Hindawi
Discrete Dynamics in Nature and Society
Volume 2018, Article ID 5232147, 11 pages
https://doi.org/10.1155/2018/5232147

Research Article

The Asymptotic Behavior of Solutions for a Class of Nonlinear Fractional Difference Equations with Damping Term

Zhihong Bai and Run Xu

School of Mathematical Sciences, Qufu Normal University, Qufu, Shandong 273165, China

Correspondence should be addressed to Run Xu; xurun_2005@163.com

Received 20 October 2017; Accepted 2 January 2018; Published 29 January 2018

Academic Editor: Chris Goodrich

Based on generalized Riccati transformation and some inequalities, some oscillation results are established for a class of nonlinear fractional difference equations with damping term. An example is given to illustrate the validity of the established results.

1. Introduction

During the past two decades, fractional calculus arose in the field of viscoelasticity, electrochemistry, physics, control, porous media, electromagnetism, and so forth; see [1–11] and the references therein. Fractional differential equations and difference equations can be used to describe some complex systems more accurately, and we may get the corresponding equations from those phenomena. However, as we all know, it is usually difficult to find the exact solutions to fractional differential or difference equations. In recent years, the study on qualitative properties of solutions of fractional differential equations, such as the existence, uniqueness, boundedness, oscillation, and other asymptotic behaviors, attracted much attention and some excellent results were obtained; we refer the reader to see [12–29] and the references cited therein.

In [12], Marian et al. studied the oscillation of fractional nonlinear difference equations of the form

$$\Delta^{\alpha} x(t) + f_1(t, x(t+\alpha)) = v(t) + f_2(t, x(t+\alpha)), \quad t \in N_0, \; 0 < \alpha \le 1. \tag{1}$$

Sagayaraj et al. [13] and Selvam et al. [14] investigated the oscillation of the following nonlinear fractional difference equations:

$$\begin{gathered}\Delta\left(p(t)\left(\Delta^{\alpha} x(t)\right)^{\gamma}\right) + q(t) f\left(\sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s)\right) = 0, \quad t \in N_{t_0+1-\alpha}, \\ \Delta\left(c(t)\left(\Delta^{\alpha} x(t)\right)^{\gamma}\right) + p(t)\left(\Delta^{\alpha} x(t)\right)^{\gamma} + q(t)\left(\sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s)\right)^{\gamma} = 0, \quad t \in N_{t_0+1-\alpha},\end{gathered} \tag{2}$$

where $0 < \alpha \le 1$ and $\gamma > 0$ is a quotient of odd positive integers.

In [15], Sagayaraj et al. studied oscillatory behavior of the following fractional difference equations:

$$\Delta\left(p(t)\Delta\left(\left[r(t)\Delta^{\alpha} x(t)\right]^{\eta}\right)\right) + F\left(t, \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s)\right) = 0, \quad t \in N_{t_0+1-\alpha}, \tag{3}$$

where $0 < \alpha \le 1$ and $\eta > 0$ is a quotient of odd positive integers.

In [16], Li investigated the oscillation of forced fractional difference equations with damping term of the form

$$\begin{aligned}(1+p(t))\Delta(\Delta^{\alpha}x(t))+p(t)\Delta^{\alpha}x(t)+f(t,x(t))\\=g(t),\quad t\in N_0,\end{aligned}\tag{4}$$

with initial condition $\Delta^{\alpha-1}x(t)|_{t=0}=x_0$, where $0<\alpha<1$.

In [24], Secer and Adiguzel established the oscillation results for a class of nonlinear fractional difference equations of the form

$$\begin{aligned}&\Delta\left(a(t)\left[\Delta\left(r(t)(\Delta^{\alpha}x(t))^{\gamma_1}\right)\right]^{\gamma_2}\right)\\&\quad+q(t)f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0,\\&\qquad t\in N_{t_0+1-\alpha},\end{aligned}\tag{5}$$

where $0<\alpha\le 1$ and γ_1 and γ_2 are the quotients of two odd positive numbers.

Motivated by the idea in [24], in this paper, we are concerned with the oscillation of a class of nonlinear fractional difference equations with damping term of the form

$$\begin{aligned}&\Delta\left(c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\right)+p(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\\&\quad+q(t)f\left(\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)=0,\\&\qquad t\in N_{t_0},\end{aligned}\tag{6}$$

where $\gamma\ge 1$ is a quotient of two odd positive integers, $0<\alpha\le 1$ is a constant, Δ^{α} denotes the Riemann-Liouville fractional difference operator of order α, and $N_{t_0}=\{t_0,t_0+1,t_0+2,\ldots\}$.

By a solution of (6), we mean a real-valued sequence $x(t)$ satisfying (6) for $t\in N_{t_0}$. A nontrivial solution $x(t)$ of (6) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (6) is called oscillatory if all of its solutions are oscillatory.

Throughout this paper, we assume that the following conditions hold:

(A) $c(t)$, $r(t)$, $p(t)$, and $q(t)$ are positive sequences, $c(t)>p(t)$.

(B) f is a monotone decreasing function satisfying $xf(x)>0$; $f(x)/x^{\gamma}\ge L>0$ for $x\ne 0$.

For convenience, in the rest of this paper, we set

$$\begin{aligned}G(t)&=\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s),\\\delta(t,t_1)&=\sum_{s=t_1}^{t-1}\frac{1}{(u(s)c(s))^{1/\gamma}}.\end{aligned}\tag{7}$$

2. Preliminaries and Lemmas

In this section, the definitions of the Riemann-Liouville fractional sum and difference are given; then some basic lemmas are presented, which will be used in the following proof.

Definition 1 (see [28]). The vth fractional sum of f, for $v>0$, is defined by

$$\Delta^{-v}f(t)=\frac{1}{\Gamma(v)}\sum_{s=a}^{t-v}(t-s-1)^{(v-1)}f(s),\tag{8}$$

where f is defined for $s=a\bmod(1)$, $\Delta^{-v}f$ is defined for $t=(a+v)\bmod(1)$, and $t^{(v)}=\Gamma(t+1)/\Gamma(t+1-v)$. The fractional sum Δ^{-v} is a map from N_a to N_{a+v}, where $N_t=\{t,t+1,t+2,\ldots\}$.

Definition 2 (see [28]). Let $\mu>0$ and $m-1<\mu<m$, where m denotes a positive integer; $m=\lceil\mu\rceil$. Set $v=m-\mu$; then μth fractional difference is defined as

$$\Delta^{\mu}f(t)=\Delta^{m-v}f(t)=\Delta^{m}\Delta^{-v}f(t).\tag{9}$$

Lemma 3 (see [14]). *Let $x(t)$ be a solution of (6), $G(t)=\sum_{s=t_0}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)$, and then*

$$\Delta G(t)=\Gamma(1-\alpha)\Delta^{\alpha}x(t).\tag{10}$$

Lemma 4 (see [29]). *The product and quotient rules of the difference operator Δ are as follows:*

$$\Delta[x(t)y(t)]=x(t+1)\Delta y(t)+\Delta x(t)\cdot y(t)\tag{11}$$

$$=\Delta x(t)\cdot y(t+1)+x(t)\Delta y(t),\tag{12}$$

$$\Delta\left[\frac{x(t)}{y(t)}\right]=\frac{\Delta x(t)\cdot y(t)-x(t)\Delta y(t)}{y(t)y(t+1)}\tag{13}$$

$$=\frac{\Delta x(t)\cdot y(t+1)-x(t+1)\Delta y(t)}{y(t)y(t+1)},\tag{14}$$

where $\Delta x(t)=x(t+1)-x(t)$.

