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Existence of solutions of a system of 3D axisymmetric inviscid stagnation flows

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Abstract

A system of two integral equations is presented to describe the system of 3D axisymmetric inviscid stagnation flows related to Navier-Stokes equations and existence of its solutions is studied. Utilizing it, we construct analytically the similarity solutions of the 3D system. A nonexistence result is obtained. Previous study was only supported by numerical results.

MSC: 34B18

Keywords: Navier-Stokes equations; 3D flows; similarity solutions; integral systems; existence results

1 Introduction

The following system of two differential equations arising in the boundary layer problems in fluid mechanics

$$f'''(\eta) + \big(f(\eta) + \lambda g(\eta)\big)f''(\eta) + \big(1 - f'^2(\eta)\big) = 0 \quad \text{on } [0,\infty), \tag{1.1}$$

$$g'''(\eta) + \big(f(\eta) + \lambda g(\eta)\big)g''(\eta) + \lambda\big(1 - g'^2(\eta)\big) = 0 \quad \text{on } [0,\infty) \tag{1.2}$$

with boundary conditions

$$\begin{aligned} &f(0) = 0, \qquad f'(0) = 0, \qquad f'(\infty) = 1, \\ &g(0) = 0, \qquad g'(0) = 0, \qquad g'(\infty) = 1 \end{aligned} \tag{1.3}$$

has been used to describe the system of 3D axisymmetric inviscid stagnation flow [1, 2], which consists of three partial differential equations [2, 3], where λ is a parameter related to the external flow components.

A solution of (1.1)-(1.3) is called a similarity solution and can be used to express the solutions of the 3D system. Regarding the study of (1.1)-(1.3), Howarth [3] presented a numerical study for the case $0 < \lambda < 1$ which can be applied to the stagnation region of an ellipsoid. Davey [2] investigated numerically the stagnation region near a saddle point ($-1 < \lambda < 0$). The two-dimensional cases, $\lambda = g = 0$ or $\lambda = 1$ and $g = f$, and the special cases of the Falkner-Skan equation were solved by Hiemenz [4] and by Homann [5], respectively. Regarding the Falkner-Skan problems, further analytical study can be found in [6–10]. Also, one may refer to recent review of similarity solutions of the Navier-Stokes equations [11].

However, up to now, there has been very little analytical study on the existence of solutions of (1.1)-(1.3).

The main aim of this paper is to study the existence of solutions of (1.1)-(1.3) analytically for the case of $|\lambda| < 1$. The method is to present a system of two integral equations and study the existence of its solutions and then use it to construct the solutions of (1.1)-(1.3). Also, a nonexistence result is obtained.

2 A system of two integral equations related to (1.1)-(1.3)

In this section, we present a system of two integral equations to describe a system of (1.1)-(1.3) under suitable conditions, which will be utilized in Section 4.

Let

$$\begin{aligned} &Q_1 = \{x \in C[0,1) : x(t) > 0, t \in [0,1)\},\\ &Q_2 = \{y \in C[0,1] \cap C^1[0,1) : y(t) \geq 0, t \in [0,1)\},\\ &Q = Q_1 \times Q_2 \end{aligned}$$

and

$$\Gamma = \{(f,g) \in C^3[0,\infty) \times C^3[0,\infty) : f'(\eta) \geq 0, g''(\eta) > 0, \eta \in [0,\infty)\}.$$

Lemma 2.1 *If* $(f,g) \in \Gamma$ *is a solution of* (1.1)-(1.3), *then* $g''(\infty) = 0$.

Proof Since $g'(+\infty) = 1$, we have

$$\liminf_{\eta\to\infty} g''(\eta) = 0. \tag{2.1}$$

Notice that $(f,g) \in \Gamma$, $f(\eta) = \int_0^\eta f'(s)\,ds \geq 0$, $g'(\eta) = \int_0^\eta g''(s)\,ds \geq 0$, $g(\eta) = \int_0^\eta g'(s)\,ds > 0$ and $1 > g'(\eta) > 0$ for $\eta \in (0,+\infty)$.

If $\lambda \geq 0$, we know $g'''(\eta) = -(f(\eta)+\lambda g(\eta))g''(\eta) - \lambda(1-g'^2(\eta)) \leq 0$ and then g'' is decreasing on $[0,+\infty)$, which implies that $\lim_{\eta\to\infty} g''(\eta)$ exists. Hence, $g''(\infty) = 0$ by (2.1).

If $\lambda < 0$, we have $g'''(0) = -\lambda > 0$ by (1.2). By (2.1), there exists $\eta_0 > 0$ such that $g''(\eta_0) < g''(0)$ and then there exists η^* such that $g''(\eta^*) = \max\{g''(\eta) : \eta \in [0,\eta_0]\}$. Obviously, $\eta^* \in (0,\eta_0]$ by $g'''(0) > 0$. We prove that g'' is decreasing on (η^*,∞).

In fact, if there exist $\eta_1, \eta_2 \in (\eta^*,+\infty)$ with $\eta_1 < \eta_2$ such that $g''(\eta_1) < g''(\eta_2)$. Let $\eta_* \in [\eta^*,\eta_2]$ such that $g''(\eta_*) = \min\{g''(\eta) : \eta \in [\eta^*,\eta_2]\} > 0$, then $g'''(\eta_*) = 0$ and $g^{(4)}(\eta_*) \geq 0$.

Differentiating (1.2) with η, we have

$$g^{(4)}(\eta) = (\lambda g'(\eta) - f'(\eta))g''(\eta) - (f(\eta) + \lambda g(\eta))g'''(\eta),$$

then

$$g^{(4)}(\eta_*) = (\lambda g'(\eta_*) - f'(\eta_*))g''(\eta_*) < 0,$$

a contradiction. Hence, $g''(\eta)$ is decreasing on $(\eta^*,+\infty)$ and then $g''(\infty) = 0$.

