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Research Article

An Analytic Solution for a Vasicek Interest Rate Convertible Bond Model

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This paper provides the analytic solution to the partial differential equation for the value of a convertible bond. The equation assumes a Vasicek model for the interest rate and a geometric Brownian motion model for the stock price. The solution is obtained using integral transforms.

This work corrects errors in the original paper by Mallier and Deakin [1] on the Green's function for the Vasicek convertible bond equation. One error involves subtle points of the inverse Laplace transform. We show that the solution of

$$\frac{\partial V}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma c S \frac{\partial^2 V}{\partial S \partial r} + \frac{1}{2}c^2 \frac{\partial^2 V}{\partial r^2} + rS\frac{\partial V}{\partial S} + (a - br)\frac{\partial V}{\partial r} - rV \tag{1}$$

in the log stock variables $x = \log S$ and $\tilde{x} = \log \tilde{S}$ is

$$V(S, r, \tau) = \iint_{-\infty}^{\infty} V_0\left(e^{\tilde{x}}, \tilde{r}\right) G(r, \tilde{r}, x - \tilde{x})\, d\tilde{r}\, d\tilde{x}, \tag{2}$$

where $V = V_0(S, r)$ at $\tau = 0$ and the Green's function (GF) is

$$G(r, \tilde{r}, x - \tilde{x}) = \exp(F) N(w, \Xi) N(\alpha, \Phi). \tag{3}$$

The normal distribution with variance w and argument Ξ is here denoted by

$$N(w,\Xi) = (2\pi w)^{-1/2}\exp\left[-\frac{\Xi^2}{(2w)}\right], \tag{4}$$

and the coefficients are

$$w = \frac{(1-e^{-2b\tau})c^2}{2b}, \qquad \Xi = \tilde{r} - re^{-b\tau} - B\left(a - \frac{Bc^2}{2}\right), \tag{5}$$

$$F = A - Br, \qquad \Phi = \tilde{x} - x - D - \frac{\Xi(2\rho\sigma/c + B)}{1+e^{-b\tau}}, \tag{6}$$

$$\alpha = \tau\sigma^2\left(1-\rho^2\right) + \left(\frac{c}{b} + \rho\sigma\right)^2\left(\tau - \frac{2}{b}\tanh\left(\frac{b\tau}{2}\right)\right), \tag{7}$$

$$F + D + \frac{v}{2} = 0, \qquad A = \frac{(B-\tau)(2ab-c^2)}{2b^2} - \frac{c^2B^2}{4b}, \tag{8}$$

$$B = \frac{1-e^{-b\tau}}{b}, \qquad v = \tau\sigma^2 + \frac{(\tau-B)(2\rho\sigma b + c)c}{b^2} - \frac{c^2B^2}{2b}. \tag{9}$$

In the case of the convertible bond, the initial condition V_0 in (2) is independent of $\tilde{r}$. Integrating (2) in $\tilde{r}$, we obtain the simpler Green's function

$$G(r,\tau,x-\tilde{x}) = \exp(F(r,\tau))N(v(\tau),\tilde{x} - x - D(r,\tau)). \tag{10}$$

The parameters in the solution have the range of values: $\sigma > 0$, $c > 0$, $|\rho| < 1$, while a and b are arbitrary since the solutions are analytic in a and b.

To prove (3), we assume V to be bounded as $S \to 0$ and $S^{c_0}V$, where c_o is a positive constant, is bounded as $S \to \infty$ so that the Mellin transform of V exists. Once the solution is determined, the initial condition may be extended to include the more general case where the integral (2) exists (e.g., $V_0 = \max(S,1)$). In the derivation of the solution, the condition $b > 0$ is assumed in (1).

To solve for V in (1), the Mellin and Laplace transform $\widehat{V}(p) := \mathcal{M}[V]$ and $\overline{V}(z) := \mathcal{L}[\widehat{V}]$ (equations (2.6), (2.7) in [1]) are applied to obtain the ODE

$$\left(\frac{c^2}{2}\right)\overline{V}_{rr} + (a - \rho c\sigma p - br)\overline{V}_r + \left[\left(2^{-1}\sigma^2 p - r\right)(1+p) - z\right]\overline{V} = -\mathcal{M}[V_0(S,r)]. \tag{11}$$