Lemma 5. *If $\gamma\ge 1$ is a quotient of two odd positive integers, then the following two inequalities are established:*

$$\text{if } G(t+1)>G(t)>0,\quad\text{then } \Delta G^{\gamma}(t)\ge(\Delta G(t))^{\gamma},\tag{15}$$

$$\text{if } G(t+1)<G(t)<0,\quad\text{then } \Delta G^{\gamma}(t)\le(\Delta G(t))^{\gamma}.\tag{16}$$

Proof. Using the inequality (see [30])

$$|x-y|^{p}\le|x^{p}-y^{p}|\quad\text{where } x\ge 0,\ y\ge 0,\ p\ge 1,\tag{17}$$

we have the following results.

If $G(t+1)>G(t)>0$, let $x=G(t+1)$, $y=G(t)$, and $p=\gamma$; then $x>y>0$; it follows from (17) that $\Delta G^{\gamma}(t)=G^{\gamma}(t+1)-G^{\gamma}(t)\ge(G(t+1)-G(t))^{\gamma}=(\Delta G(t))^{\gamma}$, so $\Delta G^{\gamma}(t)\ge(\Delta G(t))^{\gamma}$.

If $G(t+1) < G(t) < 0$, then $-G(t+1) > -G(t) > 0$; let $x = -G(t+1)$, $y = -G(t)$, and $p = \gamma$; then $x > y > 0$; by (17), we obtain

$$(-G(t+1))^{\gamma} - (-G(t))^{\gamma} \geq (-G(t+1) + G(t))^{\gamma}. \quad (18)$$

Since $\gamma \geq 1$ is a quotient of two odd positive integers, then

$$\begin{aligned}(-G(t+1))^{\gamma} - (-G(t))^{\gamma} &= -G^{\gamma}(t+1) + G^{\gamma}(t) \\ &= -\Delta G^{\gamma}(t), \\ (-G(t+1) + G(t))^{\gamma} &= (-\Delta G(t))^{\gamma} = -(\Delta G(t))^{\gamma}.\end{aligned} \quad (19)$$

Substituting (19) into (18), we have $-\Delta G^{\gamma}(t) \geq -(\Delta G(t))^{\gamma}$, which means $\Delta G^{\gamma}(t) \leq (\Delta G(t))^{\gamma}$. □

Lemma 6. *Let $a > 0$, $b, X \in R$; then $bX - aX^2 \leq b^2/4a$.*

Proof.

$$bX - aX^2 = \frac{b^2}{4a} - \left(\sqrt{a}X - \frac{b}{2\sqrt{a}}\right)^2 \leq \frac{b^2}{4a}. \quad (20)$$

We define the following sequence:

$$u(t) = \prod_{s=t_0}^{t-1} \frac{c(s)}{c(s) - p(s)}; \quad (21)$$

then

$$\begin{aligned}&u(t) > 0, \\ &u(t+1) = \frac{c(t)}{c(t) - p(t)} u(t), \\ &\Delta u(t) = u(t+1) - u(t) = \frac{p(t)}{c(t) - p(t)} u(t).\end{aligned} \quad (22)$$

□

3. Main Results

Lemma 7. *Assume that $x(t)$ is an eventually positive solution of (6) and*

$$\sum_{s=t_0}^{\infty} \frac{1}{(u(s)c(s))^{1/\gamma}} = \infty, \quad (23)$$

$$\sum_{s=t_0}^{\infty} \frac{1}{r(s)} = \infty, \quad (24)$$

$$\sum_{\xi=t_0}^{\infty} \frac{1}{r(\xi)} \sum_{\tau=\xi}^{\infty} \left[\frac{1}{u(\tau)c(\tau)} \sum_{s=\tau}^{\infty} u(s+1)q(s)\right]^{1/\gamma} = \infty; \quad (25)$$

then, there exists a sufficiently large $T \in N_{t_0}$ such that $\Delta(r(t)\Delta^{\alpha}x(t)) > 0$ on $[T, \infty)$ and one of the following two conditions holds: (i) $\Delta^{\alpha}x(t) > 0$ on $[T, \infty)$ and (ii) $\Delta^{\alpha}x(t) < 0$ on $[T, \infty)$ and $\lim_{t\to\infty} G(t) = 0$.

Proof. Since $x(t)$ is an eventually positive solution of (6), then there exists a sufficiently large t_1 such that $x(t) > 0$, $t \in [t_1, \infty)$. So $G(t) > 0$, $t \in [t_1, \infty)$. Noting assumption (B), from (6) we obtain

$$\begin{aligned}&\Delta\left(c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\right) \\ &\quad + p(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma} = -q(t)f(G(t)) \\ &\quad \leq -Lq(t)G^{\gamma}(t).\end{aligned} \quad (26)$$

Therefore, it follows from the definition of $u(t)$ and the product rule (11) of the difference operator Δ that

$$\begin{aligned}&\Delta\left(u(t)c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\right) = u(t+1) \\ &\quad \cdot \Delta\left(c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\right) + \Delta u(t) \cdot c(t) \\ &\quad \cdot \left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma} = u(t+1) \\ &\quad \cdot \Delta\left(c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\right) + \frac{p(t)}{c(t) - p(t)} \\ &\quad \cdot \frac{c(t) - p(t)}{c(t)} u(t+1)c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma} \\ &\quad = u(t+1)\left[\Delta\left(c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\right)\right. \\ &\quad \left. + p(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\right] \leq -Lu(t+1)q(t) \\ &\quad \cdot G^{\gamma}(t) < 0.\end{aligned} \quad (27)$$

Then, $u(t)c(t)[\Delta(r(t)\Delta^{\alpha}x(t))]^{\gamma}$ is strictly decreasing on $[t_1, \infty)$, and thus $\Delta(r(t)\Delta^{\alpha}x(t))$ is eventually of one sign. For $t_2 > t_1$ is sufficiently large, we claim that $\Delta(r(t)\Delta^{\alpha}x(t)) > 0$ on $[t_2, \infty)$. Otherwise, assume that there exists a sufficiently large $t_3 > t_2$ such that $\Delta(r(t_3)\Delta^{\alpha}x(t_3)) < 0$; then, for $t \in (t_3, \infty)$, we get

$$\begin{aligned}&u(t)c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma} \\ &\quad < u(t_3)c(t_3)\left[\Delta\left(r(t_3)\Delta^{\alpha}x(t_3)\right)\right]^{\gamma} = K < 0,\end{aligned} \quad (28)$$

that is,

$$\Delta\left(r(t)\Delta^{\alpha}x(t)\right) < \left[\frac{K}{u(t)c(t)}\right]^{1/\gamma} < 0, \quad t \in (t_3, \infty); \quad (29)$$

So we can get $\Delta(r(t)\Delta^{\alpha}x(t)) < 0$ on $[t_3, \infty)$. From these terms, for $t \in [t_3, \infty)$, we have

$$\begin{aligned}&r(t)\Delta^{\alpha}x(t) - r(t_3)\Delta^{\alpha}x(t_3) \\ &\quad = \sum_{s=t_3}^{t-1} \frac{(u(s)c(s))^{1/\gamma}\Delta\left(r(s)\Delta^{\alpha}x(s)\right)}{(u(s)c(s))^{1/\gamma}} \\ &\quad \leq \left(u(t_3)c(t_3)\right)^{1/\gamma}\Delta\left(r(t_3)\Delta^{\alpha}x(t_3)\right) \\ &\quad \cdot \sum_{s=t_3}^{t-1} \frac{1}{(u(s)c(s))^{1/\gamma}}.\end{aligned} \quad (30)$$

By (23), we obtain $\lim_{t\to\infty} r(t)\Delta^{\alpha}x(t) = -\infty$, which means, for some sufficiently large $t_4 > t_3$, $\Delta^{\alpha}x(t) < 0$ on $[t_4, \infty)$. By Lemma 3, we have

$$\begin{aligned} G(t) - G(t_4) &= \sum_{s=t_4}^{t-1} \Delta G(s) = \Gamma(1-\alpha) \sum_{s=t_4}^{t-1} \Delta^{\alpha} x(s) \\ &= \Gamma(1-\alpha) \sum_{s=t_4}^{t-1} \frac{r(s)\Delta^{\alpha}x(s)}{r(s)} \\ &\le \Gamma(1-\alpha) r(t_4) \Delta^{\alpha} x(t_4) \sum_{s=t_4}^{t-1} \frac{1}{r(s)}. \end{aligned} \tag{31}$$

By (24), we obtain $\lim_{t\to\infty} G(t) = -\infty$, which contradicts $G(t) > 0$, $t \in [t_1, \infty)$. Therefore, $\Delta(r(t)\Delta^{\alpha}x(t)) > 0$, $t \in [t_2, \infty)$. Thus, $\Delta^{\alpha}x(t)$ is eventually of one sign. There are two possibilities: (i) $\Delta^{\alpha}x(t) > 0$ on $[T, \infty)$ and (ii) $\Delta^{\alpha}x(t) < 0$ on $[T, \infty)$, where T is sufficiently large.

