Research Article

New Exact Solutions for the (3+1)-Dimensional Generalized BKP Equation

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The Wronskian technique is used to investigate a (3+1)-dimensional generalized BKP equation. Based on Hirota's bilinear form, new exact solutions including rational solutions, soliton solutions, positon solutions, negaton solutions, and their interaction solutions are formally derived. Moreover we analyze the strangely mechanical behavior of the Wronskian determinant solutions. The study of these solutions will enrich the variety of the dynamics of the nonlinear evolution equations.

1. Introduction

In recent years, the problem of finding exact solutions of nonlinear evolution equations (NLEEs) is very popular for both mathematicians and physicists. Because seeking exact solutions of NLEEs is of great significance in nonlinear dynamics, many methods such as the inverse scattering transformation [1], Hirota's bilinear method [2], the Darboux transformation [3], the sine-cosine method [4], G'/G-expansion method [5, 6], and the transformed rational function method [7] have been proposed. The Wronskian method which is based on the bilinear form of the NLEEs was proposed by Freeman and Nimmo in [8, 9]. It is a fairly powerful tool to construct exact solutions of NLEEs in terms of the Wronskian determinant. By means of the method, the exact solutions of some NLEEs are obtained [10–16].

The study of the BKP equation has attracted a considerable size of research work. These equations were studied using the Hirota method, the multiple exp-function algorithm, the Pfaffian technique, Riemann theta functions, the extended homoclinic test approach, and Bäcklund transformation by many authors [17–26]. In this paper, based on the Wronskian method, the new exact solutions including rational solutions, soliton solutions, positon solutions, negaton solutions, and their interaction solutions of the (3+1)-dimensional generalized BKP equations are investigated.

In this paper, we will consider the following (3+1)-dimensional generalized BKP equation:

$$u_{yt} - u_{xxxy} - 3\left(u_x u_y\right)_x + 6u_{xx} - 3u_{zz} = 0. \tag{1}$$

When $z = x$, this (3+1)-dimensional generalized BKP equation reduces to the BKP equation [27, 28]:

$$u_{yt} - u_{xxxy} - 3\left(u_x u_y\right)_x + 3u_{xx} = 0. \tag{2}$$

By the dependent variable transformation

$$u = 2\left(\ln f\right)_x = \frac{2f_x}{f}, \tag{3}$$

the (3+1)-dimensional generalized BKP equation (1) becomes a bilinear form

$$\left(D_y D_t - D_x^3 D_y + 6D_x^2 - 3D_z^2\right) f \cdot f = 0, \tag{4}$$

where D_t, D_x, D_y, and D_z are the Hirota operators [2]:

$$f\left(f_{yt} - f_{xxxy} + 6f_{xx} - 3f_{zz}\right) - f_y f_t + f_{xxx} f_y \\ + 3f_{xxy} f_x - 3f_{xx} f_{xy} - 6f_x^2 + 3f_z^2 = 0. \tag{5}$$

We will show this (3+1)-dimensional generalized BKP equation has a class of Wronskian solutions with all generating functions for matrix entries satisfying a linear system of partial differential equations involving a free parameter. Rational solutions, solitons, positons, negatons, and interaction solutions to (1) among Wronskian determinant solutions are constructed and a few plots of particular solutions are made.

The paper is organized as follows. In Section 2, we derive a Wronskian formulation for the (3+1)-dimensional generalized BKP equation. In Section 3, Wronskian solutions to the (3+1)-dimensional generalized BKP equation are obtained. Section 4 presents the conclusion.

2. A Wronskian Formulation

The Wronskian technique is a powerful tool to construct exact solutions to bilinear differential or difference equations. To use the Wronskian technique, we adopt the compact notation introduced by Freeman and Nimmo [8, 9]:

$$W(\phi_1, \phi_2, \ldots, \phi_N) = (\widehat{N-1}; \Phi) = |\widehat{N-1}|$$

$$= \begin{vmatrix} \phi_1^{(0)} & \phi_1^{(1)} & \cdots & \phi_1^{(N-1)} \\ \phi_2^{(0)} & \phi_2^{(1)} & \cdots & \phi_2^{(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_N^{(0)} & \phi_N^{(1)} & \cdots & \phi_N^{(N-1)} \end{vmatrix}, \tag{6}$$

where

$$\Phi = (\phi_1, \ldots, \phi_N)^T,$$

$$\phi_i^{(0)} = \phi_i,$$

$$\phi_i^{(j)} = \frac{\partial^j}{\partial x^j} \phi_i, \tag{7}$$

$$j \geq 1, \ 1 \leq i \leq N.$$

Solutions determined by $u = 2(\ln f)_x$ with $f = |\widehat{N-1}|$ to the (3+1)-dimensional generalized BKP equation (1) are called Wronskian solutions.

Theorem 1. *Assuming that a group of functions* $\phi_i = \phi_i(x, y, z, t)$, $1 \leq i \leq N$, *satisfies the following linear conditions*

$$-\phi_{i,xx} = \sum_{j=1}^{N} \lambda_{ij}(t) \phi_j, \tag{8}$$

$$\phi_{i,y} = k\phi_{i,x}, \tag{9}$$

$$\phi_{i,z} = \sqrt{2}\phi_{i,x}, \tag{10}$$

$$\phi_{i,t} = 4\phi_{i,xxx}, \tag{11}$$

where k is an arbitrary nonzero constant, then the Wronskian determinant $f = |\widehat{N-1}|$ *defined by (6) solves the bilinear equation (5).*

