Research Article

Phase Transition of Black Holes in Brans–Dicke Born–Infeld Gravity through Geometrical Thermodynamics

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Using the geometrical thermodynamic approach, we study phase transition of Brans–Dicke Born–Infeld black holes. We apply introduced methods and describe their shortcomings. We also use the recently proposed new method and compare its results with those of canonical ensemble. By considering the new method, we find that its Ricci scalar diverges in the places of phase transition and bound points. We also show that the bound point can be distinguished from the phase transition points through the sign of thermodynamical Ricci scalar around its divergencies.

1. Introduction

General relativity is accepted as a standard theory of gravitation and is able to pass more observational tests [1]. Although this theory is successful in various domains, it cannot describe some experimental evidences such as the accelerating expansion of the Universe [2–4]. Moreover, the general relativity theory is not consistent with Mach's principle nor Dirac's large number hypothesis [5, 6]. In addition, one needs further accurate observations to fully confirm (or disprove) the validity of general relativity in the high curvature regime such as black hole systems and other massive objects. Therefore, in recent years, more attentions have been focused on alternative theories of gravity. The most considerable alternative theories of gravity are the scalar-tensor theories. One of the good examples of these theories is Brans–Dicke (BD) theory which was introduced in 1961 to combine Mach's principle with Einstein's theory of gravity [7]. It is worthwhile to mention that BD theory is one of the modified theories of general relativity which can be used for several cosmological problems like inflation, cosmic acceleration, and dark energy modeling [8–10]. Also, it has a customizable parameter (ω) which indicates the strength of coupling between the matter and scalar fields. The action of 4-dimensional BD theory can be written as

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(\Phi R - \frac{\omega}{\Phi} (\nabla\Phi)^2 \right), \tag{1}$$

where R and Φ are, respectively, the Ricci scalar and self-gravitating scalar field. It is interesting to note that 4-dimensional stationary vacuum BD solution is just the Kerr solution with a trivial scalar field [11]. In addition, Cai and Myung proved that 4-dimensional solution of BD-Maxwell theory reduces to the Reissner–Nordström solution with a constant scalar field [12–15]. However, the solutions of BD-Maxwell gravity in higher dimensions will be reduced to the Reissner–Nordström solutions with a nontrivial scalar field because of the fact that higher dimensional stress energy tensor of Maxwell field is not traceless (conformally invariant). One of the most prominent problems which makes BD theory nonstraightforward is the fact that the field equations of this theory are highly nonlinear. To deal with this issue, one could apply conformal transformation on known solutions of other modified theories like dilaton gravity [16]. For instance, nonlinearly charged dilatonic black hole solutions and their BD counterpart in an energy dependent space-time have been obtained by applying a conformal transformation [17].

The first attempt for modifying the Maxwell theory to a consistent theory for describing point charges was made in

1912 by Mie [18, 19]. After that, Born and Infeld introduced a gauge-invariant nonlinear electrodynamic model to find a classical theory of point-like charges with finite energy density [20]. Born–Infeld (BI) theory was more interesting since it was obtained by using loop correction analysis of Quantum Field Theory. Recently, Tseytlin has shown that BI theory can be derived as an effective theory of some string theory models [21–26]. Nowadays, the effects of BI electrodynamics coupled to various gravity theories have been considered by many authors in the context of black holes [27–51], rotating black branes [14, 52–57], wormholes [58–61], superconductors [62–67], and other aspects of physics [68, 69].

On the other side, black hole thermodynamics became an interesting topic after the works of Hawking and Bekenstein [70–75]. Besides, based on the AdS/CFT correspondence, black hole thermodynamics was considered as the first step for constructing quantum gravity. In recent years, phase transition and critical behavior of the black holes have attracted more attention among physicists. Generally, at the critical point where phase transition occurs, one may find a discontinuity of state space variable such as heat capacity [76]. In addition to heat capacity, there are various approaches for studying phase transition. One of such interesting methods is based on geometrical technique. Geometrical thermodynamic method was started by Gibbs and Caratheodory [77]. Regarding this method, one could build a phase space by employing thermodynamical potential and its corresponding extensive parameter. Meanwhile, divergence points of Ricci scalar of thermodynamical metric provide information about phase transition points of thermodynamical systems.

For the first time, Weinhold introduced a new metric on the equilibrium thermodynamical phase space [78, 79] and after that another thermodynamical metric was defined by Ruppeiner from a different point of view [80, 81]. It is worthwhile to mention that there is a conformal relation between Ruppeiner and Weinhold metrics with the inverse of temperature as a conformal factor [82]. None of Weinhold and Ruppeiner metrics were invariant under Legendre transformation. Recently, Quevedo [83, 84] removed some problems of Weinhold and Ruppeiner methods by proposing a Legendre invariant thermodynamical metric. Although Quevedo could solve some problems which previous metrics were involved with, it has been confronted with another problems in some specific systems. To solve these problems, a new method was proposed in [85–87] which is known as HPEM metric. It was shown that HPEM metric is completely consistent with the results of the heat capacity in canonical ensemble in different gravitational systems.

In this paper, we are going to consider black hole solutions of BD-BI as well as Einstein-BI-dilaton gravity and study their phase transition based on geometrical thermodynamic methods. We compare our results with those of other methods such as extended phase space thermodynamics.

2. Field Equation and Conformal Transformations

The $(n + 1)$-dimensional BD-BI theory action containing a scalar field Φ and a self-interacting potential $V(\Phi)$ is as follows:

$$I_{\text{BD-BI}} = -\frac{1}{16\pi} \int_M d^{n+1}x \\ \cdot \sqrt{-g} \left(\Phi R - \frac{\omega}{\Phi}(\nabla\Phi)^2 - V(\Phi) + \mathscr{L}(\mathscr{F}) \right), \tag{2}$$

where ω is a coupling constant and $\mathscr{L}(\mathscr{F})$ is the BI theory Lagrangian

$$\mathscr{L}(\mathscr{F}) = 4\beta^2 \left(1 - \sqrt{1 + \frac{\mathscr{F}}{2\beta^2}} \right), \tag{3}$$

in which β and $\mathscr{F} = F_{\mu\nu}F^{\mu\nu}$ are BI parameter and Maxwell invariant, respectively. It is worth mentioning that $\mathscr{L}(\mathscr{F})$ will be reduced to the standard Maxwell form $\mathscr{L}(\mathscr{F}) = -\mathscr{F}$ as $\beta \to \infty$. The field equations of gravitational, scalar, and electromagnetic fields can be obtained by varying the action (2):

$$G_{\mu\nu} = \frac{\omega}{\Phi^2} \left(\nabla_\mu\Phi\nabla_\nu\Phi - \frac{1}{2}g_{\mu\nu}(\nabla\Phi)^2 \right) - \frac{V(\Phi)}{2\Phi}g_{\mu\nu} \\ + \frac{1}{\Phi} \left(\nabla_\mu\nabla_\nu\Phi - g_{\mu\nu}\nabla^2\Phi \right) + \frac{2}{\Phi} \left(\frac{F_{\mu\lambda}F_\nu^{\ \lambda}}{\sqrt{1 + \mathscr{F}/2\beta^2}} \right. \\ \left. + \frac{1}{4}g_{\mu\nu}\mathscr{L}(\mathscr{F}) \right), \tag{4}$$

$$\nabla^2\Phi = \frac{1}{2[(n-1)\omega + n]} \left((n-1)\Phi\frac{dV(\Phi)}{d\Phi} \right. \\ \left. - (n+1)V(\Phi) + (n+1)\mathscr{L}(\mathscr{F}) + \frac{4\mathscr{F}}{\sqrt{1 + \mathscr{F}/2\beta^2}} \right), \tag{5}$$

$$\nabla_\mu \left(\frac{F^{\mu\nu}}{\sqrt{1 + \mathscr{F}/2\beta^2}} \right) = 0. \tag{6}$$

