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Research Article

Some Surfaces with Zero Curvature in $\mathbb{H}^2 \times \mathbb{R}$

Dae Won Yoon

Department of Mathematics Education and RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea

Correspondence should be addressed to Dae Won Yoon; dwyoon@gnu.ac.kr

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We study surfaces defined as graph of the function $z = f(x, y)$ in the product space $\mathbb{H}^2 \times \mathbb{R}$. In particular, we completely classify flat or minimal surfaces given by $f(x, y) = u(x) + v(y)$, where $u(x)$ and $v(y)$ are smooth functions.

1. Introduction

Homogenous geometries have main roles in the modern theory of manifolds. Homogenous spaces are, in a sense, the nicest examples of Riemannian manifolds and have applications in physics [1]. To underline their importance from the mathematical point of view we roughly cite the famous Thurston conjecture. This conjecture asserts that every compact orientable 3-dimensional manifold has a canonical decomposition into pieces, each of which admits a canonical geometric structure from among the eight maximal simple connected homogenous Riemannian 3-dimensional geometries [2]. The Riemannian product space $\mathbb{H}^2 \times \mathbb{R}$ is one of the eight model spaces.

Constant mean curvature and constant Gaussian curvature surfaces are one of the main objects which have drawn geometers' interest for a very long time. Recently, the study of the geometry of surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is growing very rapidly, and the interest is mainly focused on minimal and constant mean curvature surfaces [3–9].

The purpose of this paper is to study surfaces defined as graph of the function $z = f(x, y)$ in the product space $\mathbb{H}^2 \times \mathbb{R}$. In Sections 4 and 5 we classify minimal and flat surfaces defined as $f(x, y) = u(x) + v(y)$, where $u(x)$ and $v(y)$ are smooth functions.

2. Preliminaries

Let $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the upper half plane model of the hyperbolic plane endowed with the metric, of constant Gaussian curvature -1, given by

$$g_{\mathbb{H}} = \frac{\left(dx^2 + dy^2\right)}{y^2}. \tag{1}$$

The hyperbolic space $\mathbb{H}^2$, with the group structure derived by the composition of proper affine maps, is a Lie group and the metric $g_{\mathbb{H}}$ is left invariant. Therefore, the product space $\mathbb{H}^2 \times \mathbb{R}$ is a Lie group with the left invariant product metric

$$g = \frac{dx^2 + dy^2}{y^2} + dz^2. \tag{2}$$

On the other hand, an orthonormal basis of left invariant vector fields on $\mathbb{H}^2 \times \mathbb{R}$ is

$$E_1 = y\frac{\partial}{\partial x}, \qquad E_2 = y\frac{\partial}{\partial y}, \qquad E_3 = \frac{\partial}{\partial z} \tag{3}$$

with the only nontrivial commutator relation $[E_1, E_2] = -E_1$. It follows that the Levi-Civita connection $\widetilde{\nabla}$ of $\mathbb{H}^2 \times \mathbb{R}$ is expressed as

$$\begin{aligned}
&\widetilde{\nabla}_{E_1} E_1 = E_2, \qquad \widetilde{\nabla}_{E_1} E_2 = -E_1, \qquad \widetilde{\nabla}_{E_1} E_3 = 0, \\
&\widetilde{\nabla}_{E_2} E_1 = 0, \qquad \widetilde{\nabla}_{E_2} E_2 = 0, \qquad \widetilde{\nabla}_{E_2} E_3 = 0, \\
&\widetilde{\nabla}_{E_3} E_1 = 0, \qquad \widetilde{\nabla}_{E_3} E_2 = 0, \qquad \widetilde{\nabla}_{E_3} E_3 = 0.
\end{aligned} \tag{4}$$

For any vectors $X = x_1E_1 + y_1E_2 + z_1E_3$ and $Y = x_2E_1 + y_2E_2 + z_2E_3$ in $\mathbb{H}^2 \times \mathbb{R}$ the cross-product $\times$ is defined by

$$\begin{aligned}
X \times Y = &\left(y_1 z_2 - y_2 z_1\right) E_1 + \left(x_2 z_1 - x_1 z_2\right) E_2 \\
&+ \left(x_1 y_2 - x_2 y_1\right) E_3.
\end{aligned} \tag{5}$$

