Research Article

Discrete Subspace Multiwindow Gabor Frames and Their Duals

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This paper addresses discrete subspace multiwindow Gabor analysis. Such a scenario can model many practical signals and has potential applications in signal processing. In this paper, using a suitable Zak transform matrix we characterize discrete subspace mixed multi-window Gabor frames (Riesz bases and orthonormal bases) and their duals with Gabor structure. From this characterization, we can easily obtain frames by designing Zak transform matrices. In particular, for usual multi-window Gabor frames (i.e., all windows have the same time-frequency shifts), we characterize the uniqueness of Gabor dual of type I (type II) and also give a class of examples of Gabor frames and an explicit expression of their Gabor duals of type I (type II).

1. Introduction

Let $\mathscr{H}$ be a separable Hilbert space. An at most countable sequence $\{h_i\}_{i \in \mathscr{J}}$ in $\mathscr{H}$ is called a *frame* for $\mathscr{H}$ if there exist $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{i \in \mathscr{J}} |\langle f, h_i \rangle|^2 \le B\|f\|^2 \quad \text{for } f \in \mathscr{H}, \tag{1}$$

where A and B are called *frame bounds*. The sequence $\{h_i\}_{i \in \mathscr{J}}$ is called a *Bessel sequence* in $\mathscr{H}$ if the right-hand side inequality in (1) holds. In this case B is called a *Bessel bound*. A frame for $\mathscr{H}$ is said to be a *Riesz basis* if it ceases to be a frame for $\mathscr{H}$ whenever an arbitrary element is removed. And in this case, the frame bounds are also called *Riesz bounds*. The fundamentals of frames can be found in [1–4]. We denote by $\mathbb{Z}$ the set of integers, by $\mathbb{N}$ the set of positive integers, by $\{\mathbf{e}_i\}$ the vector in Euclidean spaces with the ith component being 1 and others being zero, and by $\mathbb{N}_\gamma$ the set $\{0, 1, \ldots, \gamma - 1\}$ for $\gamma \in \mathbb{N}$. This paper addresses Gabor systems $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ of the form

$$G(\mathbf{g}, \mathbf{N}, \mathbf{M}) = \left\{ E_{m/M} T_{nN_l} g_l : 1 \le l \le L, m \in \mathbb{N}_M, n \in \mathbb{Z} \right\}, \tag{2}$$

where L is a fixed positive integer, $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ with each $g_l \in l^2(\mathbb{Z})$, $\mathbf{N} = (N_1, N_2, \ldots, N_L)$ with each $N_l \in \mathbb{N}$, and $\mathbf{M} = (M_1, M_2, \ldots, M_L)$ with each $M_l \in \mathbb{N}$. Throughout this paper, we work under the following assumptions.

Assumption 1. L is a positive integer.

Assumption 2. $M_1 = M_2 = \cdots = M_L = M$, and $N_l/M = p_l/q_l$ with p_l and q_l being relatively prime positive integers for $1 \le l \le L$.

We always denote by N the least common multiple of N_l with $1 \le l \le L$, by λ_l the positive integer satisfying $N = \lambda_l N_l$ for each $1 \le l \le L$, by p and q two relatively prime positive integers satisfying $N/M = p/q$, and by Q the number $Q = q \sum_{l=1}^{L} \lambda_l$. Obviously, they are all uniquely determined by Assumptions 1 and 2.

Remark 3. We do not lose generality by making Assumptions 1 and 2. Let us check a general Gabor system of the form

$$G(\mathbf{g}, \mathbf{N}, \mathbf{M}) = \left\{ E_{m_l/M_l} T_{nN_l} g_l : 1 \le l \le L, m_l \in \mathbb{N}_{M_l}, n \in \mathbb{Z} \right\} \tag{3}$$

with $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ satisfying $g_l \in l^2(\mathbb{Z})$, $\mathbf{N} = (N_1, N_2, \ldots, N_L)$, and $\mathbf{M} = (M_1, M_2, \ldots, M_L)$ ($M_1, M_2, \ldots, M_L$ are not necessarily all the same). Suppose M is the greatest common divisor of $M_1, M_2, \ldots, M_L$, and $M_l = \alpha_l M$ with $\alpha_l \in \mathbb{N}$ for $1 \le l \le L$. Define $g_l^{(\beta_l)}(\cdot) = e^{2\pi i(\beta_l/M_l)\cdot} g_l(\cdot)$ for $1 \le l \le L$ and $\beta_l \in \mathbb{N}_{\alpha_l}$. Then, by a simple computation, $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is a frame for a closed subspace $\mathscr{M}$ of $l^2(\mathbb{Z})$ if and only if the Gabor system

$$\left\{ E_{m/M} T_{nN_l} g_l^{(\beta_l)} : 1 \le l \le L, \beta_l \in \mathbb{N}_{\alpha_l}, m \in \mathbb{N}_M, n \in \mathbb{Z} \right\} \tag{4}$$

is a frame for $\mathcal{M}$. Therefore, the study of (3) is reduced to the study of (4).

We denote by $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ the closed linear span of $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$. A Gabor system $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ of the form (2) is called a *mixed multi-window Gabor system* if $L > 1$ since $N_1, N_2, \ldots$ and N_L are not necessarily all the same. (In particular, this system is the usual multi-window Gabor system when $N_1 = N_2 = \cdots = N_L$). It is called a *subspace mixed multi-window Gabor frame* if it is a frame for $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ and $L > 1$. Similarly, a Gabor system $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is called a *single-window Gabor system* if $L = 1$ and called a *single-window subspace Gabor frame* if it is a frame for $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ and $L = 1$.

For a Bessel sequence $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ in $l^2(\mathbb{Z})$ of the form (2), we define the associated *synthesis operator* $\mathcal{T}_{\mathbf{g}} : l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L) \to l^2(\mathbb{Z})$ by

$$\mathcal{T}_{\mathbf{g}} c = \sum_{l=1}^{L} \sum_{m \in \mathbb{N}_M} \sum_{n \in \mathbb{Z}} c_{l,m,n} E_{m/M} T_{nN_l} g_l \qquad (5)$$

for $c = (c_1, c_2, \ldots, c_L) \in l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$. Then it is a bounded operator, and its adjoint operator $\mathcal{T}_{\mathbf{g}}^*$ (so-called *analysis operator*) is given by

$$\mathcal{T}_{\mathbf{g}}^* f = (c_1(f), c_2(f), \ldots, c_L(f)) \quad \text{for } f \in l^2(\mathbb{Z}), \qquad (6)$$

where $c_l(f) = \{\langle f, E_{m/M} T_{nN_l} g_l \rangle\}_{m \in \mathbb{N}_M, n \in \mathbb{Z}}$ for each $1 \leq l \leq L$. For a Bessel sequence $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ in $l^2(\mathbb{Z})$ with $\mathbf{h} = (h_1, h_2, \ldots, h_L)$, we associate it with $\mathcal{T}_{\mathbf{h}}$ similarly. Write $\mathcal{S}_{\mathbf{h}, \mathbf{g}} = \mathcal{T}_{\mathbf{g}} \mathcal{T}_{\mathbf{h}}^*$; that is,

$$\mathcal{S}_{\mathbf{h}, \mathbf{g}} f = \sum_{l=1}^{L} \sum_{m \in \mathbb{N}_M} \sum_{n \in \mathbb{Z}} \langle f, E_{m/M} T_{nN_l} h_l \rangle E_{m/M} T_{nN_l} g_l \qquad (7)$$

for $f \in l^2(\mathbb{Z})$. Let $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ be a frame for $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ and let $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ be a Bessel sequence in $l^2(\mathbb{Z})$. $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ is called an *oblique Gabor dual* for $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ if

$$\mathcal{S}_{\mathbf{h}, \mathbf{g}} f = f \quad \text{for } f \in \mathcal{M}(\mathbf{g}, N, M). \qquad (8)$$

Here it is not required that each h_l belongs to $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$. In particular, an oblique Gabor dual $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ for $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is said to be a *Gabor dual of type I* for $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ if $h_l \in \mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ for each $1 \leq l \leq L$ and is said to be a *Gabor dual of type II* for $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ if $\text{range}(\mathcal{T}_{\mathbf{h}}^*) \subset \text{range}(\mathcal{T}_{\mathbf{g}}^*)$. These notions of duals are a generalization of the ones in [5, 6]. They are borrowed from [7, 8] which dealt with Gabor frames in $L^2(\mathbb{R})$. For a Gabor dual of type II, it is not required to be in $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$, but a containment relation between the ranges of analysis operators is required. Observe that the *canonical*

daul $G(\mathcal{S}_{\mathbf{g},\mathbf{g}}^{-1} \mathbf{g}, \mathbf{N}, \mathbf{M})$ for $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ belongs to any one of the three Gabor duals.

In the past more than twenty years, the theory of frames has been growing rapidly. Gabor frames are a class of important frames among all kinds of frames. For continuous Gabor frames, single-window Gabor frames for $L^2(\mathbb{R})$ have been studied extensively [2, 3, 9, 10]; multi-window Gabor frames for $L^2(\mathbb{R})$ were firstly studied by Zibulski and Zeevi in [11], and then by others in [12–17]; single-window subspace (of $L^2(\mathbb{R})$) Gabor frames were studied in [7, 8, 18–23]. In digital signal processing, one usually encounters discrete signals instead of continuous ones. One can obtain discrete Gabor frames for $l^2(\mathbb{Z})$ via Gabor frames for $L^2(\mathbb{R})$ through sampling under certain additional assumptions [24–26], though these assumptions are artificial and too technical. So it should be more reasonable to consider Gabor frames in $l^2(\mathbb{Z})$ without referring to frames in $L^2(\mathbb{R})$. Intuitively, the general theory of discrete Gabor analysis is similar to the continuous case. Sometimes major differences occur. In 1989, Heil in [27] showed that while Gabor frames in the continuous case are bases only if they are generated by functions that are not smooth or have poor decay, it is possible in the discrete case to construct Gabor frames that are bases and are generated by sequences with good decay. The sampled Gaussian gives an example of such a signal. Also due to its good potential for digital signal processing, since then, Gabor analysis on $l^2(\mathbb{Z})$ has attracted many researchers (see [2, 3, 24–34] and the references therein). In [6, 35], single-window Gabor frames on discrete periodic subsets of $\mathbb{Z}$ were investigated. It is well known that a single-window Gabor expansion is not enough to analyze the dynamic time-frequency contents of signals that contain a wide range of spatial and frequency components. A multi-window Gabor expansion is capable of extracting local frequencies in an adaptive manner, in which wide windows are responsible for slow-changing components and narrow windows are designed to extract transient and rapid-changing components of a signal. See [15] and the references therein for details. In addition, Example 38 and the arguments before it in the last section show that, for Gabor duals, multi-window Gabor frames behave differently from single-window ones. Motivated by the above works, we in this paper study subspace Gabor frame of the form (2) under Assumptions 1 and 2.

The rest of this paper is organized as follows. Section 2 is an auxiliary one to following sections. In this section, we introduce the notion of Zak transform matrix associated with a Gabor system of the form (2) and investigate its properties. In terms of Zak transform matrices, we in Section 3 characterize subspace multi-window Gabor frames (Riesz bases, orthonormal bases) and in Section 4 Gabor duals of type I (type II) for subspace multi-window Gabor frames. Section 5 focuses on subspace multi-window Gabor frames with all the windows having the same time-frequency shifts. We characterize the uniqueness of Gabor duals of type I (type II) and obtain a class of examples of subspace multi-window Gabor frames (Riesz bases, orthonormal bases) and their Gabor duals of type I (type II) (see Theorems 36 and 37).

2. Zak Transforms and Zak Transform Matrices

This section focuses on some properties of Zak transforms and Zak transform matrices, which is an auxiliary one to following sections. We start with some notations and notions. For $s, t \in \mathbb{N}$, we denote by $\mathscr{M}_{s,t}$ the set of $s \times t$ complex matrices. Given $\gamma \in \mathbb{N}$, two subsets S_1 and S_2 of $\mathbb{Z}$ are said to be $\gamma\mathbb{Z}$ congruent if there exists a sequence of subsets $\{S_{1,k}\}_{k\in\mathbb{Z}}$ of S_1 such that $\bigcup_{k\in\mathbb{Z}} S_{1,k} = S_1$, $S_{1,k_1} \cap S_{1,k_2} = \emptyset$ for $k_1 \neq k_2$ in $\mathbb{Z}$ and that $S_2 = \bigcup_{k\in\mathbb{Z}}(S_{1,k} + \gamma k)$, $(S_{1,k_1} + \gamma k_1) \cap (S_{1,k_2} + \gamma k_2) = \emptyset$ for $k_1 \neq k_2$ in $\mathbb{Z}$. If S_1 or S_2 is a finite set in addition, only finitely many $S_{1,k}$ in the above are nonempty, while the others are empty. For two vectors x and y in a Hilbert space $\mathscr{H}$, we always write their inner product as $\langle x, y \rangle$ instead of $\langle x, y \rangle_{\mathscr{H}}$ when it causes no confusions. Let E be a finite set in $\mathbb{Z}$ and let $\Gamma \in \mathbb{N}$. We denote by $L^2(E \times [0,1), \mathbb{C}^{\Gamma})$ the Hilbert space of vector-valued functions f satisfying $f(j, \cdot) \in L^2([0,1), \mathbb{C}^{\Gamma})$ for each $j \in E$ endowed with the inner product

$$\langle f, \tilde{f} \rangle = \sum_{j \in E} \int_0^1 \langle f(j, v), \tilde{f}(j, v) \rangle \, dv \qquad (9)$$

for $f, \tilde{f} \in L^2(E \times [0,1), \mathbb{C}^{\Gamma})$, where $\langle f(j, v), \tilde{f}(j, v) \rangle$ denotes their inner product in $\mathbb{C}^{\Gamma}$; by $l^2(E \times \mathbb{Z}, \mathbb{C}^{\Gamma})$ the Hilbert space of sequences $c = (c_1, c_2, \ldots, c_{\Gamma})$ satisfying $c_{\gamma} \in l^2(E \times \mathbb{Z})$ endowed with the inner product

$$\langle c, \tilde{c} \rangle = \sum_{\gamma=1}^{\Gamma} \langle c_{\gamma}, \tilde{c}_{\gamma} \rangle \qquad (10)$$

for $c, \tilde{c} \in l^2(E \times \mathbb{Z}, \mathbb{C}^{\Gamma})$, where $\langle c_{\gamma}, \tilde{c}_{\gamma} \rangle$ denotes their inner product in $l^2(E \times \mathbb{Z})$ and by $l_0(E \times \mathbb{Z}, \mathbb{C}^{\Gamma})$ the set of sequences $c = (c_1, c_2, \ldots, c_{\Gamma})$ with each c_{γ} being a finitely supported sequence defined on $E \times \mathbb{Z}$. For simplicity, we write $L^2(E \times [0,1))$ for $L^2(E \times [0,1), \mathbb{C}^{\Gamma})$, $l^2(E \times \mathbb{Z})$ for $l^2(E \times \mathbb{Z}, \mathbb{C}^{\Gamma})$, and $l_0(E \times \mathbb{Z})$ for $l_0(E \times \mathbb{Z}, \mathbb{C}^{\Gamma})$ when $\Gamma = 1$.