Proof. Obviously, we have

$$f_x = |\widehat{N-2}, N|,$$

$$f_{xx} = |\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|,$$

$$f_{xxx} = |\widehat{N-4}, N-2, N-1, N| \tag{12}$$

$$+ 2|\widehat{N-3}, N-1, N+1|$$

$$+ |\widehat{N-2}, N+2|.$$

Using conditions (9), (10), and (11), we get that

$$f_y = k|\widehat{N-2}, N|,$$

$$f_{xy} = k\left(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|\right),$$

$$f_{xxy} = k\left(|\widehat{N-4}, N-2, N-1, N|\right.$$

$$+ 2|\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|\left.\right),$$

$$f_{xxxy} = k\left(|\widehat{N-5}, N-3, N-2, N-1, N|\right.$$

$$+ 3|\widehat{N-4}, N-2, N-1, N+1|$$

$$+ 2|\widehat{N-3}, N, N+1| + 3|\widehat{N-3}, N-1, N+2|$$

$$+ |\widehat{N-2}, N+3|\left.\right), \tag{13}$$

$$f_z = \sqrt{2}|\widehat{N-2}, N|,$$

$$f_{zz} = 2\left(|\widehat{N-3}, N-1, N| + |\widehat{N-2}, N+1|\right),$$

$$f_t = 4\left(|\widehat{N-4}, N-2, N-1, N|\right.$$

$$- |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|\left.\right),$$

$$f_{yt} = 4k\left(|\widehat{N-5}, N-3, N-2, N-1, N|\right.$$

$$- |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|\left.\right).$$

$$\square$$

Under (8), it is not difficult to obtain [10]

$$|\widehat{N-1}| \sum_{i=1}^{N} \lambda_{ii}(t) \left(\sum_{i=1}^{N} \lambda_{ii}(t) |\widehat{N-1}| \right)$$

$$= \left(\sum_{i=1}^{N} \lambda_{ii}(t) |\widehat{N-1}| \right)^2 = \left(|\widehat{N-2}, N+1|\right.$$

$$- |\widehat{N-3}, N-1, N|\right)^2 = |\widehat{N-1}| \tag{14}$$

$$\cdot \left(|\widehat{N-5}, N-3, N-2, N-1, N|\right.$$

$$- |\widehat{N-4}, N-2, N-1, N+1|$$

$$+ 2|\widehat{N-3}, N, N+1| - |\widehat{N-3}, N-1, N+2|$$

$$+ |\widehat{N-2}, N+3|\left.\right).$$

Therefore,

$$f\left(f_{yt} - f_{xxxy} + 6f_{xx} - 3f_{zz}\right) = 3k\left|\widehat{N-1}\right|$$
$$\cdot\left(\left|\widehat{N-5}, N-3, N-2, N-1, N\right|\right.$$
$$- \left|\widehat{N-4}, N-2, N-1, N+1\right|$$
$$- 2\left|\widehat{N-3}, N, N+1\right| - \left|\widehat{N-3}, N-1, N+2\right|$$
$$+ \left|\widehat{N-2}, N+3\right|\right) = 3k\left(\left|\widehat{N-2}, N+1\right|\right.$$
$$- \left|\widehat{N-3}, N-1, N\right|\right)^2 - 12k\left|\widehat{N-3}, N, N+1\right|$$
$$\cdot\left|\widehat{N-1}\right| - f_y f_t + f_{xxx}f_y + 3f_{xxy}f_x - 6f_x^2 + 3f_z^2$$
$$= 12k\left|\widehat{N-3}, N-1, N+1\right|\left|\widehat{N-2}, N\right| - 3f_{xx}f_{xy}$$
$$= -3k\left(\left|\widehat{N-2}, N+1\right| - \left|\widehat{N-3}, N-1, N\right|\right)^2$$
$$- 12k\left|\widehat{N-3}, N-1, N\right|\left|\widehat{N-2}, N+1\right|.$$

(15)

Substitution of the above results into (4) finally leads to the following Plücker relation:

$$\left(D_y D_t - D_x^3 D_y + 6D_x^2 - 3D_z^2\right)f \cdot f$$
$$= -12k\left|\widehat{N-3}, N, N+1\right|\left|\widehat{N-1}\right|$$
$$+ 12k\left|\widehat{N-3}, N-1, N+1\right|\left|\widehat{N-2}, N\right|$$
$$- 12k\left|\widehat{N-3}, N-1, N\right|\left|\widehat{N-2}, N+1\right| = 0.$$

(16)

Theorem 1 tells us that if a group of functions $\phi_i = \phi_i(x, y, z, t)$, $1 \le i \le N$, satisfies the linear conditions in (8)–(11), then we can get a solution $f = \left|\widehat{N-1}\right|$ to the bilinear BKP equation (4). The corresponding solution of (1) is

$$u = 2\left(\ln f\right)_x = \frac{2f_x}{f} = \frac{2\left|\widehat{N-2}, N\right|}{\left|\widehat{N-1}\right|}.$$

(17)

Remark 1. From the compatibility conditions $\phi_{i,xxt} = \phi_{i,txx}$, $1 \le i \le N$, of conditions (8)–(11), we have the equality

$$\sum_{j=1}^{N}\lambda_{ij}(t)\phi_j = 0, \quad 1 \le i \le N,$$

(18)

and thus it is easy to see that the Wronskian determinant $W(\phi_1, \phi_2, \ldots, \phi_N)$ becomes zero if there is at least one entry λ_{ij} satisfying $\lambda_{ij}(t) \ne 0$.