3. Graphs in $\mathbb{H}^2 \times \mathbb{R}$

Let us consider a surface Σ parametrized by

$$\phi(x,y) = (x, y, f(x,y)), \quad (x,y) \in \Omega, \tag{6}$$

where Ω is a domain in $\mathbb{H}^2$ and $f : \Omega \to \mathbb{R}$ is a smooth function. Then Σ is a surface defined as graph of the function f defined on $\Omega \subset \mathbb{H}^2$. In this case, we have

$$\begin{aligned} e_1 &:= \phi_x = (1, 0, f_x) = \frac{1}{y}E_1 + f_x E_3, \\ e_2 &:= \phi_y = (0, 1, f_y) = \frac{1}{y}E_2 + f_y E_3. \end{aligned} \tag{7}$$

It follows that the coefficients of the first fundamental form of Σ are given by

$$\begin{aligned} E &= g(\phi_x, \phi_x) = f_x^2 + \frac{1}{y^2}, \\ F &= g(\phi_x, \phi_y) = f_x f_y, \\ G &= g(\phi_y, \phi_y) = f_y^2 + \frac{1}{y^2}. \end{aligned} \tag{8}$$

Also, the unit normal vector field U to Σ is given by

$$U(x,y) = -\frac{f_x}{\omega y}E_1 - \frac{f_y}{\omega y}E_2 + \frac{1}{\omega y^2}E_3, \tag{9}$$

where

$$\omega = \frac{1}{y^2}\sqrt{y^2\left(f_x^2 + f_y^2\right) + 1}. \tag{10}$$

By a straightforward calculation, we obtain

$$\begin{aligned} \widetilde{\nabla}_{e_1} e_1 &= \frac{1}{y^2}E_2 + f_{xx}E_3, \\ \widetilde{\nabla}_{e_1} e_2 &= -\frac{1}{y^2}E_1 + f_{xx}E_3, \\ \widetilde{\nabla}_{e_2} e_2 &= -\frac{1}{y^2}E_2 + f_{yy}E_3, \end{aligned} \tag{11}$$

which imply that the coefficients of the second fundamental form of Σ are

$$\begin{aligned} L &= g\left(\widetilde{\nabla}_{e_1} e_1, U\right) = \frac{y f_{xx} - f_y}{\omega y^3}, \\ M &= g\left(\widetilde{\nabla}_{e_1} e_2, U\right) = \frac{y f_{xy} + f_x}{\omega y^3}, \\ N &= g\left(\widetilde{\nabla}_{e_2} e_2, U\right) = \frac{y f_{yy} + f_y}{\omega y^3}. \end{aligned} \tag{12}$$

Thus, from (8) and (12) the Gaussian curvature K and the mean curvature H are, respectively,

$$\begin{aligned} K &= \frac{1}{\omega^4 y^6}\left(\left(y f_{xx} - f_y\right)\left(y f_{yy} + f_y\right) - \left(y f_{xy} + f_x\right)^2\right), \\ H &= \frac{1}{2\omega^3 y^4}\left(\left(1 + y^2 f_y^2\right) f_{xx} - y\left(f_x^2 + f_y^2\right) f_y \right. \\ &\quad \left. - 2y^2 f_x f_y f_{xy} + \left(1 + y^2 f_x^2\right) f_{yy}\right). \end{aligned} \tag{13}$$

Proposition 1. *Let Σ be a surface defined as graph of the function $f : \Omega \subset \mathbb{H}^2 \to \mathbb{R}$. Then Σ is a minimal surface if and only if*

$$\begin{aligned} &\left(1 + y^2 f_y^2\right) f_{xx} - y\left(f_x^2 + f_y^2\right) f_y - 2y^2 f_x f_y f_{xy} \\ &\quad + \left(1 + y^2 f_x^2\right) f_{yy} = 0. \end{aligned} \tag{14}$$