This completes the proof. □

Theorem 2.1 *If* $(f,g) \in \Gamma$ *is a solution of* (1.1)-(1.2), *then*

$$x(t) = \int_t^1 \frac{(2\lambda s + \lambda + y(s))(1-s)}{x(s)}\,ds + (1-t)\int_0^t \frac{\lambda s + y(s)}{x(s)}\,ds, \tag{2.2}$$

$$y(t) = \int_0^1 G_{0,1}(t,s)\frac{\lambda(s^2-1)y'(s) + (1-y^2(s))}{x^2(s)}\,ds + t \tag{2.3}$$

has a solution $(x,y) \in Q$, *where* $G_{0,1}(t,s)$ *denotes the Green function for* $u''(t) = 0$ *with* $u(0) = 0$ *and* $u(b) = 0$ *defined by*

$$G_{0,b}(t,s) = \begin{cases} t(b-s)/b, & 0 \le t \le s \le b, \\ s(b-t)/b, & 0 \le s \le t \le b. \end{cases} \tag{2.4}$$

Proof Assume that $(f,g) \in \Gamma$. Let $\eta := \eta(t) = (g')^{-1}(t)$ for $t \in [0,1)$ be the inverse function to $t = g'(\eta) : [0,\infty) \to [0,1)$. It follows that g' is strictly increasing on $[0,+\infty)$ and $\eta(t) = (g')^{-1}(t) : [0,1) \to [0,\infty)$ with $(g')^{-1}(0) = 0$, $\lim_{t\to 1^-}(g')^{-1}(t) = \infty$. Let $x(t) = g''(\eta) > 0$ for $t \in [0,1)$, by Lemma 2.1, $x(1) = \lim_{\eta\to\infty} g''(\eta) = 0$. This implies that $x(t) > 0$ for $t \in [0,1)$ and x is continuous on $[0,1)$. By Lemma 2.1, we see that x is continuous from the left at 1. Hence, we have $x(t) \in C[0,1]$ and $x(1) = 0$, *i.e.*, $x(t) \in Q_1$.

Using the chain rule to $x(t) = g''(\eta)$, we obtain $g'''(\eta)\frac{d\eta}{dt} = x'(t)$ and by the inverse function theorem, we have

$$\frac{d\eta}{dt} = \frac{1}{g''(\eta)} = \frac{1}{x(t)} \quad \text{for } t \in [0,1).$$

This, together with $g'(\eta) = t$, implies

$$g'''(\eta) = x'(t)x(t), \qquad \eta = \int_0^t \frac{1}{x(s)}\,ds \quad \text{and} \quad g'(\eta)\frac{d\eta}{dt} = \frac{t}{x(t)} \quad \text{for } t \in [0,1).$$

Integrating the last equality from 0 to t implies

$$g\big(\eta(t)\big) = \int_0^t \frac{s}{x(s)}\,ds \quad \text{for } t \in [0,1).$$

Let

$$y(t) = f'(\eta) = f'\left(\int_0^t \frac{1}{x(s)}\,ds\right) \quad \text{for } t \in [0,1).$$

Then $y(0) = 0$. By $f'(\infty) = 1$, we know that y is continuous from the left at 1 and then $y(1) = 1$.

Notice that $f'(\eta)\frac{d\eta}{dt} = \frac{y(t)}{x(t)}$, $t \in [0,1)$, we have $f(\eta) = \int_0^t \frac{y(s)}{x(s)}\,ds$.

Differentiating $y(t)$ with t, we have

$$y'(t) = f''(\eta)\frac{d\eta}{dt} = \frac{f''(\eta)}{x(t)} \quad \text{for } t \in [0,1).$$

From this, we have $f''(\eta) = y'(t)x(t)$ for $\eta \in [0,\infty)$ and $y \in Q_2$.

Differentiating $f''(\eta)$ with t and utilizing $\frac{d\eta}{dt} = \frac{1}{x(t)}$, we have

$$\frac{f'''(\eta)}{x(t)} = y''(t)x(t) + y'(t)x'(t).$$

Hence,

$$f'''(\eta) = y''(t)x^2(t) + y'(t)x(t)x'(t).$$

Substituting g, g', g'', g''' and f into (1.2) implies

$$x'(t) = -\int_0^t \frac{y(s) + \lambda s}{x(s)}\,ds + \frac{\lambda(t^2 - 1)}{x(t)}, \quad t \in [0,1). \tag{2.5}$$

Integrating (2.5) from t to 1, we have

$$\begin{aligned}
x(1) - x(t) &= -\int_t^1 \int_0^\sigma \frac{y(s) + \lambda s}{x(s)}\,ds\,d\sigma + \int_t^1 \frac{\lambda(s^2 - 1)}{x(s)}\,ds \\
&= \int_t^1 \frac{\lambda(s^2 - 1)}{x(s)}\,ds - \int_0^t \left(\int_t^1 \frac{y(s) + \lambda s}{x(s)}\,d\sigma\right) ds - \int_t^1 \left(\int_s^1 \frac{y(s) + \lambda s}{x(s)}\,d\sigma\right) ds \\
&= \int_t^1 \frac{\lambda(s^2 - 1)}{x(s)}\,ds - \int_0^t \frac{y(s) + \lambda s}{x(s)}(1 - t)\,ds - \int_t^1 \frac{(y(s) + \lambda s)(1 - s)}{x(s)}\,ds \\
&= \int_t^1 \frac{\lambda(s^2 - 1) - (\lambda s + y(s))(1 - s)}{x(s)}\,ds - (1 - t)\int_0^t \frac{\lambda s + y(s)}{x(s)}\,ds \\
&= \int_t^1 \frac{(2\lambda s + \lambda + y(s))(s - 1)}{x(s)}\,ds - (1 - t)\int_0^t \frac{\lambda s + y(s)}{x(s)}\,ds.
\end{aligned}$$

By $x(1) = 0$, then

$$x(t) = \int_t^1 \frac{(2\lambda s + \lambda + y(s))(1 - s)}{x(s)}\,ds + (1 - t)\int_0^t \frac{\lambda s + y(s)}{x(s)}\,ds.$$

Substituting f, f', f'', f''' and g into (1.1) implies

$$y''(t)x^2(t) + y'(t)x(t)x'(t) + y'(t)x(t)\int_0^t \frac{\lambda s + y(s)}{x(s)}\,ds + \left(1 - y^2(t)\right) = 0.$$

By $\int_0^t \frac{\lambda s + y(s)}{x(s)}\,ds = \frac{\lambda(t^2-1)}{x(t)} - x'(t)$, we have

$$y''(t) + \frac{\lambda(t^2 - 1)y'(t) + (1 - y^2(t))}{x^2(t)} = 0.$$

Therefore,

$$y(t) = \int_0^1 G_{0,1}(t,s)\frac{\lambda(s^2 - 1)y'(s) + (1 - y^2(s))}{x^2(s)}\,ds + t, \quad t \in [0,1),$$

where $G_{0,1}(t,s)$ is defined by (2.4). Hence, (x,y) is a solution of (2.2)-(2.3) in Q. □

3 Positive solutions of the system (2.2)-(2.3)

In this section, we will use the fixed point theorem to study the existence of positive solutions of the system (2.2)-(2.3).