Remark 2. If the coefficient matrix $\Lambda = (\lambda_{ij})$ is similar to another matrix $M = (\mu_{ij})$ under an invertible constant matrix P, let us say $\Lambda = P^{-1}MP$, then $\widetilde{\Phi} = P\Phi$ solves

$$-\widetilde{\Phi}_{xx} = M\widetilde{\Phi},$$
$$\widetilde{\Phi}_y = k\widetilde{\Phi}_x,$$

$$\widetilde{\Phi}_z = \sqrt{2}\widetilde{\Phi}_x,$$
$$\widetilde{\Phi}_t = 4\widetilde{\Phi}_{xxx},$$

(19)

and the resulting Wronskian solutions to (1) are the same:

$$u(\Lambda) = 2\partial_x \ln\left|\Phi^{(0)}, \Phi^{(1)}, \ldots, \Phi^{(N-1)}\right|$$
$$= 2\partial_x \ln\left|P\Phi^{(0)}, P\Phi^{(1)}, \ldots, P\Phi^{(N-1)}\right| = u(M).$$

(20)

Based on Remark 1, we only need to consider case of (8)–(11) under $d\Lambda/dt = 0$, that is, the following conditions:

$$-\phi_{i,xx} = \sum_{j=1}^{N}\lambda_{ij}(t)\phi_j,$$
$$\phi_{i,y} = k\phi_{i,x},$$
$$\phi_{i,z} = \sqrt{2}\phi_{i,x},$$
$$\phi_{i,t} = 4\phi_{i,xxx},$$

(21)

where $\Lambda = (\lambda_{ij})$ is an arbitrary real constant matrix. Moreover, Remark 2 tells us that an invertible constant linear transformation on Φ in the Wronskian determinant does not change the corresponding Wronskian solution, and thus, we only have to solve (21) under the Jordan form of Λ.

3. Wronskian Solutions

In principle, we can construct general Wronskian solutions of (1) associated with two types of Jordan blocks of the coefficient matrix Λ. But it is not easy. In this section we will present a few special Wronskian solutions to the generalized BKP equation, together with examples of exact solutions.

It is well known that the corresponding Jordan form of a real matrix

$$\Lambda = \begin{bmatrix} J(\lambda_1) & & & 0 \\ 1 & J(\lambda_2) & & \\ & \ddots & \ddots & \\ 0 & & 1 & J(\lambda_m) \end{bmatrix}_{n\times n}$$

(22)

has the following two types of blocks:

(I)

$$J(\lambda_i) = \begin{bmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{bmatrix}_{k_i \times k_i}$$

(23)

(II)

$$J(\lambda_i) = \begin{bmatrix} \Lambda_i & & & 0 \\ I_2 & \Lambda_i & & \\ & \ddots & \ddots & \\ 0 & & I_2 & \Lambda_i \end{bmatrix}_{l_i \times l_i},$$

(24)

$$\Lambda_i = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where λ_i, α_i, β_i are all real constants. The first type of blocks has the real eigenvalue λ_i with algebraic multiplicity $k_i (\Sigma_{i=1}^{m} k_i = N)$, and the second type of blocks has the complex eigenvalue $\lambda_i^{\pm} = \alpha_i \pm \beta_i \sqrt{-1}$ with algebraic multiplicity l_i.

3.1. Rational Solutions.

Suppose Λ has the first type of Jordan blocks. Without loss of generality, let

$$
J(\lambda_1) = \begin{bmatrix} \lambda_1 & & & 0 \\ 1 & \lambda_1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_1 \end{bmatrix}_{k_1 \times k_1}. \tag{25}
$$

In this case, if the eigenvalue $\lambda_1 = 0$, $J(\lambda_1)$ becomes of the following form:

$$
\begin{bmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}_{k_1 \times k_1}. \tag{26}
$$

From condition (21), we get

$$
\begin{aligned}
\phi_{1,xx} &= 0, \\
-\phi_{i+1,xx} &= \phi_i, \\
\phi_{i,y} &= k\phi_{i,x}, \\
\phi_{i,z} &= \sqrt{2}\phi_{i,x}, \\
\phi_{i,t} &= 4\phi_{i,xxx}, \\
i &\geq 1.
\end{aligned} \tag{27}
$$

Such functions ϕ_i $(i \geq 1)$ are all polynomials in x, y, z, and t, and a general Wronskian solution to the (3+1)-dimensional generalized BKP equation (1)

$$
u = 2\partial_x \ln W \left(\phi_1, \phi_2, \ldots, \phi_{k_1} \right) \tag{28}
$$

is rational and is called a rational Wronskian solution of order $k_1 - 1$.

From (27), we solve $\phi_{1,xx} = 0$, $\phi_{1,y} = k\phi_{1,x}$, $\phi_{1,z} = \sqrt{2}\phi_{1,x}$, $\phi_{1,t} = 4\phi_{1,xxx}$ and have

$$
\phi_1 = c_1 + c_2 \left(x + ky + \sqrt{2}z \right), \tag{29}
$$

where c_1, c_2, and $k \neq 0$ are all real constants. Similarly, by solving $-\phi_{i+1,xx} = \phi_1$, $\phi_{i+1,y} = k\phi_{i+1,x}$, $\phi_{i+1,z} = \sqrt{2}\phi_{i+1,x}$, $\phi_{i+1,t} = 4\phi_{i+1,xxx}$, $i \geq 1$, then two special rational solutions of lower-order are obtained after setting some integral constants to be zero.