For $f \in l^2(\mathbb{Z})$, define the *discrete Zak transform* $\mathscr{Z}_{qN} f$ of f by

$$(\mathscr{Z}_{qN} f)(j, v) := \sum_{\ell \in \mathbb{Z}} f(j + \ell qN) e^{2\pi i \ell v} \qquad (11)$$

for $j \in \mathbb{Z}$ and a.e. $v \in \mathbb{R}$. It is easy to check that $\mathscr{Z}_{qN}$ has *quasi-periodicity*:

$$(\mathscr{Z}_{qN} f)(j + kqN, v + \ell) = e^{-2\pi i k v} (\mathscr{Z}_{qN} f)(j, v) \qquad (12)$$

for $k, \ell, j \in \mathbb{Z}$ and a.e. $v \in \mathbb{R}$.

Definition 4. For $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ with each $g_l \in l^2(\mathbb{Z})$, we associate it with a matrix-valued function $\Psi_{\mathbf{g}}(j, v) : \mathbb{Z} \times \mathbb{R} \to \mathscr{M}_{Q,p}$ by

$$\Psi_{\mathbf{g}}(j, v) = \begin{pmatrix} G_1(j, v) \\ G_2(j, v) \\ \vdots \\ G_L(j, v) \end{pmatrix}, \qquad (13)$$

where $Q := q \sum_{l=1}^{L} \lambda_l$ and $G_l(j, v)$ is a block matrix of the form

$$G_l(j, v) = \begin{pmatrix} \mathscr{G}_l(j, v) \\ \mathscr{G}_l(j - N_l, v) \\ \vdots \\ \mathscr{G}_l(j - (\lambda_l - 1) N_l, v) \end{pmatrix} \qquad (14)$$

with $\mathscr{G}_l(j, v) : \mathbb{Z} \times \mathbb{R} \to \mathscr{M}_{q,p}$ for $1 \leq l \leq L$ and

$$\mathscr{G}_l(j, v)_{r,k} = \mathscr{Z}_{qN} g_l(j - rN + kM, v) \qquad (15)$$

for $r \in \mathbb{N}_q$, $k \in \mathbb{N}_p$.

By the quasi-periodicity of $\mathscr{Z}_{qN}$, an arbitrary $f \in l^2(\mathbb{Z})$ is uniquely determined by the values of $(\mathscr{Z}_{qN} f)(\cdot, \cdot)$ on $S \times [0,1)$ with S being a set $qN\mathbb{Z}$ congruent to $\mathbb{N}_{qN}$. So an arbitrary function $F \in L^2(S \times [0,1))$ determines a unique $f \in l^2(\mathbb{Z})$ by

$$(\mathscr{Z}_{qN} f)(j, v) = F(j, v) \quad \text{for } (j, v) \in S \times [0,1). \qquad (16)$$

Let us make another assumption that $N_1 = N_2 = \cdots = N_L$ in Definition 4. Then $\Psi_{\mathbf{g}}(j, v)$ is an $Lq \times p$-matrix-valued function of the form

$$\Psi_{\mathbf{g}}(j, v) = \begin{pmatrix} \mathscr{G}_1(j, v) \\ \mathscr{G}_2(j, v) \\ \vdots \\ \mathscr{G}_L(j, v) \end{pmatrix}, \qquad (17)$$

$$\mathscr{G}_l(j, v)_{r,k} = \mathscr{Z}_{qN} g_l(j - rN + kM, v)$$

for $1 \leq l \leq L$, $r \in \mathbb{N}_q$, and $k \in \mathbb{N}_p$. Also observe that $\mathbb{N}_{M/q} - N\mathbb{N}_q + M\mathbb{N}_p$ is $qN\mathbb{Z}$ congruent to $\mathbb{N}_{qN}$. We have if $N_1 = N_2 = \cdots = N_L$, then an arbitrary function $M(j, v) : \mathbb{N}_{M/q} \times [0,1) \to \mathscr{M}_{Lq,p}$ with all entries in $L^2(\mathbb{N}_{M/q} \times [0,1))$ determines a unique $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ with each $g_l \in l^2(\mathbb{Z})$ by

$$\Psi_{\mathbf{g}}(j, v) = M(j, v) \quad \text{for } j \in \mathbb{N}_{M/q}, \text{ a.e. } v \in [0,1). \qquad (18)$$

But if N_l, $1 \leq l \leq L$, are not all the same, a function $M(j, v) : \mathbb{N}_{M/q} \times [0,1) \to \mathscr{M}_{Q,p}$ with all entries in $L^2(\mathbb{N}_{M/q} \times [0,1))$ does not necessarily determine a $\mathbf{g}$. We show it by an example.

In this case, there exists $1 \le l \le L$ such that $\lambda_l > 1$. We may as well assume that $\lambda_1 > 1$. Choose

$$M(j,v) = \begin{pmatrix} M_1(j,v) \\ M_2(j,v) \\ \vdots \\ M_L(j,v) \end{pmatrix},$$

$$M_l(j,v) = \begin{pmatrix} M_{l,0}(j,v) \\ M_{l,1}(j,v) \\ \vdots \\ M_{l,\lambda_l-1}(j,v) \end{pmatrix} \tag{19}$$

for $1 \le l \le L$ and $(j,v) \in \mathbb{N}_{M/q} \times [0,1)$ such that every entry of $M(j,v)$ belongs to $L^2(\mathbb{N}_{M/q} \times [0,1))$ and that

$$M_{1,0}(j,v) \ne 0, \quad M_{1,1}(j,v) = 0 \\ \text{for } (j,v) \in \mathbb{N}_{M/q} \times [0,1). \tag{20}$$

Suppose there exists $\mathbf{g}$ such that $\Psi_{\mathbf{g}}(j,v) = M(j,v)$ for $j \in \mathbb{N}_{M/q}$ and a.e. $v \in [0,1)$. Then

$$\mathscr{G}_1(j,v) = M_{1,0}(j,v), \tag{21}$$

$$\mathscr{G}_1(j - N_1, v) = M_{1,1}(j,v) \tag{22}$$

for $j \in \mathbb{N}_{M/q}$ and a.e. $v \in [0,1)$, where

$$\mathscr{G}_1(j,v)_{r,k} = \mathscr{Z}_{qN} g_1(j - rN + kM, v), \\ \mathscr{G}_1(j - N_1, v)_{r,k} = \mathscr{Z}_{qN} g_1(j - N_1 - rN + kM, v) \tag{23}$$

for $(r,k) \in \mathbb{N}_q \times \mathbb{N}_p$. Since the sets $\mathbb{N}_{M/q} - N\mathbb{N}_q + M\mathbb{N}_p$ and $\mathbb{N}_{M/q} - N\mathbb{N}_q + M\mathbb{N}_p - N_1$ are both $qN\mathbb{Z}$ congruent to $\mathbb{N}_{qN}$, we have $\mathscr{Z}_{qN} g_1(j_0, v) \ne 0$ for some $j_0 \in \mathbb{Z}$ and $v \in E \subset [0,1)$ with $|E| > 0$ by (20), (21), and the quasi-periodicity of $\mathscr{Z}_{qN}$, while $\mathscr{Z}_{qN} g_1(j,v) = 0$ for $j \in \mathbb{Z}$ and a.e. $v \in [0,1)$ by (20), (22), and the quasi-periodicity of $\mathscr{Z}_{qN}$. This is a contradiction. Therefore, we must be careful when we define $\mathbf{g}$ by a function $M(j,v) : \mathbb{N}_{M/q} \times [0,1) \to \mathscr{M}_{Q,p}$ if N_l, $1 \le l \le L$, are not all the same.

Definition 5. Define $\mathbf{Z}_{qN} : l^2(\mathbb{Z}) \to L^2(\mathbb{N}_M \times [0,1), \mathbb{C}^p)$ by

$$\mathbf{Z}_{qN} f(j,v) = \begin{pmatrix} \mathscr{Z}_{qN} f(j,v) \\ \mathscr{Z}_{qN} f(j+M,v) \\ \vdots \\ \mathscr{Z}_{qN} f(j+(p-1)M,v) \end{pmatrix} \tag{24}$$

for $(j,v) \in \mathbb{N}_M \times [0,1)$.

By the quasi-periodicity of $\mathscr{Z}_{qN}$ and [35, Theorem 2.1], we have the following lemmas.

Lemma 6. $\mathbf{Z}_{qN}$ *is a unitary operator from* $l^2(\mathbb{Z})$ *onto* $L^2(\mathbb{N}_M \times [0,1), \mathbb{C}^p)$, *and* $\mathscr{Z}_{qN}$ *is a unitary operator from* $l^2(\mathbb{Z})$ *onto* $L^2(E \times [0,1))$ *for an arbitrary subset* E *of* $\mathbb{Z}$ *which is* $qN\mathbb{Z}$ *congruent to* $\mathbb{N}_{qN}$.

Definition 7. Define the *Fourier transform* $\mathscr{F} : l^2(\mathbb{N}_M \times \mathbb{Z}) \to L^2(\mathbb{N}_M \times [0,1))$ by

$$\mathscr{F}c(j,v) = \frac{1}{\sqrt{M}} \sum_{m \in \mathbb{N}_M} \sum_{n \in \mathbb{Z}} c_{m,n} e^{2\pi i(m/M)j} e^{2\pi i nv} \tag{25}$$

for $c \in l^2(\mathbb{N}_M \times \mathbb{Z})$, $j \in \mathbb{N}_M$, and a.e. $v \in [0,1)$.

Definition 8. Define $\mathscr{J} : l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L) \to L^2(\mathbb{N}_M \times [0,1), \mathbb{C}^Q)$ by

$$\mathscr{J}c(j,v) = \mathscr{C}(j,v) \tag{26}$$

for $c = (c_1, c_2, \ldots, c_L) \in l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$, $j \in \mathbb{N}_M$, and a.e. $v \in \mathbb{R}$, where

$$\mathscr{C}(j,v) = \begin{pmatrix} \mathscr{C}_1(j,v) \\ \mathscr{C}_2(j,v) \\ \vdots \\ \mathscr{C}_L(j,v) \end{pmatrix},$$

$$\mathscr{C}_l(j,v) = \begin{pmatrix} \mathscr{C}_l^{(0)}(j,v) \\ \mathscr{C}_l^{(1)}(j,v) \\ \vdots \\ \mathscr{C}_l^{(\lambda_l-1)}(j,v) \end{pmatrix},$$

$$\mathscr{C}_l^{(\tau_l)}(j,v) = \begin{pmatrix} \sum_{m \in \mathbb{N}_M} \sum_{n \in \mathbb{Z}} c_{l,m,nq\lambda_l+\tau_l} e^{2\pi i(m/M)j} e^{2\pi i nv} \\ \sum_{m \in \mathbb{N}_M} \sum_{n \in \mathbb{Z}} c_{l,m,(nq+1)\lambda_l+\tau_l} e^{2\pi i(m/M)j} e^{2\pi i nv} \\ \vdots \\ \sum_{m \in \mathbb{N}_M} \sum_{n \in \mathbb{Z}} c_{l,m,(nq+q-1)\lambda_l+\tau_l} e^{2\pi i(m/M)j} e^{2\pi i nv} \end{pmatrix} \tag{27}$$

for $1 \le l \le L$ and $\tau_l \in \mathbb{N}_{\lambda_l}$.

Similarly, for an arbitrary $d \in l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$, we associate it with $\mathscr{D}(j,v)$.

Lemma 9. (i) $\mathscr{F}$ *is a unitary operator from* $l^2(\mathbb{N}_M \times \mathbb{Z})$ *onto* $L^2(\mathbb{N}_M \times [0,1))$;
(ii) $(1/\sqrt{M})\mathscr{J}$ *is a unitary operator from* $l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$ *onto* $L^2(\mathbb{N}_M \times [0,1), \mathbb{C}^Q)$.

Proof. Since $\{(1/\sqrt{M})e^{2\pi i(m/M)j}e^{2\pi inv} : m \in \mathbb{N}_M, n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{N}_M \times [0, 1))$, we have (i). (ii) is an immediate consequence of (i). $\qquad\square$

Lemma 10. *For* $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ *with* $g_l \in l^2(\mathbb{Z})$ *for* $1 \leq l \leq L$, *one has*

(i) $\mathcal{Z}_{qN}(E_{m/M}T_{nqN+\tau_l N_l + rN}g_l)(j, v) = e^{2\pi i(m/M)j}e^{2\pi inv}$
$\mathcal{Z}_{qN}g_l(j - \tau_l N_l - rN, v)$ *for* $r \in \mathbb{N}_q$;

(ii)

$$\mathbf{Z}_{qN}f(j, v) = \overline{\Psi_{\mathbf{g}}^*(j, v)}\mathcal{J}c(j, v) \qquad (28)$$

for $f = \mathcal{T}_{\mathbf{g}}c$ *with* $c \in l_0(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$, *where* $\mathcal{T}_{\mathbf{g}}c$ *is as in* (5).

Proof. (i) can be obtained by a direct computation. Applying (i), we have

$\mathcal{Z}_{qN}f(j, v)$

$$= \mathcal{Z}_{qN}\left(\sum_{l=1}^{L}\sum_{\tau_l \in \mathbb{N}_{\lambda_l}}\sum_{r \in \mathbb{N}_q}\sum_{m \in \mathbb{N}_M}\sum_{n \in \mathbb{Z}}c_{l,m,(nq+r)\lambda_l + \tau_l}\right.$$

$$\left. \times E_{m/M}T_{nqN+\tau_l N_l + rN}g_l\right)$$

$$= \sum_{l=1}^{L}\sum_{\tau_l \in \mathbb{N}_{\lambda_l}}\sum_{r \in \mathbb{N}_q}\mathscr{C}_l^{(\tau_l)}(j, v)_r\mathcal{Z}_{qN}g_l(j - \tau_l N_l - rN, v) \qquad (29)$$

for $(j, v) \in \mathbb{Z} \times \mathbb{R}$. This implies (ii). $\qquad\square$

Remark 11. When $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ in Lemma 10 is a Bessel sequence in $l^2(\mathbb{Z})$, by the same procedure as the above we can prove (28) holds for $f = \mathcal{T}_{\mathbf{g}}c$ with $c \in l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$.