Now, we assume that $\Delta^{\alpha}x(t) < 0$, $t \in [t_5, \infty)$, where $t_5 > t_4$ is sufficiently large. Then, by Lemma 3, we have $\Delta G(t) = \Gamma(1-\alpha)\Delta^{\alpha}x(t) < 0$, $t \in [t_5, \infty)$. Since $G(t) > 0$, $t \in [t_1, \infty)$, we have $\lim_{t\to\infty} G(t) = \beta \ge 0$. We claim that $\beta = 0$. Otherwise, assume that $\beta > 0$. Then, $G(t) \ge \beta$, $t \in [t_5, \infty)$. By (27), we have

$$\begin{aligned} &\Delta\left(u(t)c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\right) \\ &\quad \le -Lu(t+1)q(t)G^{\gamma}(t) \le -Lu(t+1)q(t)\beta^{\gamma}. \end{aligned} \tag{32}$$

Substituting t with s in (32), a summation for (32) with respect to s from t to ∞ yields

$$\begin{aligned} &\sum_{s=t}^{\infty} \Delta\left(u(s)c(s)\left[\Delta\left(r(s)\Delta^{\alpha}x(s)\right)\right]^{\gamma}\right) \\ &\quad \le -L\beta^{\gamma} \sum_{s=t}^{\infty} u(s+1)q(s), \end{aligned} \tag{33}$$

which implies

$$\begin{aligned} &-u(t)c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma} \\ &\quad \le -\lim_{s\to\infty} u(s)c(s)\left[\Delta\left(r(s)\Delta^{\alpha}x(s)\right)\right]^{\gamma} \\ &\quad - L\beta^{\gamma}\sum_{s=t}^{\infty} u(s+1)q(s) < -L\beta^{\gamma}\sum_{s=t}^{\infty} u(s+1)q(s); \end{aligned} \tag{34}$$

therefore,

$$\begin{aligned} &\Delta\left(r(t)\Delta^{\alpha}x(t)\right) \\ &\quad > L^{1/\gamma}\beta\left[\frac{1}{u(t)c(t)}\sum_{s=t}^{\infty} u(s+1)q(s)\right]^{1/\gamma}. \end{aligned} \tag{35}$$

Substituting t with τ in (35), a summation for (35) with respect to τ from t to ∞ yields

$$\begin{aligned} &\sum_{\tau=t}^{\infty} \Delta\left(r(\tau)\Delta^{\alpha}x(\tau)\right) \\ &\quad > L^{1/\gamma}\beta\sum_{\tau=t}^{\infty}\left[\frac{1}{u(\tau)c(\tau)}\sum_{s=\tau}^{\infty} u(s+1)q(s)\right]^{1/\gamma}; \end{aligned} \tag{36}$$

namely,

$$\begin{aligned} &-r(t)\Delta^{\alpha}x(t) \\ &\quad > -\lim_{\tau\to\infty} r(\tau)\Delta^{\alpha}x(\tau) \\ &\quad + L^{\frac{1}{\gamma}}\beta\sum_{\tau=t}^{\infty}\left[\frac{1}{u(\tau)c(\tau)}\sum_{s=\tau}^{\infty} u(s+1)q(s)\right]^{\frac{1}{\gamma}} \\ &\quad > L^{\frac{1}{\gamma}}\beta\sum_{\tau=t}^{\infty}\left[\frac{1}{u(\tau)c(\tau)}\sum_{s=\tau}^{\infty} u(s+1)q(s)\right]^{\frac{1}{\gamma}}; \end{aligned} \tag{37}$$

therefore,

$$\begin{aligned} &\Delta^{\alpha}x(t) \\ &\quad < -L^{\frac{1}{\gamma}}\beta\frac{1}{r(t)}\sum_{\tau=t}^{\infty}\left[\frac{1}{u(\tau)c(\tau)}\sum_{s=\tau}^{\infty} u(s+1)q(s)\right]^{\frac{1}{\gamma}}; \end{aligned} \tag{38}$$

that is,

$$\begin{aligned} &\Delta G(t) < -\Gamma(1-\alpha)L^{1/\gamma}\beta\frac{1}{r(t)} \\ &\quad \cdot \sum_{\tau=t}^{\infty}\left[\frac{1}{u(\tau)c(\tau)}\sum_{s=\tau}^{\infty} u(s+1)q(s)\right]^{1/\gamma}. \end{aligned} \tag{39}$$

Substituting t with ξ in (39), a summation for (39) with respect to ξ from t_5 to $t-1$ yields

$$\begin{aligned} &\sum_{\xi=t_5}^{t-1}\Delta G(\xi) < -\Gamma(1-\alpha)L^{\frac{1}{\gamma}}\beta\sum_{\xi=t_5}^{t-1}\frac{1}{r(\xi)} \\ &\quad \cdot \sum_{\tau=\xi}^{\infty}\left[\frac{1}{u(\tau)c(\tau)}\sum_{s=\tau}^{\infty} u(s+1)q(s)\right]^{\frac{1}{\gamma}}; \end{aligned} \tag{40}$$

then

$$\begin{aligned} &G(t) - G(t_5) < -\Gamma(1-\alpha)L^{1/\gamma}\beta\sum_{\xi=t_5}^{t-1}\frac{1}{r(\xi)} \\ &\quad \cdot \sum_{\tau=\xi}^{\infty}\left[\frac{1}{u(\tau)c(\tau)}\sum_{s=\tau}^{\infty} u(s+1)q(s)\right]^{1/\gamma}. \end{aligned} \tag{41}$$

By (25), it follows from (41) that $\lim_{t\to\infty} G(t) = -\infty$, which contradicts $G(t) > 0$, $t \in [t_1, \infty)$. Then we get that $\beta = 0$, which is $\lim_{t\to\infty} G(t) = 0$. This completes the proof of Lemma 7. □

By the same proof as above, if $x(t)$ is an eventually negative solution of (6), we can obtain $\Delta(u(t)c(t)[\Delta(r(t)\Delta^{\alpha}x(t))]^{\gamma}) > 0$ and Lemma 8 holds.