It is not easy to solve (4)–(6) because there exists second order of scalar field in the denominator of field equation (4). In order to overcome such a problem, we can use a suitable conformal transformation and convert the BD-BI theory

to the Einstein-BI-dilaton gravity. The suitable conformal transformation is as follows:

$$\overline{g}_{\mu\nu} = \Phi^{2/(n-1)} g_{\mu\nu},$$

$$\overline{\Phi} = \frac{n-3}{4\alpha} \ln\Phi, \tag{7}$$

$$\alpha = \frac{(n-3)}{\sqrt{4(n-1)\omega + 4n}}.$$

The Einstein-BI-dilaton gravity action and its related field equations can be obtained from the BD-BI action and its related field equations by applying the mentioned conformal transformation [17]:

$$\overline{I}_G = -\frac{1}{16\pi} \int_{\mathcal{M}} d^{n+1}x$$
$$\cdot \sqrt{-\overline{g}} \left\{ \overline{\mathcal{R}} - \frac{4}{n-1} \left(\overline{\nabla}\overline{\Phi} \right)^2 - \overline{V}\left(\overline{\Phi}\right) + \overline{L}\left(\overline{\mathcal{F}}, \overline{\Phi}\right) \right\}, \tag{8}$$

$$\overline{\mathcal{R}}_{\mu\nu} = \frac{4}{n-1} \left(\overline{\nabla}_\mu \overline{\Phi} \overline{\nabla}_\nu \overline{\Phi} + \frac{1}{4} \overline{V}\left(\overline{\Phi}\right) \overline{g}_{\mu\nu} \right) - \frac{1}{n-1}$$
$$\cdot \overline{L}\left(\overline{\mathcal{F}}, \overline{\Phi}\right) \overline{g}_{\mu\nu} \tag{9}$$
$$+ \frac{2e^{-4\alpha\overline{\Phi}/(n-1)}}{\sqrt{1+\overline{Y}}} \left(\overline{F}_{\mu\eta} \overline{F}_\nu^\eta - \frac{\overline{\mathcal{F}}}{n-1} \overline{g}_{\mu\nu} \right),$$

$$\overline{\nabla}^2 \overline{\Phi} = \frac{n-1}{8} \frac{\partial \overline{V}\left(\overline{\Phi}\right)}{\partial \overline{\Phi}}$$
$$+ \frac{\alpha}{2(n-3)} \left((n+1) \overline{L}\left(\overline{\mathcal{F}}, \overline{\Phi}\right) + \frac{4e^{-4\alpha\overline{\Phi}/(n-1)} \overline{\mathcal{F}}}{\sqrt{1+\overline{Y}}} \right), \tag{10}$$

$$\overline{\nabla}_\mu \left(\frac{e^{-4\alpha\overline{\Phi}/(n-1)}}{\sqrt{1+\overline{Y}}} \overline{F}^{\mu\nu} \right) = 0, \tag{11}$$

where $\overline{\nabla}$ is the covariant differentiation with respect to the metric $\overline{g}_{\mu\nu}$ and $\overline{\mathcal{R}}$ is its Ricci scalar. The potential $\overline{V}(\overline{\Phi})$ and the Lagrangian $\overline{L}(\overline{F}, \overline{\Phi})$ will take the following forms [17]:

$$\overline{V}\left(\overline{\Phi}\right) = \Phi^{-(n+1)/(n-1)} V\left(\Phi\right), \tag{12}$$

$$\overline{L}\left(\overline{\mathcal{F}}, \overline{\Phi}\right) = 4\beta^2 e^{-4\alpha(n+1)\overline{\Phi}/[(n-1)(n-3)]} \left(1 - \sqrt{1 + \frac{e^{16\alpha\overline{\Phi}/[(n-1)(n-3)]} \overline{\mathcal{F}}}{2\beta^2}} \right). \tag{13}$$

In the limits of $\beta \to \infty$ and $\beta \to 0$, the Lagrangian will be $\overline{L}(\overline{\mathcal{F}}, \overline{\Phi}) = -e^{-4\alpha\overline{\Phi}/(n-1)} \overline{\mathcal{F}}$ and $\overline{L}(\overline{\mathcal{F}}, \overline{\Phi}) \to 0$, respectively, as

expected. In previous equations, we have used the following notations:

$$\overline{L}\left(\overline{\mathcal{F}}, \overline{\Phi}\right) = 4\beta^2 e^{-4\alpha(n+1)\overline{\Phi}/[(n-1)(n-3)]} \overline{L}\left(\overline{Y}\right),$$

$$\overline{L}\left(\overline{Y}\right) = 1 - \sqrt{1+\overline{Y}}, \tag{14}$$

$$\overline{Y} = \frac{e^{16\alpha\overline{\Phi}/[(n-1)(n-3)]} \overline{\mathcal{F}}}{2\beta^2}.$$

By considering the conformal relation between these two theories, it can be understood that if $(\overline{g}_{\mu\nu}, \overline{F}_{\mu\nu}, \overline{\Phi})$ are the solutions to the field equations of Einstein-BI-dilaton gravity (9)–(11), then the solutions of BD-BI theory could be obtained by the following form:

$$\left[g_{\mu\nu}, F_{\mu\nu}, \Phi \right]$$
$$= \left[\exp\left(-\frac{8\alpha\overline{\Phi}}{(n-1)(n-3)} \right) \overline{g}_{\mu\nu}, \overline{F}_{\mu\nu}, \exp\left(\frac{4\alpha\overline{\Phi}}{n-3} \right) \right]. \tag{15}$$