Lemma 12. *For* $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ *with* $g_l \in l^2(\mathbb{Z})$ *for* $1 \leq l \leq L$, *one has*

$$\Psi_{\mathbf{g}}\left(j + \frac{M}{q}n, v\right) = e^{-2\pi i m_n v}\mathscr{C}_n(v)\Psi_{\mathbf{g}}(j, v)\mathscr{D}_n(v) \qquad (30)$$

for $j \in \mathbb{Z}$, $n = k_n q + (m_n q - r_n)p$ *with* $(k_n, r_n, m_n) \in \mathbb{N}_p \times \mathbb{N}_q \times \mathbb{Z}$, *and a.e.* $v \in \mathbb{R}$, *where* $\mathscr{D}_n(v) = \begin{pmatrix} 0 & e^{-2\pi iv}I_{k_n} \\ I_{p-k_n} & 0 \end{pmatrix}$, $\mathscr{C}_n(v) = $ diag$(\mathscr{C}_{1,n}(v), \mathscr{C}_{2,n}(v), \ldots, \mathscr{C}_{L,n}(v))$, *and* $\mathscr{C}_{l,n}(v)$ *denotes the block matrix (with* λ_l *blocks) of the form* diag$(C_n(v), C_n(v), \ldots, C_n(v))$ *with* $C_n(v) = \begin{pmatrix} 0 & I_{q-r_n} \\ e^{2\pi iv}I_{r_n} & 0 \end{pmatrix}$.

Proof. By [6, Lemma 5], we have

$$\mathscr{G}_l\left(j + \frac{M}{q}n, v\right) = e^{-2\pi i m_n v}C_n(v)\mathscr{G}_l(j, v)\mathscr{D}_n(v) \qquad (31)$$

for $1 \leq l \leq L$, $j \in \mathbb{Z}$, $n = k_n q + (m_n q - r_n)p$ *with* $(k_n, r_n, m_n) \in \mathbb{N}_p \times \mathbb{N}_q \times \mathbb{Z}$, *and a.e.* $v \in \mathbb{R}$. This leads to the lemma. $\qquad\square$

3. Frame Characterization

This section is devoted to characterization of subspace Gabor frames $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ of the form (2).

Theorem 13. *For any* $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ *with each* $g_l \in l^2(\mathbb{Z})$, *the following are equivalent.*

(i) $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *is a Bessel sequence in* $l^2(\mathbb{Z})$ *with Bessel bound* B.

(ii) $\Psi_{\mathbf{g}}(j, v)\Psi_{\mathbf{g}}^*(j, v) \leq (B/M)I$ *for* $j \in \mathbb{N}_{M/q}$ *and a.e.* $v \in [0, 1)$.

(iii) $\Psi_{\mathbf{g}}(j, v)\Psi_{\mathbf{g}}^*(j, v) \leq (B/M)I$ *for* $j \in \mathbb{Z}$ *and a.e.* $v \in \mathbb{R}$.

Proof. By Lemma 12, conditions (ii) and (iii) are equivalent. So, to finish the proof, we only need to prove that (i) holds if and only if

$$\Psi_{\mathbf{g}}(j, v)\Psi_{\mathbf{g}}^*(j, v) \leq \frac{B}{M}I \quad \text{for } j \in \mathbb{N}_M, \text{ a.e. } v \in [0, 1), \qquad (32)$$

since $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is a Bessel sequence with Bessel bound B if and only if

$$\|\mathcal{T}_{\mathbf{g}}c\|^2 \leq B\|c\|^2 \qquad (33)$$

for $c \in l_0(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$, which can be rewritten as

$$\sum_{j \in \mathbb{N}_M}\int_0^1\left\|\overline{\Psi_{\mathbf{g}}^*(j, v)}\mathcal{J}c(j, v)\right\|^2 dv$$

$$\leq \frac{B}{M}\sum_{j \in \mathbb{N}_M}\int_0^1\|\mathcal{J}c(j, v)\|^2 dv \qquad (34)$$

by Lemmas 6–10.

Next we prove the equivalence between (32) and (34) to finish the proof. It is obvious that (32) implies (34). Now we turn to the converse implication. Suppose (34) holds. Since every entry in $\Psi_{\mathbf{g}}(j, \cdot)$ belongs to $L^2([0, 1))$ and thus belongs to $L^1([0, 1))$ for $j \in \mathbb{N}_M$, almost every point in $(0, 1)$ is a Lebesgue point for every entry in $\Psi_{\mathbf{g}}^*(j, v)$ and $j \in \mathbb{N}_M$. Let v_0 be such a point and $j_0 \in \mathbb{N}_M$. Fix $x \in \mathbb{C}^Q$ and $\epsilon > 0$ with $(v_0 - \epsilon, v_0 + \epsilon) \subset (0, 1)$. Define $c \in l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$ by

$$\mathcal{J}c(j, v) = \frac{1}{\sqrt{2\epsilon}}\chi_{\{j_0\} \times (v_0 - \epsilon, v_0 + \epsilon)}(j, v)\overline{x} \qquad (35)$$

for $j \in \mathbb{N}_M, v \in [0, 1)$. By Lemma 9 and the density of $l_0(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$ in $l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$, (34) holds for $c \in l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$. Substituting (35) into (34), we obtain that

$$\frac{1}{2\epsilon}\int_{(v_0 - \epsilon, v_0 + \epsilon)}\left\|\Psi_{\mathbf{g}}^*(j_0, v)x\right\|^2 dv \leq \frac{B}{M}\|x\|^2. \qquad (36)$$

Letting $\epsilon \to 0$ leads to

$$\left\|\Psi_{\mathbf{g}}^*(j_0, v_0)x\right\|^2 \leq \frac{B}{M}\|x\|^2; \qquad (37)$$

that is,

$$\left\langle \Psi_{\mathbf{g}}\left(j_0, v_0\right) \Psi_{\mathbf{g}}^*\left(j_0, v_0\right) x, x \right\rangle \le \frac{B}{M} \|x\|^2. \tag{38}$$

This gives (32) by the arbitrariness of j_0, v_0, and x. The proof is completed. $\square$

Lemma 14 ([1, Theorem A.6.5]). *Let U_1, U_2, U_3 be self-adjoint operators on a Hilbert space $\mathcal{H}$. If $U_1 \le U_2$, and $U_3 \ge 0$, and U_3 commutes with U_1 and U_2, then $U_1 U_3 \le U_2 U_3$.*

Remark 15. The inequality $\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \le (B/M)I$ in Theorem 13 can be replaced by any one of the following inequalities:

(i) $\left(\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v)\right)^2 \le (B/M)\Psi_{\mathbf{g}}(j, v)\Psi_{\mathbf{g}}^*(j, v)$;

(ii) $\Psi_{\mathbf{g}}^*(j, v)\Psi_{\mathbf{g}}(j, v) \le (B/M)I$;

(iii) $\left(\Psi_{\mathbf{g}}^*(j, v)\Psi_{\mathbf{g}}(j, v)\right)^2 \le (B/M)\Psi_{\mathbf{g}}^*(j, v)\Psi_{\mathbf{g}}(j, v)$.

Indeed, we have (i) by Lemma 14 if $\Psi_{\mathbf{g}}(j, v)\Psi_{\mathbf{g}}^*(j, v) \le (B/M)I$. From (i), we have

$$\left\langle \Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) x, \Psi_{\mathbf{g}}^*(j, v) x \right\rangle \\ \le \frac{B}{M} \left\langle \Psi_{\mathbf{g}}^*(j, v) x, \Psi_{\mathbf{g}}^*(j, v) x \right\rangle \tag{39}$$

for $x \in \mathbb{C}^Q$, and thus $\Psi_{\mathbf{g}}^*(j, v)\Psi_{\mathbf{g}}(j, v) \le (B/M)I$ on range$(\Psi_{\mathbf{g}}^*(j, v))$. This implies (ii) due to the fact that $\mathbb{C}^p = $ range$(\Psi_{\mathbf{g}}^*(j, v)) \oplus \ker(\Psi_{\mathbf{g}}(j, v))$. It is obvious that (ii) implies (iii) by Lemma 14. By the same procedure as in "(i) $\Rightarrow$ (ii)", we can prove that (iii) implies $\Psi_{\mathbf{g}}(j, v)\Psi_{\mathbf{g}}^*(j, v) \le (B/M)I$.

Remark 16. Condition (iii) in Theorem 13 is equivalent to $\|\Psi_{\mathbf{g}}(j, v)\| \le \sqrt{B/M}$ for a.e. $(j, v) \in \mathbb{Z} \times \mathbb{R}$ when $\Psi_{\mathbf{g}}(j, v)$ is understood as an operator from $\mathbb{C}^p$ into $\mathbb{C}^Q$. However, such norm is equivalent to the one obtained by taking the maximum of the absolute values of all entries of a matrix. So $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is a Bessel sequence if and only if $\mathscr{Z}_{qN} g_l \in L^\infty(\mathbb{Z} \times \mathbb{R})$ for $1 \le l \le L$ by Theorem 13.

Lemma 17. *For $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ with each $g_l \in l^2(\mathbb{Z})$, let $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ be a Bessel sequence in $l^2(\mathbb{Z})$. For $f \in l^2(\mathbb{Z})$, write $d^{(f)} = \mathscr{T}_{\mathbf{g}}^* f$. Then*

$$\mathscr{J} d^{(f)}(j, v) = M \overline{\Psi_{\mathbf{g}}(j, v)} \mathbf{Z}_{qN} f(j, v) \tag{40}$$

for $(j, v) \in \mathbb{N}_M \times \mathbb{R}$ when either $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is a Bessel sequence or $f \in l_0(\mathbb{Z})$.

Proof. By Lemmas 6 and 10, we have

$$d^{(f)}_{l,m,(nq+r)\lambda_l+\tau_l}$$

$$= \sum_{k \in \mathbb{N}_p} \sum_{j \in \mathbb{N}_M} \int_0^1 \mathscr{Z}_{qN} f(j + kM, v)$$

$$\times \overline{\mathscr{Z}_{qN} g_l(j - \tau_l N_l - rN + kM, v)}$$

$$\times e^{-2\pi i n v} dv e^{-2\pi i (m/M)j} \tag{41}$$

$$= \sum_{j \in \mathbb{N}_M} \int_0^1 \left(\overline{\mathscr{G}_l(j - \tau_l N_l, v)} \mathbf{Z}_{qN} f(j, v)\right)_r$$

$$\times e^{-2\pi i n v} dv e^{-2\pi i (m/M)j}$$

for each $r \in \mathbb{N}_q$, $\tau_l \in \mathbb{N}_{\lambda_l}$, and $1 \le l \le L$. When either $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is a Bessel sequence or $f \in l_0(\mathbb{Z})$, the integrand in (41) belongs to $L^2([0, 1))$ as a function about v by Remark 16. It follows that

$$\mathscr{D}^{(\tau_l)}(j, v) = M \overline{\mathscr{G}_l(j - \tau_l N_l, v)} \mathbf{Z}_{qN} f(j, v) \tag{42}$$

for $(j, v) \in \mathbb{N}_M \times \mathbb{R}$. The lemma therefore follows. $\square$

Theorem 18. *Let $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ and $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ be both Bessel sequences in $l^2(\mathbb{Z})$. Then*

$$\mathbf{Z}_{qN} \mathcal{S}_{\mathbf{h}, \mathbf{g}} f(j, v) = M \overline{\Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{h}}(j, v)} \mathbf{Z}_{qN} f(j, v) \tag{43}$$

for $j \in \mathbb{N}_M$ and a.e. $v \in [0, 1)$.

Proof. Write $d = \mathscr{T}_{\mathbf{h}}^* f$. Then

$$\mathbf{Z}_{qN} \mathcal{S}_{\mathbf{h}, \mathbf{g}} f(j, v) = \overline{\Psi_{\mathbf{g}}^*(j, v)} \mathscr{J} d(j, v) \tag{44}$$

by Remark 11. Applying Lemma 17 to $\mathbf{h}$, we have

$$\mathscr{J} d(j, v) = M \overline{\Psi_{\mathbf{h}}^*(j, v)} \mathbf{Z}_{qN} f(j, v), \tag{45}$$

and thus (43) holds by (44). $\square$

Theorem 19. *For $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ with each $g_l \in l^2(\mathbb{Z})$, the following are equivalent:*

(i) *$G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is a frame for $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ with frame bounds A and B;*

(ii) *$(A/M)\Psi_{\mathbf{g}}(j, v)\Psi_{\mathbf{g}}^*(j, v) \le \left(\Psi_{\mathbf{g}}(j, v)\Psi_{\mathbf{g}}^*(j, v)\right)^2 \le (B/M)\Psi_{\mathbf{g}}(j, v)\Psi_{\mathbf{g}}^*(j, v)$ for $j \in \mathbb{N}_{M/q}$ and a.e. $v \in [0, 1)$;*

(iii) *$(A/M)\Psi_{\mathbf{g}}(j, v)\Psi_{\mathbf{g}}^*(j, v) \le \left(\Psi_{\mathbf{g}}(j, v)\Psi_{\mathbf{g}}^*(j, v)\right)^2 \le (B/M)\Psi_{\mathbf{g}}(j, v)\Psi_{\mathbf{g}}^*(j, v)$ for $j \in \mathbb{Z}$ and a.e. $v \in \mathbb{R}$.*

Proof. By Theorem 13 and Remark 15, we may as well assume that $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is a Bessel sequence, and we need to treat the "lower frame bound" part under this assumption. By a similar argument to beginning proof of Theorem 13, we only need to prove that

$$\left\langle \mathcal{S}_{\mathbf{g}, \mathbf{g}} f, f \right\rangle \ge A \|f\|^2 \quad \text{for } f \in \mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M}) \tag{46}$$

if and only if

$$\left(\Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v)\right)^2 \tag{47}$$
$$\geq \frac{A}{M}\Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v) \quad \text{for } j \in \mathbb{N}_M, \ v \in [0,1).$$

Since the linear span of $G(\mathbf{g},\mathbf{N},\mathbf{M})$ is dense in $\mathscr{M}(\mathbf{g},\mathbf{N},\mathbf{M})$, (46) holds if and only if

$$\langle \mathscr{S}_{\mathbf{g},\mathbf{g}}f, f \rangle \geq A\|f\|^2 \tag{48}$$

for $f = \mathscr{T}_{\mathbf{g}}c$ with $c \in l_2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$ by [1, Lemma 5.1.7]. By Lemma 10, Theorem 18, and Remark 11, (48) can be rewritten as

$$M\left\langle \overline{\left(\Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v)\right)^2} \mathscr{J}c(j,v), \mathscr{J}c(j,v) \right\rangle \tag{49}$$
$$\geq A\left\langle \overline{\Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v)} \mathscr{J}c(j,v), \mathscr{J}c(j,v) \right\rangle,$$

equivalently,

$$M\left\langle \left(\Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v)\right)^2 \overline{\mathscr{J}c(j,v)}, \overline{\mathscr{J}c(j,v)} \right\rangle \tag{50}$$
$$\geq A\left\langle \Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v)\overline{\mathscr{J}c(j,v)}, \overline{\mathscr{J}c(j,v)} \right\rangle$$

for $j \in \mathbb{N}_M$, $v \in [0,1)$, and $c \in l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$. By the same procedure as in Theorem 13, we can prove the equivalence between (47) and (50). The proof is completed. □

Remark 20. By an argument similar to Remark 15, $(\Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v))^2$ and $\Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v)$ in Theorem 19 can be replaced by $(\Psi_{\mathbf{g}}^*(j,v)\Psi_{\mathbf{g}}(j,v))^2$ and $\Psi_{\mathbf{g}}^*(j,v)\Psi_{\mathbf{g}}(j,v)$, respectively.