(1) Zero-Order. When $c_1 = 0$, $c_2 = 1$, $\phi_1 = x + ky + \sqrt{2}z$, we have the corresponding Wronskian determinant $f = W(\phi_1) = x + ky + \sqrt{2}z$ and the associated rational Wronskian solution of zero-order:

$$
u = 2\partial_x \ln W(\phi_1) = \frac{2}{x + ky + \sqrt{2}z}. \tag{30}
$$

(2) First-Order. Taking $c_1 = 0$, $c_2 = -1$, $\phi_1 = -(x + ky + \sqrt{2}z)$, we have $\phi_2 = (1/6)(x + ky + \sqrt{2}z)^3 + 4t$. In this case, the corresponding Wronskian determinant is $f = W(\phi_1, \phi_2) = -(1/3)(x + ky + \sqrt{2}z)^3 + 4t$, and the rational Wronskian solution of first-order reads

$$
u = 2\partial_x \ln W(\phi_1, \phi_2) = \frac{2\left(x + ky + \sqrt{2}z\right)^2}{(1/3)\left(x + ky + \sqrt{2}z\right)^3 - 4t}. \tag{31}
$$

(3) Second-Order. Taking $\phi_1 = x + ky + \sqrt{2}z$, $\phi_2 = -(1/6)(x + ky + \sqrt{2}z)^3 - 4t$, we have $\phi_3 = (1/120)(x + ky + \sqrt{2}z)^5 + 2(x + ky + \sqrt{2}z)^2 t$. Then the Wronskian determinant is $f = W(\phi_1, \phi_2, \phi_3) = -(1/45)(x + ky + \sqrt{2}z)^6 + (4/3)(x + ky + \sqrt{2}z)^3 t + 16t^2$, and the rational Wronskian solution of second-order is given by

$$
\begin{aligned}
u &= 2\partial_x \ln W(\phi_1, \phi_2, \phi_3) \\
&= \frac{-(2/15)\left(x + ky + \sqrt{2}z\right)^5 + 4\left(x + ky + \sqrt{2}z\right)^2 t}{-(1/45)\left(x + ky + \sqrt{2}z\right)^6 + (4/3)\left(x + ky + \sqrt{2}z\right)^3 t + 16t^2}.
\end{aligned} \tag{32}
$$

3.2. Solitons, Positons, and Negatons.

If the eigenvalue $\lambda_i \neq 0$, $J(\lambda_i)$ becomes of the following form:

$$
\begin{bmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{bmatrix}_{k_i \times k_i}. \tag{33}
$$

We start from the eigenfunction $\phi_i(\lambda_i)$ determined by

$$
\begin{aligned}
-\left(\phi_i(\lambda_i)\right)_{xx} &= \lambda_i \phi_i(\lambda_i), \\
\left(\phi_i(\lambda_i)\right)_y &= k\left(\phi_i(\lambda_i)\right)_x, \\
\left(\phi_i(\lambda_i)\right)_z &= \sqrt{2}\left(\phi_i(\lambda_i)\right)_x, \\
\left(\phi_i(\lambda_i)\right)_t &= 4\left(\phi_i(\lambda_i)\right)_{xxx}.
\end{aligned} \tag{34}
$$

General solutions to this system in two cases of $\lambda_i > 0$ and $\lambda_i < 0$ read as

$$
\begin{aligned}
\phi_i(\lambda_i) &= C_{1i} \sin\left(\sqrt{\lambda_i}\left(x + ky + \sqrt{2}z - 4t\lambda_i\right)\right) \\
&\quad + C_{2i} \cos\left(\sqrt{\lambda_i}\left(x + ky + \sqrt{2}z - 4t\lambda_i\right)\right), \\
\phi_i(\lambda_i) &= C_{3i} \sinh\left(\sqrt{-\lambda_i}\left(x + ky + \sqrt{2}z - 4t\lambda_i\right)\right) \\
&\quad + C_{4i} \cosh\left(\sqrt{-\lambda_i}\left(x + ky + \sqrt{2}z - 4t\lambda_i\right)\right),
\end{aligned} \tag{35}
$$

respectively, where C_{1i}, C_{2i}, C_{3i}, and C_{4i} are arbitrary real constants. By an inspection, we find that

$$-\begin{bmatrix} \phi_i(\lambda_i) \\ \frac{1}{1!}\partial_{\lambda_i}\phi_i(\lambda_i) \\ \vdots \\ \frac{1}{(k_i-1)!}\partial_{\lambda_i}^{k_i-1}\phi_i(\lambda_i) \end{bmatrix}_{xx}$$

$$= \begin{bmatrix} \lambda_i & & & 0 \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ 0 & & 1 & \lambda_i \end{bmatrix}_{k_i \times k_i} \begin{bmatrix} \phi_i(\lambda_i) \\ \frac{1}{1!}\partial_{\lambda_i}\phi_i(\lambda_i) \\ \vdots \\ \frac{1}{(k_i-1)!}\partial_{\lambda_i}^{k_i-1}\phi_i(\lambda_i) \end{bmatrix}, \quad (36)$$

$$\left(\frac{1}{j!}\partial_{\lambda_i}^j\phi_i(\lambda_i)\right)_y = k\left(\frac{1}{j!}\partial_{\lambda_i}^j\phi_i(\lambda_i)\right)_x,$$

$$\left(\frac{1}{j!}\partial_{\lambda_i}^j\phi_i(\lambda_i)\right)_z = \sqrt{2}\left(\frac{1}{j!}\partial_{\lambda_i}^j\phi_i(\lambda_i)\right)_x,$$

$$\left(\frac{1}{j!}\partial_{\lambda_i}^j\phi_i(\lambda_i)\right)_t = 4\left(\frac{1}{j!}\partial_{\lambda_i}^j\phi_i(\lambda_i)\right)_{xxx},$$

$$0 \le j \le k_i - 1.$$

Therefore, through this set of eigenfunctions, we obtain a Wronskian solution to (1):

$$u = 2\partial_x \ln W\left(\phi_i(\lambda_i), \frac{1}{1!}\partial_{\lambda_i}\phi_i(\lambda_i), \ldots, \frac{1}{(k_i-1)!} \right.$$
$$\left. \cdot \partial_{\lambda_i}^{k_i-1}\phi_i(\lambda_i)\right), \quad (37)$$

which corresponds to the first type of Jordan blocks with a nonzero real eigenvalue.