2.1. Black Hole Solutions in Einstein-BI-Dilaton Gravity and BD-BI Theory

2.1.1. Einstein Frame. In this section, we briefly obtain the Einstein-BI-dilaton gravity solutions and then by using the conformal transformation, we calculate the solutions of BD-BI theory [88]. We assume the following metric with various horizon topology:

$$d\overline{s}^2 = -Z(r) dt^2 + \frac{dr^2}{Z(r)} + r^2 R^2(r) d\Omega_k^2, \tag{16}$$

where $d\Omega_k^2$ is an $(n-1)$-dimensional hypersurface of Euclidean metric with constant curvature $(n-1)(n-2)k$ and volume ϖ_{n-1} with the following explicit form:

$$d\Omega_k^2$$
$$= \begin{cases} d\theta_1^2 + \sum_{i=2}^{n-1} \prod_{j=1}^{i-1} \sin^2\theta_j d\theta_i^2 & k = 1 \\ d\theta_1^2 + \sinh^2\theta_1 d\theta_2^2 + \sinh^2\theta_1 \sum_{i=3}^{n-1} \prod_{j=2}^{i-1} \sin^2\theta_j d\theta_i^2 & k = -1 \\ \sum_{i=1}^{n-1} d\phi_i^2 & k = 0. \end{cases} \tag{17}$$

In order to obtain consistent solutions, we should consider a suitable functional form for the potential, $\overline{V}(\overline{\Phi})$. It was shown that the proper potential is a Liouville-type one with both topological and BI correction terms, as [17]

$$\overline{V}\left(\overline{\Phi}\right) = 2\Lambda \exp\left(\frac{4\alpha\overline{\Phi}}{n-1} \right)$$
$$+ \frac{k(n-1)(n-2)\alpha^2}{b^2(\alpha^2-1)} \exp\left(\frac{4\overline{\Phi}}{(n-1)\alpha} \right) \tag{18}$$
$$+ \frac{W(r)}{\beta^2}.$$

It is notable to mention that, in the limit of $\alpha \to 0$ (absence of dilaton field) and $\beta \to \infty$, $\overline{\mathbf{V}}(\Phi)$ reduces to 2Λ, as expected [13]. Now, regarding the field equations (9)–(11), metric (16), and the potential $\overline{\mathbf{V}}(\Phi)$, it is a matter of calculation to show that

$$F_{tr} = E(r)$$

$$= \frac{qe^{(4\alpha\overline{\Phi}(r)/(n-1))}}{[rR(r)]^{(n-1)}\sqrt{1 + e^{(8\alpha\overline{\Phi}(r)/(n-3))}q^2[rR(r)]^{-2(n-1)}/\beta^2}},$$

$$\overline{\Phi} = \frac{(n-1)\alpha}{2(1+\alpha^2)}\ln\left(\frac{b}{r}\right),$$

$$W(r) = \frac{4q(n-1)\beta^2 R(r)}{(1+\alpha^2)r^\gamma b^{n\gamma}}\int\frac{E(r)}{r^{n(1-\gamma)-\gamma}}dr$$

$$+ \frac{4\beta^4}{R(r)^{2(n+1)/(n-3)}}\left(1 - \frac{E(r)R(r)^{(n-3)}}{qr^{1-n}}\right)$$

$$- \frac{4q\beta^2 E(r)}{r^{n-1}}\left(\frac{r}{b}\right)^{\gamma(n-1)},$$

$$Z(r) = -\frac{k(n-2)(\alpha^2+1)^2(r/b)^{2\gamma}}{(\alpha^2+n-2)(\alpha^2-1)} + \left(\frac{(1+\alpha^2)^2 r^2}{(n-1)}\right)$$

$$\cdot\frac{2\Lambda(r/b)^{-2\gamma}}{(\alpha^2-n)} - \frac{m}{r^{(n-1)(1-\gamma)-1}}$$

$$- \frac{4(1+\alpha^2)^2 q^2(r/b)^{2\gamma(n-2)}}{(n-\alpha^2)r^{2(n-2)}}\left(\frac{1}{2(n-1)}F_1(\eta)\right.$$

$$\left. - \frac{1}{\alpha^2+n-2}F_2(\eta)\right),$$

$$R(r) = \exp\left(\frac{2\alpha\overline{\Phi}}{n-1}\right) = \left(\frac{b}{r}\right)^\gamma,$$ (19)

where m is an integration constant related to mass and b is another constant related to scalar field, and

$$F_1(\eta)$$

$$= {}_2F_1\left(\left[\frac{1}{2}, \frac{(n-3)\Upsilon}{\alpha^2+n-2}\right], \left[1 + \frac{(n-3)\Upsilon}{\alpha^2+n-2}\right], -\eta\right),$$

$$F_2(\eta)$$

$$= {}_2F_1\left(\left[\frac{1}{2}, \frac{(n-3)\Upsilon}{2(n-1)}\right], \left[1 + \frac{(n-3)\Upsilon}{2(n-1)}\right], -\eta\right),$$

$$\Upsilon = \frac{\alpha^2+n-2}{2\alpha^2+n-3},$$ (20)

$$\eta = \frac{q^2(r/b)^{2\gamma(n-1)(n-5)/(n-3)}}{\beta^2 r^{2(n-1)}},$$

$$\gamma = \frac{\alpha^2}{1+\alpha^2}.$$

It is worthwhile to mention that the dilatonic Maxwell solutions [89] can be achieved from the obtained solutions in the limit of $\beta \to \infty$. The divergencies of scalar curvatures at the origin guarantee the existence of singularity. We interpret such a singularity as black hole since it is covered by an event horizon [17].

2.1.2. Jordan Frame. To obtain the black hole solutions of BD-BI theory, first, by using the conformal transformation (12), $\mathbf{V}(\Phi)$ would be

$$\mathbf{V}(\Phi) = 2\Lambda\Phi^2$$

$$+ \frac{k(n-1)(n-2)\alpha^2}{b^2(\alpha^2-1)}\Phi^{[(n+1)(1+\alpha^2)-4]/[(n-1)\alpha^2]}$$ (21)

$$+ \Phi^{(n+1)/(n-1)}\frac{W(r)}{\beta^2}.$$

Also, by considering the following $(n+1)$-dimensional metric,

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 H^2(r)d\Omega_k^2,$$ (22)

one can find the following solutions through conformal transformation:

$$A(r) = \left(\frac{r}{b}\right)^{4\gamma/(n-3)}Z(r),$$

$$B(r) = \left(\frac{r}{b}\right)^{-4\gamma/(n-3)}Z(r),$$

$$H(r) = \left(\frac{r}{b}\right)^{-\gamma((n-5)/(n-3))},$$ (23)

$$\Phi(r) = \left(\frac{r}{b}\right)^{-2\gamma(n-1)/(n-3)}.$$

It is notable that, like Einstein frame, these solutions can be interpreted as black holes which are covered by event horizon.

