} \left(\lambda_j^{(j)} - a_i\right) - p_{j-1}\left(\lambda_j^{(j)}\right) p_{s-1}\left(\lambda_1^{(j)}\right) \tag{16}$$

$$\times \prod_{i=s+1}^{j-1} \left(\lambda_1^{(j)} - a_i\right)$$

of the coefficient matrix of the system (15) is nonzero, then the system has unique solutions a_j and b_{j-1}^2, $j = s+1, \ldots, n$. In this case, from Lemma 6 we have $\tilde{h}_j > 0$. By solving the system (15) we obtain

$$a_j = \left(\lambda_1^{(j)} p_{j-1}\left(\lambda_1^{(j)}\right) p_{s-1}\left(\lambda_j^{(j)}\right) \right.$$

$$\times \prod_{i=s+1}^{j-1} \left(\lambda_j^{(j)} - a_i\right) - \lambda_j^{(j)} p_{j-1}\left(\lambda_j^{(j)}\right) p_{s-1}\left(\lambda_1^{(j)}\right)$$

$$\left. \times \prod_{i=s+1}^{j-1} \left(\lambda_1^{(j)} - a_i\right) \right) \left(h_j\right)^{-1}, \tag{17}$$

$$b_{j-1}^2 = \frac{\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) p_{j-1}\left(\lambda_1^{(j)}\right) p_{j-1}\left(\lambda_j^{(j)}\right)}{h_j}.$$

Since

$$(-1)^{j-1} \left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) p_{j-1}\left(\lambda_1^{(j)}\right) p_{j-1}\left(\lambda_j^{(j)}\right) \geq 0; \tag{18}$$

then b_{j-1} is a real number and therefore, there exists A with the spectral properties required.

Now we will show that, if $h_j = 0$, the system (15) still has a solution. We do this by induction by showing that the rank of the coefficients matrix is equal to the rank of the augmented matrix.

Let $j = s+1$. If $h_{s+1} = 0$, then

$$\tilde{h}_{s+1} = (-1)^s h_{s+1}$$

$$= (-1)^s \left(p_s\left(\lambda_1^{(s+1)}\right) p_{s-1}\left(\lambda_{s+1}^{(s+1)}\right) \right. \tag{19}$$

$$\left. - p_s\left(\lambda_{s+1}^{(s+1)}\right) p_{s-1}\left(\lambda_1^{(s+1)}\right) \right) = 0,$$

which, from Lemma 5, is equivalent to

$$p_s\left(\lambda_1^{(s+1)}\right) p_{s-1}\left(\lambda_{s+1}^{(s+1)}\right) = 0,$$

$$p_s\left(\lambda_{s+1}^{(s+1)}\right) p_{s-1}\left(\lambda_1^{(s+1)}\right) = 0. \tag{20}$$

In this case the augmented matrix is

$$\begin{pmatrix} p_s\left(\lambda_1^{(s+1)}\right) & p_{s-1}\left(\lambda_1^{(s+1)}\right) & \lambda_1^{(s+1)} p_s\left(\lambda_1^{(s+1)}\right) \\ p_s\left(\lambda_{s+1}^{(s+1)}\right) & p_{s-1}\left(\lambda_{s+1}^{(s+1)}\right) & \lambda_{s+1}^{(s+1)} p_s\left(\lambda_{s+1}^{(s+1)}\right) \end{pmatrix}, \tag{21}$$

and the ranks of both matrices, the coefficient matrix and the augmented matrix, are equal. Hence A_{s+1} exists.

Now we consider $j \geq s + 2$. If $h_j = 0$, then

$$
\tilde{h}_j = (-1)^{j-1} h_j
$$

$$
= (-1)^{j-1} \left(p_{j-1}\left(\lambda_1^{(j)}\right) p_{s-1}\left(\lambda_j^{(j)}\right) \right.
$$

$$
\times \prod_{i=s+1}^{j-1} \left(\lambda_j^{(j)} - a_i\right) - p_{j-1}\left(\lambda_j^{(j)}\right) p_{s-1}\left(\lambda_1^{(j)}\right)
$$

$$
\left. \times \prod_{i=s+1}^{j-1} \left(\lambda_1^{(j)} - a_i\right) \right) = 0.
$$

(22)

From Lemma 5

$$
p_{j-1}\left(\lambda_1^{(j)}\right) = 0 \bigvee p_{s-1}\left(\lambda_j^{(j)}\right) \prod_{i=s+1}^{j-1} \left(\lambda_j^{(j)} - a_i\right) = 0,
$$