Lemma 8. *Assume that $x(t)$ is an eventually negative solution of (6) and (23)–(25) hold. Then, there exists a sufficiently large $T \in N_{t_0}$ such that $\Delta(r(t)\Delta^{\alpha}x(t)) < 0$ on $[T,\infty)$ and one of the following two conditions holds: (i) $\Delta^{\alpha}x(t) < 0$ on $[T,\infty)$ and (ii) $\Delta^{\alpha}x(t) > 0$ on $[T,\infty)$ and $\lim_{t\to\infty} G(t) = 0$.*

Lemma 9. *Assume that $x(t)$ is an eventually positive solution of (6) such that $\Delta(r(t)\Delta^{\alpha}x(t)) > 0$, $\Delta^{\alpha}x(t) > 0$ on $[t_1,\infty)$, where $t_1 \geq t_0$ is sufficiently large. Then*

$$\Delta G(t) \geq \frac{\Gamma(1-\alpha)\,\delta(t,t_1)(u(t)c(t))^{1/\gamma}\,\Delta(r(t)\Delta^{\alpha}x(t))}{r(t)}. \tag{42}$$

Proof. Assume that $x(t)$ is an eventually positive solution of (6); then we obtain that $u(t)c(t)[\Delta(r(t)\Delta^{\alpha}x(t))]^{\gamma}$ is strictly decreasing on $[t_1,\infty)$; by (27) we have,

$$\begin{aligned} r(t)\Delta^{\alpha}x(t) &\geq r(t)\Delta^{\alpha}x(t) - r(t_1)\Delta^{\alpha}x(t_1) \\ &= \sum_{s=t_1}^{t-1} \frac{(u(s)c(s))^{1/\gamma}\,\Delta(r(s)\Delta^{\alpha}x(s))}{(u(s)c(s))^{1/\gamma}} \\ &\geq (u(t)c(t))^{1/\gamma}\,\Delta(r(t)\Delta^{\alpha}x(t)) \sum_{s=t_1}^{t-1}\frac{1}{(u(s)c(s))^{1/\gamma}} \\ &= \delta(t,t_1)(u(t)c(t))^{1/\gamma}\,\Delta(r(t)\Delta^{\alpha}x(t)). \end{aligned} \tag{43}$$

By Lemma 3, we obtain

$$\Delta G(t) \geq \frac{\Gamma(1-\alpha)\,\delta(t,t_1)(u(t)c(t))^{1/\gamma}\,\Delta(r(t)\Delta^{\alpha}x(t))}{r(t)}. \tag{44}$$

Then the proof is complete. □

With the same proof as that in Lemma 9, we can obtain the following.

Lemma 10. *Assume that $x(t)$ is an eventually negative solution of (6) such that $\Delta(r(t)\Delta^{\alpha}x(t)) < 0$, $\Delta^{\alpha}x(t) < 0$ on $[t_1,\infty)$, where $t_1 \geq t_0$ is sufficiently large. Then*

$$\begin{aligned} \Delta G(t) &\leq \frac{\Gamma(1-\alpha)\,\delta(t,t_1)(u(t)c(t))^{1/\gamma}\,\Delta(r(t)\Delta^{\alpha}x(t))}{r(t)} \\ &< \frac{\Gamma(1-\alpha)\,\delta(t,t_1)(u(t+1)c(t+1))^{1/\gamma}\,\Delta(r(t+1)\Delta^{\alpha}x(t+1))}{r(t)}. \end{aligned} \tag{45}$$

Theorem 11. *Assume that (23)–(25) hold. If*

$$\limsup_{t\to\infty}\sum_{s=T}^{t-1}\left(Lu(s)q(s) - \frac{(\Delta u(s))^2}{4u(s)u(s+1)R(s)}\right) = \infty, \tag{46}$$

where T is sufficiently large, $u(t)$ is defined as in (21), and $R(t) = [\Gamma(1-\alpha)\delta(t,t_1)/r(t)]^{\gamma}$, then (6) is oscillatory or satisfies $\lim_{t\to\infty} G(t) = 0$.

Proof. Suppose to the contrary that (6) has a nonoscillatory solution $x(t)$, $t \in N_{t_0}$; then $x(t)$ is either eventually positive or eventually negative.

In the case when $x(t)$ is eventually positive, we assume that $x(t) > 0$ on $[t_1,\infty)$, where $t_1 \in N_{t_0}$ is sufficiently large; then $G(t) > 0$. By Lemma 7, we obtain $\Delta(r(t)\Delta^{\alpha}x(t)) > 0$ on $[t_2,\infty)$, where $t_2 > t_1$ is sufficiently large, and either $\Delta^{\alpha}x(t) > 0$ on $[t_2,\infty)$ or $\lim_{t\to\infty} G(t) = 0$.

If $\Delta^{\alpha}x(t) > 0$ on $[t_2,\infty)$, then the conclusion of Lemma 9 holds.

Define the generalized Riccati function as follows:

$$w(t) = u(t)\frac{c(t)[\Delta(r(t)\Delta^{\alpha}x(t))]^{\gamma}}{G^{\gamma}(t)}, \quad t \in [t_2,\infty). \tag{47}$$

It is clear that $w(t) > 0$. By the product rule (12) and the quotient rule (13), for $t \in [t_2,\infty)$, we have

$$\begin{aligned} \Delta w(t) &= \Delta u(t)\cdot\frac{w(t+1)}{u(t+1)} + u(t)\,\Delta\left(\frac{c(t)[\Delta(r(t)\Delta^{\alpha}x(t))]^{\gamma}}{G^{\gamma}(t)}\right) \\ &= \Delta u(t)\cdot\frac{w(t+1)}{u(t+1)} + u(t)\frac{\Delta\left(c(t)[\Delta(r(t)\Delta^{\alpha}x(t))]^{\gamma}\right)\cdot G^{\gamma}(t) - c(t)[\Delta(r(t)\Delta^{\alpha}x(t))]^{\gamma}\,\Delta G^{\gamma}(t)}{G^{\gamma}(t)G^{\gamma}(t+1)} \end{aligned}$$

$$
\begin{aligned}
&= \Delta u(t) \cdot \frac{w(t+1)}{u(t+1)} + u(t) \frac{-p(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma} - q(t) f(G(t))}{G^{\gamma}(t+1)} - \frac{u(t)c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\Delta G^{\gamma}(t)}{G^{\gamma}(t)G^{\gamma}(t+1)} \\
&= \frac{\Delta u(t)}{u(t+1)} w(t+1) - \frac{u(t)p(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}}{G^{\gamma}(t+1)} - u(t)q(t)\frac{f(G(t))}{G^{\gamma}(t+1)} \\
&\quad - \frac{u(t)c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\Delta G^{\gamma}(t)}{G^{\gamma}(t)G^{\gamma}(t+1)}.
\end{aligned} \tag{48}
$$

From $\Delta(r(t)\Delta^{\alpha}x(t)) > 0$ and Lemma 3, we have

$$
\frac{u(t)p(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}}{G^{\gamma}(t+1)} > 0, \tag{49}
$$

and $\Delta G(t) > 0$; then $G(t+1) > G(t) > 0$; it follows from (B) that

$$
\frac{f(G(t))}{G^{\gamma}(t+1)} > \frac{f(G(t+1))}{G^{\gamma}(t+1)} \geq L, \tag{50}
$$

and $\Delta G^{\gamma}(t) \geq (\Delta G(t))^{\gamma}$ by (15).

Using Lemma 9 and the fact that $u(t)c(t)[\Delta(r(t)\Delta^{\alpha}x(t))]^{\gamma}$ is strictly decreasing, we have

$$
\begin{aligned}
&\frac{u(t)c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\Delta G^{\gamma}(t)}{G^{\gamma}(t)G^{\gamma}(t+1)} \geq \frac{u(t)c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}(\Delta G(t))^{\gamma}}{G^{\gamma}(t)G^{\gamma}(t+1)} \\
&\geq \frac{u(t)c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\left[\Gamma(1-\alpha)\delta(t,t_1)/r(t)\right]^{\gamma} u(t)c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}}{G^{\gamma}(t)G^{\gamma}(t+1)} \\
&> \frac{\left(u(t)c(t)\left[\Delta\left(r(t)\Delta^{\alpha}x(t)\right)\right]^{\gamma}\right)^{2} R(t)}{G^{2\gamma}(t+1)} > \frac{\left(u(t+1)c(t+1)\left[\Delta\left(r(t+1)\Delta^{\alpha}x(t+1)\right)\right]^{\gamma}\right)^{2} R(t)}{G^{2\gamma}(t+1)} = R(t)w^{2}(t+1).
\end{aligned} \tag{51}
$$

Now substituting (49), (50), and (51) into (48), we obtain

$$
\begin{aligned}
\Delta w(t) &< \frac{\Delta u(t)}{u(t+1)} w(t+1) - Lu(t)q(t) \\
&\quad - R(t)w^{2}(t+1).
\end{aligned} \tag{52}
$$

