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Research Article

Sharp One-Parameter Mean Bounds for Yang Mean

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We prove that the double inequality $J_\alpha(a,b) < U(a,b) < J_\beta(a,b)$ holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha \le \sqrt{2}/(\pi-\sqrt{2}) = 0.8187\cdots$ and $\beta \ge 3/2$, where $U(a,b)=(a-b)/[\sqrt{2}\arctan((a-b)/\sqrt{2ab})]$, and $J_p(a,b)=p(a^{p+1}-b^{p+1})/[(p+1)(a^p-b^p)]$ $(p\neq 0,-1)$, $J_0(a,b)=(a-b)/(\log a-\log b)$, and $J_{-1}(a,b)=ab(\log a-\log b)/(a-b)$ are the Yang and pth one-parameter means of a and b, respectively.

1. Introduction

Let $p\in\mathbb{R}$ and $a,b>0$ with $a\neq b$. Then the pth one-parameter mean $J_p(a,b)$, pth power mean $M_p(a,b)$, harmonic mean $H(a,b)$, geometric mean $G(a,b)$, logarithmic mean $L(a,b)$, first Seiffert mean $P(a,b)$, identric mean $I(a,b)$, arithmetic mean $A(a,b)$, Yang mean $U(a,b)$, second Seiffert mean $T(a,b)$, and quadratic mean $Q(a,b)$ are, respectively, defined by

$$J_p(a,b)=\begin{cases}\dfrac{p\left(a^{p+1}-b^{p+1}\right)}{(p+1)\left(a^p-b^p\right)}, & p\neq 0,-1,\\[2mm] \dfrac{a-b}{\log a-\log b}, & p=0,\\[2mm] \dfrac{ab(\log a-\log b)}{a-b}, & p=-1,\end{cases}$$

$$M_p(a,b)=\left[\frac{a^p+b^p}{2}\right]^{1/p}\quad (p\neq 0),$$

$$M_0(a,b)=\sqrt{ab},$$

$$H(a,b)=\frac{2ab}{a+b},$$

$$G(a,b)=\sqrt{ab},$$

$$L(a,b)=\frac{b-a}{\log b-\log a},$$

$$P(a,b)=\frac{a-b}{2\arcsin((a-b)/(a+b))},$$

$$I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},$$

$$A(a,b)=\frac{a+b}{2},$$

$$U(a,b)=\frac{a-b}{\sqrt{2}\arctan\left((a-b)/\sqrt{2ab}\right)},$$

$$T(a,b)=\frac{a-b}{2\arctan((a-b)/(a+b))},$$

$$Q(a,b)=\sqrt{\frac{a^2+b^2}{2}}. \tag{1}$$

It is well known that both the means $J_p(a,b)$ and $M_p(a,b)$ are continuous and strictly increasing with respect to $p\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$. Recently, the one-parameter mean $J_p(a,b)$ and Yang mean $U(a,b)$ have attracted the attention of many researchers.

Alzer [1] proved that the inequalities

$$G(a,b) < \sqrt{J_p(a,b)J_{-p}(a,b)} < L(a,b) < \frac{J_p(a,b)+J_{-p}(a,b)}{2} < A(a,b) \tag{2}$$

hold for all $a,b>0$ with $a\neq b$ and $p\neq 0$.

In [2, 3], the authors discussed the monotonicity and logarithmic convexity properties of the one-parameter mean $J_p(a,b)$.

In [4, 5], the authors proved that the double inequalities

$$\begin{aligned} J_{p_1}(a,b) &< \alpha A(a,b)+(1-\alpha)L(a,b) < J_{q_1}(a,b) \\ J_{p_2}(a,b) &< \alpha A(a,b)+(1-\alpha)H(a,b) < J_{q_2}(a,b), \end{aligned} \tag{3}$$

hold for all $a,b>0$ with $a\neq b$ and $\alpha\in(0,1)$ if and only if $p_1\le \alpha/(2-\alpha)$, $q_1\ge\alpha$, $p_2\le 3\alpha-2$, and $q_2\ge \alpha/(2-\alpha)$.