Proposition 2. *Let Σ be a surface defined as graph of the function $f : \Omega \subset \mathbb{H}^2 \to \mathbb{R}$. Then Σ is flat if and only if*

$$\left(y f_{xx} - f_y\right)\left(y f_{yy} + f_y\right) - \left(y f_{xy} + f_x\right)^2 = 0. \tag{15}$$

Remark 3. Some examples are satisfying the ODE (14) studied in [7]. Also, examples in Lorentz product space $\mathbb{H}^2 \times \mathbb{R}_1$ can be found in [10].

4. Minimal Surfaces Defined by $f(x,y) = u(x) + v(y)$

Let Σ be a surface in $\mathbb{H}^2 \times \mathbb{R}$ parametrized by

$$\phi(x,y) = (x, y, u(x) + v(y)) \tag{16}$$

for all $y > 0$, where $u(x)$ and $v(y)$ are smooth functions. We suppose that Σ is a minimal surface. Then, from (14) we have the following minimal surface equation:

$$\begin{aligned} &\left(1 + y^2(v')^2\right) u'' - y\left((u')^2 + (v')^2\right) v' \\ &\quad + \left(1 + y^2(u')^2\right) v'' = 0. \end{aligned} \tag{17}$$

In order to solve it, divide first by $1 + y^2(v')^2 \neq 0$; then we get

$$u'' - \frac{y\left((u')^2 + (v')^2\right)}{1 + y^2(v')^2}v' + \frac{1 + y^2(u')^2}{1 + y^2(v')^2}v'' = 0, \tag{18}$$

for all $x, y \in \Omega$. Differentiating with respect to x, we obtain

$$u''' + 2\left(\frac{y^2 v'' - y v'}{1 + y^2(v')^2}\right) u' u'' = 0. \tag{19}$$

First of all, we suppose that $u'' = 0$ on an open interval; that is, $u(x) = ax + b$, $a, b \in \mathbb{R}$. In this case, from (17) we obtain

$$v'' - \frac{a^2 y}{1 + a^2 y^2}v' - \frac{y}{1 + a^2 y^2}(v')^3 = 0. \tag{20}$$

We put $v'(y) = p(y)$. Then the last equation can be written as

$$p' - \frac{y}{1+a^2y^2}\left(a^2p + p^3\right) = 0. \tag{21}$$

Its general solution is given by

$$p = \pm\frac{c_1 a\sqrt{1+a^2y^2}}{\sqrt{1-c_1^2\left(1+a^2y^2\right)}}. \tag{22}$$

From this, we thus have

$$v(y) = \pm\int\frac{c_1 a\sqrt{1+a^2y^2}}{\sqrt{1-c_1^2\left(1+a^2y^2\right)}}dy, \tag{23}$$

where $c_1 \in \mathbb{R}$.

Now, we assume that $u'' \neq 0$ on an open interval, and divide (19) by $u'u''$. It follows that

$$\frac{u'''}{u'u''} + 2\frac{y^2v'' - yv'}{1+y^2(v')^2} = 0. \tag{24}$$

Hence we deduce the existence of a real number $k \in \mathbb{R}$ such that

$$u''' = 2ku'u'', \qquad y^2v'' - yv' = -k\left(1+y^2\left(v'\right)^2\right). \tag{25}$$

Let us distinguish the following cases according to k.

Case 1. If $k = 0$, then $u''' = 0$ and $yv'' - v' = 0$. It follows that $u(x) = a_1x^2 + b_1x + c_1$ $(a_1 \neq 0, b_1, c_1 \in \mathbb{R})$. If $v' = 0$, then $v(y) = a_2$ $(a_2 \in \mathbb{R})$. In this case, from (17) we obtain $a_1 = 0$; it is a contradiction. If $v' \neq 0$, then we get $v(y) = (1/2)b_2y^2 + c_2$ $(b_2 \neq 0, c_2 \in \mathbb{R})$. In such case, (17) is polynomial equation on x and y. From the coefficients of y^4 and the constant term we have $2a_1 - b_2 = 0$ and $2a_1 + b_2 = 0$, which imply $a_1 = 0$ and $b_2 = 0$. It is a contradiction.