Definition 21. Given $\mathbf{g} = (g_1, g_2, \dots, g_L)$ with each $g_l \in l^2(\mathbb{Z})$, let $G(\mathbf{g},\mathbf{N},\mathbf{M})$ be a Bessel sequence in $l^2(\mathbb{Z})$. We say that $G(\mathbf{g},\mathbf{N},\mathbf{M})$ has Riesz property if, for $c \in l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$, we must have $c = 0$ whenever $\mathscr{T}_{\mathbf{g}}c = 0$.

By an easy application of the spectral theorem of self-adjoint matrices, we have the following lemma (see also [36, page 978]).

Lemma 22. *Given a measurable set E in $\mathbb{R}$ with $|E| > 0$, let $\mathscr{A} : E \to \mathscr{M}_{s,t}$ be a matrix-valued measurable function. Define by $\mathscr{P}(v)$ the orthogonal projection of $\mathbb{C}^t$ onto $\ker(\mathscr{A}(v))$. Then*

$$\mathscr{P}(v) = \lim_{n\to\infty} \exp\left(-n\mathscr{A}^*(v)\mathscr{A}(v)\right) \tag{51}$$

for $v \in E$, and thus $\mathscr{P}(v)$ is measurable.

Theorem 23. *For $\mathbf{g} = (g_1, g_2, \dots, g_L)$ with each $g_l \in l^2(\mathbb{Z})$, the following are equivalent:*

(i) *$G(\mathbf{g},\mathbf{N},\mathbf{M})$ has Riesz property;*

(ii) *$\mathrm{rank}(\Psi_{\mathbf{g}}(j,v)) = Q$ for $j \in \mathbb{N}_{M/q}$ and a.e. $v \in [0,1)$;*

(iii) *$\mathrm{rank}(\Psi_{\mathbf{g}}(j,v)) = Q$ for $j \in \mathbb{Z}$ and a.e. $v \in \mathbb{R}$.*

Proof. By Lemma 12, (ii) and (iii) are equivalent. So we only need to prove that $c = 0$ is a unique solution to

$$\mathscr{T}_{\mathbf{g}}c = 0 \tag{52}$$

in $l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$ if and only if

$$\mathrm{rank}\left(\Psi_{\mathbf{g}}(j,v)\right) = Q \quad \text{for } j \in \mathbb{N}_M, \ v \in [0,1). \tag{53}$$

By Remark 11, (52) can be rewritten as

$$\overline{\Psi_{\mathbf{g}}^*(j,v)}\mathscr{J}c(j,v) = 0 \quad \text{for } j \in \mathbb{N}_M, \ v \in [0,1). \tag{54}$$

So $c = 0$ is a unique solution to (52) in $l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$ if and only if $d(j,v) = 0$ is a unique solution to

$$\overline{\Psi_{\mathbf{g}}^*(j,v)}d(j,v) = 0 \quad \text{for } j \in \mathbb{N}_M, \text{ a.e. } v \in [0,1) \tag{55}$$

in $L^2(\mathbb{N}_M \times [0,1), \mathbb{C}^Q)$ by Lemma 9. It is obvious that (53) implies that $d(j,v) = 0$ is a unique solution to (55) in $L^2(\mathbb{N}_M \times [0,1), \mathbb{C}^Q)$. Next we prove the converse implication. Suppose there exist $j_0 \in \mathbb{N}_M$ and $E \subset [0,1)$ with $|E| > 0$ such that $\mathrm{rank}(\Psi_{\mathbf{g}}(j_0,v)) < Q$ on E. Let $\mathscr{P}(j_0,v)$ be the orthogonal projection of $\mathbb{C}^Q$ onto $\ker(\Psi_{\mathbf{g}}^*(j_0,v))$. Then there exist $i_0 \in \mathbb{N}_Q$ and $E' \subset E$ with $|E'| > 0$ such that $\mathscr{P}(j_0,v)\mathbf{e}_{i_0} \neq 0$ for $v \in E'$. Define $d(j,v) \in L^2(\mathbb{N}_M \times [0,1), \mathbb{C}^Q)$ by

$$d(j,v) = \begin{cases} \mathscr{P}(j,v)\mathbf{e}_{i_0}, & \text{if } j = j_0, \ v \in E', \\ 0, & \text{otherwise} \end{cases} \tag{56}$$

for $j \in \mathbb{N}_M$ and $v \in [0,1)$. Then $d(j,v)$ is well defined, and $\|d(j,v)\| \leq 1$ by Lemma 22. It follows that $d(j,v)$ is a nonzero solution to (55) in $L^2(\mathbb{N}_M \times [0,1), \mathbb{C}^Q)$. This is a contradiction. The proof is completed. □

Since a Riesz basis is exactly a frame having Riesz property, and an orthonormal basis is exactly a Riesz basis with Riesz bound 1, we have the following theorem by Theorems 19 and 23.

Theorem 24. *For $\mathbf{g} = (g_1, g_2, \dots, g_L)$ with each $g_l \in l^2(\mathbb{Z})$, the following are equivalent:*

(i) *$G(\mathbf{g},\mathbf{N},\mathbf{M})$ is a Riesz basis for $G(\mathbf{g},\mathbf{N},\mathbf{M})$ with Riesz bounds A and B (an orthonormal basis);*

(ii) *$(A/M)I \leq \Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v) \leq (B/M)I(\Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v) = (1/M)I)$ for $j \in \mathbb{N}_{M/q}$ a.e. $v \in [0,1)$;*

(iii) *$(A/M)I \leq \Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v) \leq (B/M)I(\Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v) = (1/M)I)$ for $j \in \mathbb{Z}$ and a.e. $v \in \mathbb{R}$.*

Next we turn to examples of Theorems 19 and 24. Suppose

$$\mathbf{N} = (6,4), \qquad \mathbf{M} = (2,2). \tag{57}$$

Then $N/M = 6$ with $p = 6$, $q = 1$, and $Q = 5$. For $\mathbf{g} = (g_1, g_2)$ with $g_1, g_2 \in l^2(\mathbb{Z})$, we associate it with $\Psi_{\mathbf{g}}$ as in Definition 4. Then

$$\Psi_{\mathbf{g}}(j, v) = \begin{pmatrix} G_1(j, v) \\ G_2(j, v) \end{pmatrix} \quad \text{with}$$

$$G_1(j, v) = \begin{pmatrix} \mathcal{G}_1(j, v) \\ \mathcal{G}_1(j - 6, v) \end{pmatrix}, \quad G_2(j, v) = \begin{pmatrix} \mathcal{G}_2(j, v) \\ \mathcal{G}_2(j - 4, v) \\ \mathcal{G}_2(j - 8, v) \end{pmatrix},$$

(58)

where

$$\mathcal{G}_l(j, v) = \left(\mathcal{Z}_{12} g_l(j + 2k, v) \right)_{0,k} \in \mathcal{M}_{1,6}$$

$$\text{with } j \in \mathbb{N}_2, \ k \in \mathbb{N}_6$$

(59)

for $l = 1, 2$. By the quasi-periodicity of Zak transform, we have

$$\mathcal{G}_1(j - 6, v)$$

$$= \left(e^{2\pi i v} \mathcal{Z}_{12} g_1(j + 6, v), e^{2\pi i v} \mathcal{Z}_{12} g_1(j + 8, v), \right.$$

$$e^{2\pi i v} \mathcal{Z}_{12} g_1(j + 10, v), \mathcal{Z}_{12} g_1(j, v),$$

$$\left. \mathcal{Z}_{12} g_1(j + 2, v), \mathcal{Z}_{12} g_1(j + 4, v) \right),$$

$$\mathcal{G}_2(j - 4, v)$$

$$= \left(e^{2\pi i v} \mathcal{Z}_{12} g_2(j + 8, v), e^{2\pi i v} \mathcal{Z}_{12} g_2(j + 10, v), \right.$$

$$\mathcal{Z}_{12} g_2(j, v), \mathcal{Z}_{12} g_2(j + 2, v),$$

$$\left. \mathcal{Z}_{12} g_2(j + 4, v), \mathcal{Z}_{12} g_2(j + 6, v) \right),$$

$$\mathcal{G}_2(j - 8, v)$$

$$= \left(e^{2\pi i v} \mathcal{Z}_{12} g_2(j + 4, v), e^{2\pi i v} \mathcal{Z}_{12} g_2(j + 6, v), \right.$$

$$e^{2\pi i v} \mathcal{Z}_{12} g_2(j + 8, v), \mathcal{Z}_{12} g_2(j + 10, v),$$

$$\left. \mathcal{Z}_{12} g_2(j, v), \mathcal{Z}_{12} g_2(j + 2, v) \right).$$

(60)

Thus for each $j \in \mathbb{N}_2$ and a.e. $v \in [0, 1)$, $\mathcal{G}_1(j - 6, v)$, and $\mathcal{G}_2(j - 4, v)$, $\mathcal{G}_2(j - 8, v)$ are uniquely determined by $\mathcal{G}_1(j, v)$ and $\mathcal{G}_2(j, v)$, respectively. Observe that $\mathbb{N}_2 + 2\mathbb{N}_6$ is $12\mathbb{Z}$ congruent to $\mathbb{N}_{12}$. It follows that $\mathbf{g}$ is uniquely determined by the values of $\Psi_{\mathbf{g}}(j, v)$ for $j \in \mathbb{N}_2$ and a.e. $v \in [0, 1)$. Therefore, an arbitrary matrix 5×6 matrix-valued function $\mathcal{K}(j, v)$ for $j \in \mathbb{N}_2$ and a.e. $v \in [0, 1)$ with all entries being in $L^2(\mathbb{N}_2 \times [0, 1))$ determines a unique $\mathbf{g}$ by

$$\Psi_{\mathbf{g}}(j, v) = \mathcal{K}(j, v) \quad \text{for } j \in \mathbb{N}_2, \text{ a.e. } v \in [0, 1). \quad (61)$$

Let

$$\mathcal{K}(j, v)$$

$$= \begin{pmatrix} a(j, v) & \lambda(j, v) a(j, v) & \lambda^2(j, v) a(j, v) & 0 & 0 & 0 \\ 0 & 0 & 0 & a(j, v) & \lambda(j, v) a(j, v) & \lambda^2(j, v) a(j, v) \\ -\overline{\lambda(j, v) a(j, v)} & \overline{a(j, v)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{\lambda(j, v) a(j, v)} & \overline{a(j, v)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\overline{\lambda(j, v) a(j, v)} & \overline{a(j, v)} \end{pmatrix}$$

(62)

for $j \in \mathbb{N}_2$ and a.e. $v \in [0, 1)$. Define $\mathbf{g} = (g_1, g_2)$ by

$$\Psi_{\mathbf{g}}(j, v) = \mathcal{K}(j, v) \quad \text{for } j \in \mathbb{N}_2, \text{ a.e. } v \in [0, 1]. \quad (63)$$

Then we obtain the following example.

Example 25. Let $\mathbf{N}$ and $\mathbf{M}$ be defined as in (57) and define $\mathbf{g} = (g_1, g_2)$ by (63), where $\lambda(j, \cdot)$, $a(j, \cdot)$ are continuous on $[0, 1]$ for $j \in \{0, 1\}$. Assume that

$$|\lambda(j, v)| < 1,$$

$$|a(j, v)|^2 \left(1 - 2|\lambda(j, v)|^4 - 2|\lambda(j, v)|^6 - |\lambda(j, v)|^8 \right)$$

$$< 1 - |\lambda(j, v)|^3,$$

$$|a(j, v)|^2 \left(1 + 2|\lambda(j, v)|^3 + 2|\lambda(j, v)|^5 + |\lambda(j, v)|^7 \right)$$

$$< 1 + |\lambda(j, v)|^3$$

(64)

for $(j, v) \in \{0, 1\} \times [0, 1]$ satisfying $a(j, v) \neq 0$. Then

$$\text{supp}(g_1) \subset \mathbb{N}_6 + 12\mathbb{Z}, \qquad \text{supp}(g_2) \subset \mathbb{N}_4 + 12\mathbb{Z},$$

(65)

and $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is a frame for $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$.