Xia et al. [6] proved that the double inequality

$$J_{(3\alpha-1)/2}(a,b) < \alpha A(a,b)+(1-\alpha)G(a,b) < J_{\alpha/(2-\alpha)}(a,b) \tag{4}$$

holds for all $a,b>0$ with $a\neq b$ if $\alpha\in(0,2/3)$, and inequality (4) is reversed if $\alpha\in(2/3,1)$.

Gao and Niu [7] presented the best possible parameters p and q such that the double inequality $J_p(a,b) < A^{\alpha}(a,b)G^{\beta}(a,b)H^{1-\alpha-\beta}(a,b) < J_q(a,b)$ holds for all $a,b>0$ with $a\neq b$ and $\alpha+\beta\in(0,1)$.

In [8, 9], the authors proved that the double inequalities

$$\begin{aligned} J_{\lambda_1}(a,b) &< T(a,b) < J_{\mu_1}(a,b), \\ J_{\lambda_2}(a,b) &< I(a,b) < J_{\mu_2}(a,b) \end{aligned} \tag{5}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\lambda_1\le 2/(2-\pi)$, $\mu_1\ge 2$, $\lambda_2\le 1/2$, and $\mu_2\ge 1/(e-1)$.

Xia et al. [10] found that $M_{(1+2p)/3}(a,b)$ is the best possible lower power mean bound for the one-parameter mean $J_p(a,b)$ if $p\in(-2,-1/2)\cup(1,\infty)$ and $M_{(1+2p)/3}(a,b)$ is the best possible upper power mean bound for the one-parameter mean $J_p(a,b)$ if $p\in(-\infty,-2)\cup(-1/2,1)$.

For all $a,b>0$ with $a\neq b$, Yang [11] provided the bounds for the Yang mean $U(a,b)$ in terms of other bivariate means as follows:

$$\begin{aligned} & P(a,b) < U(a,b) < T(a,b), \\ & \frac{G(a,b)T(a,b)}{A(a,b)} < U(a,b) < \frac{P(a,b)Q(a,b)}{A(a,b)}, \\ & Q^{1/2}(a,b)\left[\frac{2G(a,b)+Q(a,b)}{3}\right]^{1/2} < U(a,b) < Q^{2/3}(a,b)\left[\frac{G(a,b)+Q(a,b)}{2}\right]^{1/3}, \\ & \frac{G(a,b)+Q(a,b)}{2} < U(a,b) < \left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^{1/2}+\frac{1}{3}Q^{1/2}(a,b)\right]^2. \end{aligned} \tag{6}$$

In [12, 13], the authors proved that the double inequalities

$$\begin{aligned} & \left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^p+\frac{1}{3}Q^p(a,b)\right]^{1/p} < U(a,b) < \left[\frac{2}{3}\left(\frac{G(a,b)+Q(a,b)}{2}\right)^q+\frac{1}{3}Q^q(a,b)\right]^{1/q} \\ & \frac{2^{1-\lambda}(G(a,b)+Q(a,b))^{\lambda}Q(a,b)+G(a,b)Q^{\lambda}(a,b)}{2^{1-\lambda}(G(a,b)+Q(a,b))^{\lambda}+Q^{\lambda}(a,b)} < U(a,b) < \frac{2^{1-\mu}(G(a,b)+Q(a,b))^{\mu}Q(a,b)+G(a,b)Q^{\mu}(a,b)}{2^{1-\mu}(G(a,b)+Q(a,b))^{\mu}+Q^{\mu}(a,b)}, \\ & M_{\alpha}(a,b) < U(a,b) < M_{\beta}(a,b), \end{aligned} \tag{7}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $p\le p_0$, $q\ge 1/5$, $\lambda\ge 1/5$, $\mu\le p_1$, $\alpha\le 2\log 2/(2\log\pi-\log 2)$, and $\beta\ge 4/3$, where $p_0=0.1941\cdots$ is the unique solution of the equation $p\log(2/\pi)-\log(1+2^{1-p})+\log 3=0$ on the interval $(1/10,\infty)$, and $p_1=\log(\pi-2)/\log 2=0.1910\cdots$.

