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Research Article

Antiplane Problem of Periodically Stacked Parallel Cracks in an Infinite Orthotropic Plate

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The antiplane problem of the periodic parallel cracks in an infinite linear elastic orthotropic composite plate is studied in this paper. The antiplane problem is turned into the boundary value problem of partial differential equation. By constructing proper Westergaard stress function and using the periodicity of the hyperbolic function, the antiplane problem of the periodic parallel cracks degenerates into an algebra problem. Using the complex variable function method and the undetermined coefficients method, as well as with the help of boundary conditions, the boundary value problem of partial differential equation can be solved, and the analytic expressions for stress intensity factor, stress, and displacement near the periodical parallel cracks tip are obtained. When the cracks spacing tends to infinity, the antiplane problem of the periodic parallel cracks degenerates into the case of the antiplane problem of a single central crack.

1. Introduction

Composite materials are a very promising class of structural materials and widely used in many fields. Defects in the composite materials are easier to cause singular stress and cracks. However, periodic crack is the important model to study the problem of multiple cracks. For simplicity, we can consider the agminate cracks as periodic cracks ideally. And the research on periodic cracks problem contributes to making an intensive understanding of failure mechanism of composite materials; therefore, it is very important to study the periodic cracks problem.

Over the past few decades, the antiplane problem of periodic cracks was investigated by many researchers. For example, by using Fourier transforms method, Erdogan, Ozturk, Chen, and Ding [1–4] studied the antiplane problem in functionally graded materials containing a periodic array of collinear cracks. By using Laplace transform and Fourier transform, Wang and Mai [5] analyzed the dynamic antiplane problem of periodic parallel cracks in an infinite functionally graded material, and the stress intensity factors were obtained. By using distributed dislocation method, Pak and Goloubeva [6] studied the antiplane problem in piezoelectric materials containing a periodic array of parallel cracks. The stress and the electric displacement intensity factors were obtained. By using complex variable function method, Tong et al. [7] studied the antiplane problem in piezoelectric materials containing a doubly periodic cracks of unequal size, and a closed form solution of stress intensity factor was obtained. By using the method of conformal mapping, Hao and Wu [8, 9] considered the antiplane problem on parallel periodical cracks of finite length starting from the interface of two half-planes, and the stress intensity factor was obtained. By using the complex variable function method and the undetermined coefficients method, Lekhnitskii [10] studied the antiplane problem of collinear periodic cracks in an infinite orthotropic fiber reinforcement composite plate, and the analytic expressions for stress intensity factors, stress field, and displacement field of the collinear periodic cracks tip were achieved.

The antiplane problem of the periodic parallel cracks in an infinite linear elastic orthotropic composite plate is studied in this paper. The antiplane problem is turned into the boundary value problem of partial differential equation. By constructing proper Westergaard stress function and using the periodicity of the hyperbolic function, the antiplane problem of the periodic parallel cracks degenerates into an algebra problem. The analytic expressions for stress intensity factor, stress, and displacement near the periodical parallel cracks tip are obtained.

2. Mechanical Model

As seen in Figure 1, we consider an infinite linear elastic orthotropic composite plate with periodic parallel cracks of mode III. The crack length is $2a$, the crack spacing is ω, and the antiplane shear force is τ.

The relations between the strain and the stress are as follows [10]:

$$\tau_{yz} = Q_{44}\gamma_{yz}, \qquad \tau_{zx} = Q_{55}\gamma_{zx}, \qquad \gamma_{yz} = \frac{\partial w}{\partial y}, \qquad \gamma_{zx} = \frac{\partial w}{\partial x}, \tag{1}$$

where Q_{44} and Q_{55} are the principal directions of the elasticity, w is the displacement, γ_{yz} and γ_{zx} are the strain, and τ_{yz} and τ_{zx} are the stress.

The balancing equation is

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0. \tag{2}$$

Substituting (1) into (2), the governing equation of the antiplane problem can be obtained as follows:

$$Q_{55}\frac{\partial^2 w}{\partial x^2} + Q_{44}\frac{\partial^2 w}{\partial y^2} = 0. \tag{3}$$

As can be seen in Figure 1, the boundary conditions of the periodic parallel cracks of mode III are as follows:

$$y \longrightarrow \infty : \tau_{yz} = \tau, \qquad -a < x < a, \quad y = n\omega \quad (n = 0, \pm 1, \pm 2, \ldots) : \quad \tau_{yz} = 0. \tag{4}$$

An analysis of antiplane problem near periodic parallel cracks tip can be turned to find the solution of the boundary value problem of partial differential equations (3) and (4).