When $\lambda_i > 0$, we get positon solutions [29], and when $\lambda_i < 0$, we get negaton solutions [30]. If we suppose Λ have n different nonzero real eigenvalues, in which there are l positive real eigenvalues and $n - l$ negative real eigenvalues,

then a more general positon can be obtained by combining l sets of eigenfunctions associated with different $\lambda_i > 0$:

$$u = 2\partial_x \ln W\left(\phi_1(\lambda_1), \frac{1}{1!}\partial_{\lambda_1}\phi_1(\lambda_1), \ldots, \frac{1}{(k_1-1)!}\right.$$
$$\cdot \partial_{\lambda_1}^{k_1-1}\phi_1(\lambda_1); \ldots; \phi_l(\lambda_l), \frac{1}{1!}\partial_{\lambda_i}\phi_l(\lambda_l), \ldots, \frac{1}{(k_l-1)!} \quad (38)$$
$$\left. \cdot \partial_{\lambda_l}^{k_l-1}\phi_l(\lambda_l)\right).$$

Similarly, a more general negaton can be obtained by combining $n - l$ sets of eigenfunctions associated with different $\lambda_i < 0$:

$$u = 2\partial_x \ln W\left(\phi_1(\lambda_1), \frac{1}{1!}\partial_{\lambda_1}\phi_1(\lambda_1), \ldots, \frac{1}{(k_1-1)!}\right.$$
$$\cdot \partial_{\lambda_1}^{k_1-1}\phi_1(\lambda_1); \ldots; \phi_{n-l}(\lambda_{n-l}), \frac{1}{1!} \quad (39)$$
$$\left. \cdot \partial_{\lambda_{n-l}}\phi_{n-l}(\lambda_{n-l}), \ldots, \frac{1}{(k_{n-l}-1)!}\partial_{\lambda_{n-l}}^{k_{n-l}-1}\phi_{n-l}(\lambda_{n-l})\right).$$

This solution is called an l-positon of order $(k_1 - 1, k_2 - 1, \ldots, k_l-1)$ or $n-l$-negaton of order $(k_1-1, k_2-1, \ldots, k_{n-l}-1)$. If $l = n$ or $l = 0$, we simply say that it is an n-positon of order n or an n-negaton of order n.

(1) Solitons. An n-soliton solution is a special n-negaton:

$$u = 2\partial_x \ln W(\phi_1, \phi_2, \ldots, \phi_n), \quad (40)$$

with ϕ_i being given by

$$\phi_i = \cosh\left(\sqrt{-\lambda_i}\left(x + ky + \sqrt{2}z - 4t\lambda_i\right) + \gamma_i\right),$$
$$i \text{ odd},$$
$$\phi_i = \sinh\left(\sqrt{-\lambda_i}\left(x + ky + \sqrt{2}z - 4t\lambda_i\right) + \gamma_i\right), \quad (41)$$
$$i \text{ even},$$

where $\lambda_1 < \lambda_2 < \cdots < \lambda_n < 0$ and γ_i $(1 \le i \le n)$ are arbitrary real constants. For example, a 1-soliton to (1) is given by

$$u = 2\partial_x \ln W(\phi_1) = 2\partial_x$$
$$\cdot \ln\left(\cosh\left(\sqrt{-\lambda_1}\left(x + ky + \sqrt{2}z - 4t\lambda_1\right) + \gamma_1\right)\right) \quad (42)$$
$$= 2\sqrt{-\lambda_1}\tanh(\theta_1),$$

where $\theta_1 = \sqrt{-\lambda_1}(x + ky + \sqrt{2}z - 4t\lambda_1) + \gamma_1$.
Similarly, we have a 2-soliton to (1):

$$u = 2\partial_x \ln W(\cosh(\theta_1), \sinh(\theta_2)) = \frac{2(\lambda_1 - \lambda_2)(\sinh(\theta_1 - \theta_2) - \sinh(\theta_1 + \theta_2))}{(\sqrt{-\lambda_1} - \sqrt{-\lambda_2})\cosh(\theta_1 + \theta_2) - (\sqrt{-\lambda_1} + \sqrt{-\lambda_2})\cosh(\theta_1 - \theta_2)}, \quad (43)$$
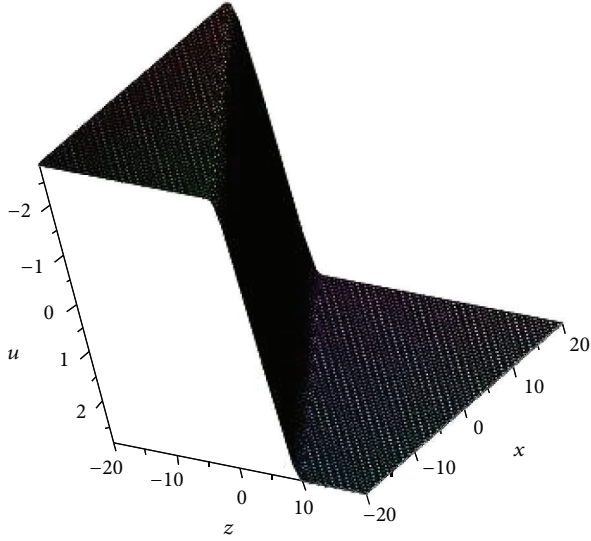
FIGURE 1: The shape of the 1-soliton to (1) with $\lambda_1 = -2$, $y = 0$, $t = 1$, $\gamma_1 = 0$.
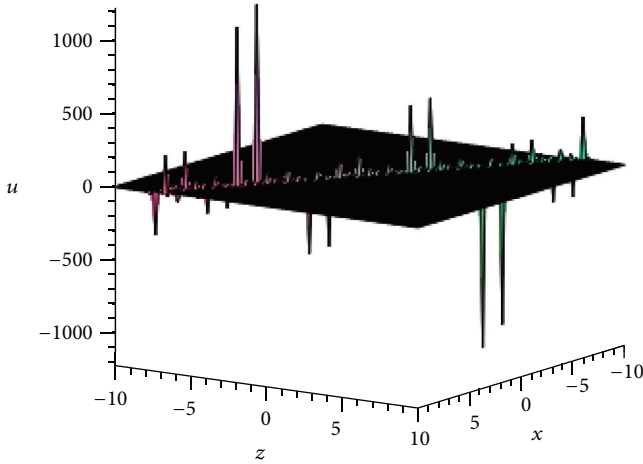