(23)

$$
p_{j-1}\left(\lambda_j^{(j)}\right) = 0 \bigvee p_{s-1}\left(\lambda_1^{(j)}\right) \prod_{i=s+1}^{j-1} \left(\lambda_1^{(j)} - a_i\right) = 0.
$$

Then $h_j = 0$ leads us to the following cases:

(i) $\lambda_1^{(j)} = \lambda_1^{(j-1)} \wedge \lambda_{j-1}^{(j-1)} = \lambda_j^{(j)}$,

(ii) $\lambda_1^{(j)} = \lambda_1^{(j-1)} \wedge p_{s-1}(\lambda_1^{(j)}) \prod_{i=s+1}^{j-1}(\lambda_1^{(j)} - a_i) = 0$,

(iii) $p_{s-1}(\lambda_j^{(j)}) \prod_{i=s+1}^{j-1}(\lambda_j^{(j)} - a_i) = 0 \wedge \lambda_{j-1}^{(j-1)} = \lambda_j^{(j)}$,

(iv) $p_{s-1}(\lambda_j^{(j)}) \prod_{i=s+1}^{j-1}(\lambda_j^{(j)} - a_i) = 0 \wedge p_{s-1}(\lambda_1^{(j)}) \prod_{i=s+1}^{j-1}(\lambda_1^{(j)} - a_i) = 0$,

and the augmented matrix is

$$
\begin{pmatrix}
p_{j-1}\left(\lambda_1^{(j)}\right) & p_{s-1}\left(\lambda_1^{(j)}\right) \prod_{i=s+1}^{j-1} \left(\lambda_1^{(j)} - a_i\right) & \lambda_1^{(j)} p_{j-1}\left(\lambda_1^{(j)}\right) \\
p_{j-1}\left(\lambda_j^{(j)}\right) & p_{s-1}\left(\lambda_j^{(j)}\right) \prod_{i=s+1}^{j-1} \left(\lambda_j^{(j)} - a_i\right) & \lambda_j^{(j)} p_{j-1}\left(\lambda_j^{(j)}\right)
\end{pmatrix}.
$$

(24)

By replacing conditions (i)–(iii) in (24), it is clear that the coefficients matrix and the augmented matrix have the same rank. From condition (iv), the system of (15) becomes

$$
p_{j-1}\left(\lambda_1^{(j)}\right) a_j = \lambda_1^{(j)} p_{j-1}\left(\lambda_1^{(j)}\right),
$$

$$
p_{j-1}\left(\lambda_j^{(j)}\right) a_j = \lambda_j^{(j)} p_{j-1}\left(\lambda_j^{(j)}\right).
$$

(25)

If $p_{j-1}(\lambda_1^{(j)}) \neq 0$ and $p_{j-1}(\lambda_j^{(j)}) \neq 0$, then $a_j = \lambda_1^{(j)} = \lambda_j^{(j)}$ and from (13)

$$
\lambda_1^{(j)} = \lambda_1^{(j-1)} = \cdots = \lambda_1^{(1)} = \cdots = \lambda_{j-1}^{(j-1)} = \lambda_j^{(j)}.
$$

(26)

Thus, $p_{j-1}(\lambda_1^{(j)}) = p_{j-1}(\lambda_j^{(j)}) = 0$, which is a contradiction. Hence, under condition (iv) $p_{j-1}(\lambda_1^{(j)}) = 0$ or $p_{j-1}(\lambda_j^{(j)}) = 0$

and therefore the coefficients matrix and the augmented matrix have also the same rank. By taking $b_{j-1}^2 \geq 0$, there exists a $j \times j$ matrix A_j with the required spectral properties. The necessity comes from the Cauchy interlacing property. □

We have seen in the proof of Theorem 8 that if the determinant h_j of the coefficients matrix of the system (15) is nonzero, then the Problem 1 has a unique solution except for the sign of the b_i entries.

Now we solve the Problem 1 in the case that the b_i entries are required to be positive. We need the following lemma.

Lemma 9. *Let A be a matrix of the form* (1) *with $b_i \neq 0$, $i = 1, 2, \ldots, n - 1$. Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, respectively, be the minimal and maximal eigenvalues of the leading principal submatrix A_j, $j = 1, 2, \ldots, n$, of A. Then*

$$
\lambda_1^{(j)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_j^{(j)},
$$

$$
\lambda_1^{(j)} < a_i < \lambda_j^{(j)}, \quad i = 2, 3, \ldots, j
$$

(27)

for each $j = 2, 3, \ldots, n$.