3. Thermodynamic Properties: Dilatonic-BI versus BD-BI Black Holes

3.1. Thermodynamic Quantities. In the following, we give a brief review regarding thermodynamic quantities of the black hole solutions in both frames. The Hawking temperature of the black hole can be obtained by using the surface gravity interpretation (κ) through the following relation:

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi}\sqrt{-\frac{1}{2}\left(\nabla_\mu\chi_\nu\right)\left(\nabla^\mu\chi^\nu\right)}$$

$$= \begin{cases} \dfrac{Z'(r_+)}{4\pi}, & \text{dilatonic BI} \\[2ex] \dfrac{1}{4\pi}\sqrt{\dfrac{B(r)}{A(r)}}A'(r_+), & \text{BD-BI}, \end{cases}$$ (24)

in which $\chi = \partial/\partial t$ is the time like null Killing vector. It is easy to show that the Hawking temperature in both frames is uniform as

$$T = \frac{(\alpha^2 + 1)}{2\pi(n-1)}\left[\frac{-k(n-2)(n-1)}{2(\alpha^2-1)r_+}\left(\frac{b}{r_+}\right)^{-2\gamma}\right.$$
$$\left. - \Lambda r_+\left(\frac{b}{r_+}\right)^{2\gamma} + \Gamma_+\right], \quad \text{dilatonic BI \& BD-BI,} \tag{25}$$

where

$$\Gamma_+ = -\frac{(\alpha^2+1)^2 q^2}{2\pi(n-1)}\left(\frac{r_+}{b}\right)^{2\gamma(n-2)} r_+^{3-2n} F_1(\eta_+). \tag{26}$$

Following [12, 17], the finite mass and entropy of the black hole in both Einstein and Jordan frames are

$$M = \frac{\varpi_{n-1}b^{(n-1)\gamma}}{16\pi}\left(\frac{n-1}{1+\alpha^2}\right)m,$$
$$S = \frac{\varpi_{n-1}b^{(n-1)\gamma}}{4}r_+^{(n-1)(1-\gamma)}. \tag{27}$$

In addition, the electric charge Q of the black holes can be obtained via Gauss's law

$$Q = \frac{q}{4\pi}. \tag{28}$$

Also, one can obtain the electric potential as

$$U = \left(\frac{r_+}{b}\right)^{4\gamma+1}\frac{b\beta(\alpha^2+1)}{(5\alpha^2+1)}{}_2F_1\left(\left[\frac{1}{2}, \frac{5\alpha^2+1}{6(2\alpha^2+1)}\right],\right.$$
$$\left.\left[\frac{17\alpha^2+7}{6(2\alpha^2+1)}\right], \frac{\beta^2 b^6}{q^2}\left(\frac{r_+}{b}\right)^{6\gamma+6}\right). \tag{29}$$

It is straightforward to show that the mentioned conserved and thermodynamical quantities satisfy the first law of thermodynamics as

$$dM = TdS + UdQ. \tag{30}$$

3.2. Heat Capacity and Thermal Stability. Here, we want to investigate thermal stability of the black holes. Due to the set of state functions and thermodynamic variables of a system, one may study the thermodynamic stability from different points of view through various ensembles. One of the common methods to study phase transition is regarding the canonical ensemble. In this ensemble, thermal stability of a system will be ensured by positivity of the heat capacity. One can obtain the heat capacity relation with fixed charge as

$$C_Q = \frac{(\partial M/\partial S)_Q}{(\partial^2 M/\partial S^2)_Q} = \frac{M_S}{M_{SS}} = T\left(\frac{\partial S}{\partial T}\right)_Q, \tag{31}$$

where $M_S = \partial M/\partial S$ and $M_{SS} = \partial^2 M/\partial S^2$.

From the nominator of heat capacity, it is evident that the temperature (M_S) has crucial role on the sign of C_Q. In addition, divergence points of heat capacity are indicating second-order phase transition. Hence, these divergencies are utilized for calculating critical values and investigating the critical behavior of the black hole. Now, for studying phase transition, we introduce various geometrical thermodynamic methods and compare their results with those arisen from the heat capacity.

3.3. Geometrical Study of the Phase Transition. One of the basic motivations for considering the geometrical thermodynamics comes from the fact that this formalism helps us to describe in an invariant way the thermodynamic properties of a given thermodynamical system in terms of geometric structures. Also, this method is a strong machinery for describing phase transition of the black holes. Another motivation is to give an independent picture regarding thermodynamical aspects of a system. In addition to some useful information about bound points, phase transitions, and thermal stability conditions, this method contains information regarding molecular interaction around phase transitions for thermodynamical systems. In other words, by studying the sign of thermodynamical Ricci scalar around phase transition points, one can extract information whether interaction is repulsive or attractive. Based on such motivations, it will be interesting to investigate black hole phase transition in the context of geometrical thermodynamics, as an independent approach.

In order to study the phase transition, one can employ thermodynamical quantities to build geometrical space-time. There are several metrics in the context of geometrical thermodynamics which one can use to study phase transition and critical behavior. The well-known thermodynamical metrics are Weinhold, Ruppeiner, Quevedo, and HPEM as the recently proposed method. As we mentioned, in some specific types of systems, the Weinhold, Ruppeiner, and Quevedo metrics are not applicable and they will face some problems. Here, we want to discuss these thermodynamical metrics and their possible mismatched problems.

Thermodynamical metric was first introduced by Weinhold [78, 79]. This thermodynamical metric is given by

$$dS_W^2 = g_{ab}^W dX^a dX^b, \tag{32}$$

where $g_{ab}^W = \partial^2 M(X^c)/\partial X^a \partial X^b$, $X^a \equiv X^a(S, N^i)$, and N^i denotes other extensive variables of the system. By calculating M as a function of extensive quantities (such as entropy and electric charge) and using Weinhold metric (32), one can find the Ricci scalar. It is expected that the singular points of the Weinhold Ricci scalar match the root or divergence points of the heat capacity, to indicate the bound point or the phase transition ones. We plot Figure 1 to investigate the mentioned behavior.

After that, Ruppeiner [80, 81] has defined another thermodynamical metric with the following form:

$$dS_R^2 = g_{ab}^R dX^a dX^b, \tag{33}$$

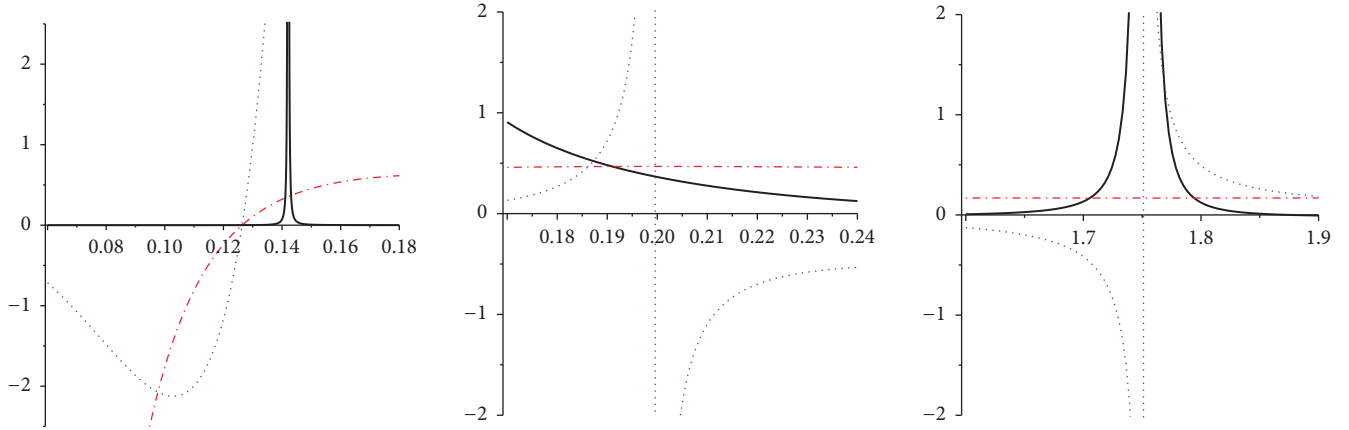where $g_{ab}^R = -\partial^2 S(X^c)/\partial X^a \partial X^b$ and $X^a \equiv X^a(M, N^i)$.