Proof. Equation (65) is an immediate consequence of (63). Next we prove that $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is a frame for $\mathscr{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$. By a simple computation, we have

$$
\begin{aligned}
&\left\langle \left(\Psi_{\mathbf{g}}\left(j,v\right)\Psi_{\mathbf{g}}^{*}\left(j,v\right)\right)^{2}x,x\right\rangle \\
&\qquad = \left|a\left(j,v\right)\right|^{2}\sum_{l=1}^{5}\alpha_{l}\left(j,v\right)\left|x_{l}\right|^{2}+\beta\left(j,v\right), \\
&\left\langle \Psi_{\mathbf{g}}\left(j,v\right)\Psi_{\mathbf{g}}^{*}\left(j,v\right)x,x\right\rangle \\
&\qquad = \left|a\left(j,v\right)\right|^{2}\sum_{l=1}^{5}\widetilde{\alpha}_{l}\left(j,v\right)\left|x_{l}\right|^{2}+\widetilde{\beta}\left(j,v\right)
\end{aligned}
\tag{66}
$$

for $(j,v)\in\{0,1\}\times[0,1]$ and $x\in\mathbb{C}^{5}$, where

$$
\begin{aligned}
\alpha_{1}\left(j,v\right) &= \left|\lambda\left(j,v\right)\right|^{3} \\
&\quad + \left(1+\left|\lambda\left(j,v\right)\right|^{2}+\left|\lambda\left(j,v\right)\right|^{4}\right)^{2}\left|a\left(j,v\right)\right|^{2}, \\
\alpha_{2}\left(j,v\right) &= 1+\left(1+\left|\lambda\left(j,v\right)\right|^{2}+\left|\lambda\left(j,v\right)\right|^{4}\right)^{2} \\
&\quad \times \left|a\left(j,v\right)\right|^{2}, \\
\alpha_{4}\left(j,v\right) &= 1+\left|\lambda\left(j,v\right)\right|^{3}+\left(1+\left|\lambda\left(j,v\right)\right|^{2}\right)^{2}\left|a\left(j,v\right)\right|^{2}, \\
\alpha_{4}\left(j,v\right) &= 1+\left|\lambda\left(j,v\right)\right|^{3}+\left(1+\left|\lambda\left(j,v\right)\right|^{2}\right)^{2}\left|a\left(j,v\right)\right|^{2}, \\
\alpha_{3}\left(j,v\right) &= \alpha_{5}\left(j,v\right)=\left(1+\left|\lambda\left(j,v\right)\right|^{2}\right)^{2}\left|a\left(j,v\right)\right|^{2}, \\
\beta\left(j,v\right) &= 2\operatorname{Re}\left(\left[\lambda(j,v)^{3}\right.\right. \\
&\qquad \times \left(2+2\left|\lambda\left(j,v\right)\right|^{2}+\left|\lambda\left(j,v\right)\right|^{4}\right) \\
&\qquad \left.\times a(j,v)^{2}\left|a\left(j,v\right)\right|^{2}\right]\overline{x}_{1}x_{4}\right) \\
&\quad + 2\operatorname{Re}\left(\left[\left(2+2\left|\lambda\left(j,v\right)\right|^{2}+\left|\lambda\left(j,v\right)\right|^{4}\right)\right.\right. \\
&\qquad \left.\left.\times a(j,v)^{2}\left|a\left(j,v\right)\right|^{2}\right]\overline{x}_{2}x_{4}\right) \\
&\quad - 2\operatorname{Re}\left(\lambda(j,v)^{3}\left|a\left(j,v\right)\right|^{2}\overline{x}_{1}x_{2}\right), \\
\widetilde{\alpha}_{1}\left(j,v\right) &= \widetilde{\alpha}_{2}\left(j,v\right)=1+\left|\lambda\left(j,v\right)\right|^{2}+\left|\lambda\left(j,v\right)\right|^{4}, \\
\widetilde{\alpha}_{3}\left(j,v\right) &= \widetilde{\alpha}_{4}\left(j,v\right)=\widetilde{\alpha}_{5}\left(j,v\right)=1+\left|\lambda\left(j,v\right)\right|^{2}, \\
\widetilde{\beta}\left(j,v\right) &= -2\operatorname{Re}\left(\lambda(j,v)^{3}a(j,v)^{2}\overline{x}_{1}x_{4}\right) \\
&\quad + 2\operatorname{Re}\left(a(j,v)^{2}\overline{x}_{2}x_{4}\right).
\end{aligned}
\tag{67}
$$

It is easy to check that

$$
\begin{aligned}
\left|\beta\left(j,v\right)\right| &\leq \left|a\left(j,v\right)\right|^{2}\left[\beta_{1}\left(j,v\right)\left|x_{1}\right|^{2}+\beta_{2}\left(j,v\right)\left|x_{2}\right|^{2}\right. \\
&\qquad \left.+\beta_{4}\left(j,v\right)\left|x_{4}\right|^{2}\right],
\end{aligned}
\tag{68}
$$

where

$$
\begin{aligned}
\beta_{1}\left(j,v\right) &= \left|\lambda\left(j,v\right)\right|^{3}+\left|a\left(j,v\right)\right|^{2} \\
&\quad \times \left(2\left|\lambda\left(j,v\right)\right|^{3}+2\left|\lambda\left(j,v\right)\right|^{5}+\left|\lambda\left(j,v\right)\right|^{7}\right), \\
\beta_{2}\left(j,v\right) &= \left|\lambda\left(j,v\right)\right|^{3}+\left|a\left(j,v\right)\right|^{2} \\
&\quad \times \left(2+2\left|\lambda\left(j,v\right)\right|^{2}+\left|\lambda\left(j,v\right)\right|^{4}\right), \\
\beta_{4}\left(j,v\right) &= \left|a\left(j,v\right)\right|^{2} \\
&\quad \times \left(2+2\left|\lambda\left(j,v\right)\right|^{2}+2\left|\lambda\left(j,v\right)\right|^{3}\right. \\
&\qquad +\left|\lambda\left(j,v\right)\right|^{4}+2\left|\lambda\left(j,v\right)\right|^{5} \\
&\qquad \left.+\left|\lambda\left(j,v\right)\right|^{7}\right).
\end{aligned}
$$

$$
\begin{aligned}
\left|\widetilde{\beta}\left(j,v\right)\right| &\leq \left|a\left(j,v\right)\right|^{2} \\
&\quad \times \left[\widetilde{\beta}_{1}\left(j,v\right)\left|x_{1}\right|^{2}+\widetilde{\beta}_{2}\left(j,v\right)\left|x_{2}\right|^{2}+\widetilde{\beta}_{4}\left(j,v\right)\left|x_{4}\right|^{2}\right],
\end{aligned}
\tag{69}
$$

where

$$
\begin{aligned}
\widetilde{\beta}_{1}\left(j,v\right) &= \left|\lambda\left(j,v\right)\right|^{3}, \qquad \widetilde{\beta}_{2}\left(j,v\right)=1, \\
\widetilde{\beta}_{4}\left(j,v\right) &= 1+\left|\lambda\left(j,v\right)\right|^{3}.
\end{aligned}
\tag{70}
$$

Write

$$
\begin{aligned}
C_{l}\left(j,v\right) &= \alpha_{l}\left(j,v\right)+\beta_{l}\left(j,v\right), \\
\widetilde{C}_{l}\left(j,v\right) &= \alpha_{l}\left(j,v\right)-\beta_{l}\left(j,v\right) \quad \text{for } l=1,2,4, \\
C\left(j,v\right) &= \left(1+\left|\lambda\left(j,v\right)\right|^{2}\right)^{2}\left|a\left(j,v\right)\right|^{2}, \\
D_{l}\left(j,v\right) &= \widetilde{\alpha}_{l}\left(j,v\right)+\widetilde{\beta}_{l}\left(j,v\right), \\
\widetilde{D}_{l}\left(j,v\right) &= \widetilde{\alpha}_{l}\left(j,v\right)-\widetilde{\beta}_{l}\left(j,v\right) \quad \text{for } l=1,2,4, \\
D\left(j,v\right) &= \left|\lambda\left(j,v\right)\right|^{2}+1.
\end{aligned}
\tag{71}
$$

Take

$$
\begin{aligned}
\widetilde{C} &= \min\left\{C\left(j,v\right),\widetilde{C}_{l}\left(j,v\right):l=1,2,4,j\in\mathbb{N}_{2},v\in[0,1]\right\}, \\
C &= \max\left\{C\left(j,v\right),C_{l}\left(j,v\right):l=1,2,4,j\in\mathbb{N}_{2},v\in[0,1]\right\}, \\
\widetilde{D} &= \min\left\{D\left(j,v\right),\widetilde{D}_{l}\left(j,v\right):l=1,2,4,j\in\mathbb{N}_{2},v\in[0,1]\right\}, \\
D &= \max\left\{D\left(j,v\right),D_{l}\left(j,v\right):l=1,2,4,j\in\mathbb{N}_{2},v\in[0,1]\right\}.
\end{aligned}
\tag{72}
$$

Then

$$\left\langle \left(\Psi_{\mathbf{g}}(j,v) \Psi_{\mathbf{g}}^*(j,v) \right)^2 x, x \right\rangle$$

$$\leq |a(j,v)|^2 \left[C_1(j,v) |x_1|^2 + C_2(j,v) |x_2|^2 \right.$$

$$+ C_4(j,v) |x_4|^2 + C(j,v)$$

$$\left. \times \left(|x_3|^2 + |x_5|^2 \right) \right]$$

$$\leq C |a(j,v)|^2 \|x\|^2,$$

$$\left\langle \left(\Psi_{\mathbf{g}}(j,v) \Psi_{\mathbf{g}}^*(j,v) \right)^2 x, x \right\rangle$$

$$\geq |a(j,v)|^2 \left[\widetilde{C}_1(j,v) |x_1|^2 + \widetilde{C}_2(j,v) |x_2|^2 \right.$$

$$+ \widetilde{C}_4(j,v) |x_4|^2 + C(j,v)$$

$$\left. \times \left(|x_3|^2 + |x_5|^2 \right) \right]$$

$$\geq \widetilde{C} |a(j,v)|^2 \|x\|^2,$$

$$\left\langle \Psi_{\mathbf{g}}(j,v) \Psi_{\mathbf{g}}^*(j,v) x, x \right\rangle \qquad (73)$$

$$\leq |a(j,v)|^2 \left[D_1(j,v) |x_1|^2 + D_2(j,v) |x_2|^2 \right.$$

$$+ D_4(j,v) |x_4|^2 + D(j,v)$$

$$\left. \times \left(|x_3|^2 + |x_5|^2 \right) \right]$$

$$\leq D |a(j,v)|^2 \|x\|^2,$$

$$\left\langle \Psi_{\mathbf{g}}(j,v) \Psi_{\mathbf{g}}^*(j,v) x, x \right\rangle$$

$$\geq |a(j,v)|^2 \left[\widetilde{D}_1(j,v) |x_1|^2 \right.$$

$$+ \widetilde{D}_2(j,v) |x_2|^2 + \widetilde{D}_4(j,v) |x_4|^2$$

$$\left. + D(j,v) \left(|x_3|^2 + |x_5|^2 \right) \right]$$

$$\geq \widetilde{D} |a(j,v)|^2 \|x\|^2.$$

It follows that

$$\frac{A}{2} \left\langle \Psi_{\mathbf{g}}(j,v) \Psi_{\mathbf{g}}^*(j,v) x, x \right\rangle \leq \left\langle \left(\Psi_{\mathbf{g}}(j,v) \Psi_{\mathbf{g}}^*(j,v) \right)^2 x, x \right\rangle$$

$$\leq \frac{B}{2} \left\langle \Psi_{\mathbf{g}}(j,v) \Psi_{\mathbf{g}}^*(j,v) x, x \right\rangle, \qquad (74)$$

for $x \in \mathbb{C}^5$, $j \in \mathbb{N}_2$, and a.e. $v \in [0,1)$, and thus

$$\frac{A}{2} \Psi_{\mathbf{g}}(j,v) \Psi_{\mathbf{g}}^*(j,v) \leq \left(\Psi_{\mathbf{g}}(j,v) \Psi_{\mathbf{g}}^*(j,v) \right)^2$$

$$\leq \frac{B}{2} \Psi_{\mathbf{g}}(j,v) \Psi_{\mathbf{g}}^*(j,v) \qquad (75)$$

for $j \in \mathbb{N}_2$ and a.e. $v \in [0,1)$, where $A = 2\widetilde{C}/D$, $B = 2C/\widetilde{D}$. By Theorem 19, $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is a frame for $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ with frame bounds A and B.   □

Remark 26. By a simple computation, we have rank$(\Psi_{\mathbf{g}}(j, v)) = 5$ if and only if

$$a(0,v) a(1,v) \lambda(0,v) \lambda(1,v) \neq 0 \quad \text{for a.e. } v \in [0,1). \qquad (76)$$

Also observe that rank$(\Psi_{\mathbf{g}}(j,v)) = 5$ is equivalent to $\Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v)$ being invertible, and that (75) can be reduced to

$$\frac{AI}{2} \leq \Psi_{\mathbf{g}}(j,v) \Psi_{\mathbf{g}}^*(j,v) \leq \frac{BI}{2} \quad \text{for } j \in \mathbb{N}_2, \text{ a.e. } v \in [0,1) \qquad (77)$$

in this case. Therefore, $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is a Riesz basis for $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ with Riesz bounds A and B if and only if (76) holds.

4. Gabor Dual Characterization

Let $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ be a frame for $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$. In this section, we discuss three kinds of duals with Gabor structure. We establish characterizations of Gabor duals of type I and type II and obtain a sufficient condition for Gabor duals of type II and oblique Gabor duals.

By an argument similar to [5, Lemma 2.6], we have the following lemmas.

Lemma 27. *For $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ and $\mathbf{h} = (h_1, h_2, \ldots, h_L)$ with each $g_l, h_l \in l^2(\mathbb{Z})$, the following are equivalent:*

(i) *$h_l \in \mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ for each $1 \leq l \leq L$;*

(ii) *there exists a function $A : \mathbb{N}_{M/q} \times [0,1) \rightarrow \mathcal{M}_{Q,Q}$ such that*

$$\Psi_{\mathbf{h}}(j,v) = A(j,v) \Psi_{\mathbf{g}}(j,v) \qquad (78)$$

for $j \in \mathbb{N}_{M/q}$ and a.e. $v \in [0,1)$;

(iii) *there exists a function $A : \mathbb{Z} \times \mathbb{R} \rightarrow \mathcal{M}_{Q,Q}$ such that*

$$\Psi_{\mathbf{h}}(j,v) = A(j,v) \Psi_{\mathbf{g}}(j,v) \qquad (79)$$

for $j \in \mathbb{Z}$ and a.e. $v \in \mathbb{R}$.