The displacement is

$$w = U\left(x + s\left(y - n\omega\right)\right) \quad (n = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

Substituting (5) into (3), the characteristic is obtained [11]:

$$Q_{44}s^2 + Q_{55} = 0. \tag{6}$$

The solutions of the characteristic equation (6) can be set [11] as

$$s_1 = i\sqrt{\frac{Q_{55}}{Q_{44}}} = i\beta \qquad s_2 = -i\beta. \tag{7}$$

Let

$$z_1 = x + s_1\left(y - n\omega\right) + in\omega = x_1 + iy_1, \tag{8}$$

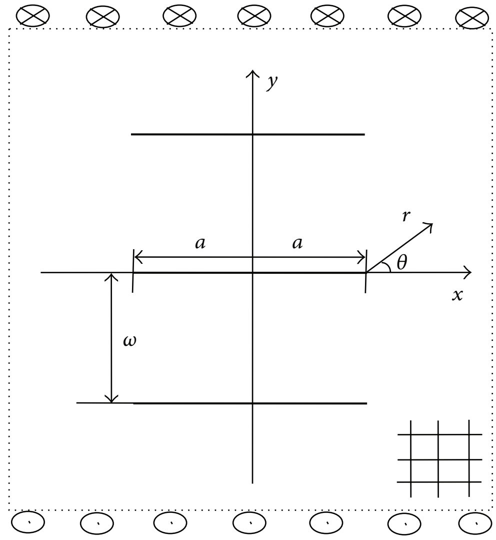

Figure 1: Orthotropic plate with periodic parallel cracks of mode III.

where

$$x_1 = x, \qquad y_1 = \beta\left(y - n\omega\right) + n\omega. \tag{9}$$

By using formula (8), we can know that the governing equation (3) can be rewritten as a generalized bi-harmonic equation:

$$\nabla_1^2 U = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right) U = 0. \tag{10}$$

By the theory of complex variable, the solution for partial differential equation (3) may be chosen as [11]

$$w = U = a_1 \operatorname{Re}\left(\overline{U}_1\right) + b_1 \operatorname{Im}\left(\overline{U}_1\right), \tag{11}$$

where a_1 and b_1 are undetermined real parameters, U_1 is an analytic function of z_1, and

$$\frac{d\overline{U}_1}{dz_1} = U_1 = U_1\left(z_1\right). \tag{12}$$

Substituting (11) and (12) into (1), the stress expressions can be written as [11]

$$\tau_{yz} = Q_{44}\beta\left[-a_1 \operatorname{Im}\left(U_1\right) + b_1 \operatorname{Re}\left(U_1\right)\right], \qquad \tau_{zx} = Q_{55}\left[b_1 \operatorname{Im}\left(U_1\right) + a_1 \operatorname{Re}\left(U_1\right)\right]. \tag{13}$$

3. Westergaard Stress Function

Considering the boundary value problem of partial differential equations (3) and (4), we can select the Westergaard stress function as follows [12]:

$$U_1\left(z_1\right) = \frac{\tau \operatorname{sech}\left(2\pi\left(a + n\omega i\right)/\omega\right) \cdot \tanh\left(2\pi z_1/\omega\right)}{\sqrt{\tanh^2\left(2\pi z_1/\omega\right) - \tanh^2\left(2\pi\left(a + n\omega i\right)/\omega\right)}}, \tag{14}$$

when $y \rightarrow +\infty : U_1(z_1) = \tau$.

When $-a < x < a$, $y = n\omega$ $(n = 0, \pm 1, \pm 2, \ldots)$,

$$U_1(z_1) = \frac{-\tau \operatorname{sech}(2\pi a/\omega) \cdot \tanh(2\pi x/\omega)}{\sqrt{\tanh^2(2\pi a/\omega) - \tanh^2(2\pi x/\omega)}} \cdot i. \tag{15}$$

Substituting (13) and (15) into boundary conditions (4), the unique solution can be derived as follows:

$$a_1 = 0, \qquad b_1 = \frac{1}{\beta Q_{44}}. \tag{16}$$

Therefore, by substituting (14) and (16) into (11), we can obtain the real analytic solution w, which meets the governing equation (3) and the boundary conditions (4).

4. Stress Intensity Factor

According to the distribution of cracks and the loadings of orthotropic composite plate, we select the stress intensity factor as follows [13]:

$$K_{\mathrm{III}} = \lim_{z_1 \to a + n\omega i} \left[2\pi\left(z_1 - (a + n\omega i)\right)\right]^{1/2} U_1(z_1). \tag{17}$$

Substituting (14) into (17), we obtain

$$K_{\mathrm{III}} = \tau \sqrt{\frac{\omega \cdot \tanh(2\pi a/\omega)}{2}} = Y\tau\sqrt{\pi a}, \tag{18}$$

where $Y = \sqrt{(\omega/2\pi a) \cdot \tanh(2\pi a/\omega)}$ is called the shape factor and the stress intensity factor depends on Y. Labeling $K'_{\mathrm{III}} = \tau\sqrt{\pi a}$, K'_{III} is the stress intensity factor of a single central crack.