Proof. From Lemma 7, (27) hold for $j \leq s$. For $j = s + 1$, we have from (5)

$$
p_{s+1}(\lambda) = (\lambda - a_{s+1}) p_s - b_s^2 p_{s-1}(\lambda).
$$

(28)

As $b_s \neq 0$, then from Lemma 7, we have $p_{s+1}(\lambda_1^{(s)}) \neq 0$, $p_{s+1}(\lambda_s^{(s)}) \neq 0$, and

$$
\lambda_1^{(s+1)} < \lambda_1^{(s)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_s^{(s)} < \lambda_{s+1}^{(s+1)}.
$$

(29)

If $\lambda_1^{(s+1)} = a_{s+1}$ or $\lambda_{s+1}^{(s+1)} = a_{s+1}$, then

$$
0 = p_{s+1}(a_{s+1}) = -b_s^2 p_{s-1}(a_{s+1})
$$

(30)

contradicts $b_s \neq 0$ or (29) and from (7) we have

$$
\lambda_1^{(s+1)} < a_{s+1} < \lambda_{s+1}^{(s+1)}.
$$

(31)

Let $j = s + 2$. Then from (5)

$$
p_{s+2}\left(\lambda_1^{(s+1)}\right) = -b_{s+1}^2 \left(\lambda_1^{(s+1)} - a_{s+1}\right) p_{s-1}\left(\lambda_1^{(s+1)}\right) \neq 0.
$$

(32)

In the same way $p_{s+2}(\lambda_{s+1}^{(s+1)}) \neq 0$. Hence $\lambda_1^{(s+1)}$ and $\lambda_{s+1}^{(s+1)}$ are not zeroes of $p_{s+2}(\lambda)$ and from (6)

$$
\lambda_1^{(s+2)} < \lambda_1^{(s+1)} < \lambda_1^{(s)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)}
$$

$$
< \lambda_2^{(2)} < \cdots < \lambda_s^{(s)} < \lambda_{s+1}^{(s+1)} < \lambda_{s+2}^{(s+2)}.
$$

(33)

Now suppose that $\lambda_1^{(s+2)} = a_{s+2}$. Then

$$
0 = p_{s+2}(a_{s+2}) = -b_{s+1}^2 (a_{s+2} - a_{s+1}) p_{s-1}(a_{s+2})
$$

$$
= -b_{s+1}^2 \left(\lambda_1^{s+2} - a_{s+1}\right) p_{s-1}(a_{s+2})
$$

(34)

contradicts the inequalities (31) and (33). The same occurs if we assume that $\lambda_{s+2}^{(s+2)} = a_{s+2}$. Then from (7) and **Lemma 7** we have

$$\lambda_1^{(s+2)} < a_i < \lambda_{s+2}^{(s+2)}, \quad i = 2, 3, \ldots, s + 2. \tag{35}$$

Now, suppose that (27) hold for $s + 3 \leq j \leq n - 1$ and consider

$$p_{j+1}(\lambda) = \left(\lambda - a_{j+1}\right) p_j(\lambda) - b_j^2 \prod_{i=s+1}^{j} \left(\lambda - a_i\right) p_{s-1}(\lambda). \tag{36}$$

Since $b_j \neq 0$ and $\lambda_1^{(j)} < a_i < \lambda_j^{(j)}$, $i = 2, 3, \ldots, j$, then $\prod_{i=s+1}^{j}(\lambda_1^j - a_i) \neq 0$ and $\prod_{i=s+1}^{j}(\lambda_j^j - a_i) \neq 0$. Hence neither λ_1^j nor λ_j^j are zeroes of $p_{j+1}(\lambda)$. Then from (6) we have

$$\lambda_1^{(j+1)} < \lambda_1^{(j)} < \cdots < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \cdots < \lambda_j^{(j)} < \lambda_{j+1}^{(j+1)}. \tag{37}$$

Finally, if $\lambda_1^{(j+1)} = a_{j+1}$, then

$$0 = p_{j+1}\left(a_{j+1}\right) = -b_j^2 \prod_{i=s+1}^{j} \left(a_{j+1} - a_i\right) p_{s-1}\left(a_{j+1}\right) \tag{38}$$
$$= -b_j^2 \prod_{i=s+1}^{j} \left(\lambda_1^{j+1} - a_i\right) p_{s-1}\left(\lambda_1^{j+1}\right)$$

contradicts (33). Then

$$\lambda_1^{(j+1)} < a_i < \lambda_{j+1}^{(j+1)}, \quad i = 2, 3, \ldots, j + 1. \tag{39}$$

□

The following theorem solves Problem 1 with $b_j > 0$.