FIGURE 1: Weinhold metric. $\mathscr{R}$ (continuous line), C_Q (dotted line), and T (dot dashed) versus r_+ for $\Lambda = -1, n = 4, q = 0.1, b = 1, \omega = 10$, and $\beta = 1.5$. "Note: all three panels are plotted with the same parameters, but different regions and scales."
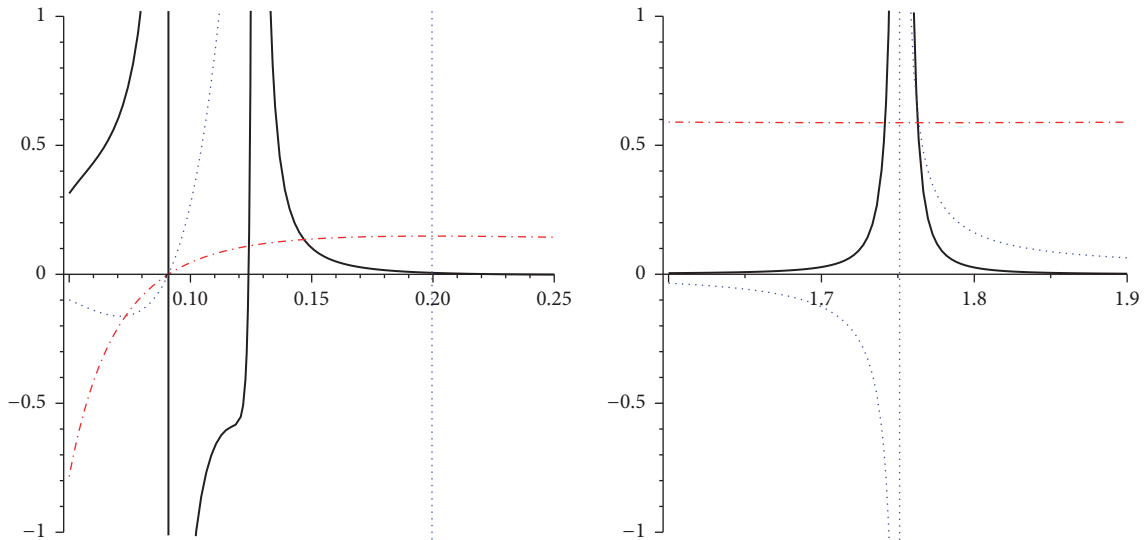

FIGURE 2: Ruppeiner metric. $\mathscr{R}$ (continuous line), C_Q (dotted line), and T (dot dashed) versus r_+ for $\Lambda = -1, n = 4, q = 0.1, b = 1, \omega = 10$, and $\beta = 1.5$. "Note: both panels are plotted with the same parameters, but different regions and scales."

In the Ruppeiner metric, thermodynamical potential is entropy. It is worthwhile mentioning that these two metrics are conformally related to each other [82]. We plot Figure 2 to show that the Ruppeiner Ricci scalar divergencies are not matched with those of heat capacity.

As we have shown, calculating thermodynamical Ricci scalar of these two thermodynamical metrics indicates that the results were not completely consistent with the results of heat capacity in the canonical ensemble. In order to remove some failures of the Weinhold and Ruppeiner metrics, recently, another metric which is Legendre invariant has been introduced by Quevedo [83, 84]. The Quevedo metric has the following form:

$$ds_Q^2 = \Omega \left(-M_{SS}dS^2 + M_{QQ}dQ^2\right), \quad (34)$$

where the conformal coefficient Ω is

$$\Omega = \left(SM_S + QM_Q\right). \quad (35)$$

Considering Figures 1–3, we find that, by using these three well-known metrics, there is at least a mismatch between heat capacity divergencies and thermodynamical Ricci scalar divergencies (of these three metrics). Therefore, these metrics are not appropriate tools for investigation of our black hole phase transitions and related critical behavior. In other words, the method of geometrical thermodynamics which has been reported in [90] is not an applicable method in the scalar field theory.

Very recently, a new metric was proposed by Hendi et al. (HPEM metric) to solve this problem. This method is applied for various gravitating systems and it is shown that the root and divergence points of the heat capacity coincide with the divergence points of the HPEM Ricci scalar (see
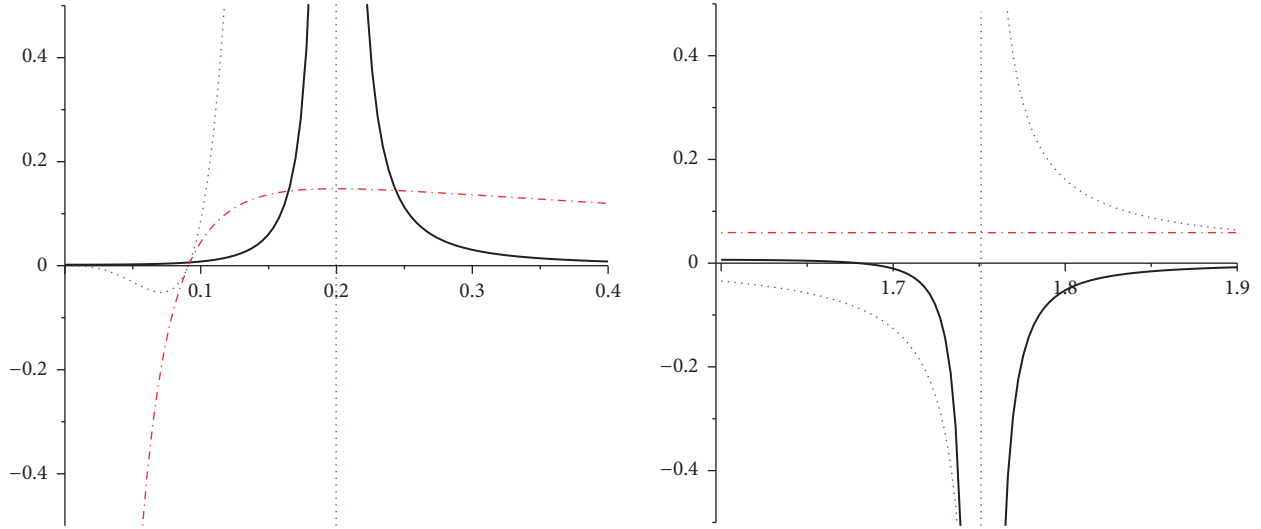
FIGURE 3: Quevedo metric. $\mathcal{R}$ (continuous line), C_Q (dotted line), and T (dot dashed) versus r_+ for $\Lambda = -1, n = 4, q = 0.1, b = 1, \omega = 10$, and $\beta = 1.5$. "Note: both panels are plotted with the same parameters, but different regions."