Let

$$\delta = \delta(\lambda) = -\frac{\lambda}{2\lambda + 1}, \quad \lambda \in \left(-\frac{1}{3}, 0\right].$$

It is easy to verify

$$0 < \delta < 1 \quad \text{if and only if} \quad -\frac{1}{3} < \lambda < 0.$$

We define some functions

$$h(\lambda) = \int_\delta^1 (2\lambda s + \lambda + s)(1 - s)\,ds + (1 - \delta)\int_0^\delta (\lambda s + s)\,ds = \frac{(3\lambda + 1)^3}{6(2\lambda + 1)^2} + \frac{\lambda^2(\lambda + 1)(3\lambda + 1)}{2(2\lambda + 1)^3},$$

$$\sigma(\lambda) = \frac{\sqrt{\frac{3+5\lambda}{3}} + \sqrt{\frac{3-7\lambda}{3}}}{2},$$

$$l(\lambda) = -\lambda\int_0^\delta \left(1 - s^2\right) ds = \frac{\lambda^2(11\lambda^2 + 12\lambda + 3)}{3(2\lambda + 1)^3},$$

$$\omega(\lambda) = \frac{h^2(\lambda)}{\sigma^2(\lambda)} - 2l(\lambda).$$

By computation, $\omega(0) = \frac{1}{36}$, $\omega(-\frac{1}{3}) = -\frac{4}{9}$, there exists $\lambda_0 \in (-\frac{1}{3}, 0)$ such that $\omega(\lambda) > 0$ for $\lambda \in (\lambda_0, 0]$ and $\omega(\lambda_0) = 0$.

In order to study the existence of solutions of (2.2)-(2.3) in Q for $\lambda \in (\lambda_0, 1)$, we denote the norm of the Banach space $C[0,1] \times C^1[0,1]$ by

$$\|(x, y)\| = \|x\| + \|y\| + \|y'\|,$$

where $\|x\| = \max\{|x(t)| : t \in [0,1]\}$.

Let $(x, y) \in C[0,1] \times C^1[0,1]$ and $n > 0$ be a natural number, we define

$$\varphi x(t) = \max\{x(t), c(t)\}, \qquad \varphi_n x(t) = \max\left\{x(t), c(t), \frac{1}{n}\right\}, \qquad \theta y(t) = \max\{y(t), t\},$$

where $c(t) = c_\lambda(1 - t)$, $t \in [0,1]$,

$$c_\lambda = \begin{cases} \frac{1}{n}, & \lambda \ge 0, \\ \min\{\sqrt{h(\lambda)}, \sqrt{\omega(\lambda)}, \frac{(1+\lambda)(1-\delta)\delta^2}{4\sigma(\lambda)}\}, & \lambda_0 < \lambda < 0. \end{cases} \tag{3.1}$$

Notation

$$\alpha(y)(t) = 2\lambda t + \lambda + y(t),$$

$$\beta(y)(t) = \lambda t + y(t),$$

$$h(y)(t) = \lambda\left(t^2 - 1\right)y'(t) + \left(1 - \left(\theta y(t)\right)^2\right)$$

and

$$B_n(x,y)(t) = \int_0^1 G_{0,1}(t,s)\frac{h(y)(s)}{(\varphi_n x(s))^2}\,ds + t,$$
$$S_n(x,y)(t) = \int_t^1 \frac{\alpha(y)(s)(1-s)}{\varphi_n x(s)}\,ds,$$
$$T_n(x,y)(t) = \int_0^t \frac{\beta(y)(s)}{\varphi_n x(s)}\,ds,$$

where $G_{0,1}(t,s)$ is defined by (2.4).

Let $(x,y) \in C[0,1] \times C^1[0,1]$, we define an operator F as follows:

$$F_n(x,y)(t) = \big(A_n(x,y)(t), B_n(x,y)(t)\big),$$

where

$$A_n(x,y)(t) = S_n(x,y)(t) + (1-t)T_n(x,y) + \frac{1}{n}.$$

It is easy to verify that φ_n, θ are continuous operators from $C[0,1]$ into $C[0,1]$ and $\varphi_n x(t) \geq \frac{1}{n}$, $t \in [0,1]$, we know the following proposition holds:

Lemma 3.1 *F_n is a continuous and compact operator from $C[0,1] \times C^1[0,1]$ to $C[0,1] \times C^1[0,1]$.*

Lemma 3.2 *Let $(\lambda, z, w) \in (-1,1) \times C[0,1] \times C^1[0,1]$ and $0 < \mu \leq 1$ such that*

$$x(t) = \mu A_n(x,y)(t), \tag{3.2}$$
$$y(t) = \mu B_n(x,y)(t). \tag{3.3}$$

Then the following assertions hold:

(i) $\mu t \leq y(t) \leq 1$ *for* $t \in [0,1]$.
(ii) $\int_0^1 |y'(s)|\,ds \leq 2$ *and* $V_0^1(y) \leq 2$, *where* $V_0^1(y)$ *is a total variation of* y *on* $[0,1]$.
(iii) *If* $\mu = 1$, *then* $y(t)$ *is increasing on* $(0,1)$ *and then* $\theta y(t) = y(t)$ *for* $t \in [0,1]$.

Proof We shall use the basic fact: let $u(t) \in C[a,b] \times C^2(a,b)$ and $u(\xi)$ ($\xi \in (a,b)$) be local minimum (maximum), then $u''(\xi) \geq 0$ (≤ 0).

(i) If there exists $t_0 \in (0,1)$ such that $y(t_0) > 1$, by $y(0) = 0 < \mu = y(1)$, we know that there exists $t_* \in (0,1)$ such that $y(t_*) = \max\{y(t) : t \in [0,1]\} > 1$. Differentiating (3.3) with t twice, we have

$$y''(t) = -\mu\frac{h(y)(t)}{(\varphi_n x(t))^2}. \tag{3.4}$$

By $y'(t_*) = 0$ and (3.4), we have

$$y''(t_*) = -\frac{\mu(1-y^2(t_*))}{(\varphi_n x(t_*))^2} > 0,$$

a contradiction. Hence, $y(t) \leq 1$ for $t \in (0,1)$.