Theorem 10. *Let the real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$, be given. Then there exists a unique $n \times n$ matrix A of the form (1), with $a_j \in \mathbb{R}$ and $b_j > 0$, such that $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ are, respectively, the minimal and maximal eigenvalues of its leading principal submatrix A_j, $j = 1, 2, \ldots, n$, if and only if*

$$\lambda_1^{(n)} < \cdots < \lambda_1^{(3)} < \lambda_1^{(2)} < \lambda_1^{(1)} < \lambda_2^{(2)} < \lambda_3^{(3)} < \cdots < \lambda_n^{(n)}. \tag{40}$$

Proof. The proof is quite similar to the proof of **Theorem 8:** Let $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 2, \ldots, n$, satisfy (40). From **Theorem 3**, there exists A_j, $j = 2, \ldots, s$ with $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$ as its minimal and maximal eigenvalues, respectively. To show the existence of A_j, $j = s + 1, \ldots, n$ with the required spectral properties, is equivalent to showing that the system of (15) has real solutions a_j and b_{j-1}, with $b_{j-1} > 0$, $j = s + 1, s + 2, \ldots, n$.

To do this it is enough to show that the determinant of the coefficients matrix

$$h_j = p_{j-1}\left(\lambda_1^{(j)}\right) p_{s-1}\left(\lambda_j^{(j)}\right)$$
$$\times \prod_{i=s+1}^{j-1} \left(\lambda_j^{(j)} - a_i\right) - p_{j-1}\left(\lambda_j^{(j)}\right) p_{s-1}\left(\lambda_1^{(j)}\right) \tag{41}$$
$$\times \prod_{i=s+1}^{j-1} \left(\lambda_1^{(j)} - a_i\right)$$

is nonzero.

From **Lemmas 6 and 9** it follows that $\tilde{h}_j = (-1)^{j-1} h_j > 0$. Hence $h_j \neq 0$ and the system (15) has real and unique solutions:

$$a_j = \left(\lambda_1^{(j)} p_{j-1}\left(\lambda_1^{(j)}\right) p_{s-1}\left(\lambda_j^{(j)}\right)\right.$$
$$\times \prod_{i=s+1}^{j-1} \left(\lambda_j^{(j)} - a_i\right) - \lambda_j^{(j)} p_{j-1}\left(\lambda_j^{(j)}\right) p_{s-1}\left(\lambda_1^{(j)}\right)$$
$$\left.\times \prod_{i=s+1}^{j-1} \left(\lambda_1^{(j)} - a_i\right)\right) \left(h_j\right)^{-1}, \tag{42}$$
$$b_{j-1}^2 = \frac{\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) p_{j-1}\left(\lambda_1^{(j)}\right) p_{j-1}\left(\lambda_j^{(j)}\right)}{h_j},$$

where

$$(-1)^{j-1}\left(\lambda_j^{(j)} - \lambda_1^{(j)}\right) p_{j-1}\left(\lambda_1^{(j)}\right) p_{j-1}\left(\lambda_j^{(j)}\right) > 0. \tag{43}$$

Then it is clear that $b_{j-1}^2 > 0$. Therefore, the b_{j-1} can be chosen positive and then there exists a unique matrix A_j with the required spectral properties. The necessity of the result comes from **Lemma 9**. □

4. Examples

Now we give an algorithm to construct the solution A of Problem 1.

Algorithm.

(1) Input a positive integer s and real numbers $\lambda_1^{(j)}$ and $\lambda_j^{(j)}$, $j = 1, 2, \ldots, n$;

(2) let $a_1 = \lambda_1^{(1)}$. $p_1(\lambda) = \lambda - a_1$;

(3) for $j = 2, \ldots, n$, calculate $p_j(\lambda)$ according to (5);

(4) compute a_j and b_{j-1} according to (17).