Case 2. If $k \neq 0$, then from the first equation in (25) we have

$$u'' = e^{2ku+d_1}, \tag{26}$$

where $d_1 \in \mathbb{R}$. Let

$$u = \frac{1}{2k}\left(-d_1 + \ln g\right) \tag{27}$$

be any solution of (26), where g is a smooth function. Then (26) can be rewritten as

$$gg'' - \left(g'\right)^2 = 2kg^3. \tag{28}$$

We put $p = g'$. Then, we have

$$\frac{dp}{dg} - \frac{1}{g}p = 2kg^2p^{-1}. \tag{29}$$

We again put $t = p^2$. In this case the above equation becomes

$$\frac{dt}{dg} - \frac{2}{g}t = 4kg^2 \tag{30}$$

and its general solution is given by

$$t = g^2\left(4kg + c_1\right). \tag{31}$$

Thus, we get

$$\frac{dg}{dx} = \pm g\sqrt{4kg + c_1}. \tag{32}$$

After an integration, we can find

$$g = \frac{c_1}{4k}\tan^2\left(8k^2\sqrt{c_1}\left(\pm x + c_2\right)\right) - \frac{c_1}{4k}, \tag{33}$$

where $c_2 \in \mathbb{R}$. By combining (27) and (33), we thus have

$$u(x) = \frac{1}{2k}\left[-d_1 + \ln\left(\frac{c_1}{4k}\tan^2\left(8k^2\sqrt{c_1}\left(\pm x + c_2\right)\right) - \frac{c_1}{4k}\right)\right]. \tag{34}$$

Now, we consider the second equation in (25). Since $y > 0$, we yield

$$v'' + \frac{k}{y^2} - \frac{1}{y}v' + k\left(v'\right)^2 = 0. \tag{35}$$

We put $p = v'$. Then, the above equation becomes

$$p' + \frac{k}{y^2} - \frac{1}{y}p + kp^2 = 0. \tag{36}$$

Since $k \neq 0$, without loss of generality we take $k = 1$ or $k = -1$.

Subcase i. Let $k = 1$. We do the change

$$p = \frac{1}{y} + \frac{1}{h(y)}, \tag{37}$$

where h is a nonzero smooth function. Then, (36) can be rewritten as the form

$$h' - \frac{1}{y}h = 1. \tag{38}$$

Thus, its general solution is

$$h(y) = y\left(\ln y + c_1\right), \tag{39}$$

where $c_1 \in \mathbb{R}$. So, $p = (1/y) + (1/y(\ln y + c_1))$ and from its integration we can obtain

$$v(y) = \ln\left(c_2 y \ln\left(y + c_1\right)\right), \tag{40}$$

where $c_2 \in \mathbb{R}$.

Subcase ii. Let $k = -1$. We put

$$p = -\frac{1}{y} + \frac{1}{h(y)}, \tag{41}$$

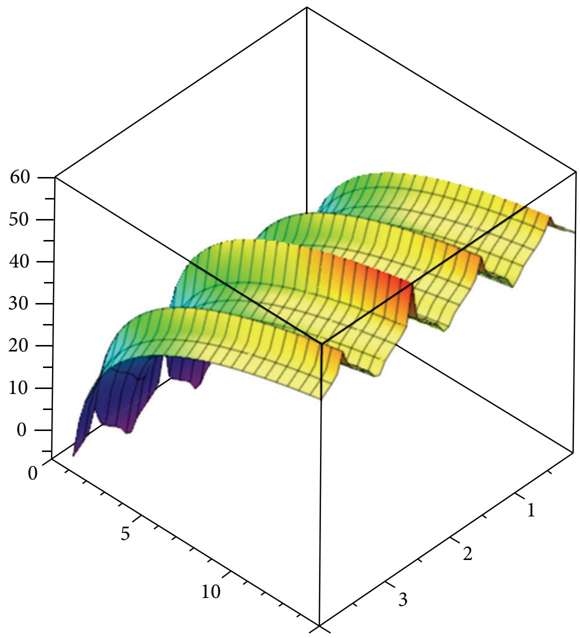

Figure 1: A minimal surface defined by (34) and (44).