If there exists $t_0 \in [0,1]$ such that $\mu t_0 > y(t_0)$, let $\tau(t) = \mu t - y(t)$, by $\tau(0) = 0 = \tau(1)$ and $\tau(t_0) > 0$, we may assume $t_* \in (0,1)$ such that $\tau(t_*) = \max\{\tau(t) : t \in [0,1]\}$. This implies $\tau'(t_*) = 0$, *i.e.*, $y'(t_*) = \mu$, and $\tau''(t_*) \leq 0$. By (3.4) and $\theta y(t_*) = t_*$, we know

$$h(y)(t_*) = \lambda\left(t_*^2 - 1\right)\mu + \left(1 - t_*^2\right) = (1 - \lambda\mu)\left(1 - t_*^2\right) > 0,$$

then

$$\tau''(t_*) = -y''(t_*) = \frac{\mu h(y)(t_*)}{(\varphi_n x(t_*))^2} > 0,$$

a contradiction. Hence, (i) holds.

(ii) Let $\tilde{t} \in [0,1]$ such that $y(\tilde{t}) = \max\{y(t) : t \in [0,1]\}$ and $\gamma = \sup\{\tilde{t}\}$. If $\gamma < 1$, we prove that $y(t)$ is increasing on $(0,\gamma)$ and decreasing on $(\gamma,1)$.

Since $y(0) = 0$ and $y(1) = \mu > 0$, then $\gamma > 0$. Let $\gamma < 1$. If there exist $t_1, t_2 \in (0,\gamma)$ with $t_1 < t_2$ such that $y(t_1) > y(t_2)$, let $t_* \in (t_1,\gamma)$ such that $y(t_*) = \min\{y(t) : t \in [t_1,\gamma]\}$, then $y(t_*) < 1$ by (i). From $y'(t_*) = 0$, $t_* \leq \theta y(t_*) < 1$ and (3.4), we know

$$y''(t_*) = -\frac{\mu(1 - (\theta y(t_*))^2)}{(\varphi_n x(t_*))^2} < 0,$$

a contradiction.

If there exist $t_1, t_2 \in (\gamma,1)$ with $t_1 < t_2$ such that $y(t_1) < y(t_2)$, let $t_* \in (\gamma,t_2)$ such that $y(t_*) = \min\{y(t) : t \in [\gamma,t_2]\}$, then $y(t_*) < 1$ by (i). Analogously, we know easily

$$y''(t_*) = -\frac{\mu(1 - (\theta y(t_*))^2)}{(\varphi_n x(t_*))^2} < 0,$$

a contradiction. Hence,

$$\int_0^1 \left|y'(s)\right| ds = \int_0^\gamma y'(s)\, ds - \int_\gamma^1 y'(s)\, ds = 2y(\gamma) - \mu \leq 2,$$

and $V_0^1(y) = \int_0^1 |y'(s)|\, ds \leq 2$, *i.e.*, (ii) holds.

(iii) Let $\mu = 1$. By (i) and $y(1) = 1$, we know $\gamma = 1$ and then $y(t)$ is increasing on $(0,1)$ and then $\theta y(t) = y(t)$ for $t \in [0,1]$. Hence, (iii) holds. □

Lemma 3.3 [12] *Let E be a Banach space, D be a bounded open set of E and $\theta \in D$, $F : \overline{D} \to E$ is compact. If $x \neq \mu Fx$ for any $0 < \mu < 1$ and $x \in \partial D$, then F has a fixed point in $\overline{D}$.*

Lemma 3.4 *Let $\lambda \in (-1,1)$, then F has a fixed point (x_n, y_n) in $C[0,1] \times C^1[0,1]$, i.e., there exists $(x_n, y_n) \in C[0,1] \times C^1[0,1]$ such that*

$$x_n(t) = A(x_n, y_n)(t), \tag{3.5}$$

$$y_n(t) = B(x_n, y_n)(t) \tag{3.6}$$

hold.

Proof Let

$$\Omega = \{(x,y) : (x,y) \in C[0,1] \times C^1[0,1], \|(x,y)\| < R\},$$

where $R = 16n^2$. We prove $(x,y) \neq \mu F(x,y)$ for $0 < \mu < 1$ and with $\|(x,y)\| = R$.

In fact, if there exist (x,y) and μ with $\|(x,y)\| = R$ and $0 < \mu < 1$ such that $(x,y) \neq \mu F(x,y)$, by Lemma 3.2(i) and (iii), we have $\|y\| \leq 1$.

Since $|\alpha(y)(s)| \leq (2|\lambda| + |\lambda| + 1) = 3|\lambda| + 1$ and $|\beta(y)(s)| \leq |\lambda| + 1$ for $s \in [0,1]$, this, together with $1 - t \leq 1 - s$ for $s \leq t$ and $\varphi_n x(t) \geq \frac{1}{n}$, implies

$$\begin{aligned} |S_n(x,y)(t)| &\leq n \int_0^1 |\alpha(y)(s)|\, ds \leq (3|\lambda| + 1)n, \\ (1-t)|T(x,y)(t)| &\leq n \int_0^1 |\beta(s)|\, ds \leq (|\lambda| + 1)n. \end{aligned}$$

And then $|x(t)| \leq |S_n(x,y)(t)| + (1-t)|T_n(x,y)(t)| + 1 \leq 2(2|\lambda| + 1)n + 1$, *i.e.*, $\|x\| \leq 2(2|\lambda| + 1)n + 1$.