The general homogeneous solution ([2, 3] Section V.I, page 249) of (11) is

$$V_h = \exp\left(-\frac{(1+p)r}{b}\right)\mathcal{F}\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{u^2}{2}\right), \tag{12}$$

$$-\nu = \frac{z}{b} + 2E, \quad u(r) = \sqrt{\frac{2}{b^3c^2}}\left(rb^2 - ab + c^2 + p\left(cb\sigma\rho + c^2\right)\right), \tag{13}$$

$$E = \frac{(1+p)\left(2ab - c^2 - p\Lambda\right)b^{-3}}{4}, \quad \Lambda = \left(c + b\sigma\rho\right)^2 + (b\sigma)^2\left(1-\rho^2\right), \tag{14}$$

and $\mathcal{F}$ is the general solution of the confluent hypergeometric equation ([2, 3] Section V.I). The general solution (12) in terms of the parabolic cylinder function $D_\nu(u)$ ([2, 3] Section V.II, page 117), with arbitrary constants C_1 and C_2 ($\nu \neq 0, 1, \ldots$), is

$$V_h = \exp\left(-\frac{(1+p)r}{b}\right)2^{-\nu/2}e^{(u^2/4)}\left(C_1D_\nu(u) + C_2D_\nu(-u)\right). \tag{15}$$

Replacing $\mathcal{M}[V_0(S,r)]$ in (11) by the delta function $\delta(r-\tilde{r})$ (c.f., (20) for details), the GF for (11) has the form

$$G_1(r,\tilde{r}) = 2c^{-2}h_1(r)h_2(\tilde{r})W^{-1}[h_1(\tilde{r}), h_2(\tilde{r})], \quad r > \tilde{r}, \tag{16}$$

where h_j are appropriate homogeneous solutions in (15), W is the Wronskian, and G_1 for $r < \tilde{r}$ is defined by interchanging r and $\tilde{r}$ in h_j, but not in W.

For the existence and the evaluation of the inverse Laplace transform (ILT) of G_1, the asymptotic expansion, valid for large $(-\nu)$ in the sector $|\arg(-\nu)| < \pi$,

$$\Gamma(-\nu)D_\nu(v(r))D_\nu(-w(\tilde{r})) \sim \left(-\frac{\nu 2}{\pi}\right)^{-1/2}\exp\left(-(-\nu)^{1/2}(v(r) - w(\tilde{r}))\right) \tag{17}$$

is required where $v(r) = \pm u(r)$ and $w(\tilde{r}) = \pm u(\tilde{r})$. The expansion for the Gamma function is given in ([2, 3] Section V.I, page 47). The expansion with a restricted domain for the parabolic cylinder function appears in [2, 3] (Section V.I, page 249 (8)) and the general case is proved by applying the Method of Steepest Descent to the integral representation ([4, 5], page 349). The solutions h_i in (16) must be chosen such that G_1 has an ILT that exists for all r and $\tilde{r}$. For the general case, we define h_i in (15) by replacing C_j by C_{ij}. There are four terms in (16), only one for which the ILT exists: $C_{12} = C_{21} = 0$, $v - w = (2b)^{1/2}|r - \tilde{r}|/c$ in (17). Thus,

$$G_1 = g_1(r)g_2(\tilde{r})c^{-1}(b\pi)^{-1/2}\Gamma(-\nu)D_\nu(u(r))D_\nu(-u(\tilde{r})), \quad r > \tilde{r}, \tag{18}$$

where $g_j(r) = \exp[(-1)^j((1+p)r/b - u^2(r)/4)]$. For $r < \tilde{r}$, G_1 is defined by interchanging r and $\tilde{r}$ in D_ν. However, to explain the results in [1], we compare (2.16) to (16, 20) so that $h_1 \propto \mathcal{V}_2$ and $h_2 \propto \mathcal{V}_1$ in (2.13) (change sign on RHS of (2.14), (2.16)). Consequently, h_1 and h_2 are defined in (15) by taking ($C_1 = 0, C_2 = 1$) and ($C_1 = -1, C_2 = 1$), respectively. The modified

GF is $G_1^m := -G_1^* + G_1^s$ where G_1^* and G_1^s are defined from G_1 by changing u to $-u$ and $u(r)$ to $-u(r)$, respectively.