Figures 4–6, for more details). The generalized HPEM metric with n extensive variables ($n \geq 2$) has the following form [85–87]:

$$ds_{\text{HPEM}}^2 = \frac{SM_S}{\left(\prod_{i=2}^{n} \left(\partial^2 M/\partial\chi_i^2\right)\right)^3} \left(-M_{SS}dS^2 + \sum_{i=2}^{n} \left(\frac{\partial^2 M}{\partial\chi_i^2}\right) d\chi_i^2\right),$$
(36)

where χ_i's ($\chi_i \neq S$) are extensive parameters. It is notable that HPEM metric is the same as that presented by Quevedo (with the same "$-, +, +, \ldots$" signature), but with different conformal factor, and therefore it is expected to enjoy Legendre invariance. In what follows, we will investigate the stability and phase transition of the physical BD-BI black holes in the context of the heat capacity and geometrical thermodynamics by using HPEM metric.

As the first significant point which must be taken into deep consideration, one should regard the sign of the temperature. The positivity of the temperature denotes a physical black hole, whereas the negativity of T represents a nonphysical system. The temperature behavior has been shown in the figures, too. As we can see, there is a lower bound for the horizon radius (r_0), in which for $r_+ < r_0$, we encounter a nonphysical black hole, owing to negative sign of temperature. In contrast, in the case of $r_+ > r_0$, we confront a physical system due to the positivity of the temperature. In other words, the horizon radius of physical black holes is located in this region.

Figure 4 shows that, for the special values of the electric charge, nonlinearity parameter, and BD-coupling coefficient, we can obtain three characteristic points. One of them refers to the root of heat capacity (or temperature) which is known as r_0 and others are related to the divergence points of heat capacity which are denoted as r_{d_1} and r_{d_2} ($r_{d_1} < r_{d_2}$). We also find that all divergence points of the

TABLE 1: Critical points of BD-BI theory for $q = 0.1, \Lambda = -1, \omega = 10$, $b = 1$, and $\beta = 1.5$.

n	r_0	r_{d_1}	r_{d_2}
5	0.2052	0.3820	2.5814
6	0.2842	0.4772	3.5605
7	0.3422	0.5390	4.7619

Ricci scalar (for HPEM metric) coincide with these three points. Here, we use some tables to study the influences of different parameters (dimensions, nonlinearity parameter, and BD-coupling coefficient) on the mentioned characteristic points.

These tables provide information regarding the lower bound of horizon radius, two points of phase transition (for the case of BD-BI), and their dependencies to the variation of dimensions, nonlinearity parameter, and coupling coefficient. Regarding the tables and Figures 4–7, it is evident that one root and two divergence points for the heat capacity are almost observed. It is worthwhile to mention that the region of $r_0 < r_+ < r_{d_1}$ (positive sign of heat capacity) shows the stability of the system. In contrast, one can find that, for the region of $r_{d_1} < r_+ < r_{d_2}$, the heat capacity has negative sign which indicates instability. In addition, at region $r_+ > r_{d_2}$, the system is in the stable state due to the positive sign of heat capacity (see Figures 4–6 for more details). According to Table 1, one can conclude that the lower bound radius and two divergence points are increasing functions of the dimensions. Also, according to Table 2, the lower bound of horizon radius and the first divergence point (r_{d_1}) will increase by increasing β (the nonlinearity parameter), whereas the second point of divergency remains steady over this change. Considering figures and Table 2, it is obvious that, by increasing β, root and the first divergence point of heat capacity will increase up to a point and then any increment in this parameter would have negligible effect on these values.
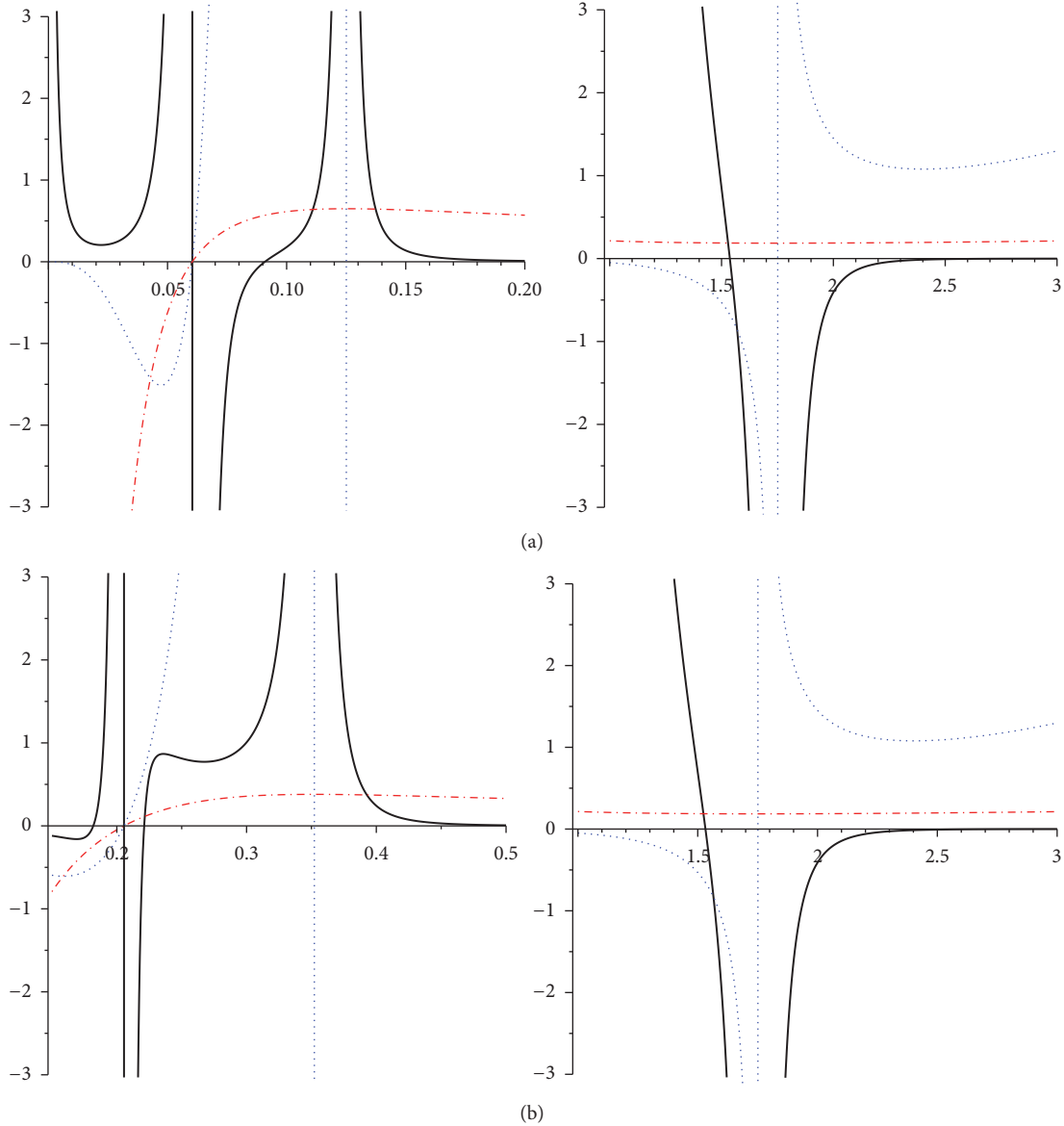
(a)