By (3.3), we have

$$y'(t) = -\int_0^t s\frac{h(y)(s)}{(\varphi_n x(s))^2}\, ds + \int_t^1 (1-s)\frac{h(y)(s)}{(\varphi_n x(s))^2}\, ds + 1. \tag{3.7}$$

Noticing that $|h(y)(s)| \leq |\lambda||y'(s)| + 1$ and $\varphi_n x(s) \geq \frac{1}{n}$ for $s \in [0,1]$, we obtain $|\frac{h(y)(s)}{(\varphi_n x(s))^2}| \leq n^2(|\lambda||y'(s)| + 1)$ for $s \in [0,1]$. This, together with (3.7) and Lemma 3.2(ii), implies

$$\begin{aligned} |y'(t)| &\leq \int_0^1 \left|\frac{h(y)(s)}{(\varphi_n x(s))^2}\right| ds + \int_0^1 \left|\frac{h(y)(s)}{(\varphi_n x(s))^2}\right| ds + 1 \\ &\leq 2(2|\lambda| + 1)n^2 + 1, \end{aligned}$$

i.e., $\|y'\| \leq 2(2|\lambda| + 1)n^2 + 1$. Hence,

$$\begin{aligned} \|(x,y)\| &= \|x\| + \|y\| + \|y'\| \\ &\leq 2(2|\lambda| + 1)n + 1 + 2(2|\lambda| + 1)n^2 + 1 < R, \end{aligned}$$

a contradiction.

By Lemmas 3.1 and 3.3, F has a fixed point (x_n, y_n) in $C[0,1] \times C^1[0,1]$. □

Lemma 3.5 *Let* (x_n, y_n) *be in Lemma* 3.4, *then*

(i) $\{x_n(t)\}$ *is bounded on* $[0,1]$.

(ii) $\{x_n'(t)\}$ *is bounded on* $[0,b]$ *for any* $b \in (\frac{1}{2}, 1)$.

Proof By Lemma 3.3(i), we know $0 \leq y_n(t) \leq 1$. By (3.5), we have

$$x_n'(t) = \frac{-\lambda(1-t^2)}{\varphi x_n(t)} - \int_0^t \frac{y_n(s) + \lambda s}{\varphi x_n(s)}\, ds, \quad t \in [0,1). \tag{3.8}$$

(i) For $\lambda \geq 0$, we know $x_n'(t) < 0$ for $t \in [0,1]$, *i.e.*, $x_n(t)$ is decreasing in $[0,1]$, by $x_n(1) = \frac{1}{n}$, $\varphi x_n(t) = x_n(t)$ for $t \in [0,1]$. By $\alpha(y_n)(t) \geq t$ for $t \in [0,1]$ and (3.5), we have

$$x_n(t) \geq S_n(x_n, y_n)(t) \geq \int_t^1 \frac{s}{\varphi_n x(s)}\,ds \geq \frac{t}{x_n(t)} \int_t^1 (1-s)\,ds. \tag{3.9}$$

And then $x_n(t) \geq \frac{(1-t)\sqrt{2t}}{2}$ for $t \in [0,1]$. Obviously, $x_n(t) \geq \frac{1-t}{2}$ for $t \in [\frac{1}{2}, 1]$. This, together with the decrease in x_n, implies

$$x_n(t) \geq \frac{1-t}{4} \quad \text{for } t \in [0,1]. \tag{3.10}$$

Let $c^*(t) = \mu(1-t)$, μ defined by

$$\mu = \begin{cases} \frac{1}{4} & \text{if } \lambda \geq 0, \\ c_\lambda & \text{if } \lambda < 0, \end{cases}$$

where c_λ defined in (3.1).

It is easy to verify $\varphi_n x_n(t) \geq c^*(t)$ for $t \in [0,1]$. And then

$$\begin{aligned} \left|S_n(x_n, y_n)(t)\right| &\leq \int_0^1 \frac{|\alpha(y_n)(s)|(1-s)}{c^*(s)}\,ds \leq \int_0^1 \frac{3|\lambda|+1}{\mu}\,ds < +\infty, \\ (1-t)\left|T_n(x_n, y_n)(t)\right| &\leq (1-t)\int_0^t \frac{|\beta(y_n)(s)|}{c^*(s)}\,ds \leq \int_0^1 \frac{1+|\lambda|}{\mu}\,ds < +\infty. \end{aligned}$$

The last two inequalities imply that $\{x_n(t)\}$ is bounded on $[0,1]$.

(ii) By (3.8),

$$\left|x_n'(t)\right| \leq \frac{|\lambda|(1+t)}{\mu} + \int_0^t \frac{2}{c^*(s)}\,ds, \quad t \in [0,1),$$

we know that $\{x_n'(t)\}$ is bounded on $[0,b]$ for any $b \in (\frac{1}{2}, 1)$. □

Lemma 3.6 *Let* (x_n, y_n) *be in Lemma* 3.4, *then*

(i) $t \leq y_n(t) \leq 1$ *and* $y_n(t)$ *is increasing in* $[0,1]$.

(ii) $\{y_n'(t)\}$ *is bounded and equicontinuous in* $[0,b]$ *for any* $b \in (\frac{1}{2}, 1)$.

Proof

(i) Lemma 3.2(i) and (iii) imply the desired results.

(ii) For $b \in (\frac{1}{2}, 1)$, let $t_b \in [0,b]$ such that $y_n'(t_b) = \min\{y_n'(t) : t \in [0,b]\}$. Since $y_n'(t) \geq 0$ on $[0,1]$, by Lemma 3.2(ii), $y_n'(t_b)b \leq \int_0^b y_n'(s)\,ds \leq \int_0^1 y_n'(s)\,ds \leq 2$, we obtain $y_n'(t_b) \leq \frac{2}{b}$.

Differentiating (3.6) with t twice, we have $y_n''(t) = -\frac{h(y_n)(t)}{(\varphi_n x_n(t))^2}$. Integrating this equality from 0 to $t \leq b$, we have

$$y_n'(t) - y_n'(t_b) = -\int_{t_b}^t \frac{h(y_n)(s)}{(\varphi_n x_n(s))^2}\,ds.$$

Noticing that $|h(y_n)(t)| \leq |\lambda| y_n'(t) + 1$ and $c^*(t) \geq c^*(b)$ for $t \in [0,b]$ and Lemma 3.2(ii), we know

$$\left|y_n'(t)\right| \leq \frac{2|\lambda| + 1}{(c^*(b))^2} + \left|y_n'(t_b)\right| \leq \frac{2|\lambda| + 1}{(c^*(b))^2} + \frac{2}{b},$$

i.e., $\{y_n'(t)\}$ is bounded on $[0,b]$. Let $M_b = \sup\{M_n\}$ (where $M_n = \max\{y_n'(t) : t \in [0,b]\}$), we know

$$\left|y_n''(t)\right| = \frac{|h(y_n)(t)|}{(\varphi x_n(t))^2} \leq \frac{|\lambda| M_b + 1}{(c^*(b))^2} < +\infty \quad \text{for } 0 \leq t \leq b.$$