Taking $a = R(t) > 0$, $b = \Delta u(t)/u(t+1)$, and $X = w(t+1)$ in (52), using Lemma 6, we obtain

$$
\Delta w(t) < -Lu(t)q(t) + \frac{(\Delta u(t))^{2}}{4R(t)(u(t+1))^{2}}. \tag{53}
$$

Substituting t with s in (53), a summation for (53) with respect to s from t_2 to $t-1$ yields

$$
\begin{aligned}
&\sum_{s=t_2}^{t-1}\left(Lu(s)q(s) - \frac{(\Delta u(s))^{2}}{4R(s)(u(s+1))^{2}}\right) < -\sum_{s=t_2}^{t-1}\Delta w(s) \\
&= w(t_2) - w(t) < w(t_2) < \infty.
\end{aligned} \tag{54}
$$

Since $u(s+1) > u(s) > 0$, it is clear that

$$
\frac{(\Delta u(s))^{2}}{4u(s)u(s+1)R(s)} > \frac{(\Delta u(s))^{2}}{4(u(s+1))^{2}R(s)}; \tag{55}
$$

therefore,

$$
\begin{aligned}
&\sum_{s=t_2}^{t-1}\left(Lu(s)q(s) - \frac{(\Delta u(s))^{2}}{4(u(s+1))^{2}R(s)}\right) \\
&\quad > \sum_{s=t_2}^{t-1}\left(Lu(s)q(s) - \frac{(\Delta u(s))^{2}}{4u(s)u(s+1)R(s)}\right),
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
&\limsup_{t\to\infty}\sum_{s=t_2}^{t-1}\left(Lu(s)q(s) - \frac{(\Delta u(s))^{2}}{4(u(s+1))^{2}R(s)}\right) \\
&\quad \geq \limsup_{t\to\infty}\sum_{s=t_2}^{t-1}\left(Lu(s)q(s) - \frac{(\Delta u(s))^{2}}{4u(s)u(s+1)R(s)}\right).
\end{aligned} \tag{57}
$$

From (46) and (57), we obtain that

$$
\limsup_{t\to\infty}\sum_{s=t_2}^{t-1}\left(Lu(s)q(s) - \frac{(\Delta u(s))^{2}}{4(u(s+1))^{2}R(s)}\right) = \infty. \tag{58}
$$

Taking lim sup in (54) as $t \to \infty$, we have

$$
\begin{aligned}
&\limsup_{t\to\infty}\sum_{s=t_2}^{t-1}\left(Lu(s)q(s) - \frac{(\Delta u(s))^{2}}{4R(s)(u(s+1))^{2}}\right) \\
&\quad \leq w(t_2) < \infty,
\end{aligned} \tag{59}
$$

which contradicts (58).

If $\Delta^{\alpha}x(t) < 0$ on $[t_2, \infty)$, then, from Lemma 7, we get that $\lim_{t\to\infty} G(t) = 0$.

In the case when $x(t)$ is eventually negative, we assume that $x(t) < 0$ on $[t_1, \infty)$, where $t_1 \in N_{t_0}$ is sufficiently large; then $G(t) < 0$. By Lemma 8, we obtain $\Delta(r(t)\Delta^{\alpha}x(t)) < 0$ on $[t_2, \infty)$, where $t_2 > t_1$ is sufficiently large, and either $\Delta^{\alpha}x(t) < 0$ on $[t_2, \infty)$ or $\lim_{t\to\infty} G(t) = 0$.

If $\Delta^{\alpha}x(t) < 0$ on $[t_2, \infty)$, then the conclusion of Lemma 10 holds.

Define $w(t)$ as in (47); since $\Delta(r(t)\Delta^{\alpha}x(t)) < 0, G(t) < 0$, then $w(t) > 0$.

Using the product rule (12) and the quotient rule (14), for $t \in [t_2, \infty)$, we get

$$\begin{aligned}\Delta w(t) = {} & \frac{\Delta u(t)}{u(t+1)} w(t+1) \\ & - \frac{u(t)p(t)[\Delta(r(t)\Delta^{\alpha}x(t))]^{\gamma}}{G^{\gamma}(t)} \\ & - u(t)q(t)\frac{f(G(t))}{G^{\gamma}(t)} \\ & - \frac{u(t)c(t+1)[\Delta(r(t+1)\Delta^{\alpha}x(t+1))]^{\gamma}\Delta G^{\gamma}(t)}{G^{\gamma}(t)G^{\gamma}(t+1)}.\end{aligned} \tag{60}$$

By $\Delta(r(t)\Delta^{\alpha}x(t)) < 0$ and $G(t) < 0$, we have

$$\frac{u(t)p(t)[\Delta(r(t)\Delta^{\alpha}x(t))]^{\gamma}}{G^{\gamma}(t)} > 0, \tag{61}$$

and $\Delta G(t) < 0$ by Lemma 3; then $G(t+1) < G(t) < 0$; from (B) and (16), we obtain that

$$\frac{f(G(t))}{G^{\gamma}(t)} \geq L, \tag{62}$$

$$\Delta G^{\gamma}(t) \leq (\Delta G(t))^{\gamma}. \tag{63}$$

Proceeding the proof of Lemma 7, we have

$$\begin{aligned}& \Delta\left(u(t)c(t)[\Delta(r(t)\Delta^{\alpha}x(t))]^{\gamma}\right) \\ & \quad \geq -Lu(t+1)q(t)G^{\gamma}(t) > 0,\end{aligned} \tag{64}$$

which implies $u(t)c(t)[\Delta(r(t)\Delta^{\alpha}x(t))]^{\gamma}$ is strictly increasing. By virtue of Lemma 10, $\Delta G^{\gamma}(t) \leq (\Delta G(t))^{\gamma}$, and

$$\frac{u(t)c(t+1)[\Delta(r(t+1)\Delta^{\alpha}x(t+1))]^{\gamma}}{G^{\gamma}(t)G^{\gamma}(t+1)} < 0, \tag{65}$$

we have

$$\begin{aligned}\frac{u(t)c(t+1)[\Delta(r(t+1)\Delta^{\alpha}x(t+1))]^{\gamma}\Delta G^{\gamma}(t)}{G^{\gamma}(t)G^{\gamma}(t+1)} & \geq \frac{u(t)c(t+1)[\Delta(r(t+1)\Delta^{\alpha}x(t+1))]^{\gamma}(\Delta G(t))^{\gamma}}{G^{\gamma}(t)G^{\gamma}(t+1)} \\ & > \frac{u(t)u(t+1)\left(c(t+1)[\Delta(r(t+1)\Delta^{\alpha}x(t+1))]^{\gamma}\right)^2[\Gamma(1-\alpha)\delta(t,t_1)/r(t)]^{\gamma}}{G^{\gamma}(t)G^{\gamma}(t+1)} \\ & = \frac{u(t)u(t+1)\left(c(t+1)[\Delta(r(t+1)\Delta^{\alpha}x(t+1))]^{\gamma}\right)^2 R(t)}{G^{\gamma}(t)G^{\gamma}(t+1)} = \frac{u(t)G^{2\gamma}(t+1)R(t)}{u(t+1)G^{\gamma}(t)G^{\gamma}(t+1)}w^2(t+1) \\ & > \frac{u(t)R(t)}{u(t+1)}w^2(t+1),\end{aligned} \tag{66}$$

where $G^{\gamma}(t+1) < G^{\gamma}(t) < 0$ and $G^{\gamma}(t+1)/G^{\gamma}(t) > 1$.