As outlined in [1], the ILT $G_2 := \mathcal{L}^{-1}(G_1)$ ((2.17), [1]) is equal to the contributions from the simple poles of $\Gamma(-\nu)$ at $\nu = n$ $(n = 0, 1 \ldots)$. G_2 is equal to a sum involving Hermite polynomials ([2, 3] Section V.II, page 194 (22)) so that

$$G_2 = N(\eta, \tilde{r} - r) \exp\left[\frac{\sqrt{2b}}{4c}(r - \tilde{r})s_1 - s_2\frac{\lambda b\tau}{8} + \frac{b\tau}{2} - 2bE\tau - \frac{(1+p)}{b}(r - \tilde{r})\right], \tag{19}$$

where $s_m = u^m(r) + u^m(\tilde{r})$, $\eta = \tau c^2 \sinh(b\tau)/(b\tau)$, $\lambda = (2/(b\tau))\tanh(b\tau/2)$. The semicircle's contribution to G_2 goes to zero as the radius goes to infinity follows from the approximation of G_1 in (18) via (17). For the modified GF, $G_2^m := -G_2^* + G_2^s$ where G_2^* and G_2^s are formally defined by the contributions from the poles: $G_2^* = G_2$, $G_2^s = G_2 \exp(-u(r)u(\tilde{r})/\sinh(b\tau))$.

The last step is to evaluate the inverse Mellin transforms (IMT; (2.18), [1]) $G_3 := \mathcal{M}^{-1}G_2$ and, for the modified GF, $G_3^m := -G_3 + G_3^s$, where $G_3^s := \mathcal{M}^{-1}G_2^s$. To do this, the argument of the exponential in G_2 and G_2^s is expressed in the form $\alpha p^2/2 + \beta p + \gamma$, and formula (2.29) in [1] is applied. For G_2, α is given by (7). For G_2^s, $\alpha := \alpha^s$ is given by (7) where tanh is replaced by coth. Correcting the error in [1] (page 228, L.-4, (+) to (−)), then $2\alpha_+ = \alpha$ and $2\alpha_- = \alpha^s$, where $\alpha_\pm$ appear in (2.27) and (2.33). Assuming that $(c/b + \rho\sigma) \neq 0$, then there is a positive number τ_o such that $\alpha_- < 0$ for $0 < \tau < \tau_o$. Thus the IMT of G_2^s does not exist for $0 < \tau < \tau_o$, and G_1 in (18) is the correct Green's function. For G_3, we have $G_3 = \exp\gamma N(\eta, \tilde{r} - r)N(\alpha, \beta - \log S)$. The variables $(\overline{V}, V_0, G_1)$ and (V, V_0, G_3) are connected by

$$\overline{V} = \int_{-\infty}^{\infty} \mathcal{M}[V_0(S, \tilde{r})]G_1 d\tilde{r}, \quad V = \int_{-\infty}^{\infty} \mathcal{M}^{-1}[\mathcal{M}[V_0]\mathcal{M}[G_3]]d\tilde{r}. \tag{20}$$

Using the convolution theorem ((2.30), [1]), the solution is (2), where

$$G(r, \tilde{r}, x - \tilde{x}) = \exp(\gamma)N(\eta, \tilde{r} - r)N(\alpha, \tilde{x} - x + \beta), \tag{21}$$

$$\alpha = \tau\left\{\sigma^2\left(1 - \rho^2\right) + \left(\frac{c}{b}\right)^2(1 - \lambda)\phi^2\right\}, \quad \phi = 1 + \frac{\rho\sigma b}{c}, \tag{22}$$

$$2\beta = \frac{2(r - \tilde{r})\rho\sigma}{c} + \tau\left\{\left(\frac{c}{b}\right)^2(1 - \lambda)d_1\phi + \sigma^2 - (r + \tilde{r})\phi + \frac{2\sigma a\rho}{c}\right\} \tag{23}$$

$$\begin{aligned} 2\gamma = {} & \frac{(r - \tilde{r})((r + \tilde{r})b - 2a)}{c^2} \\ & + \tau\left\{b - \left(\frac{a}{c}\right)^2 + \frac{(c/b)^2(1 - \lambda)d_2}{2} - (r + \tilde{r})\left(1 - \frac{ab}{c^2}\right) - \frac{(r^2 + \tilde{r}^2)(b/c)^2}{2}\right\}, \end{aligned} \tag{24}$$

$d_m = q^m(r) + q^m(\tilde{r})$, and $q(r) = (rb^2 - ab + c^2)/c^2$. Extensive algebraic manipulations are required to express G in (21) in the final form (3). The Green's function in (3) has the expected property: $G \to \delta(r - \tilde{r})\delta(x - \tilde{x})$ and $V(S, r, \tau) \to V_0(S, r)$ as $\tau \to 0$ in (2).

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