This implies that $\{y_n'(t)\}$ is equicontinuous on $[0,b]$. □

Theorem 3.1 *There exists* $(x,y) \in C[0,1] \times (C[0,1] \cap C^1[0,1))$ *such that*

$$x(t) = S(x,y)(t) + (1-t)T(x,y), \tag{3.11}$$

$$y(t) = B(x,y)(t) \tag{3.12}$$

hold, where

$$S(x,y)(t) = \int_t^1 \frac{\alpha(y)(s)(1-s)}{\varphi x(s)}\, ds,$$

$$T(x,y)(t) = \int_0^t \frac{\beta(y)(s)}{\varphi x(s)}\, ds,$$

$$B(x,y)(t) = \int_0^1 G_{0,1}(t,s) \frac{h(y)(s)}{(\varphi x(s))^2}\, ds + t.$$

Proof Let (x_n, y_n) be in Lemma 3.4, by Lemma 3.5(ii) and (iii), we know that $\{x_n(t)\}$ is bounded and equicontinuous on $[0,b]$ for any $b \in (\frac{1}{2}, 1)$. Letting $b = 1 - \frac{1}{k}$ $(k = 3, 4, \ldots)$, utilizing the diagonal principle and the Arzela-Ascoli theorem, we know that there exists a subsequence $\{x_{n_k}(t)\}$ of $\{x_n(t)\}$ and $x(t) \in C[0,1)$ such that $x_{n_k}(t)$ converges to $x(t)$ for $t \in [0,1)$. Without loss of generality, we assume that $\{x_{n_k}(t)\}$ is itself of $\{x_n(t)\}$.

By Lemma 3.6, we know that $\{y_n'(t)\}$ is bounded and equicontinuous on $[0,b]$ for any $b \in (\frac{1}{2}, 1)$ and then $\{y_n(t)\}$ is bounded and equicontinuous on $[0,b]$. Let $b = 1 - \frac{1}{k}$ $(k = 3, 4, \ldots)$, the diagonal principle and the Arzela-Ascoli theorem imply that there exist y and y_0 in $C[0,1)$ and two subsequences $\{y_{n_k}(t)\}$ and $\{y_{n_i}'(t)\}$ with $\{y_{n_i}(t)\} \subseteq \{y_{n_k}(t)\} \subseteq \{y_n(t)\}$ such that $y_{n_k}(t)$ converges to $y(t)$ for $t \in [0,1)$ with $y(1) = 1$ and $y_{n_i}'(t)$ converges to $y_0(t)$ for each $t \in [0,1)$. For the sake of convenience, we assume that $\{y_{n_i}(t)\}$ and $\{y_{n_k}(t)\}$ are itself of $\{y_n(t)\}$. By $y_n(t) = \int_0^t y_n'(s)\, ds$, we obtain $y(t) = \int_0^t y_0(s)\, ds$ and then $y_0(t) = y'(t)$ for $t \in [0,1)$.

Since

$$\left|\frac{\alpha(y_n)(s)(1-s)}{\varphi x_n(s)}\right| \leq \frac{3|\lambda| + 1}{c^*},$$

$$(1-t)\left|\frac{\beta(y_n)(s)}{\varphi x_n(s)}\right| \leq \frac{1 + |\lambda|}{c^*} \quad (s \leq t),$$

$\alpha(y_n)(t)$ converges to $\alpha(y)(t)$ and $\beta(y_n)(t)$ converges to $\beta(y)(t)$ for $t \in [0,1)$, by the Lebesgue dominated theorem (the dominated function $F(s) = \frac{3|\lambda|+1}{c^*}$, $s \in [0,1]$), we have that (x,y) satisfies (3.11) and $x \in Q_1$.

Fix $t \in (0,1)$ and choose $b \in (0,1)$ such that $t \leq b$, then

$$y_n(t) = \int_0^b G_{0,b} \frac{h(y_n)(s)}{(\varphi x_n(s))^2}\, ds + \frac{t}{b} y_n(b) \quad \text{for } t \in [0,b].$$

Noticing that $|h(y_n)(s)| \leq |\lambda||y_n'(s)| + 1 \leq |\lambda| M_b + 1$ and $h(y_n)(s)$ converges to $h(y)(s)$ for $s \in [0,b]$, by the Lebesgue dominated theorem (the dominated function $F(s) = \frac{M_b+1}{(c^*(b))^2}$ on $\in [0,b]$), we have

$$y(t) = \int_0^b G_{0,b} \frac{h(y)(s)}{(\varphi x(s))^2}\, ds + \frac{t}{b} y(b) \quad \text{for } t \in [0,b].$$

Differentiating the last equality twice, we know

$$y''(t) = -\frac{h(y)(t)}{(\varphi x(t))^2} \quad \text{for } t \in [0,1).$$

By (i), we know $t \leq y(t) \leq 1$ and $\lim_{t\to 1} y(t) = 1 = y(1)$ and then $y \in C[0,1] \cap C^1[0,1)$. This, together with (2.4), implies that $y(t)$ satisfies (3.12). Clearly, $(x,y) \in Q$. □

Theorem 3.2 *For $\lambda \in (\lambda_0, 1)$, the system* (2.2)-(2.3) *has at least a solution (x,y) in Q.*

Proof Let (x,y) in Theorem 3.1. It is clear that we only prove $\varphi x(t) = x(t)$. If $\lambda \geq 0$, by (3.10), we obtain $x(t) \geq \frac{1-t}{4}$ for $t \in [0,1]$ and then $\varphi x(t) = x(t)$. Next, we prove $x(t) \geq c_\lambda(1-t)$ for $t \in [0,1]$ for $\lambda_0 < \lambda < 0$.