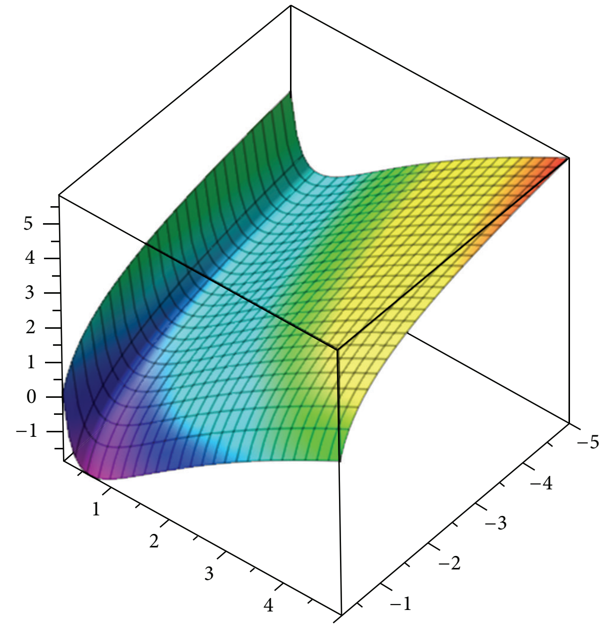

Figure 2: A flat surface defined by (52) and (55).

where h is a nonzero smooth function. Then, (36) becomes

$$h' - \frac{1}{y}h = -1 \tag{42}$$

and its general solution is given by

$$h(y) = -y(\ln y + c_1), \tag{43}$$

where $c_1 \in \mathbb{R}$. Thus, we have

$$v(y) = -\ln(c_2 y \ln(y + c_1)), \tag{44}$$

where $c_2 \in \mathbb{R}$. The surface given by (34) and (44) is shown in Figure 1.

Consequently, we have the following.

Theorem 4. *Let Σ be a surface defined as graph of the function $f(x,y) = u(x) + v(y)$. If Σ is a minimal surface, then Σ is parametrized as*

$$\phi(x,y) = (x, y, u(x) + v(y)), \tag{45}$$

where

(1) $u(x) = ax + b$ *and* $v(y) = \pm\int (c_1 a\sqrt{1+a^2y^2}/\sqrt{1-c_1^2(1+a^2y^2)})dy$ *with* $a, b, c_1 \in \mathbb{R}$, *or*

(2) $u(x) = (1/2k)[-c_3 + \ln((c_1/4k)\tan^2(8k^2\sqrt{c_1}(\pm x + c_2)) - (c_1/4k))]$ *and* $v(y) = \pm\ln(d_1 y \ln(y + d_2))$ *with* $k \neq 0, c_1, c_2, c_3, d_1, d_2 \in \mathbb{R}$.

5. Flat Surfaces Defined by $f(x,y) = u(x) + v(y)$

Let Σ be a surface defined by (16). Assume that Σ is a flat surface. Then, from (15) we have the following flat surface equation:

$$y(yv'' + v')u'' - (yv'' + v')v' - (u')^2 = 0. \tag{46}$$

In order to solve it, differentiating with respect to x, we have

$$y(yv'' + v')\frac{d}{dx}(u'') - \frac{d}{dx}((u')^2) = 0. \tag{47}$$

Thus, there exists a nonzero real number k such that

$$\frac{d}{dx}(u'') = k\frac{d}{dx}((u')^2), \qquad y(yv'' + v') = \frac{1}{k}. \tag{48}$$