(b)

FIGURE 4: HPEM metric. $\mathscr{R}$ (continuous line), C_Q (dotted line), and T (dot dashed) versus r_+ for $\Lambda = -1, n = 4, q = 0.1, b = 1$, and $\omega = 10$. $\beta = 1$ (a) and $\beta = 5$ (b). "Note: both panels in the same line are plotted with the same parameters, but different regions and scales."

To put in other words, it can be interpreted that, in large β, we will face the Brans-Dicke-Maxwell behavior [91]. For large β, the obtained values for lower bound horizon radius and divergence point are the same as the obtained values for the Brans-Dicke-Maxwell case [91]. It is notable that the unstable region (between two divergencies, where the heat capacity is negative) is larger in small β than the large one (Brans-Dicke-Maxwell case) as it would be expected, which is due to the nature of nonlinearity that would cause the instability of system to increase. Meanwhile, r_0 and r_{d_1} have ascending functions and r_{d_2} will be declined by increasing ω (see Table 3). Generally, from what has been discussed above, dimensionality n and BD-coupling coefficient ω are playing the main role in changes of the location of larger divergence point.

TABLE 2: Critical points of BD-BI theory for $q = 0.1, \Lambda = -1, \omega = 10, b = 1$, and $n = 4$.

β	r_0	r_{d_1}	r_{d_2}
0.1	0.0056	0.0112	1.7512
1.0	0.0604	0.1251	1.7512
1.5	0.0902	0.1996	1.7512
5.0	0.2056	0.3523	1.7512
100.0	0.2400	0.3600	1.7512
200.0	0.2400	0.3600	1.7512

4. Conclusion

In this paper, the main goal was studying thermodynamical behavior of the BD-BI and Einstein-BI-dilaton black hole
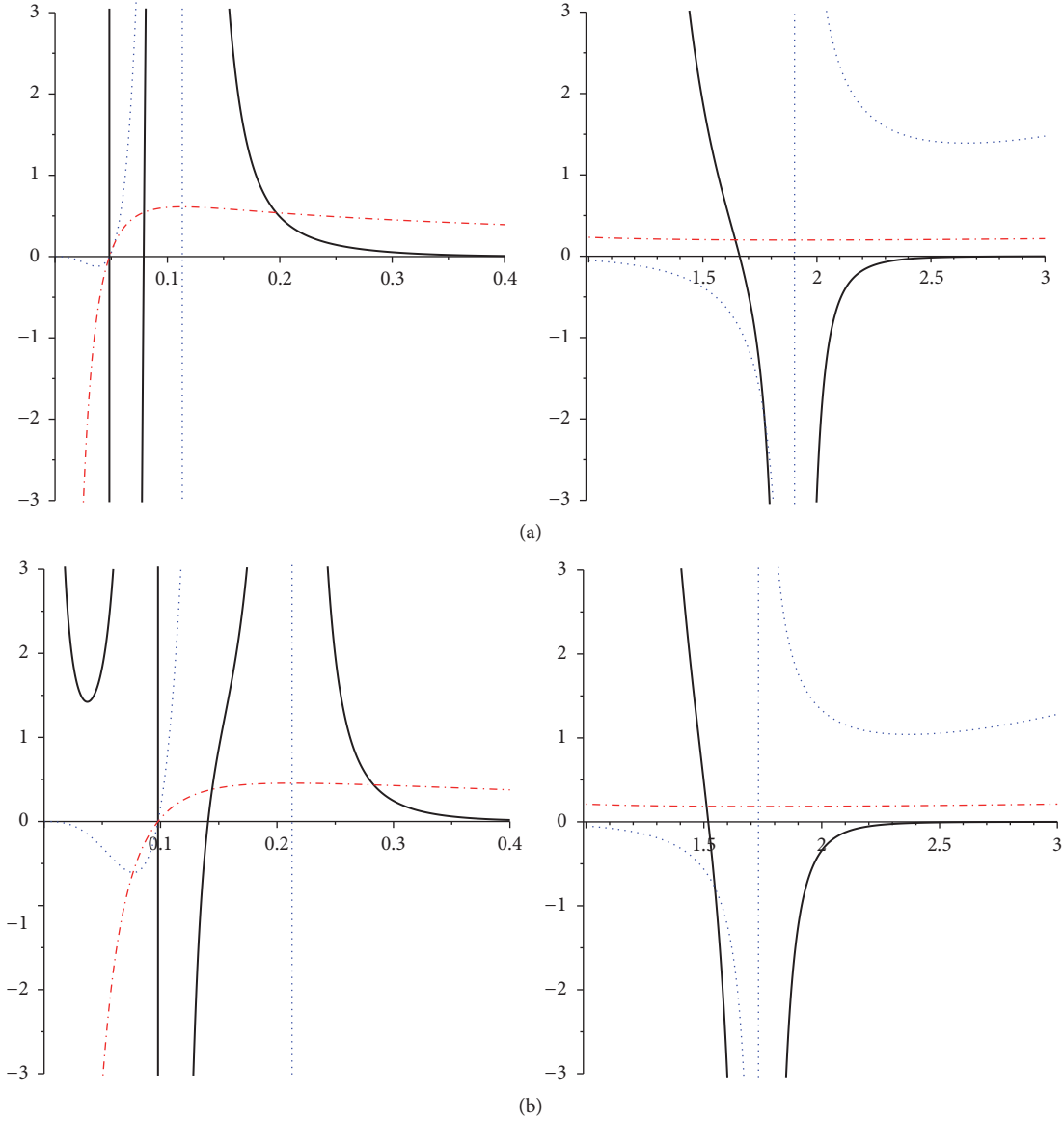
(a)

(b)

FIGURE 5: HPEM metric. $\mathcal{R}$ (continuous line), C_Q (dotted line), and T (dot dashed) versus r_+ for $\Lambda = -1$, $n = 4$, $q = 0.1$, $b = 1$, and $\beta = 1.5$. $\omega = 0.2$ (a) and $\omega = 200$ (b). "Note: both panels in the same line are plotted with the same parameters, but different regions."