Let $\gamma \in [0,1]$ such that $M = \varphi x(\gamma) = \max\{\varphi x(t) : t \in [0,1]\}$, then

$$\begin{aligned} M \geq x(\delta) &= \int_\delta^1 \frac{(2\lambda s + \lambda + y(s))(1-s)}{\varphi x(s)} + (1-\delta) \int_0^\delta \frac{\lambda s + y(s)}{\varphi x(s)}\, ds \\ &\geq \int_\delta^1 \frac{(2\lambda s + \lambda + s)(1-s)}{\max\{M, c_\lambda\}}\, ds + (1-\delta) \int_0^\delta \frac{\lambda s + s}{\max\{M, c_\lambda\}}\, ds \\ &= \frac{1}{\max\{M, c_\lambda\}} \int_\delta^1 (2\lambda s + \lambda + s)(1-s)\, ds + (1-\delta) \int_0^\delta (\lambda s + s)\, ds \\ &= \frac{h(\lambda)}{\max\{M, c_\lambda\}}. \end{aligned}$$

From this and $c_\lambda \leq \sqrt{h(\lambda)}$, we obtain $M \geq \sqrt{h(\lambda)}$ and $x(\gamma) = \varphi x(\gamma) = M$.

Let $S(t) = S(x,y)(t)$ and $S = \max\{S(t) : t \in [0,1]\}$, we prove

$$S \leq \sqrt{\frac{3+5\lambda}{3}}. \tag{3.13}$$

By $\alpha(y)(0) = \lambda < 0$ and $\alpha(y)(1) = 3\lambda + 1 > 0$, there exists $t_0 \in (0,1)$ such that $\alpha(y)(t_0) = 0$. Since $\alpha(y)''(t) = y''(t) \leq 0$ for $t \in [0,1]$, *i.e.*, $\alpha(y)(t)$ is concave down on $[0,1]$, then $\alpha(y)(s) \leq 0$ for $s \in [0,t_0]$ and $\alpha(y)(s) \geq 0$ for $s \in [t_0,1]$. Hence, $S = S(t_0)$.

By (3.11), we have

$$\varphi x(t) \geq x(t) \geq S(t) \quad \text{for } t \in [t_0,1],$$

we know

$$S(t)\bigl(-S'(t)\bigr) = \frac{S(t)\alpha(y)(t)}{\varphi x(t)} \le 2\lambda t + \lambda + 1 \quad \text{for } t \in [t_0, 1].$$

Integrating the last inequality from t_0 to 1 and utilizing $S(1) = 0$, we have

$$\frac{S^2(t_0)}{2} \le \int_{t_0}^1 (2\lambda s + \lambda + 1)(1-s)\,ds \le \int_0^1 (2\lambda s + \lambda + 1)(1-s)\,ds = \frac{3+5\lambda}{6}.$$

Hence, (3.13) holds.

By $x'(0) > 0$, $x(\delta) > 0$ and $x(1) = 0$, we have $0 < \gamma < 1$ and $x'(\gamma) = 0$, then

$$0 = x'(\gamma) = -\frac{\lambda(1-\gamma^2)}{\varphi x(\gamma)} - \int_0^\gamma \frac{\lambda s + y(s)}{\varphi x(s)}\,ds,$$

i.e.,

$$\int_0^\gamma \frac{\lambda s + y(s)}{\varphi x(s)}\,ds = -\frac{\lambda(1-\gamma^2)}{\varphi x(\gamma)}.$$

Hence,

$$(1-\gamma)T(x,y)(\gamma) = -\frac{\lambda(1-\gamma)(1-\gamma^2)}{\varphi x(\gamma)} \le -\frac{\lambda}{M}.$$

This, together with (3.13), implies

$$M = x(\gamma) = S(x,y)(\gamma) + (1-\gamma)T(x,y)(\gamma) \le \sqrt{\frac{3+5\lambda}{3}} - \frac{\lambda}{M},$$

i.e.,

$$M \le \frac{\sqrt{\frac{3+5\lambda}{3}} + \sqrt{\frac{3-7\lambda}{3}}}{2} = \sigma(\lambda).$$

Since $\alpha(y)(t) \ge 2\lambda t + \lambda + t \ge 0$ for $t \in [\delta, 1]$, we have

$$\begin{aligned} x(t) &\ge (1-t)T(x,y)(t) \ge (1-t)T(x,y)(\delta) \\ &\ge (1-t)\int_0^\delta \frac{\lambda s + s}{\sigma(\lambda)}\,ds \ge \frac{(\lambda+1)\delta^2}{2\sigma(\lambda)}(1-t), \quad t \in [\delta, 1]. \end{aligned}$$

And then $x(t) \ge c_\lambda(t)$ for $t \in [\delta, 1]$.

Finally, we prove $x(t) \ge c_\lambda$ for $t \in [0, \delta]$.

In fact, if there exists $t \in [0, \delta]$ such that $x(t) < c_\lambda$, by $x(\delta) > c_\lambda$, there exists $t' \in (0, \delta)$ such that $x(t) > c_\lambda$ for $t \in (t', \delta]$ and $x(t') = c_\lambda$.

From

$$x(\delta) = S(x,y)(\delta) + (1-\delta)T(x,y)(\delta),$$

$S(x,y)(\delta) \geq \int_\delta^1 \frac{2\lambda s+\lambda+s}{\sigma(\lambda)}\,ds$ and $T(x,y)(\delta) \geq \int_0^\delta \frac{\lambda s+s}{\sigma(\lambda)}\,ds$, we obtain

$$x(\delta) \geq \frac{h(\lambda)}{\sigma(\lambda)}.$$

By (3.11), we have

$$x'(t) = -\frac{\lambda(1-t^2)}{x(t)} - \int_0^t \frac{\lambda s + y(s)}{\varphi x(s)}\,ds \leq -\frac{\lambda(1-t^2)}{x(t)}, \quad t \in [t', \delta],$$

i.e., $x(t)x'(t) \leq -\lambda(1-t^2)$, $t \in [t', \delta]$. Integrating this inequality from t' to δ, we have

$$\frac{x^2(\delta) - c_\lambda^2}{2} \leq \int_{t'}^{\delta} -\lambda\left(1-s^2\right) ds < \int_0^{\delta} -\lambda\left(1-s^2\right) ds$$

and then $c_\lambda^2 > x^2(\delta) + 2\int_0^\delta \lambda(1-s^2)\,ds \geq \frac{h^2(\lambda)}{\sigma^2(\lambda)} - 2l(\lambda) = \omega(\lambda)$, a contradiction.

This completes the proof. □

4 Existence of solutions of (1.1)-(1.3)

In this section, we use positive solutions obtained in Theorem 3.2 to construct the solutions of (1.1)-(1.3) in Γ.

Theorem 4.1 *For* $\lambda \in (\lambda_0, 1)$, *the system* (1.1)-(1.3) *has at least a solution* $(f,g) \in \Gamma$.