Lemma 28. *Given $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ with each $g_l \in l^2(\mathbb{Z})$, $\mathbf{h} = (h_1, h_2, \ldots, h_L)$ with each $h_l \in l^2(\mathbb{Z})$, let $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ and $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ be both Bessel sequences in $l^2(\mathbb{Z})$. Then the following are equivalent:*

(i) *range$(\mathcal{T}_{\mathbf{h}}^*) \subset \overline{\text{range}(\mathcal{T}_{\mathbf{g}}^*)}$;*

(ii) *there exists a function $B : \mathbb{N}_{M/q} \times [0,1) \rightarrow \mathcal{M}_{p,p}$ such that*

$$\Psi_{\mathbf{h}}(j,v) = \Psi_{\mathbf{g}}(j,v) B(j,v) \quad \text{for } (j,v) \in \mathbb{N}_{M/q} \times [0,1); \qquad (80)$$

(iii) *there exists a function $B : \mathbb{Z} \times \mathbb{R} \rightarrow \mathcal{M}_{p,p}$ such that*

$$\Psi_{\mathbf{h}}(j,v) = \Psi_{\mathbf{g}}(j,v) B(j,v) \quad \text{for } (j,v) \in \mathbb{Z} \times \mathbb{R}. \qquad (81)$$

Proof. If (ii) holds, we can obtain (iii) by choosing $B : \mathbb{Z} \times \mathbb{R} \to \mathcal{M}_{p,p}$ as

$$B\left(j + \frac{M}{q}n, v\right) = \mathcal{D}_n^*(v) B(j, v) \mathcal{D}_n(v) \qquad (82)$$

for $(j, v) \in \mathbb{N}_{M/q} \times \mathbb{R}$, where $\mathcal{D}_n(v)$ is defined as in **Lemma 12**. So we only need to prove that (i) holds if and only if there exists a function $B : \mathbb{N}_M \times [0, 1) \to \mathcal{M}_{p,p}$ such that

$$\Psi_{\mathbf{h}}(j, v) = \Psi_{\mathbf{g}}(j, v) B(j, v) \quad \text{for } (j, v) \in \mathbb{N}_M \times [0, 1). \qquad (83)$$

For $c \in l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$, c is orthogonal to $\text{range}(\mathcal{T}_{\mathbf{g}}^*)$ $(\text{range}(\mathcal{T}_{\mathbf{h}}^*))$ if and only if

$$\mathcal{J}c(j, v) \perp \overline{\Psi_{\mathbf{g}}(j, v)} \mathbf{Z}_{qN} f(j, v)$$
$$\left(\mathcal{J}c(j, v) \perp \overline{\Psi_{\mathbf{h}}(j, v)} \mathbf{Z}_{qN} f(j, v)\right) \qquad (84)$$

in $L^2(\mathbb{N}_M \times [0, 1), \mathbb{C}^Q)$-inner product by **Lemmas 9 and 17**, equivalently,

$$\overline{\Psi_{\mathbf{g}}^*(j, v)} \mathcal{J}c(j, v) \perp \mathbf{Z}_{qN} f(j, v)$$
$$\left(\overline{\Psi_{\mathbf{h}}^*(j, v)} \mathcal{J}c(j, v) \perp \mathbf{Z}_{qN} f(j, v)\right) \quad \text{for } f \in l^2(\mathbb{Z}) \qquad (85)$$

in $L^2(\mathbb{N}_M \times [0, 1), \mathbb{C}^P)$-inner. Again by **Lemma 6**, (85) is equivalent to

$$\overline{\Psi_{\mathbf{g}}^*(j, v)} \mathcal{J}c(j, v) = 0 \qquad \left(\overline{\Psi_{\mathbf{h}}^*(j, v)} \mathcal{J}c(j, v) = 0\right)$$
$$\text{for } j \in \mathbb{N}_M, \text{ a.e. } v \in [0, 1). \qquad (86)$$

So we only need to prove that (83) holds if and only if, for $\mathcal{J}c(j, v) \in L^2(\mathbb{N}_M \times [0, 1), \mathbb{C}^Q)$,

$$\overline{\Psi_{\mathbf{h}}^*(j, v)} \mathcal{J}c(j, v) = 0 \quad \text{for } j \in \mathbb{N}_M, \text{ a.e. } v \in [0, 1) \qquad (87)$$

whenever

$$\overline{\Psi_{\mathbf{g}}^*(j, v)} \mathcal{J}c(j, v) = 0 \quad \text{for } j \in \mathbb{N}_M, \text{ a.e. } v \in [0, 1). \qquad (88)$$

Obviously the necessity holds. Next we prove that a contradiction will occur if (83) is violated. Suppose (83) does not hold. Then there exist some $j_0 \in \mathbb{N}_M$ and $E \subset [0, 1)$ with $|E| > 0$, on which some r_0th column h_{r_0} of $\overline{\Psi_{\mathbf{h}}(j_0, v)}$ satisfies

$$h_{r_0} \notin \text{range}\left(\overline{\Psi_{\mathbf{g}}(j_0, v)}\right) \quad \text{for } v \in E. \qquad (89)$$

Let $\mathcal{P}(j_0, v)$ be the orthogonal projection of $\mathbb{C}^Q$ onto $\ker(\overline{\Psi_{\mathbf{g}}^*(j_0, v)})$. Define $\mathcal{J}c(j, v) \in L^2(\mathbb{N}_M \times [0, 1), \mathbb{C}^Q)$ by

$$\mathcal{J}c(j, v) = \begin{cases} \mathcal{P}(j_0, v) h_{r_0}(j_0, v), & \text{if } j = j_0, \ v \in E, \\ 0, & \text{otherwise} \end{cases} \qquad (90)$$

for $(j, v) \in \mathbb{N}_M \times [0, 1)$. Then $\mathcal{J}c(j, v)$ solves (88) and $\mathcal{J}c(j_0, v) \neq 0$ for $v \in E$. Also observe that the r_0th component of $\overline{\Psi_{\mathbf{h}}^*(j_0, v)} \mathcal{J}c(j_0, v)$ is exactly $\|\mathcal{J}c(j_0, v)\|^2$ for $v \in E$. It follows that $\mathcal{J}c(j, v)$ fails to solve (87). This finishes the proof. □

Lemma 29. *Given* $\mathbf{g} = (g_1, g_2, \ldots, g_L)$, $\mathbf{h} = (h_1, h_2, \ldots, h_L)$ *with each* $g_l, h_l \in l^2(\mathbb{Z})$, *let* $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *and* $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ *be both Bessel sequences in* $l^2(\mathbb{Z})$. *Then*

$$\mathcal{S}_{\mathbf{h},\mathbf{g}} f = f \qquad (91)$$

for $f \in \mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *if and only if*

$$\Psi_{\mathbf{g}}^*(j, v) = M\Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{h}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \qquad (92)$$

for $j \in \mathbb{N}_{M/q}$ *and a.e.* $v \in [0, 1)$.

Proof. By **Lemma 12**, (92) holds for $j \in \mathbb{N}_{M/q}$ and a.e. $v \in [0, 1)$ if and only if it holds for $j \in \mathbb{N}_M$ and a.e. $v \in [0, 1)$. Next we prove that (91) holds for $f \in \mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ if and only if (92) holds for $j \in \mathbb{N}_M$ and a.e. $v \in [0, 1)$.

Since $\text{range}(\mathcal{T}_{\mathbf{g}})$ is dense in $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$, (91) holds for $f \in \mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ if and only if it holds for $f \in \text{range}(\mathcal{T}_{\mathbf{g}})$, equivalently,

$$\Psi_{\mathbf{g}}^*(j, v) \mathcal{J}c(j, v)$$
$$= M\Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{h}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \mathcal{J}c(j, v) \qquad (93)$$

for $c \in l^2(\mathbb{N}_M \times \mathbb{Z}, \mathbb{C}^L)$ by **Remark 11** and **Theorem 18**. This is also equivalent to

$$\Psi_{\mathbf{g}}^*(j, v) d(j, v)$$
$$= M\Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{h}}(j, v) \Psi_{\mathbf{g}}^*(j, v) d(j, v) \qquad (94)$$

for $d(j, v) \in L^2(\mathbb{N}_M \times [0, 1), \mathbb{C}^Q)$, $j \in \mathbb{N}_M$, and a.e. $v \in [0, 1)$ by **Lemma 9**. It is obvious that (94) holds if (92) holds for $j \in \mathbb{N}_M$ and a.e. $v \in [0, 1)$. Now suppose (94) holds. For an arbitrarily fixed $x \in \mathbb{C}^Q$, choose $d(j, v)$ as

$$d(j, v) = x \quad \text{for } (j, v) \in \mathbb{N}_M \times [0, 1). \qquad (95)$$

Then $d(j, v) \in L^2(\mathbb{N}_M \times [0, 1), \mathbb{C}^Q)$, and thus $\Psi_{\mathbf{g}}^*(j, v)x = M\Psi_{\mathbf{g}}^*(j, v)\Psi_{\mathbf{h}}(j, v)\Psi_{\mathbf{g}}^*(j, v)x$ for $j \in \mathbb{N}_M$ and a.e. $v \in [0, 1)$ by (94). So (92) holds for $j \in \mathbb{N}_M$ and a.e. $v \in [0, 1)$ by the arbitrariness of x. The proof is completed. □

By **Lemmas 27–29**, we have the following theorem which characterizes the Gabor duals of type I (resp., type II).

Theorem 30. *Given* $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ *with each* $g_l \in l^2(\mathbb{Z})$, *let* $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *be a frame for* $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$. *Then, for any Bessel sequence* $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ *in* $l^2(\mathbb{Z})$, $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ *is a Gabor dual of type I (type II) for* $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *if and only if the following hold:*

(i) *there exists* $A : \mathbb{N}_{M/q} \times [0, 1) \to \mathcal{M}_{Q,Q}(B : \mathbb{N}_{M/q} \times [0, 1) \to \mathcal{M}_{p,p})$ *such that*

$$\Psi_{\mathbf{h}}(j, v) = A(j, v) \Psi_{\mathbf{g}}(j, v) \quad \left(\Psi_{\mathbf{h}}(j, v) = \Psi_{\mathbf{g}}(j, v) B(j, v)\right); \qquad (96)$$

(ii) $\Psi_{\mathbf{g}}^*(j, v) = M\Psi_{\mathbf{g}}^*(j, v)\Psi_{\mathbf{h}}(j, v)\Psi_{\mathbf{g}}^*(j, v)$ *for* $j \in \mathbb{N}_{M/q}$ *and a.e.* $v \in [0, 1)$.

Theorem 31. *Given* $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ *with each* $g_l \in l^2(\mathbb{Z})$, *let* $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *be a frame for* $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$. *Then, for any Bessel sequence* $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ *in* $l^2(\mathbb{Z})$,

(i) $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ *is a Gabor dual of type I for* $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *if and only if there exists* $\mathscr{A} : \mathbb{N}_{M/q} \times [0, 1) \to \mathcal{M}_{Q,Q}$ *such that*

$$
\begin{aligned}
\Psi_{\mathbf{h}}(j, v) = & \frac{1}{M} \left(\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \right)^\dagger \Psi_{\mathbf{g}}(j, v) \\
& \times \left[I - M \Psi_{\mathbf{g}}^*(j, v) \mathscr{A}(j, v) \Psi_{\mathbf{g}}(j, v) \right] \\
& + \mathscr{A}(j, v) \Psi_{\mathbf{g}}(j, v)
\end{aligned}
\tag{97}
$$

for $j \in \mathbb{N}_{M/q}$ *and a.e.* $v \in [0, 1)$;

(ii) $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ *is a Gabor dual of type II for* $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *if*

$$
\begin{aligned}
\Psi_{\mathbf{h}}(j, v) = & \frac{1}{M} \Psi_{\mathbf{g}}(j, v) \left(\Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{g}}(j, v) \right)^\dagger \\
& \times \left[I - M \Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{g}}(j, v) \mathscr{A}(j, v) \right] \\
& + \Psi_{\mathbf{g}}(j, v) \mathscr{A}(j, v)
\end{aligned}
\tag{98}
$$

for $j \in \mathbb{N}_{M/q}$ *and a.e.* $v \in [0, 1)$;

(iii) $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ *is an oblique dual of* $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *if one of the following conditions holds:*

(a) $\Psi_{\mathbf{h}}(j, v) = (1/M) \Psi_{\mathbf{g}}(j, v) (\Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{g}}(j, v))^\dagger [I - M \Psi_{\mathbf{g}}^*(j, v) \mathscr{A}(j, v) \Psi_{\mathbf{g}}(j, v)] + \mathscr{A}(j, v) \Psi_{\mathbf{g}}(j, v)$ *for* $j \in \mathbb{N}_{M/q}$ *and a.e.* $v \in [0, 1)$;

(b) $\Psi_{\mathbf{h}}(j, v) = (1/M) (\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v))^\dagger \Psi_{\mathbf{g}}(j, v) [I - M \Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{g}}(j, v) \mathscr{A}(j, v)] + \Psi_{\mathbf{g}}(j, v) \mathscr{A}(j, v)$ *for* $j \in \mathbb{N}_{M/q}$ *and a.e.* $v \in [0, 1)$.

Proof. The items (ii) and (iii) can be proved similarly to item (i). Next we prove item (i). First we assume that (97) holds. Then

$$
\begin{aligned}
& M \Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{h}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \\
& = \Psi_{\mathbf{g}}^*(j, v) \left(\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \right)^\dagger \\
& \quad \times \Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \\
& \quad - M \Psi_{\mathbf{g}}^*(j, v) \left(\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \right)^\dagger \\
& \quad \times \Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \mathscr{A}(j, v) \\
& \quad \times \Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \\
& \quad + M \Psi_{\mathbf{g}}^*(j, v) \mathscr{A}(j, v) \Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v)
\end{aligned}
\tag{99}
$$

for $j \in \mathbb{N}_{M/q}$ *and a.e.* $v \in [0, 1)$. Also observe that $(\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v))^\dagger \Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v)$ is the projection from $\mathbb{C}^Q$ onto range$(\Psi_{\mathbf{g}}(j, v))$. It follows that $M \Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{h}}(j, v) \Psi_{\mathbf{g}}^*(j, v) = \Psi_{\mathbf{g}}^*(j, v)$, and thus $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ is a Gabor dual of type I for $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ by Theorem 30.

Now we turn to the converse implication. Suppose $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ is a Gabor dual of type I for $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$. Then there exists $B(j, v) : \mathbb{N}_{M/q} \times [0, 1) \to \mathcal{M}_{Q,Q}$ such that

$$
\begin{aligned}
\Psi_{\mathbf{h}}(j, v) & = B(j, v) \Psi_{\mathbf{g}}(j, v), \\
\Psi_{\mathbf{g}}^*(j, v) & = M \Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{h}}(j, v) \Psi_{\mathbf{g}}^*(j, v)
\end{aligned}
\tag{100}
$$

for $j \in \mathbb{N}_{M/q}$ and a.e. $v \in [0, 1)$. It follows that

$$
\begin{aligned}
& \frac{1}{M} \left(\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \right)^\dagger \Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \\
& = \left(\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \right)^\dagger \Psi_{\mathbf{g}}(j, v) \\
& \quad \times \Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{h}}(j, v) \Psi_{\mathbf{g}}^*(j, v)
\end{aligned}
\tag{101}
$$

for $j \in \mathbb{N}_{M/q}$ and a.e. $v \in [0, 1)$, and thus

$$
\begin{aligned}
& \frac{1}{M} \left(\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \right)^\dagger \Psi_{\mathbf{g}}(j, v) \\
& = \left(\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \right)^\dagger \\
& \quad \times \Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{h}}(j, v)
\end{aligned}
\tag{102}
$$

for $j \in \mathbb{N}_{M/q}$ and a.e. $v \in [0, 1)$ due to the fact that $\mathbb{C}^P = \text{range}(\Psi_{\mathbf{g}}^*(j, v)) \oplus \ker(\Psi_{\mathbf{g}}(j, v))$. Put $\mathscr{A}(j, v) = B(j, v) - (\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v))^\dagger$. Then the right-hand side of (97) equals

$$
\begin{aligned}
& \Psi_{\mathbf{h}}(j, v) + \left(\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \right)^\dagger \mathscr{P}_{\text{range}(\Psi_{\mathbf{g}}(j, v))} \Psi_{\mathbf{g}}(j, v) \\
& - \left(\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \right)^\dagger \Psi_{\mathbf{g}}(j, v)
\end{aligned}
\tag{103}
$$

by (102), and this is exactly $\Psi_{\mathbf{h}}(j, v)$ for $j \in \mathbb{N}_{M/q}$ and a.e. $v \in [0, 1)$ again by the fact that

$$
\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \left(\Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \right)^\dagger = \mathscr{P}_{\text{range}(\Psi_{\mathbf{g}}(j, v))}.
\tag{104}
$$

$\square$

5. The Case of $N_1 = N_2 = \cdots = N_L$

Theorems 30 and 31 characterize duals with Gabor structure for general $\mathbf{N} = (N_1, N_2, \ldots, N_L)$. This section deals with the case $N_1 = N_2 = \cdots = N_L$. We start with a remark on the case of N_l, $1 \le l \le L$, being not all the same.