Example 1. The following numbers [2]

$$
\begin{array}{cccccc}
\lambda_1^{(5)} & \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \lambda_1^{(1)} & \lambda_2^{(2)} \\
-11.2369 & -11.1921 & -10.9106 & -8.7760 & -6.0043 & -2.6295 \\
\lambda_3^{(3)} & \lambda_4^{(4)} & \lambda_5^{(5)} \\
1.8532 & 8.4266 & 10.4020
\end{array}
\tag{44}
$$

satisfy the necessary and sufficient condition (40) of Theorem 10. Then the doubly arrow matrix with $b_i > 0$ and $s = 3$ is

$$
A = \begin{pmatrix}
-6.0043 & 3.0584 & 5.2453 & & \\
3.0584 & -5.4011 & & & \\
5.2453 & & -2.3357 & 4.4747 & 1.8880 \\
& & 4.4747 & 6.1414 & \\
& & 1.8880 & & 9.8361
\end{pmatrix}.
\tag{45}
$$

From the above 5×5 doubly arrow matrix A, we recompute the spectrum $\sigma(A_j)$ of its submatrix A_j by MATLAB 7.0, $j = 1, 2, \ldots, 5$, and get

$$
\begin{aligned}
\sigma(A_1) \quad & \overline{\lambda_1^{(1)}} = -6.0043, \\
\sigma(A_2) \quad & \overline{\lambda_1^{(2)}} = -8.7759 \quad \overline{\lambda_2^{(2)}} = -2.6295, \\
\sigma(A_3) \quad & \overline{\lambda_1^{(3)}} = -10.9106 \quad \overline{\lambda_2^{(3)}} = -4.6837 \quad \overline{\lambda_3^{(3)}} = 1.8532, \\
\sigma(A_4) \quad & \overline{\lambda_1^{(4)}} = -11.1921 \quad \overline{\lambda_2^{(4)}} = -5.0836 \quad \overline{\lambda_3^{(4)}} = 0.2494 \quad \overline{\lambda_4^{(4)}} = 8.4266, \\
\sigma(A_5) \quad & \overline{\lambda_1^{(5)}} = -11.2369 \quad \overline{\lambda_2^{(5)}} = -5.1390 \quad \overline{\lambda_3^{(5)}} = 0.1384 \quad \overline{\lambda_4^{(5)}} = 8.0719 \quad \overline{\lambda_5^{(5)}} = 10.4020.
\end{aligned}
\tag{46}
$$

Example 2. We modify the previous example, that some given eigenvalues become equal to [2]

$$
\begin{array}{cccccc}
\lambda_1^{(5)} & \lambda_1^{(4)} & \lambda_1^{(3)} & \lambda_1^{(2)} & \lambda_1^{(1)} & \lambda_2^{(2)} \\
-11.2369 & -10.9106 & -10.9106 & -8.7760 & -6.0043 & -6.0043 \\
\lambda_3^{(3)} & \lambda_4^{(4)} & \lambda_5^{(5)} \\
1.8532 & 8.4266 & 10.4020.
\end{array}
\tag{47}
$$

These numbers satisfy the necessary and sufficient condition (13) of Theorem 8. One solution of Problem 1 with $s = 3$ is the

matrix

$$
A = \begin{pmatrix}
-6.0043 & 0 & 6.2090 & & \\
0 & -8.7760 & & & \\
6.2090 & & -3.0531 & 0 & 4.0562 \\
& & 0 & 8.4266 & \\
& & 4.0562 & & 8.9205
\end{pmatrix}.
\tag{48}
$$

Recomputing the spectrum of A, we have

$$
\begin{aligned}
\sigma(A_1) \quad & \overline{\lambda_1^{(1)}} = -6.0043, \\
\sigma(A_2) \quad & \overline{\lambda_1^{(2)}} = -8.7760 \quad \overline{\lambda_2^{(2)}} = -6.0043, \\
\sigma(A_3) \quad & \overline{\lambda_1^{(3)}} = -10.9106 \quad \overline{\lambda_2^{(3)}} = -8.7760 \quad \overline{\lambda_3^{(3)}} = 1.8532, \\
\sigma(A_4) \quad & \overline{\lambda_1^{(4)}} = -10.9106 \quad \overline{\lambda_2^{(4)}} = -8.7760 \quad \overline{\lambda_3^{(4)}} = 1.8532 \quad \overline{\lambda_4^{(4)}} = 8.4266, \\
\sigma(A_5) \quad & \overline{\lambda_1^{(5)}} = -11.2369 \quad \overline{\lambda_2^{(5)}} = -8.7760 \quad \overline{\lambda_3^{(5)}} = 0.6980 \quad \overline{\lambda_4^{(5)}} = 8.4266 \quad \overline{\lambda_5^{(5)}} = 10.4020.
\end{aligned}
\tag{49}
$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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