Very recently, Zhou et al. [14] proved that $\alpha=1/2$ and $\beta=\log 3/(1+\log 2)=0.6488\cdots$ are the best possible parameters such that the double inequality

$$\left[\frac{a^{\alpha}+(ab)^{\alpha/2}+b^{\alpha}}{3}\right]^{1/\alpha} < U(a,b) < \left[\frac{a^{\beta}+(ab)^{\beta/2}+b^{\beta}}{3}\right]^{1/\beta} \tag{8}$$

holds for all $a,b>0$ with $a\neq b$.

The aim of this paper is to present the best possible parameters α and β such that the double inequality $J_{\alpha}(a,b) < U(a,b) < J_{\beta}(a,b)$ holds for all $a,b>0$ with $a\neq b$.

2. Main Result

In order to prove our main result we need a lemma, which we present in this section.

Lemma 1. *Let $p\in\mathbb{R}$, and*

$$\begin{aligned} f(x,p) = {} & px^{4p+6}-(p+1)x^{4p+5}+px^{4p+4} \\ & -(p+1)x^{4p+1}-p(p+1)x^{2p+7} \\ & +2(p+1)^2x^{2p+5}-2px^{2p+4} \\ & -2p(p+1)x^{2p+3}-2px^{2p+2} \\ & +2(p+1)^2x^{2p+1}-p(p+1)x^{2p-1} \\ & -(p+1)x^5+px^2-(p+1)x+p. \end{aligned} \tag{9}$$

Then the following statements are true:

(1) *if $p=3/2$, then $f(x,p)>0$ for all $x\in(1,\infty)$;*

(2) *if* $p = \sqrt{2}/(\pi - \sqrt{2}) = 0.8187\cdots$, *then there exists* $\lambda \in (1,\infty)$ *such that* $f(x,p) < 0$ *for* $x \in (1,\lambda)$ *and* $f(x,p) > 0$ *for* $x \in (\lambda,\infty)$.

Proof. For part (1), if $p = 3/2$, then (9) becomes

$$f(x,p) = \frac{1}{4}(x-1)^6\left(x^2+2x+2\right)\left(2x^2+2x+1\right) \cdot \left(3x^2+4x+3\right). \tag{10}$$

Therefore, part (1) follows from (10).

For part (2), let $p = \sqrt{2}/(\pi-\sqrt{2})$, $f_1(x,p) = \partial f(x,p)/\partial x$, $f_2(x,p) = (1/2)(\partial f_1(x,p)/\partial x)$, $f_3(x,p) = (1/(p+1)x^2)(\partial f_2(x,p)/\partial x)$, $f_4(x,p) = (x^{7-2p}/2p)(\partial f_3(x,p)/\partial x)$, $f_5(x,p) = (1/2x)(\partial f_4(x,p)/\partial x)$, $f_6(x,p) = \partial f_5(x,p)/\partial x$, $f_7(x,p) = (1/2)(\partial f_6(x,p)/\partial x)$, $f_8(x,p) = \partial f_7(x,p)/\partial x$, $f_9(x,p) = (1/2(p+1))(\partial f_8(x,p)/\partial x)$, and $f_{10}(x,p) = \partial f_9(x,p)/\partial x$. Then elaborated computations lead to

$$\lim_{x\to 1} f(x,p) = 0, \quad \lim_{x\to+\infty} f(x,p) = +\infty, \tag{11}$$

$$\lim_{x\to 1} f_1(x,p) = 0, \quad \lim_{x\to+\infty} f_1(x,p) = +\infty, \tag{12}$$

$$\lim_{x\to 1} f_2(x,p) = 0, \quad \lim_{x\to+\infty} f_2(x,p) = +\infty, \tag{13}$$

$$\lim_{x\to 1} f_3(x,p) = 0, \quad \lim_{x\to+\infty} f_3(x,p) = +\infty, \tag{14}$$

$$\lim_{x\to 1} f_4(x,p) = -48(p+1)\left(\frac{3}{2}-p\right) < 0, \quad \lim_{x\to+\infty} f_4(x,p) = +\infty, \tag{15}$$