Combining (60), (61), (62), and (66), we can obtain

$$\begin{aligned}\Delta w(t) < {} & \frac{\Delta u(t)}{u(t+1)}w(t+1) - Lu(t)q(t) \\ & - \frac{u(t)R(t)}{u(t+1)}w^2(t+1).\end{aligned} \tag{67}$$

Taking

$$\begin{aligned} a &= \frac{u(t)R(t)}{u(t+1)} > 0, \\ b &= \frac{\Delta u(t)}{u(t+1)}, \\ X &= w(t+1),\end{aligned} \tag{68}$$

using Lemma 6, we have

$$\Delta w(t) < -Lu(t)q(t) + \frac{(\Delta u(t))^2}{4u(t)u(t+1)R(t)}. \tag{69}$$

Substituting t with s in (69), a summation for (69) with respect to s from t_2 to $t-1$ yields

$$\begin{aligned}& \sum_{s=t_2}^{t-1}\left(Lu(s)q(s) - \frac{(\Delta u(s))^2}{4u(s)u(s+1)R(s)}\right) \\ & \quad < \sum_{s=t_2}^{t-1} -\Delta w(s) = w(t_2) - w(t) < w(t_2) < \infty.\end{aligned} \tag{70}$$

Taking lim sup in (70) as $t \to \infty$, we have

$$\limsup_{t\to\infty} \sum_{s=t_2}^{t-1} \left(Lu(s)q(s) - \frac{(\Delta u(s))^2}{4u(s)u(s+1)R(s)} \right) \le w(t_2) < \infty, \tag{71}$$

which contradicts (46).

If $\Delta^\alpha x(t) > 0$ on $[t_2, \infty)$, then, from Lemma 8, we get that $\lim_{t\to\infty} G(t) = 0$.

The proof of Theorem 11 is complete. □

Theorem 12. *Assume that (23)–(25) hold and there exists a positive sequence $H(t,s)$ such that*

$$\begin{aligned} &H(t,t) = 0 \quad \text{for } t \ge t_0, \\ &H(t,s) > 0 \quad \text{for } t > s \ge t_0, \\ &\Delta_2 H(t,s) = H(t,s+1) - H(t,s) < 0 \\ &\qquad\qquad \text{for } t \ge s \ge t_0. \end{aligned} \tag{72}$$

If

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \cdot \sum_{s=t_0}^{t-1} \left(Lu(s)q(s)H(t,s) - \frac{h^2(t,s)u(s+1)}{4H(t,s)u(s)R(s)} \right) = \infty, \tag{73}$$

where $h(t,s) = \Delta_2 H(t,s) + H(t,s)\Delta u(s)/u(s+1)$ and $u(t)$ and $R(t)$ are the same as in Theorem 11, then (6) is oscillatory or satisfies $\lim_{t\to\infty} G(t) = 0$.

Proof. Suppose on the contrary that $x(t)$ is a nonoscillatory solution of (6); then $x(t)$ is either eventually positive or eventually negative.

In the case when $x(t)$ is eventually positive, we assume that $x(t) > 0$ on $[t_1, \infty)$, where $t_1 \in N_{t_0}$ is sufficiently large. According to the proof of Theorem 11, if $\Delta^\alpha x(t) > 0$ on $[t_2, \infty)$, then (52) holds.

Substituting t with s in (52), multiplying both sides by $H(t,s)$, and then summing with respect to s from t_2 to $t-1$ yield

$$\begin{aligned} \sum_{s=t_2}^{t-1} Lu(s)q(s)H(t,s) &< -\sum_{s=t_2}^{t-1} H(t,s)\Delta w(s) \\ &\quad + \sum_{s=t_2}^{t-1} H(t,s)\frac{\Delta u(s)}{u(s+1)} w(s+1) \\ &\quad - \sum_{s=t_2}^{t-1} H(t,s)R(s)w^2(s+1). \end{aligned} \tag{74}$$

Using summation by parts formula, we obtain

$$\begin{aligned} -\sum_{s=t_2}^{t-1} H(t,s)\Delta w(s) &= -H(t,s)w(s)\big|_{s=t_2}^{t} + \sum_{s=t_2}^{t-1} w(s+1)\Delta_2 H(t,s) \\ &= H(t,t_2)w(t_2) + \sum_{s=t_2}^{t-1} w(s+1)\Delta_2 H(t,s). \end{aligned} \tag{75}$$

Therefore,

$$\begin{aligned} \sum_{s=t_2}^{t-1} Lu(s)q(s)H(t,s) &< H(t,t_2)w(t_2) \\ &\quad + \sum_{s=t_2}^{t-1} \left[\left(\Delta_2 H(t,s) + \frac{H(t,s)\Delta u(s)}{u(s+1)} \right) w(s+1) - H(t,s)R(s)w^2(s+1) \right] = H(t,t_2)w(t_2) \\ &\quad + \sum_{s=t_2}^{t-1} \left(h(t,s)w(s+1) - H(t,s)R(s)w^2(s+1) \right). \end{aligned} \tag{76}$$

Taking $a = H(t,s)R(s) > 0$, $b = h(t,s)$, and $X = w(t+1)$ in (76), using Lemma 6, we obtain

$$\sum_{s=t_2}^{t-1} Lu(s)q(s)H(t,s) < H(t,t_2)w(t_2) + \sum_{s=t_2}^{t-1} \frac{h^2(t,s)}{4H(t,s)R(s)}, \tag{77}$$

which means

$$\sum_{s=t_2}^{t-1} \left(Lu(s)q(s)H(t,s) - \frac{h^2(t,s)}{4H(t,s)R(s)} \right) < H(t,t_2)w(t_2) < H(t,t_0)w(t_2), \tag{78}$$

for $t > t_2 > t_1 > t_0$. Then

$$\begin{aligned} &\sum_{s=t_0}^{t-1} \left(Lu(s)q(s)H(t,s) - \frac{h^2(t,s)}{4H(t,s)R(s)} \right) \\ &= \sum_{s=t_0}^{t_2-1} \left(Lu(s)q(s)H(t,s) - \frac{h^2(t,s)}{4H(t,s)R(s)} \right) + \sum_{s=t_2}^{t-1} \left(Lu(s)q(s)H(t,s) - \frac{h^2(t,s)}{4H(t,s)R(s)} \right) \\ &< \sum_{s=t_0}^{t_2-1} \left(Lu(s)q(s)H(t,s) - \frac{h^2(t,s)}{4H(t,s)R(s)} \right) \end{aligned}$$

$$\begin{aligned}&+H(t,t_0)w(t_2)\\&<H(t,t_0)\sum_{s=t_0}^{t_2-1}Lu(s)q(s)+H(t,t_0)w(t_2),\end{aligned}\tag{79}$$

which means

$$\begin{aligned}&\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left(Lu(s)q(s)H(t,s)-\frac{h^2(t,s)}{4H(t,s)R(s)}\right)\\&<\sum_{s=t_0}^{t_2-1}Lu(s)q(s)+w(t_2).\end{aligned}\tag{80}$$

Since $u(s+1)>u(s)>0$, it is obvious that

$$\frac{h^2(t,s)u(s+1)}{4H(t,s)u(s)R(s)}>\frac{h^2(t,s)}{4H(t,s)R(s)};\tag{81}$$

therefore,

$$\begin{aligned}&\frac{1}{H(t,t_0)}\sum_{s=t_0}^{t-1}\left(Lu(s)q(s)H(t,s)-\frac{h^2(t,s)}{4H(t,s)R(s)}\right)\\&>\frac{1}{H(t,t_0)}\\&\cdot\sum_{s=t_0}^{t-1}\left(Lu(s)q(s)H(t,s)-\frac{h^2(t,s)u(s+1)}{4H(t,s)u(s)R(s)}\right),\end{aligned}\tag{82}$$

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\\&\cdot\sum_{s=t_0}^{t-1}\left(Lu(s)q(s)H(t,s)-\frac{h^2(t,s)}{4H(t,s)R(s)}\right)\\&\geq\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\\&\cdot\sum_{s=t_0}^{t-1}\left(Lu(s)q(s)H(t,s)-\frac{h^2(t,s)u(s+1)}{4H(t,s)u(s)R(s)}\right).\end{aligned}\tag{83}$$

By (73) and (83), we have

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\\&\cdot\sum_{s=t_0}^{t-1}\left(Lu(s)q(s)H(t,s)-\frac{h^2(t,s)}{4H(t,s)R(s)}\right)=\infty.\end{aligned}\tag{84}$$

Taking lim sup in (80) as $t\to\infty$, we obtain

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\\&\cdot\sum_{s=t_0}^{t-1}\left(Lu(s)q(s)H(t,s)-\frac{h^2(t,s)}{4H(t,s)R(s)}\right)\\&\leq\sum_{s=t_0}^{t_2-1}Lu(s)q(s)+w(t_2)<\infty,\end{aligned}\tag{85}$$

which contradicts (84).