Remark 32. Not every subspace Gabor frame $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ admits an oblique Gabor dual.

We show it by revisiting Example 25. Let us make an additional assumption that $\lambda(0, v) = \lambda(1, v) = 0$ and $a(0, v)a(1, v) \ne 0$ for $v \in [0, 1]$. Then $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ is a frame but not a Riesz basis for $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ by Remark 26. Suppose $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ with $\mathbf{h} = (h_1, h_2)$ is an oblique Gabor dual for $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$. Then

$$
\Psi_{\mathbf{g}}^*(j, v) = 2 \Psi_{\mathbf{g}}^*(j, v) \Psi_{\mathbf{h}}(j, v) \Psi_{\mathbf{g}}^*(j, v)
\tag{105}
$$

$$
\text{for } j \in \mathbb{N}_2, \text{ a.e. } v \in [0, 1)
$$

by Theorem 30. Writing out $(0,0)$, $(3,1)$, and $(5,4)$ entries of both sides, we have

$$2\left(\overline{a\left(j,v\right)}\right)^2 \mathcal{Z}_{12}h_1\left(j,v\right) = \overline{a\left(j,v\right)}, \qquad (106)$$

$$2\left(\overline{a\left(j,v\right)}\right)^2 \mathcal{Z}_{12}h_1\left(j,v\right) \qquad (107)$$
$$+ 2\left|a\left(j,v\right)\right|^2 \mathcal{Z}_{12}h_2\left(j+2,v\right) = \overline{a(j,v)},$$

$$2\left(a\left(j,v\right)\right)^2 \mathcal{Z}_{12}h_2\left(j+2,v\right) = a\left(j,v\right) \qquad (108)$$

for $j \in \mathbb{N}_2$ and a.e. $v \in [0,1)$. By (107) and (108), we have $2\overline{a(j,v)}\mathcal{Z}_{12}h_1(j,v) = 0$ for $j \in \mathbb{N}_2$ and a.e. $v \in [0,1)$. This contradicts (106).

Observe that $N_1 \neq N_2$ in Remark 32 ($N_1 = 6$ and $N_2 = 4$). It is natural to ask the following question.

Does every subspace Gabor frame $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ admit no oblique Gabor dual whenever N_l, $1 \le l \le L$, are not all the same?

The following example gives a negative answer to this question.

Example 33. Let $\mathbf{N} = (1,2)$ and $\mathbf{M} = (3,3)$. Assume that $\mathcal{J}(v) = \left(\begin{smallmatrix} \mathcal{J}_1(v) \\ \mathcal{J}_2(v) \end{smallmatrix}\right)$, $\mathcal{E}(v) = \left(\begin{smallmatrix} \mathcal{E}_1(v) \\ \mathcal{E}_2(v) \end{smallmatrix}\right)$ have the form

$$\mathcal{J}_1(v) = \begin{pmatrix} a_{0,0}(v) & a_{0,1}(v) \\ a_{1,0}(v) & a_{1,1}(v) \\ a_{2,0}(v) & a_{2,1}(v) \\ a_{2,1}(v) & e^{-2\pi iv}a_{2,0}(v) \\ e^{2\pi iv}a_{0,1}(v) & a_{0,0}(v) \\ e^{2\pi iv}a_{1,1}(v) & a_{1,0}(v) \end{pmatrix},$$

$$\mathcal{E}_1(v) = \begin{pmatrix} c_{0,0}(v) & c_{0,1}(v) \\ c_{1,0}(v) & c_{1,1}(v) \\ c_{2,0}(v) & c_{2,1}(v) \\ c_{2,1}(v) & e^{-2\pi iv}c_{2,0}(v) \\ e^{2\pi iv}c_{0,1}(v) & c_{0,0}(v) \\ e^{2\pi iv}c_{1,1}(v) & c_{1,0}(v) \end{pmatrix},$$

$$\mathcal{J}_2(v) = \begin{pmatrix} b_{0,0}(v) & b_{0,1}(v) \\ b_{1,0}(v) & b_{1,1}(v) \\ b_{2,0}(v) & b_{2,1}(v) \end{pmatrix},$$

$$\mathcal{E}_2(v) = \begin{pmatrix} d_{0,0}(v) & d_{0,1}(v) \\ d_{1,0}(v) & d_{1,1}(v) \\ d_{2,0}(v) & d_{2,1}(v) \end{pmatrix},$$

$$(109)$$

for $v \in [0,1)$ with all entries of $\mathcal{J}(v)$ and $\mathcal{E}(v)$ in $L^\infty([0,1))$, that $(\mathcal{J}_1(v))^* \mathcal{E}_1(v)$ has the form $\left(\begin{smallmatrix} A(v) & 0 \\ 0 & A(v) \end{smallmatrix}\right)$ and $(\mathcal{J}(v))^* \mathcal{J}(v)$ has the form $\left(\begin{smallmatrix} B(v) & 0 \\ 0 & B(v) \end{smallmatrix}\right)$ for $v \in [0,1)$, and that

$$(\mathcal{J}_2(v))^* \mathcal{E}_2(v) = \begin{pmatrix} \dfrac{1}{3} - A(v) & 0 \\ 0 & \dfrac{1}{3} - A(v) \end{pmatrix} \qquad (110)$$

for $v \in [0,1)$ satisfying $B(v) \neq 0$. Define $\mathbf{g} = (g_1, g_2)$ and $\mathbf{h} = (h_1, h_2)$ by

$$\Psi_{\mathbf{g}}(0,v) = \mathcal{J}(v), \quad \Psi_{\mathbf{h}}(0,v) = \mathcal{E}(v) \quad \text{for } v \in [0,1). \qquad (111)$$

Then $\mathbf{g}$ and $\mathbf{h}$ are well defined by the quasi-periodicity of the Zak transform $\mathcal{Z}_6$ and $6\mathbb{Z}$ congruence between $\{0\} - 2\mathbb{N}_3 + 3\mathbb{N}_2$ and $\mathbb{N}_6$, and $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ and $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ are both Bessel sequences by Remark 16. A simple computation shows that

$$\Psi_{\mathbf{g}}^*(0,v)\Psi_{\mathbf{g}}(0,v) = 3\Psi_{\mathbf{g}}^*(0,v)\Psi_{\mathbf{h}}(0,v)\Psi_{\mathbf{g}}^*(0,v)\Psi_{\mathbf{g}}(0,v) \qquad (112)$$

for $v \in [0,1)$. This implies that $\Psi_{\mathbf{g}}^*(0,v) = 3\Psi_{\mathbf{g}}^*(0,v)\Psi_{\mathbf{h}}(0, v)\Psi_{\mathbf{g}}^*(0,v)$ for $v \in [0,1)$ due to the fact that $\mathbb{C}^9 = \text{range}(\Psi_{\mathbf{g}}(0,v)) \oplus \ker(\Psi_{\mathbf{g}}^*(0,v))$. So $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ is an oblique Gabor dual for $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ by Lemma 29.

Remark 32 and Example 33 show that the Gabor dual theory for subspace Gabor frames is complicated when N_l, $1 \le l \le L$, are not all the same. There are still many unresolved problems in this direction. Next we work under the following assumption.

Assumption 34. L is a positive integer, $\mathbf{M} = (M, M, \ldots, M)$, and $\mathbf{N} = (N_1, N_2, \ldots, N_L)$ with $N_1 = N_2 = \cdots = N_L = N$.

The following theorem characterizes the uniqueness of such duals.

Theorem 35. *Given $\mathbf{g} = (g_1, g_2, \ldots, g_L)$ with each $g_l \in l^2(\mathbb{Z})$, let $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ be a frame for $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$. Then the following are equivalent:*

(i) *$G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ has a unique Gabor dual of type I (type II);*

(ii) *$\text{rank}(\Psi_{\mathbf{g}}(j,v)) \in \{0, Lq\}(\text{rank}(\Psi_{\mathbf{g}}(j,v)) \in \{0, p\})$ for a.e. $(j,v) \in \mathbb{N}_{M/q} \times [0,1)$;*

(iii) *$\text{rank}(\Psi_{\mathbf{g}}(j,v)) \in \{0, Lq\}(\text{rank}(\Psi_{\mathbf{g}}(j,v)) \in \{0, p\})$ for a.e. $(j,v) \in \mathbb{Z} \times \mathbb{R}$.*

Proof. We first deal with the Gabor dual of type I. By Lemma 12, (ii) and (iii) are equivalent. By Theorem 30 and Lemma 27, we only need to prove that (ii) holds if and only if a function $A : \mathbb{N}_M \times [0,1) \to \mathcal{M}_{Lq,Lq}$ solves

$$A(j,v)\Psi_{\mathbf{g}}(j,v) = 0 \quad \text{for } j \in \mathbb{N}_M, \text{ a.e. } v \in [0,1) \qquad (113)$$

whenever it solves

$$\Psi_{\mathbf{g}}^*(j,v)A(j,v)\Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v) = 0 \qquad (114)$$
$$\text{for } j \in \mathbb{N}_M, \text{ a.e. } v \in [0,1).$$

Suppose (ii) holds and $A : \mathbb{N}_M \times [0,1) \rightarrow \mathcal{M}_{Lq,Lq}$ solves (114). If (j,v) is such that $\text{rank}(\Psi_{\mathbf{g}}(j,v)) = 0$, then $\Psi_{\mathbf{g}}(j,v) = 0$, and thus $A(j,v)\Psi_{\mathbf{g}}(j,v) = 0$. If (j,v) is such that $\text{rank}(\Psi_{\mathbf{g}}(j,v)) = Lq$, then (114) implies that

$$A(j,v)\Psi_{\mathbf{g}}(j,v)\Psi_{\mathbf{g}}^*(j,v) = 0, \tag{115}$$

which leads to $A(j,v) = 0$ and thus $A(j,v)\Psi_{\mathbf{g}}(j,v) = 0$. Conversely, suppose (ii) does not hold. Then there exist some $j_0 \in \mathbb{N}_M$ and $\mathbf{e}_k, \mathbf{e}_r$ such that

$$\mathcal{P}(j_0,v)\mathbf{e}_k \neq 0, \qquad \mathbf{e}_r^*\Psi_{\mathbf{g}}(j_0,v) \neq 0 \tag{116}$$

on some $E \subset [0,1)$ with $|E| > 0$, where $\mathcal{P}(j_0,v)$ is the orthogonal projection of $\mathbb{C}^{Lq}$ onto $\ker(\Psi_{\mathbf{g}}^*(j_0,v))$. Define

$$A(j_0,v) = \begin{cases} \mathcal{P}(j_0,v)\mathbf{e}_k\mathbf{e}_r^*, & \text{if } j = j_0, \ v \in E, \\ 0, & \text{otherwise} \end{cases} \tag{117}$$

for $(j,v) \in \mathbb{N}_M \times [0,1)$. Then (114) holds, but (113) fails to hold.

Next we turn to the Gabor duals of type II. Similarly to the above arguments, by Theorem 30 and Lemmas 27 and 28, we only need to prove that (ii) holds if and only if a function $B : \mathbb{N}_M \times [0,1) \rightarrow \mathcal{M}_{p,p}$ solves

$$\Psi_{\mathbf{g}}(j,v)B(j,v) = 0 \quad \text{for } j \in \mathbb{N}_M, \text{ a.e. } v \in [0,1) \tag{118}$$

whenever it solves

$$\Psi_{\mathbf{g}}^*(j,v)\Psi_{\mathbf{g}}(j,v)B(j,v)\Psi_{\mathbf{g}}^*(j,v) = 0$$

$$\text{for } j \in \mathbb{N}_M, \text{ a.e. } v \in [0,1). \tag{119}$$

The necessity can be proved similarly to the case of type I. Now we suppose (ii) does not hold. Then there exist some $j_0 \in \mathbb{N}_M$ and $\mathbf{e}_k, \mathbf{e}_r$ such that

$$\mathcal{P}(j_0,v)\mathbf{e}_k \neq 0, \qquad \Psi_{\mathbf{g}}(j_0,v)\mathbf{e}_r \neq 0 \tag{120}$$

on some $E \subset [0,1)$ with $|E| > 0$, where $\mathcal{P}(j_0,v)$ is the orthogonal projection of $\mathbb{C}^p$ onto $\ker(\Psi_{\mathbf{g}}(j_0,v))$. Define $B(j_0,v)$ by

$$B^*(j_0,v) = \begin{cases} \mathcal{P}(j_0,v)\mathbf{e}_k\mathbf{e}_r^*, & \text{if } j = j_0, \ v \in E, \\ 0, & \text{otherwise} \end{cases} \tag{121}$$

for $(j,v) \in \mathbb{N}_M \times [0,1)$. Then (119) holds, but (118) fails to hold. The proof is completed. □

It is well known that every scalar $k \times n$ matrix M with rank $r \geq 1$ has a decomposition

$$M = U \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} V^*, \tag{122}$$

where U and V are, respectively, $k \times k$ and $n \times n$ unitary matrices, $\left(\begin{smallmatrix} D & 0 \\ 0 & 0 \end{smallmatrix} \right)$ is a $k \times n$ block matrix in which D is an $r \times r$ diagonal matrix with positive entries in the diagonal (see [1, Theorem 1.5.4]). Observe that $\text{rank}(M)$ may change

in variables if M is a matrix-valued function. Next we restrict ourselves to Gabor systems $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ such that

$$\Psi_{\mathbf{g}}(j,v) = U(j,v) \begin{pmatrix} D(j,v) & 0 \\ 0 & 0 \end{pmatrix} V^*(j,v)$$

$$\text{for } j \in \mathbb{N}_{M/q}, \text{ a.e. } v \in [0,1), \tag{123}$$

where $U(j,v)$ and $V(j,v)$ are, respectively, $Lq \times Lq$ and $p \times p$ unitary matrices for $j \in \mathbb{N}_{M/q}$ and a.e. $v \in [0,1)$, and $\left(\begin{smallmatrix} D(j,v) & 0 \\ 0 & 0 \end{smallmatrix} \right)$ is an $Lq \times p$ block matrix in which $D(j,v)$ is of the form

$$D(j,v) = \text{diag}(\lambda_1(j,v), \lambda_2(j,v), \ldots, \lambda_m(j,v)) \tag{124}$$

with $\lambda_i(j,v) \geq 0$ for $j \in \mathbb{N}_{M/q}$ and a.e. $v \in [0,1)$ and $\bigcup_{j \in \mathbb{N}_{M/q}} \{v \in [0,1) : \lambda_i(j,v) > 0\}$ being of positive measure for each $1 \leq i \leq m$.