Suppose $\tau = 1$ and $a = 2$. As seen in Figure 2, the stress intensity factor K_{III} and the shape factor Y increase rapidly with the increase in the distance between cracks and then reach a steady state. When $\omega \to \infty$, $Y = \sqrt{(\omega/2\pi a) \cdot \tanh(2\pi a/\omega)} \to 1$ and $K_{\mathrm{III}} \to K'_{\mathrm{III}}$. In other words, when $\omega \to \infty$, the antiplane problem of periodic parallel cracks turns into the antiplane problem of a single central crack, and it is entirely consistent with the previous results.

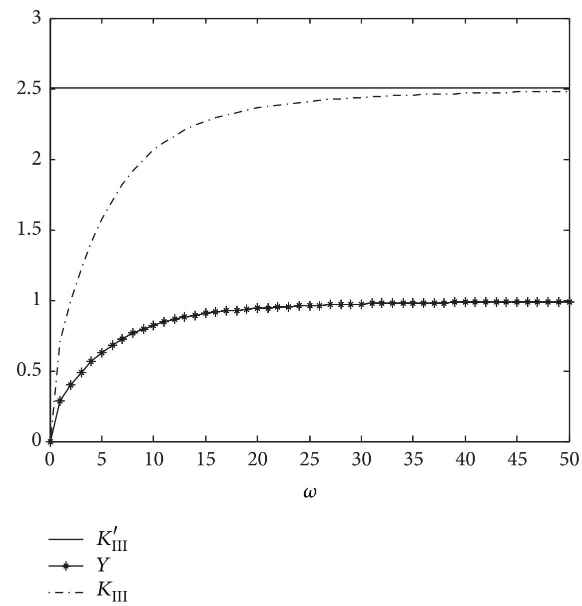

Figure 2: Variation curves of K'_{III}, Y, and K_{III} with crack spacing ω.

5. Stress Field and Displacement Field Near Crack Tip

According to (17), in the vicinity of the periodic parallel cracks tip, we can obtain [11]

$$U_1(z_1) = \frac{K_{\mathrm{III}}}{\sqrt{2\pi\left(z_1 - (a + n\omega i)\right)}}, \quad (n = 0, \pm 1, \pm 2, \ldots), \quad \text{as } z_1 \longrightarrow a + n\omega i. \tag{19}$$

Let

$$z_1 - (a + n\omega i) = r\left(\cos\theta + s_1 \sin\theta\right). \tag{20}$$

So

$$U_1(z_1) = \frac{K_{\mathrm{III}}}{\sqrt{2\pi r\left(\cos\theta + s_1 \sin\theta\right)}}, \quad \text{as } r \longrightarrow 0, \tag{21}$$

where polar radius r is the shortest distance to a point near the periodic parallel cracks tip.

Substituting (16) and (19) into (13), according to (1), we can obtain the analytic expressions for stress and displacement of the periodical parallel cracks tip:

$$\begin{aligned} \tau_{yz} &= \frac{K_{\mathrm{III}}}{\sqrt{2\pi}} \operatorname{Re}\left[\frac{1}{\sqrt{z_1 - (a + n\omega i)}}\right], \\ \tau_{zx} &= -\frac{K_{\mathrm{III}}}{\sqrt{2\pi}} \operatorname{Re}\left[\frac{s_1}{\sqrt{z_1 - (a + n\omega i)}}\right], \\ \omega &= K_{\mathrm{III}} \sqrt{\frac{2}{\pi Q_{44} Q_{55}}} \operatorname{Im}\left[\sqrt{z_1 - (a + n\omega i)}\right]. \end{aligned} \tag{22}$$

According to (20), (22) can be rewritten as follows:

$$\begin{aligned} \tau_{yz} &= \frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} \operatorname{Re}\left[\frac{1}{\sqrt{\cos\theta + s_1 \sin\theta}}\right], \\ \tau_{zx} &= -\frac{K_{\mathrm{III}}}{\sqrt{2\pi r}} \operatorname{Re}\left[\frac{s_1}{\sqrt{\cos\theta + s_1 \sin\theta}}\right], \\ \omega &= K_{\mathrm{III}} \sqrt{\frac{2r}{\pi Q_{44} Q_{55}}} \operatorname{Im}\left[\sqrt{\cos\theta + s_1 \sin\theta}\right]. \end{aligned} \tag{23}$$

6. Conclusions

In this paper, the antiplane problem of the periodic parallel cracks in an infinite linear elastic orthotropic composite plate is studied. The antiplane problem near periodic parallel cracks tip can be turned to find the solution of the boundary value problem of partial differential equation.

(1) The analytic expressions for stress intensity factor, stress, and displacement are obtained by constructing proper Westergaard stress function and using the complex variable function method and the boundary conditions.

(2) The stress intensity factor around the periodic parallel cracks tip depends on the shape factor Y.

(3) When $\omega \rightarrow \infty$, $Y \rightarrow 1$, and $K_{\mathrm{III}} \rightarrow K'_{\mathrm{III}}$, it is concluded that the antiplane problem of the periodic parallel cracks degenerates into the antiplane problem of a single central crack.

Conflict of Interests

The authors declare no direct financial relation with any commercial entities mentioned in the paper that might lead to a conflict of interests.

Acknowledgments

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