If $\Delta^\alpha x(t)<0$ on $[t_2,\infty)$, then, from Lemma 7, we have $\lim_{t\to\infty}G(t)=0$.

In the case when $x(t)$ is eventually negative, it can be proved similarly; here we omit it.

The proof of Theorem 12 is complete. □

4. Applications

In this section, an example is shown to illustrate the validity of the established results above.

Example 13. Consider the following fractional difference equation:

$$\begin{aligned}&\Delta\left(t^2\left[\Delta\left(t^{-1}\Delta^\alpha x(t)\right)\right]^3\right)+t\left[\Delta\left(t^{-1}\Delta^\alpha x(t)\right)\right]^3\\&+t^2\left(\sum_{s=2}^{t-1+\alpha}(t-s-1)^{(-\alpha)}x(s)\right)^{-1}=0,\\&\qquad t\in N_2,\end{aligned}\tag{86}$$

where $0<\alpha<1$ and $N_2=\{2,3,4,\ldots\}$. Comparing with (6), we have

$$\begin{aligned}c(t)&=t^2,\\r(t)&=t^{-1},\\p(t)&=t,\\q(t)&=t^2,\\f(x)&=x^{-1},\\\gamma&=3,\\t_0&=2,\\\frac{f(x)}{x^\gamma}&=\frac{1}{x^4}>\varepsilon=L>0,\quad x\neq0,\end{aligned}\tag{87}$$

where ε is a certain positive number.

It is clear that (A) and (B) hold. Moreover, it follows from (21) that

$$\begin{aligned}u(t)&=\prod_{s=t_0}^{t-1}\frac{c(s)}{c(s)-p(s)}=\prod_{s=2}^{t-1}\frac{s^2}{s^2-s}=t-1,\\\Delta u(t)&=u(t+1)-u(t)=1.\end{aligned}\tag{88}$$

Furthermore,

$$\begin{aligned}&\sum_{s=t_0}^{\infty}\frac{1}{(u(s)c(s))^{1/\gamma}}=\sum_{s=2}^{\infty}\frac{1}{((s-1)s^2)^{1/3}}>\sum_{s=2}^{\infty}\frac{1}{s}=\infty,\\&\sum_{s=t_0}^{\infty}\frac{1}{r(s)}=\sum_{s=2}^{\infty}s=\infty,\\&\sum_{\xi=t_0}^{\infty}\frac{1}{r(\xi)}\sum_{\tau=\xi}^{\infty}\left[\frac{1}{u(\tau)c(\tau)}\sum_{s=\tau}^{\infty}u(s+1)q(s)\right]^{1/\gamma}\\&\quad=\sum_{\xi=2}^{\infty}\xi\sum_{\tau=\xi}^{\infty}\left[\frac{1}{(\tau-1)\tau^2}\sum_{s=\tau}^{\infty}s^3\right]^{1/3}=\infty,\end{aligned}\tag{89}$$

which means (23), (24), and (25) hold.

For a sufficiently large t_1, we have

$$\begin{aligned}\delta(t,t_1) &= \sum_{s=t_1}^{t-1}\frac{1}{(u(s)c(s))^{1/\gamma}} = \sum_{s=t_1}^{t-1}\frac{1}{((s-1)s^2)^{1/3}},\\ R(t) &= \left[\frac{\Gamma(1-\alpha)\delta(t,t_1)}{r(t)}\right]^{\gamma}\\ &= \left[\Gamma(1-\alpha)t\sum_{s=t_1}^{t-1}\frac{1}{((s-1)s^2)^{1/3}}\right]^3.\end{aligned} \tag{90}$$

For $t_2 > t_1$,

$$\begin{aligned}&\sum_{s=t_2}^{t-1}\left(Lu(s)q(s) - \frac{(\Delta u(s))^2}{4u(s)u(s+1)R(s)}\right)\\ &= \sum_{s=t_2}^{t-1}\left(\varepsilon(s-1)s^2 - \frac{1}{4(s-1)s\left[\Gamma(1-\alpha)s\sum_{\xi=t_1}^{s-1}\left(1/((\xi-1)\xi^2)^{1/3}\right)\right]^3}\right)\\ &= \sum_{s=t_2}^{t-1}\left(\varepsilon(s-1)s^2 - \frac{1}{4(s-1)s^4\left[\Gamma(1-\alpha)\sum_{\xi=t_1}^{s-1}\left(1/((\xi-1)\xi^2)^{1/3}\right)\right]^3}\right)\\ &> \sum_{s=t_2}^{t-1}\left(\varepsilon(s-1)s^2 - \frac{1}{4(\Gamma(1-\alpha))^3(s-1)s^4\left(1/(t_1-1)t_1^2\right)}\right)\\ &= \sum_{s=t_2}^{t-1}\left(\varepsilon(s-1)s^2 - \frac{(t_1-1)t_1^2}{4(\Gamma(1-\alpha))^3(s-1)s^4}\right).\end{aligned} \tag{91}$$

Hence,

$$\begin{aligned}&\limsup_{t\to\infty}\sum_{s=t_2}^{t-1}\left(Lu(s)q(s) - \frac{(\Delta u(s))^2}{4u(s)u(s+1)R(s)}\right)\\ &\quad\geq \limsup_{t\to\infty}\sum_{s=t_2}^{t-1}\left(\varepsilon(s-1)s^2 - \frac{(t_1-1)t_1^2}{4(\Gamma(1-\alpha))^3(s-1)s^4}\right) = \infty,\end{aligned} \tag{92}$$

which implies that condition (46) is satisfied. Therefore, (86) is oscillatory or satisfies $\lim_{t\to\infty} G(t) = 0$ by virtue of Theorem 11.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Zhihong Bai carried out the oscillation criteria and completed the corresponding proof. Run Xu participated in Section 4. All authors read and approved the final manuscript.

Acknowledgments

This research is supported by National Science Foundation of China (11671227).

References

[1] A. Dzielinski, G. Sarwas, and D. Sierociuk, "Comparison and validation of integer and fractional order ultracapacitor models," *Advances in Difference Equations*, vol. 2011, article no. 11, 2011.

[2] W. G. Glockle and T. F. Nonnenmacher, "A fractional calculus approach to self-similar protein dynamics," *Biophysical Journal*, vol. 68, no. 1, pp. 46–53, 1995.

[3] J. A. Tenreiro Machado, M. E. Mata, and A. M. Lopes, "Fractional state space analysis of economic systems," *Entropy*, vol. 17, no. 8, pp. 5402–5421, 2015.

[4] F. M. Atici and P. W. Eloe, "Modeling with fractional difference equations," *Journal of Mathematical Analysis and Applications*, vol. 369, no. 1, pp. 1–9, 2010.

[5] A. Dzielinski and D. Sierociuk, *Fractional Order Model of Beam Heating Process and Its Experimental Verification. New Trends in Nanotechnology and Fractional Calculus Applications*, Springer, NY, USA, 2010.

[6] T. D. Lee, "Can time be a discrete dynamical variable?" *Physics Letters B*, vol. 122, no. 3-4, pp. 217–220, 1983.

[7] A. B. Malinowska and D. F. Torres, *Introduction to the fractional calculus of variations*, vol. 16, Imperial College Press, London, UK, 2012.