FIGURE 2: The shape of the 2-soliton to (1) with $\lambda_1 = -4$, $\lambda_2 = -1$, $y = 0$, $t = 0$, $\gamma_1 = \gamma_2 = 0$.

where $\theta_i = \sqrt{-\lambda_i}(x + ky + \sqrt{2}z - 4t\lambda_i) + \gamma_i$, $i = 1, 2$. Figures 1 and 2 of three-dimensional plots show the n-soliton to (1) defined by (40) on the indicated specific regions, with specific values being chosen for the parameters.

(2) Positons. Two kinds of special positons of order $k_1 - 1$ are

$$u = 2\partial_x \ln W \left(\phi, \partial_\lambda \phi, \ldots, \partial_\lambda^{k_1-1} \phi \right),$$
$$\phi = \cos \left(\sqrt{\lambda} \left(x + ky + \sqrt{2}z - 4t\lambda \right) + \gamma \left(\sqrt{\lambda} \right) \right), \quad (44)$$

$$u = 2\partial_x \ln W \left(\phi, \partial_\lambda \phi, \ldots, \partial_\lambda^{k_1-1} \phi \right),$$
$$\phi = \sin \left(\sqrt{\lambda} \left(x + ky + \sqrt{2}z - 4t\lambda \right) + \gamma \left(\sqrt{\lambda} \right) \right), \quad (45)$$
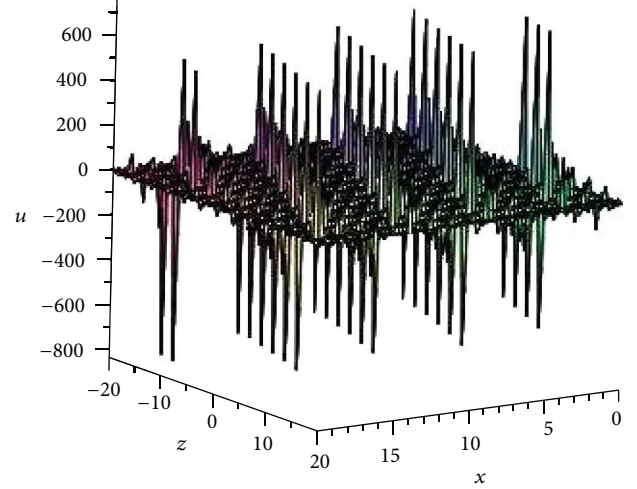

FIGURE 3: The shape of the 1-positon of zero-order to (1) with $\lambda_1 = 2$, $y = 0$, $t = 1$, $\gamma_3 = 0$.

where $\lambda > 0$ and γ is an arbitrary function of $\sqrt{\lambda}$. But these two kinds of positons are equivalent to each other, due to the existence of the arbitrary function γ.

When $\lambda_1 > 0$, a 1-positon of zero-order reads

$$u = 2\partial_x \ln W \left(\phi_1 \right)$$
$$= 2\partial_x \ln \left(\cos \left(\sqrt{\lambda_1} \left(x + ky + \sqrt{2}z - 4t\lambda_1 \right) + \gamma_1 \right) \right) \quad (46)$$
$$= -2\sqrt{\lambda_1} \tan \left(\theta_3 \right),$$

where $\theta_3 = \sqrt{\lambda_1}(x + ky + \sqrt{2}z - 4t\lambda_1) + \gamma_3$. And a 1-positon of first-order is

$$u = 2\partial_x \ln W \left(\cos \left(\theta_3 \right), \partial_{\lambda_1} \cos \left(\theta_3 \right) \right)$$
$$= \frac{4\sqrt{\lambda_1} \left(1 + \cos \left(2\theta_3 \right) \right)}{2\sqrt{\lambda_1} \left(x + ky + \sqrt{2}z - 12t\lambda_1 \right) + \sin \left(2\theta_3 \right)}. \quad (47)$$

Figures 3 and 4 of three-dimensional plots show the special positons to (1) defined by (44) on the indicated specific regions, with specific values being chosen for the parameters.

(3) Negatons. Two kinds of special negatons of order $k_1 - 1$ are

$$u = 2\partial_x \ln W \left(\phi, \partial_\lambda \phi, \ldots, \partial_\lambda^{k_1-1} \phi \right),$$
$$\phi = \cosh \left(\sqrt{-\lambda} \left(x + ky + \sqrt{2}z - 4t\lambda \right) + \gamma \left(\sqrt{-\lambda} \right) \right), \quad (48)$$

$$u = 2\partial_x \ln W \left(\phi, \partial_\lambda \phi, \ldots, \partial_\lambda^{k_1-1} \phi \right),$$
$$\phi = \sinh \left(\sqrt{-\lambda} \left(x + ky + \sqrt{2}z - 4t\lambda \right) + \gamma \left(\sqrt{-\lambda} \right) \right), \quad (49)$$

where $\lambda < 0$ and γ is an arbitrary function of $\sqrt{-\lambda}$. Similarly, these two kinds of negatons are equivalent to each other.