From the first equation in (48), we get

$$u'' = k(u')^2 + c_1, \tag{49}$$

where $c_1 \in \mathbb{R}$. We put $p = u'$, and it follows that we yield

$$\frac{dp}{du} = \frac{kp^2 + c_1}{p}. \tag{50}$$

From this, the general solution is

$$p = \pm\sqrt{\frac{1}{k}e^{2k(u+c_2)} - \frac{c_1}{k}}, \tag{51}$$

where $c_2 \in \mathbb{R}$. We can assume that $c_1 = 0$. From the last equation we can easily obtain (see Figure 2)

$$u(x) = \pm\frac{1}{k}(\ln(-\sqrt{k}(x + c_3)) + kc_2), \tag{52}$$

where $c_3 \in \mathbb{R}$.

In order to solve the second equation in (48), divide by y^2 and put $q = v'$. Then, we get

$$q' + \frac{1}{y}q = \frac{1}{ky^2} \tag{53}$$

and its general solution is given by

$$q = \frac{1}{y}\left(\frac{1}{k}\ln y + d_1\right), \tag{54}$$

where $d_1 \in \mathbb{R}$. From this, we thus obtain (see Figure 2)

$$v(y) = \frac{1}{2k}(\ln y)^2 + d_1 \ln y + d_2, \tag{55}$$

where $d_2 \in \mathbb{R}$.

As a conclusion, we have the following.

Theorem 5. *Let Σ be a surface defined as graph of the function $f(x, y) = u(x) + v(y)$. If Σ is a flat surface, then Σ is parametrized as*

$$\phi(x, y) = (x, y, u(x) + v(y)), \tag{56}$$

where $u(x) = \pm(1/k)(\ln(-\sqrt{k}(x + c_1)) + kc_2)$ and $v(y) = (1/2k)(\ln y)^2 + d_1 \ln y + d_1$ with $k \neq 0, c_1, c_2, d_1, d_2 \in \mathbb{R}$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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References

[1] J. Gegenberg, S. Vaidya, and J. F. Vázquez-Poritz, "Thurston geometries from eleven dimensions," *Classical and Quantum Gravity*, vol. 19, no. 23, pp. L199–L204, 2002.

[2] W. Thurston, *Three-Dimensional Geometry and Topology*, vol. 35 of *Princeton Mathematical Series*, Princeton University Press, Princeton, NJ, USA, 1997.

[3] J. M. Espinar, M. M. Rodríguez, and H. Rosenberg, "The extrinsic curvature of entire minimal graphs in $H^2 \times R$," *Indiana University Mathematics Journal*, vol. 59, no. 3, pp. 875–889, 2010.

[4] I. Fernández and P. Mira, "Harmonic maps and constant mean curvature surfaces in $H^2 \times R$," *The American Journal of Mathematics*, vol. 129, no. 4, pp. 1145–1181, 2007.

[5] Y. W. Kim, S. E. Koh, H. Shin, and S. D. Yang, "Helicoidal minimal surfaces in $H^2 \times R$," *Bulletin of the Australian Mathematical Society*, vol. 86, no. 1, pp. 135–149, 2012.

[6] S. Montaldo and I. I. Onnis, "Invariant CMC surfaces in $H^2 \times R$," *Glasgow Mathematical Journal*, vol. 46, no. 2, pp. 311–321, 2004.

[7] S. Montaldo and I. I. Onnis, "A note on surfaces in $H^2 \times R$," *Bollettino della Unione Matematica Italiana B*, vol. 10, no. 3, pp. 939–950, 2007.

[8] B. Nelli and H. Rosenberg, "Minimal surfaces in $H^2 \times R$," *Bulletin of the Brazilian Mathematical Society*, vol. 33, no. 2, pp. 263–292, 2002.

[9] D. W. Yoon, "Minimal translation surfaces in $H^2 \times R$," *Taiwanese Journal of Mathematics*, vol. 17, no. 5, pp. 1545–1556, 2013.

[10] A. L. Albujer, "New examples of entire maximal graphs in $H^2 \times R_1$," *Differential Geometry and its Applications*, vol. 26, no. 4, pp. 456–462, 2008.