As an immediate consequence of Theorems 19, 24, and 35, we have the following theorem.

Theorem 36. *Let* $\mathbf{0} \neq \mathbf{g} = (g_1, g_2, \ldots, g_L)$ *be defined as in* (123). *Then*

(i) $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *is a frame for* $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *with frame bounds A, B if and only if*

$$\sqrt{\frac{A}{M}} \leq \lambda_i(j,v)$$

$$\leq \sqrt{\frac{B}{M}} \quad \text{on } \{(j,v) \in \mathbb{N}_{M/q} \times [0,1) : \lambda_i(j,v) > 0\} \tag{125}$$

for $1 \leq i \leq m$;

(ii) $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *is a Riesz basis (an orthonormal basis) for* $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *with Riesz bounds A, B if and only if*

$$m = Lq, \qquad \sqrt{\frac{A}{M}} \leq \lambda_i(j,v) \leq \sqrt{\frac{B}{M}} \quad \left(\lambda_i(j,v) = \sqrt{\frac{1}{M}} \right) \tag{126}$$

for $1 \leq i \leq m$, $j \in \mathbb{N}_{M/q}$, *and a.e.* $v \in [0,1)$;

(iii) $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *is a frame for* $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *with frame bounds A and B and simultaneously admits a unique Gabor dual of type I (type II) if and only if* $m = Lq$ $(m = p)$, *and for each* $j \in \mathbb{N}_{M/q}$ *and a.e.* $v \in [0,1)$, *either* $\lambda_i(j,v) = 0$ *for each* $1 \leq i \leq m$ *or* $\sqrt{A/M} \leq \lambda_i(j,v) \leq \sqrt{B/M}$ *for each* $1 \leq i \leq m$.

Theorem 37. *Let* $\mathbf{g}$ *be defined as in* (123) *and let* $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *be a frame for* $\mathcal{M}(\mathbf{g}, \mathbf{N}, \mathbf{M})$. *Then, for* $\mathbf{h} = (h_1, h_2, \ldots, h_L)$ *with each* $h_l \in l^2(\mathbb{Z})$, *we have*

(i) $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ *is a Gabor dual of type I for* $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ *if and only if* $\Psi_{\mathbf{h}}$ *has the form*

$$\Psi_{\mathbf{h}}(j,v) = \frac{1}{M} U(j,v)C(j,v) \begin{pmatrix} D(j,v) & 0 \\ 0 & 0 \end{pmatrix} V^*(j,v)$$

$$\text{for } j \in \mathbb{N}_{M/q}, \text{ a.e. } v \in [0,1), \tag{127}$$

where $C : \mathbb{N}_{M/q} \times [0, 1) \to \mathcal{M}_{Lq,Lq}$ satisfies $C(j, v) = (C_1(j, v), 0)$ with $C_1(j, v) \in \mathcal{M}_{Lq,m}$, the entries of the ith column of $C_1(j, v)$ are essentially bounded on $\{(j, v) \in \mathbb{N}_{M/q} \times [0, 1) : \lambda_i(j, v) > 0\}$ for $1 \le i \le m$, and

$$(C(j, v))_{i,n} = \begin{cases} 0, & i \ne n, \ \lambda_i(j, v) \lambda_n(j, v) \ne 0, \\ (\lambda_i(j, v))^{-2}, & i = n, \ \lambda_i(j, v) \ne 0 \end{cases} \tag{128}$$

for $1 \le i, n \le m$;

(ii) $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ is a Gabor dual of type II for $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ if and only if $\Psi_{\mathbf{h}}$ has the form

$$\Psi_{\mathbf{h}}(j, v) = \frac{1}{M} U(j, v) \begin{pmatrix} D(j, v) & 0 \\ 0 & 0 \end{pmatrix} C(j, v) V^*(j, v) \tag{129}$$

for $j \in \mathbb{N}_{M/q}$, a.e. $v \in [0, 1)$,

where $C : \mathbb{N}_{M/q} \times [0, 1) \to \mathcal{M}_{p,p}$ satisfies $C(j, v) = \begin{pmatrix} C_1(j,v) \\ 0 \end{pmatrix}$ with $C_1(j, v) \in \mathcal{M}_{m,p}$, the entries of the ith row of $C_1(j, v)$ are essentially bounded on $\{(j, v) \in \mathbb{N}_{M/q} \times [0, 1) : \lambda_i(j, v) > 0\}$ for $1 \le i \le m$, and

$$(C(j, v))_{i,n} = \begin{cases} 0, & i \ne n, \ \lambda_i(j, v) \lambda_n(j, v) \ne 0, \\ (\lambda_i(j, v))^{-2}, & i = n, \ \lambda_i(j, v) \ne 0 \end{cases} \tag{130}$$

for $1 \le i, n \le m$.

Proof. We only prove item (i), and (ii) can be proved similarly. By Theorem 30, $G(\mathbf{h}, \mathbf{N}, \mathbf{M})$ is a Gabor dual of type I for $G(\mathbf{g}, \mathbf{N}, \mathbf{M})$ if and only if there exists a function $A : \mathbb{N}_{M/q} \times [0, 1) \to \mathcal{M}_{Lq,Lq}$ such that

$$\Psi_{\mathbf{h}}(j, v) = A(j, v) \Psi_{\mathbf{g}}(j, v), \tag{131}$$

$$\Psi_{\mathbf{g}}^*(j, v) = M \Psi_{\mathbf{g}}^*(j, v) A(j, v) \Psi_{\mathbf{g}}(j, v) \Psi_{\mathbf{g}}^*(j, v) \tag{132}$$

for $j \in \mathbb{N}_{M/q}$ and a.e. $[0, 1)$, and for each $j \in \mathbb{N}_{M/q}$ the entries of $\Psi_{\mathbf{h}}(j, v)$ belong to $L^\infty([0, 1))$. Write

$$C(j, v) = M U^*(j, v) A(j, v) U(j, v)$$
$$= (C_1(j, v), C_2(j, v)), \tag{133}$$

where $C_1(j, v) \in \mathcal{M}_{Lq,m}$. A simple computation shows that (132) is equivalent to

$$D(j, v) = D(j, v) C_{11}(j, v) D^2(j, v) \tag{134}$$

for $j \in \mathbb{N}_{M/q}$, a.e. $v \in [0, 1)$,

where $C_1(j, v) = \begin{pmatrix} C_{11}(j,v) \\ C_{21}(j,v) \end{pmatrix}$, $C_{11}(j, v) \in \mathcal{M}_{m,m}$. It is obvious that (134) is equivalent to (128), which implies that (132) is equivalent to (128). By (133), (132) can be rewritten as

$$\Psi_{\mathbf{h}}(j, v) = \frac{1}{M} U(j, v) (C_1(j, v) D(j, v), 0) V^*(j, v). \tag{135}$$

Since $C_2(j, v)$ does not appear in (135), we assume that $C_2(j, v) = 0$ without loss of generality. By the unitary properties of $U(j, v)$ and $V(j, v)$, for each $j \in \mathbb{N}_{M/q}$, the entries of $\Psi_{\mathbf{h}}(j, v)$ belong to $L^\infty([0, 1))$ if and only if the entries of $C_1(j, v) D(j, v)$ belong to $L^\infty([0, 1))$; that is, the entries of the ith column of $C_1(j, v)$ are essentially bounded on $\{(j, v) \in \mathbb{N}_{M/q} \times [0, 1) : \lambda_i(j, v) > 0\}$ for $1 \le i \le m$. The proof is completed. $\square$

Next we discuss a special case of Theorem 36. Let $\mathbf{N} = 1$, $\mathbf{M} = 2$, $\vec{\mathbf{1}} = (1, 1)$, and $\vec{\mathbf{2}} = (2, 2)$ and write $V_1(0, v) = \widetilde{V}(0, v) = 1$ and

$$U_1(0, v) = \frac{1}{\sqrt{1 + \cos^2 2\pi v}} \begin{pmatrix} -1 & \cos 2\pi v \\ \cos 2\pi v & 1 \end{pmatrix},$$

$$D_1(0, v) = \begin{pmatrix} \sqrt{1 + \cos^2 2\pi v} \\ 0 \end{pmatrix},$$

$\widetilde{U}(0, v)$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & i & -i\cos 2\pi v & -i\sin 2\pi v \\ \cos 2\pi v & i\sin 2\pi v & -i & i \\ -1 & -i & -i\sin 2\pi v & i\cos 2\pi v \\ -i\sin 2\pi v & -\cos 2\pi v & -1 & -1 \end{pmatrix}$$

$$\widetilde{D}(0, v) = \begin{pmatrix} \sqrt{3} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{136}$$

for $v \in [0, 1)$. Define $\mathbf{g}$ and $\widetilde{\mathbf{g}} = (\widetilde{g}_1, \widetilde{g}_2)$ as

$$\Psi_{\mathbf{g}}(0, v) = U_1(0, v) D_1(0, v) V_1^*(0, v),$$
$$\Psi_{\widetilde{\mathbf{g}}}(0, v) = \widetilde{U}(0, v) \widetilde{D}(0, v) \widetilde{V}^*(0, v) \tag{137}$$

by letting $U = U_1$, $D = D_1$, and $V = V_1$ and $U = \widetilde{U}$, $D = \widetilde{D}$, and $V = \widetilde{V}$ in (123), respectively. Then

$$\Psi_{\mathbf{g}}(0, v) = \begin{pmatrix} -1 \\ \cos 2\pi v \end{pmatrix},$$

$$\Psi_{\widetilde{\mathbf{g}}}(0, v) = \begin{pmatrix} -1 \\ \cos 2\pi v \\ -1 \\ -i\sin 2\pi v \end{pmatrix} \quad \text{for } v \in [0, 1). \tag{138}$$

This implies that $\mathbf{g} = \widetilde{g}_1 = -\chi_{\{0\}} + (1/2)\chi_{\{-3,1\}}$, $\widetilde{g}_2 = -\chi_{\{0\}} - (1/2)\chi_{\{1\}} + (1/2)\chi_{\{-3\}}$ and that

$$\mathcal{M}(\mathbf{g}, 1, 2) = \mathcal{M}(\widetilde{\mathbf{g}}, \vec{\mathbf{1}}, \vec{\mathbf{2}}) = l^2(\mathbb{Z}) \tag{139}$$

by [5, Theorem 3.1]. So $G(\mathbf{g}, 1, 2)$ and $G(\widetilde{\mathbf{g}}, \vec{\mathbf{1}}, \vec{\mathbf{2}})$ are both frames for $l^2(\mathbb{Z})$, but neither is a Riesz basis by Theorem 36.

Define $\mathbf{h} \in l^2(\mathbb{Z})$ and $\widetilde{\mathbf{h}} = (\widetilde{h}_1, \widetilde{h}_2)$ with $\widetilde{h}_1, \widetilde{h}_2 \in l^2(\mathbb{Z})$ by $\Psi_{\mathbf{h}}(0, \nu) = (1/2)\Psi_{\mathbf{g}}(0, \nu)(\Psi_{\mathbf{g}}^*(0, \nu)\Psi_{\mathbf{g}}(0, \nu))^{\dagger}$ and $\Psi_{\widetilde{\mathbf{h}}}(0, \nu) = (1/2)\Psi_{\widetilde{\mathbf{g}}}(0, \nu)(\Psi_{\widetilde{\mathbf{g}}}^*(0, \nu)\Psi_{\widetilde{\mathbf{g}}}(0, \nu))^{\dagger}$ for $\nu \in [0, 1)$. Then $G(\mathbf{h}, 1, 2)$ and $G(\widetilde{\mathbf{h}}, \vec{\mathbf{1}}, \vec{\mathbf{2}})$ are, respectively, the canonical duals for $G(\mathbf{g}, 1, 2)$ and $G(\widetilde{\mathbf{g}}, \vec{\mathbf{1}}, \vec{\mathbf{2}})$ by [5, Remark 5.5], and

$$\Psi_{\mathbf{h}}(0, \nu) = \frac{1}{2\left(1 + \cos^2 2\pi\nu\right)} \begin{pmatrix} -1 \\ \cos 2\pi\nu \end{pmatrix}, \qquad (140)$$

$$\Psi_{\widetilde{\mathbf{h}}}(0, \nu) = \frac{1}{6} \begin{pmatrix} -1 \\ \cos 2\pi\nu \\ -1 \\ -i \sin 2\pi\nu \end{pmatrix} \qquad (141)$$

for $\nu \in [0, 1)$. It follows that

$$\widetilde{\mathbf{h}} = \left(-\frac{1}{6}\chi_{\{0\}} + \frac{1}{12}\chi_{\{-3,1\}}, -\frac{1}{6}\chi_{\{0\}} - \frac{1}{12}\chi_{\{1\}} + \frac{1}{12}\chi_{\{-3\}} \right), \qquad (142)$$

and $\mathbf{h}$ is infinitely supported. It is interesting that $\widetilde{\mathbf{g}}$ and $\widetilde{\mathbf{h}}$ are both finitely supported, but $\mathbf{h}$ is not although $\mathbf{g}$ is the first component of $\widetilde{\mathbf{g}}$. Therefore, there exist significant differences in Gabor duals between multi-window Gabor frames and single-window ones. Now we conclude this paper by summarizing the above arguments as the following example.

Example 38. Let $\vec{\mathbf{1}} = (1, 1)$ and $\vec{\mathbf{2}} = (2, 2)$,

$$\mathbf{g} = \widetilde{g}_1 = -\chi_{\{0\}} + \frac{1}{2}\chi_{\{-3,1\}},$$
$$\widetilde{g}_2 = -\chi_{\{0\}} - \frac{1}{2}\chi_{\{1\}} + \frac{1}{2}\chi_{\{-3\}}. \qquad (143)$$

$\mathbf{h}$ and $\widetilde{\mathbf{h}}$, as in (140) and (142) and write $\widetilde{\mathbf{g}} = (\widetilde{g}_1, \widetilde{g}_2)$. Then

(i) $G(\mathbf{g}, 1, 2)$ and $G(\widetilde{\mathbf{g}}, \vec{\mathbf{1}}, \vec{\mathbf{2}})$ are both frames for $l^2(\mathbb{Z})$, but neither is a Riesz basis for $l^2(\mathbb{Z})$;

(ii) $G(\mathbf{h}, 1, 2)$ and $G(\widetilde{\mathbf{h}}, \vec{\mathbf{1}}, \vec{\mathbf{2}})$ are, respectively, the canonical duals for $G(\mathbf{g}, 1, 2)$ and $G(\widetilde{\mathbf{g}}, \vec{\mathbf{1}}, \vec{\mathbf{2}})$;

(iii) $\widetilde{\mathbf{h}}$ is finitely supported, and $\mathbf{h}$ is not.

Acknowledgments

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