When $\lambda_1 < 0$, a 1-negaton of first-order reads

$$u = 2\partial_x \ln W \left(\cosh \left(\theta_1 \right), \partial_{\lambda_1} \cosh \left(\theta_1 \right) \right)$$

$$= \frac{4\sqrt{-\lambda_1} \left(1 + \cosh \left(2\theta_1 \right) \right)}{2\sqrt{-\lambda_1} \left(x + ky + \sqrt{2}z - 4t\lambda_1 \right) + \sinh \left(2\theta_1 \right)}, \tag{50}$$

$$u = 2\partial_x \ln W \left(\cosh \left(\theta_1 \right), \partial_{\lambda_1} \cosh \left(\theta_1 \right), \partial_{\lambda_1}^2 \cosh \left(\theta_1 \right) \right)$$

$$= \frac{\left(-6\lambda_1 \delta_1 + 48t\lambda_1^2 \right) \sinh \left(\theta_1 \right) - 4\lambda_1 \sqrt{-\lambda_1} \delta_1^2 \cosh \left(\theta_1 \right) + 6\sqrt{-\lambda_1} \sinh^2 \left(\theta_1 \right) \cosh \left(\theta_1 \right)}{-2\lambda_1 \delta_1^2 \sinh \left(\theta_1 \right) + \left(24t\lambda_1 \sqrt{-\lambda_1} - \sqrt{-\lambda_1} \delta_1 \right) \cosh \left(\theta_1 \right) + \sinh \left(\theta_1 \right) \cosh^2 \left(\theta_1 \right)}, \tag{51}$$

where $\delta_1 = x + ky + \sqrt{2}z - 12t\lambda_1$. Figures 5 and 6 of three-dimensional plots show the special negatons to (1) defined by (48) on the indicated specific regions, with specific values being chosen for the parameters.

3.3. Interaction Solutions.
We are now presenting examples of Wronskian interaction solutions among different kinds of Wronskian solutions to the (3+1)-dimensional generalized BKP equation (1).

Let us assume that there are two sets of eigenfunctions

$$\phi_1 \left(\lambda \right), \phi_2 \left(\lambda \right), \ldots, \phi_l \left(\lambda \right);$$

$$\psi_1 \left(\mu \right), \psi_2 \left(\mu \right), \ldots, \psi_m \left(\mu \right) \tag{52}$$

associated with two different eigenvalues λ and μ, respectively. A Wronskian solution

$$u = 2\partial_x \ln W$$

$$\cdot \left(\phi_1 \left(\lambda \right), \phi_2 \left(\lambda \right), \ldots, \phi_l \left(\lambda \right); \psi_1 \left(\mu \right), \psi_2 \left(\mu \right), \ldots, \psi_m \left(\mu \right) \right) \tag{53}$$

is said to be a Wronskian interaction solution between two solutions determined by the two sets of eigenfunctions in (52).

In what follows, we would like to show a few special Wronskian interaction solutions. Let us first choose different sets of eigenfunctions:

$$\phi_{\text{rational}} = x + ky + \sqrt{2}z,$$

$$\phi_{\text{soliton}} = \cosh \left(\sqrt{-\lambda_1} \left(x + ky + \sqrt{2}z - 4t\lambda_1 \right) + \gamma_1 \right), \tag{54}$$

$$\phi_{\text{positon}} = \cos \left(\sqrt{\lambda_2} \left(x + ky + \sqrt{2}z - 4t\lambda_2 \right) + \gamma_2 \right),$$

where $\lambda_1 < 0$, $\lambda_2 > 0$, and γ_i ($i = 1, 2$) are arbitrary real constants.

where $\theta_1 = \sqrt{-\lambda_1}(x + ky + \sqrt{2}z - 4t\lambda_1) + \gamma_1$. And the 1-negaton of second-order is given by

Through three Wronskian interaction solutions between any two of a rational solution, a single soliton and a single positon read as

$$u_{\text{rs}} = 2\partial_x \ln W \left(\phi_{\text{rational}}, \phi_{\text{soliton}} \right)$$

$$= \frac{-2\lambda_1 \left(x + ky + \sqrt{2}z \right) \cosh \left(\theta_1 \right)}{\sqrt{-\lambda_1} \left(x + ky + \sqrt{2}z \right) \sinh \left(\theta_1 \right) - \cosh \left(\theta_1 \right)},$$

$$u_{\text{rp}} = 2\partial_x \ln W \left(\phi_{\text{rational}}, \phi_{\text{positon}} \right)$$

$$= \frac{2\lambda_2 \left(x + ky + \sqrt{2}z \right) \cos \left(\theta_2 \right)}{\sqrt{\lambda_2} \left(x + ky + \sqrt{2}z \right) \sin \left(\theta_2 \right) + \cos \left(\theta_2 \right)}, \tag{55}$$

$$u_{\text{sp}} = 2\partial_x \ln W \left(\phi_{\text{soliton}}, \phi_{\text{positon}} \right)$$

$$= \frac{2 \left(\lambda_2 - \lambda_1 \right) \cosh \left(\theta_1 \right) \cos \left(\theta_2 \right)}{\sqrt{-\lambda_1} \sinh \left(\theta_1 \right) \cos \left(\theta_2 \right) + \sqrt{\lambda_2} \cosh \left(\theta_1 \right) \sin \left(\theta_2 \right)},$$

where $\theta_1 = \sqrt{-\lambda_1}(x + ky + \sqrt{2}z - 4t\lambda_1) + \gamma_1$ and $\theta_2 = \sqrt{\lambda_2}(x + ky + \sqrt{2}z - 4t\lambda_2) + \gamma_2$.