$$\lim_{x\to 1} f_5(x,p) = -192(p+1)^2\left(\frac{3}{2}-p\right) < 0, \quad \lim_{x\to+\infty} f_5(x,p) = +\infty, \tag{16}$$

$$\lim_{x\to 1} f_6(x,p) = 2(p+1)\left(368p^3+332p^2-484p-963\right) < 0, \tag{17}$$

$$\lim_{x\to+\infty} f_6(x,p) = +\infty, \tag{18}$$

$$\lim_{x\to 1} f_7(x,p) = (p+1)\left(1024p^4+2096p^3+1844p^2-2876p-5193\right) < 0, \tag{19}$$

$$\lim_{x\to+\infty} f_7(x,p) = +\infty, \tag{20}$$

$$\lim_{x\to 1} f_8(x,p) = (p+1)\left(2560p^5+6336p^4+14176p^3+11028p^2-12680p-22005\right) < 0, \tag{21}$$

$$\lim_{x\to+\infty} f_8(x,p) = +\infty, \tag{22}$$

$$\lim_{x\to 1} f_9(x,p) = 6\left(512p^6+1184p^5+4064p^4+6372p^3+4068p^2-3495p-5775\right) = 15.2085\cdots > 0, \tag{23}$$

$$\begin{aligned} f_{10}(x,p) ={}& (p+2)(2p+1)(2p+3)^2(2p+5)(2p+7)(4p+1)(4p+5)x^{2p} - 8p(p+1)(p+2)(p+3)(2p+1)(2p+3)(4p+3)(4p+5)x^{2p-1} \\ &+ p(2p-1)(2p+1)^2(2p+3)(2p+5)(4p-1)\cdot(4p+3)x^{2p-2} - 8p(p-1)^2(p-2)(2p-1)(2p-3)\left(16p^2-1\right)x^{2p-5} - 720(p+3)(2p+5)(2p+7)x. \end{aligned} \tag{24}$$

Note that

$$\begin{aligned} &2p > 1 > 2p-1 > 0 > 2p-2 > 2p-5, \\ &1536p^7+15040p^6+59440p^5+122280p^4+137144p^3+61850p^2-49845p-72450 = 85165.4405\cdots > 0. \end{aligned} \tag{25}$$

It follows from (24) and (25) that

$$\begin{aligned} f_{10}(x,p) >{}& \big[(p+2)(2p+1)(2p+3)^2(2p+5)\cdot(2p+7)(4p+1)(4p+5) - 8p(p+1)(p+2)\cdot(p+3)(2p+1)(2p+3)(4p+3)(4p+5) \\ &- 720(p+3)(2p+5)(2p+7)\big]x + \big[p(2p-1)\cdot(2p+1)^2(2p+3)(2p+5)(4p-1)(4p+3) - 8p(p-1)^2(p-2)(2p-1)(2p-3)\left(16p^2-1\right)\big] \\ &\cdot x^{2p-2} = \left(1536p^7+15040p^6+59440p^5+122280p^4+137144p^3+61850p^2-49845p-72450\right)x \\ &+ p(2p-1)(4p-1)\left(704p^4+136p^3+1120p^2+248p-3\right)x^{2p-2} > 0, \end{aligned} \tag{26}$$

for $x \in (1,\infty)$.

From (23) and (26) we clearly see that $f_8(x,p)$ is strictly increasing with respect to x on the interval $(1,\infty)$. Then (21) and (22) lead to the conclusion that there exists $\lambda_1 > 1$ such that the function $x \to f_7(x,p)$ is strictly decreasing on $(1,\lambda_1]$ and strictly increasing on $[\lambda_1,\infty)$.

It follows from (19) and (20) together with the piecewise monotonicity of the function $x \to f_7(x,p)$ that there exists $\lambda_2 > 1$ such that the function $x \to f_6(x,p)$ is strictly decreasing on $(1,\lambda_2]$ and strictly increasing on $[\lambda_2,\infty)$.

Making use of (13)–(18) and the same method as the above we know that there exists $\lambda_i > 1$ $(i = 3,4,5,6,7)$ such that the function $x \to f_{8-i}(x,p)$ is strictly decreasing on $(1,\lambda_i]$ and strictly increasing on $[\lambda_i,\infty)$.