Proof Let $\lambda \in (\lambda_0, 1)$, by Theorem 3.2, the system (2.2)-(2.3) has at least a solution (x,y) in Q. By $x(t) \geq c_*(t)$ and (2.2), we know

$$\begin{aligned} x(t) &\leq \int_t^1 \frac{(1-s)(3|\lambda|+1)}{c_*(s)}\,ds + (1-t)\int_0^t \frac{|\lambda|+1}{c_*(s)}\,ds \\ &\leq \frac{1}{c_*}\left(\int_t^1 \left(3|\lambda|+1\right) ds + (1-t)\int_0^t \frac{1+|\lambda|}{1-s}\,ds\right) \\ &\leq \frac{1}{c_*}\left(3|\lambda| + 1 - \left(1+|\lambda|\right)\ln(1-t)\right)(1-t). \end{aligned}$$

Let $u(t) = \frac{1}{c_*}(3|\lambda| + 1 - (1+|\lambda|)\ln(1-t))$, $du = \frac{1+|\lambda|}{c_*(1-t)}\,dt$ and then

$$\int_0^1 \frac{1}{z(s)}\,ds \geq \int_0^1 \frac{1}{u(s)(1-s)}\,ds = \frac{c_*}{1+|\lambda|}\int_0^\infty \frac{du}{u} = \infty,$$

we have $\int_0^1 \frac{1}{x(s)}\,ds = \infty$.

Let

$$\eta := \eta(t) = \int_0^t \frac{1}{x(s)}\,ds, \quad 0 \leq t < 1. \tag{4.1}$$

Then $\eta(t)$ is strictly increasing on $[0,1)$ and

$$\eta(0) = 0, \qquad \eta(1-0) = \int_0^1 \frac{1}{x(s)}\,ds = +\infty.$$

Let $t = h(\eta)$ be the inverse function to $\eta = \eta(t)$, we define the function

$$g(\eta) = \int_0^\eta h(s)\,ds, \qquad f(\eta) = \int_0^\eta y\bigl(h(s)\bigr)\,ds, \quad 0 \le \eta < +\infty.$$

Then

$$g'(\eta) = h(\eta), \qquad g(0) = 0, \qquad g'(0) = 0, \qquad g'(\infty) = 1$$

and

$$f'(\eta) = y\bigl(h(\eta)\bigr), \qquad f(0) = 0, \qquad f'(0) = 0, \qquad f'(\infty) = 1.$$

From (4.1), we have

$$\eta = \eta\bigl(g'(\eta)\bigr) = \int_0^{g'(\eta)} \frac{1}{x(s)}\,ds, \quad 0 \le \eta < +\infty. \tag{4.2}$$

Differentiating (4.2) with respect to η, we have

$$g''(\eta) = x\bigl(g'(\eta)\bigr) = x(t), \quad 0 \le \eta < +\infty. \tag{4.3}$$

Then $g''(\eta) > 0$ for $0 \le \eta < +\infty$.

Differentiating (4.3) with respect to η, we have

$$g'''(\eta) = x'\bigl(g'(\eta)\bigr), \qquad g''(\eta) = x'(t)x(t), \quad 0 \le t < 1. \tag{4.4}$$

Differentiating (2.2) with respect to t, we have

$$x'(t) = -\int_0^t \frac{\lambda s + y(s)}{x(s)}\,ds + \frac{-\lambda(1-t^2)}{x(t)}, \quad 0 \le t < 1. \tag{4.5}$$

By setting $s = g'(\sigma)$ and utilizing $t = g'(\eta)$ and (4.3), we have

$$\begin{aligned}\int_0^t \frac{\lambda s + y(s)}{x(s)}\,ds &= \int_0^{g'(\eta)} \frac{\lambda s + y(s)}{x(s)}\,ds \\ &= \int_0^\eta \bigl(f'(\sigma) + \lambda g'(\sigma)\bigr)\,d\sigma = f(\eta) + \lambda g(\eta).\end{aligned} \tag{4.6}$$

By (4.3), (4.4), (4.5) and (4.6), we have

$$g''' = -(f + \lambda g)g'' + \lambda\bigl(g'^2 - 1\bigr).$$

By (4.1), we have $\frac{dt}{d\eta} = x(t)$. Differentiating $f'(\eta)$ with respect to η, we have

$$f''(\eta) = y'(t)\frac{dt}{d\eta} = y'(t)x(t), \qquad f'''(\eta) = y''(t)x^2(t) + y'(t)x'(t)x(t).$$

Differentiating (2.3) with t twice and combining (4.5) and (4.6), we obtain

$$\begin{aligned}
&f''' + (f + \lambda g)f'' + \left(1 - f'^2\right) \\
&\quad = y''(t)x^2(t) + y'(t)x'(t)x(t) + y'(t)x(t)\int_0^t \frac{\lambda s + y(s)}{x(s)}\,ds + \left(1 - y^2(t)\right) \\
&\quad = x^2(t)\left[y''(t) + \frac{\lambda(t^2 - 1)y'(t) + (1 - y^2(t))}{x^2(t)}\right] = 0.
\end{aligned}$$

This completes the proof. □

Remark 4.1 For $\lambda < -1$, by Theorem 1 [2], (1.1)-(1.3) has no solution such that $\lim_{\eta\to\infty} g'(\eta) = 1$ with $|g'(\eta)| < 1$ for $\eta \geq \eta_0$, $\eta_0 \geq 0$ is a constant.

Utilizing the system (2.2)-(2.3), we know easily that (1.1)-(1.3) has no solution in Γ for $\lambda \leq -1$.

In fact, if (1.1)-(1.3) has a solution $(f, g) \in \Gamma$ for some $\lambda \leq -1$, by Theorem 2.1, then (1.1)-(1.3) has a solution in $(x, y) \in Q$. Noticing that

$$\alpha(y)(t) = 2\lambda t + \lambda + y(t) \leq 2\lambda t + \lambda + 1 < 0 \quad \text{for } t \in (0, 1),$$

we know

$$g''(0) = x(0) = \int_0^1 \frac{\alpha(y)(s)(1 - s)}{\varphi x(s)}\,ds < 0,$$

a contradiction.

This research uses integrals of equations to investigate the existence of solutions of the 3D axisymmetric inviscid stagnation flows related to Navier-Stokes equations and supplies a gap of analytical study in this field.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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