One Wronskian interaction solution involving the three eigenfunctions is given by

$$u_{\text{rsp}} = 2\partial_x \ln W \left(\phi_{\text{rational}}, \phi_{\text{soliton}}, \phi_{\text{positon}} \right) = \frac{2q}{p}, \tag{56}$$

where

$$p = \left(x + ky + \sqrt{2}z \right) \left(\lambda_2 \sqrt{-\lambda_1} \sinh \left(\theta_1 \right) \cos \left(\theta_2 \right) \right.$$

$$\left. + \lambda_1 \sqrt{\lambda_2} \cosh \left(\theta_1 \right) \sin \left(\theta_2 \right) \right) + \left(\lambda_1 - \lambda_2 \right) \cosh \left(\theta_1 \right)$$

$$\cdot \cos \left(\theta_2 \right),$$

$$q = \left(x + ky + \sqrt{2}z \right) \sqrt{-\lambda_1 \lambda_2} \left(\lambda_1 - \lambda_2 \right) \sinh \left(\theta_1 \right)$$

$$\cdot \sin \left(\theta_2 \right) + \lambda_1 \sqrt{-\lambda_1} \sinh \left(\theta_1 \right) \cos \left(\theta_2 \right) + \lambda_2 \sqrt{\lambda_2}$$

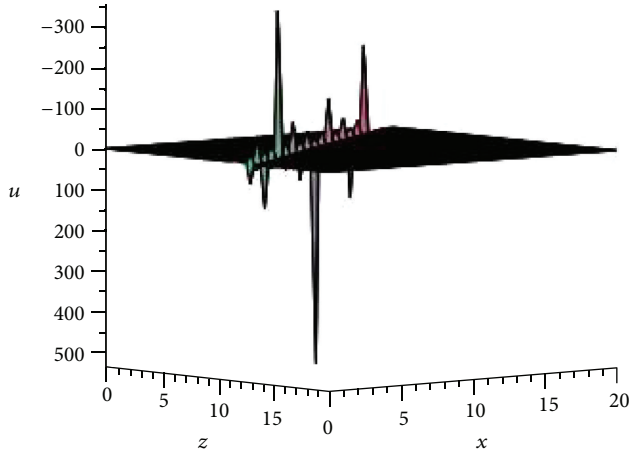$$\cdot \cosh \left(\theta_1 \right) \sin \left(\theta_2 \right),$$

FIGURE 4: The shape of the 1-positon of first-order to (1) with $\lambda_1 = 1.5$, $y = 0$, $t = 1$, $\gamma_3 = 0$.

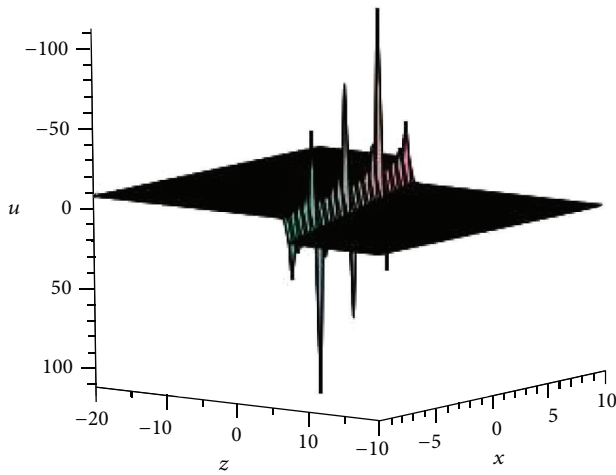FIGURE 6: The shape of the 1-negaton of second-order to (1) with $\lambda_1 = -1$, $y = 0$, $t = 0$, $\gamma_1 = 0$.

FIGURE 5: The shape of the 1-negaton of first-order to (1) with $\lambda_1 = -4$, $y = 0$, $t = 0$, $\gamma_1 = 0$.

$$\theta_1 = \sqrt{-\lambda_1}\left(x + ky + \sqrt{2}z - 4t\lambda_1\right) + \gamma_1,$$

$$\theta_2 = \sqrt{\lambda_2}\left(x + ky + \sqrt{2}z - 4t\lambda_2\right) + \gamma_2.$$

$$(57)$$

Of course, we have more general Wronskian interaction solutions among three or more kinds of solutions such as rational solutions, positons, solitons, breathers, and negatons. Roughly speaking, it increases the complexities of rational solutions, positons, solitons, and negatons, respectively, to add zero, positive, negative eigenvalues to the spectrum of the coefficient matrix.

4. Conclusion

In summary we have extended the Wronskian method to a (3+1)-dimensional generalized BKP equation by its bilinear form. Moreover, we obtained some rational solutions, solitons, positons, negatons, and their interaction solutions to
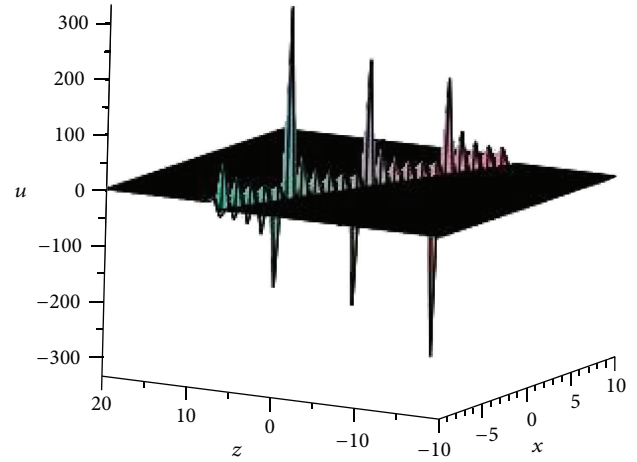
this equation by solving the systems of linear partial differential equations. All these show the richness of the solution space of the (3+1)-dimensional generalized BKP equation and the resulting solutions are expected to help understand wave dynamics in weakly nonlinear and dispersive media.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

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