It follows from (12) and the piecewise monotonicity of the function $x \to f_1(x,p)$ that there exists $\lambda^* > 1$ such that the function $x \to f(x,p)$ is strictly decreasing on $(1,\lambda^*]$ and strictly increasing on $[\lambda^*,\infty)$.

Therefore, part (2) follows easily from (11) and the piecewise monotonicity of the function $x \to f(x,p)$. □

Theorem 2. *The double inequality*

$$J_\alpha(a,b) < U(a,b) < J_\beta(a,b) \tag{27}$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\alpha \leq \sqrt{2}/(\pi-\sqrt{2}) = 0.8187\cdots$ and $\beta \geq 3/2$.

Proof. Since $U(a,b)$ and $J_p(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a = x^2 > 1$ and $b = 1$. Let $p \in \mathbb{R}$ and $p \neq 0,-1$. Then (1) lead to

$$\begin{aligned} J_p(a,b) - U(a,b) &= J_p(x^2,1) - U(x^2,1) \\ &= \frac{p(x^{2p+2}-1)}{(p+1)(x^{2p}-1)\arctan\left((x^2-1)/\sqrt{2}x\right)}F(x,p), \end{aligned} \tag{28}$$

where

$$F(x,p) = \arctan\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{(p+1)(x^2-1)(x^{2p}-1)}{\sqrt{2}p(x^{2p+2}-1)}, \tag{29}$$

$$\lim_{x\to 1} F(x,p) = 0, \tag{30}$$

$$\frac{\partial F(x,p)}{\partial x} = \frac{\sqrt{2}}{p(x^4+1)(x^{2p+2}-1)^2} f(x,p), \tag{31}$$

where $f(x,p)$ is defined by (9).

We divide the proof into four cases.

Case 1 ($p = \sqrt{2}/(\pi-\sqrt{2})$). Then it follows from Lemma 1(2), (29), and (31) that there exists $\lambda > 1$ such that the function $x \to F(x,p)$ is strictly decreasing on $(1,\lambda]$ and strictly increasing on $[\lambda,\infty)$, and

$$\lim_{x\to\infty} F(x,p) = 0. \tag{32}$$

Therefore,

$$J_{\sqrt{2}/(\pi-\sqrt{2})}(a,b) < U(a,b) \tag{33}$$

follows easily from (28), (30), and (32) together with the piecewise monotonicity of the function $x \to F(x,p)$.

Case 2 ($p > \sqrt{2}/(\pi-\sqrt{2})$). Then (1) leads to

$$\lim_{x\to\infty} \frac{J_p(x,1)}{U(x,1)} = \frac{\sqrt{2}p}{2(p+1)}\pi > 1. \tag{34}$$

Inequality (34) implies that there exists large enough $X = X(p) > 1$ such that $U(a,b) < J_p(a,b)$ for all $a,b > 0$ with $a/b \in (0,1/X) \cup (X,\infty)$.

Case 3 ($p = 3/2$). Then from Lemma 1(1) and (31) we know that the function $x \to F(x,p)$ is strictly increasing on the interval $(1,\infty)$. Therefore,

$$U(a,b) < J_{3/2}(a,b) \tag{35}$$

follows from (28) and (30) together with the monotonicity of the function $x \to F(x,p)$.

Case 4 ($0 < p < 3/2$). Let $x > 0$ and $x \to 0$; then making use of Taylor expansion we get

$$\begin{aligned} &U(1,1+x) - J_p(1,1+x) \\ &= \frac{x}{\sqrt{2}\arctan\left(x/\sqrt{2(1+x)}\right)} \\ &\quad - \frac{p\left[1-(1+x)^{p+1}\right]}{(p+1)\left[1-(1+x)^p\right]} = \frac{3-2p}{24}x^2 + o(x^2). \end{aligned} \tag{36}$$

Equation (36) implies that there exists small enough $\delta > 0$ such that $U(1,1+x) > J_p(1,1+x)$ for all $x \in (0,\delta)$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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