TABLE 3: Critical points of BD-BI theory for $q = 0.1$, $\Lambda = -1$, $n = 4$, $b = 1$, and $\beta = 1.5$.

ω	r_0	r_{d_1}	r_{d_2}
0.2	0.0482	0.1132	1.9022
2	0.0742	0.1658	1.8038
200	0.0974	0.2128	1.7318

solutions. Since both of these solutions had very similar thermodynamical behavior in the context of geometrical thermodynamics, we have just considered the BD-BI ones. We have investigated the stability and phase transition in the canonical ensemble through the use of heat capacity. We have found that, for having a physical black hole (positive temperature), there should be a restriction on the value of

the horizon radius, which leads to a physical limitation point. This point was a border between nonphysical and physical black hole horizon radius. Moreover, investigating the phase transition of the black holes exhibited that there exist second-order phase transition points. In other words, the heat capacity had one real positive root and two divergence points. It was shown that these points (the root and divergence points of heat capacity) were affected by variation of the BI parameter, BD-coupling constant, and dimensions. From the presented tables and figures, we have found that the effect of dimensions on the larger divergence point was more than other factors and in contrast, the BI parameter had no sensible effect on this value. The effect of BD-coupling constant on these three points was so small in a way that, by applying a dramatic change in this constant, we observed a small change in the value of such characteristic points.
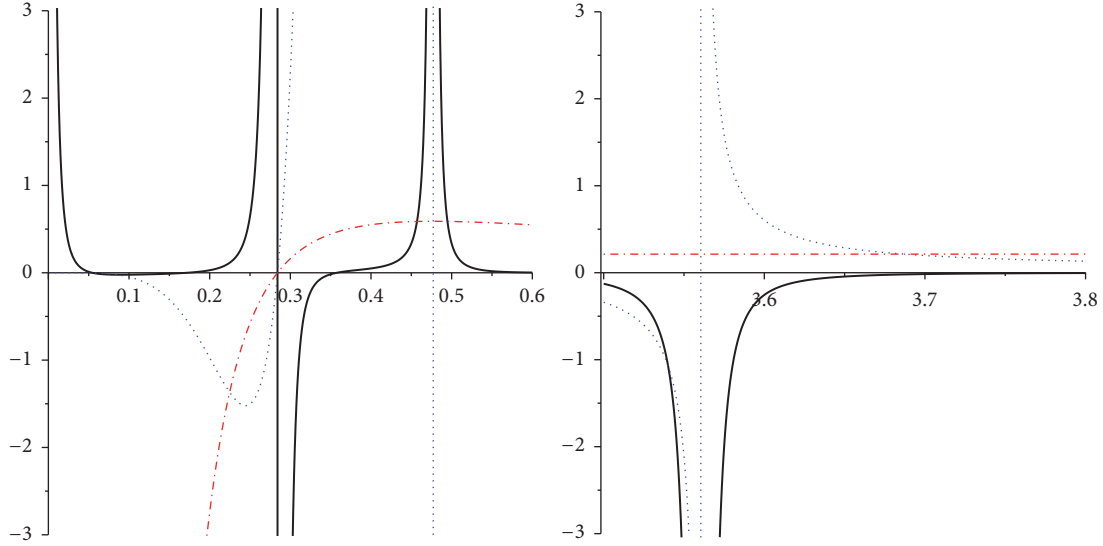
FIGURE 6: HPEM metric. $\mathcal{R}$ (continuous line), C_Q (dotted line), and T (dot dashed) versus r_+ for $\Lambda = -1$, $n = 6$, $q = 0.1$, $b = 1$, $\omega = 10$, and $\beta = 1.5$. "Note: both panels are plotted with the same parameters, but different regions and scales."
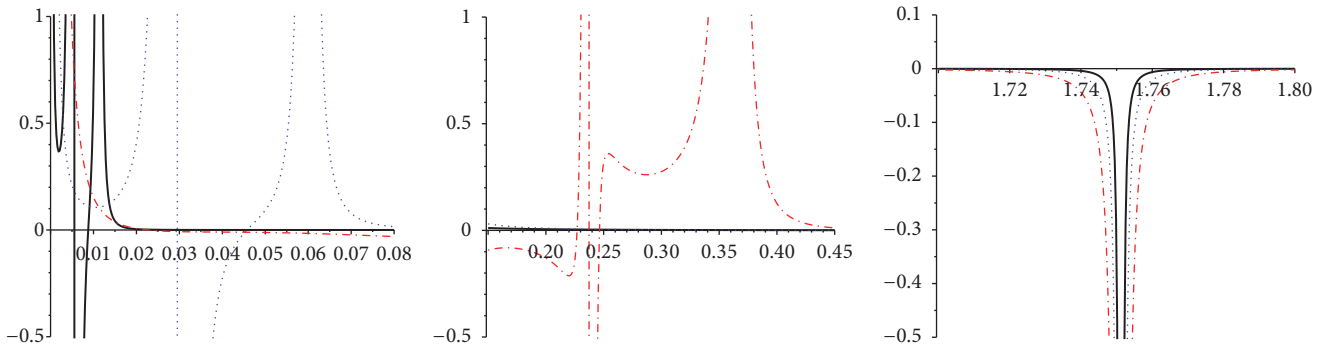

FIGURE 7: HPEM metric. $\mathcal{R}$ versus r_+ for $\Lambda = -1$, $n = 4$, $q = 0.1$, $b = 1$, $\omega = 10$, and $\beta = 0.1$ (continuous line), $\beta = 0.5$ (dotted line), and Maxwell (dot dashed). "Note: all three panels are plotted with the same parameters, but different regions and scales."

It was illustrated that in the context of thermal stability there exist four regions, specified by the root and two divergence points of the heat capacity. The root of heat capacity was referred to as the lower bound of horizon radius that separated the nonphysical black holes from the physical ones. Between the two divergencies, we encountered an unstable state and after the second divergence point black hole obtained a stable state. It is notable that, for small β, because of the nonlinearity effect, the unstable region is larger than the Maxwell case (large β) [91].

Eventually, we employed the geometrical thermodynamic method to study the phase transition. We have shown that Weinhold, Ruppeiner, and Quevedo metrics failed to provide a consistent result with the heat capacity's result. In other words, their thermodynamical Ricci scalar's divergencies did not match the root and divergencies of the heat capacity, exactly. In some of these methods, we encountered extra divergency which did not coincide with any of the phase transition points.

At last, using the HPEM metric, we achieved desirable results. It was shown that all the divergence points of the Ricci scalar of the mentioned metric covered the divergencies and root of the heat capacity. It is worth mentioning that the behavior of the curvature scalar was different near its divergence points. In other words, the divergence points of the Ricci scalar related to root of the heat capacity could be distinguished from the divergencies related to phase transition points based on the curvature scalar behavior.

Regarding the used method of this paper, it is interesting to extend obtained results to an energy dependent space-time and discuss the role of gravity's rainbow [92–95]. We leave this issue for future work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

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