[8] D. Sierociuk, A. Dzieliński, G. Sarwas, I. Petras, I. Podlubny, and T. Skovranek, "Modelling heat transfer in heterogeneous media using fractional calculus," *Philosophical Transactions of the Royal Society of London, A: Mathematical, Physical and Engineering Sciences*, vol. 371, no. 1990, Article ID 20120146, 2013.

[9] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[10] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, vol. 3 of *Series on Complexity, Nonlinearity and Chaos*, World Scientific, 2012.

[11] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[12] S. L. Marian, M. R. Sagayaraj, A. G. M. Selvam, and M. P. Loganathan, "Oscillation of fractional nonlinear difference equations," *Mathematica Aeterna*, vol. 2, no. 9-10, pp. 805–813, 2012.

[13] M. R. Sagayaraj, A. G. Selvam, and M. P. Loganathan, "On the oscillation of nonlinear fractional difference equations," *Mathematica Æterna*, vol. 4, no. 1-2, pp. 91–99, 2014.

[14] A. G. M. Selvam, M. R. Sagayaraj, and M. P. Loganathan, "Oscillatory behavior of a class of fractional difference equations with damping," *International Electronic Journal of Pure and Applied Mathematics*, vol. 7, no. 3, pp. 220–224, 2014.

[15] M. R. Sagayaraj, A. G. Selvam, and M. P. Loganathan, "Oscillation criteria for a class of discrete nonlinear fractional equations," *The Bulletin of Society for Mathematical Services and Standards*, vol. 3, pp. 37-35, 2014.

[16] W. N. Li, "Oscillation results for certain forced fractional difference equations with damping term," *Advances in Difference Equations*, vol. 2016, no. 1, article no. 70, 9 pages, 2016.

[17] R. Xu and F. Meng, "Some new weakly singular integral inequalities and their applications to fractional differential equations," *Journal of Inequalities and Applications*, vol. 2016, no. 1, article no. 78, pp. 1–16, 2016.

[18] R. Xu and X. Ma, "Some new retarded nonlinear Volterra-Fredholm type integral inequalities with maxima in two variables and their applications," *Journal of Inequalities and Applications*, vol. 2017, article no. 187, 2017.

[19] R. Xu, "Some new Volterra-Fredholm-type nonlinear discrete inequalities with two variables involving iterated sums and their applications," *Journal of Applied Mathematics*, vol. 2017, Article ID 1474052, 14 pages, 2017.

[20] R. Xu, "Some new nonlinear weakly singular integral inequalities and their applications," *Journal of Mathematical Inequalities*, vol. 11, no. 4, pp. 1007–1018, 2017.

[21] R. Xu and Y. Zhang, "Generalized Gronwall fractional summation inequalities and their applications," *Journal of Inequalities and Applications*, vol. 2015, no. 1, article no. 242, 2015.

[22] F. Meng, "A new approach for solving fractional partial differential equations," *Journal of Applied Mathematics*, vol. 2013, no. 1, Article ID 256823, pp. 1–10, 2013.

[23] F. Meng and Q. Feng, "A new fractional subequation method and its applications for space-time fractional partial differential equations," *Journal of Applied Mathematics*, vol. 2013, Article ID 481729, pp. 721–730, 2013.

[24] A. Secer and H. Adiguzel, "Oscillation of solutions for a class of nonlinear fractional difference equations," *Journal of Nonlinear Sciences and Applications*, vol. 9, no. 11, pp. 5862–5869, 2016.

[25] R. Xu, "Oscillation criteria for nonlinear fractional differential equations," *Journal of Applied Mathematics*, vol. 2013, Article ID 971357, 7 pages, 2013.

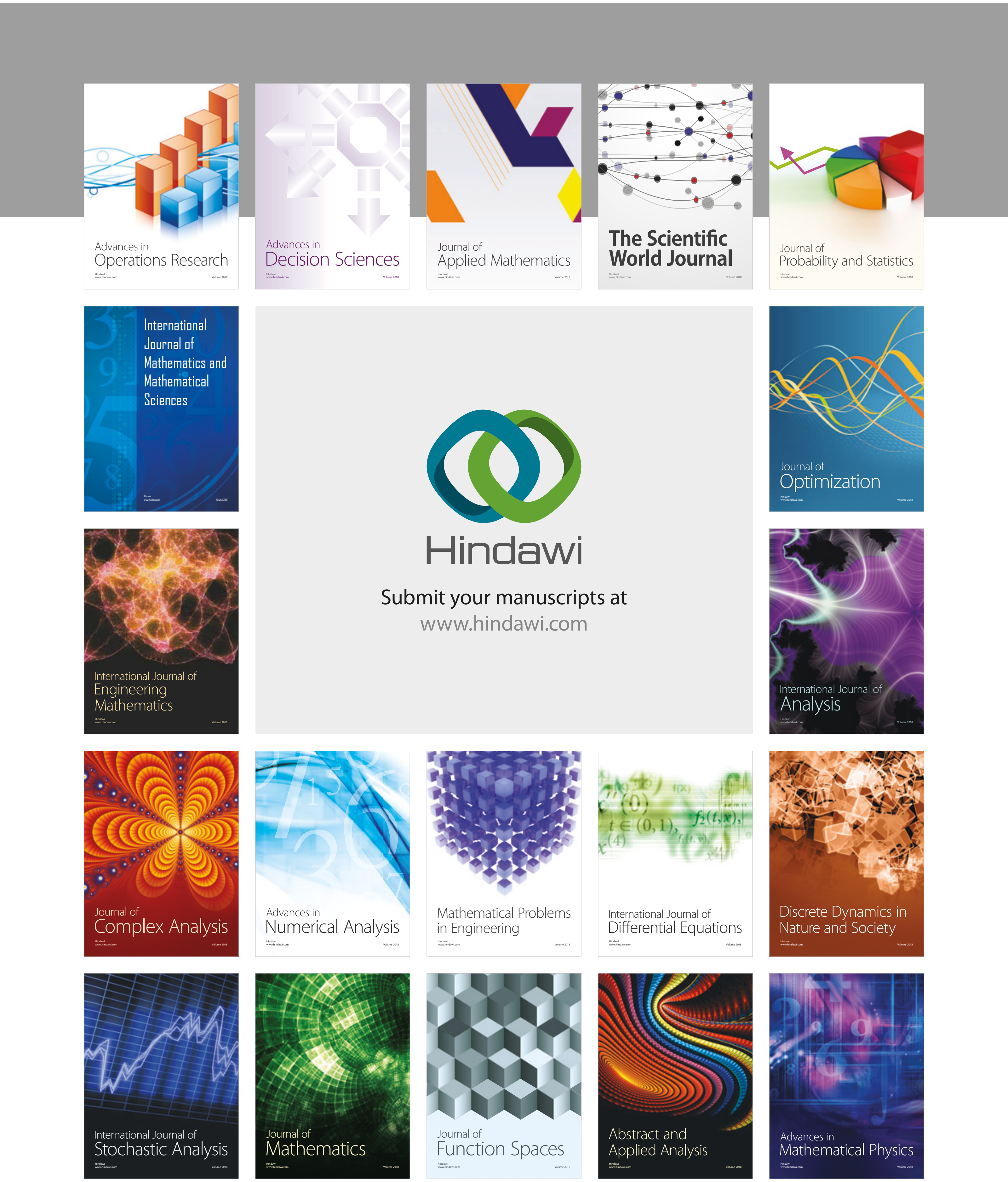
[26] J. O. Alzabut and T. Abdeljawad, "Sufficient conditions for the oscillation of nonlinear fractional difference equations," *Journal of Fractional Calculus and Applications*, vol. 5, no. 1, pp. 177–187, 2014.

[27] W. N. Li and W. Sheng, "Sufficient conditions for oscillation of a nonlinear fractional nabla difference system," *SpringerPlus*, vol. 5, no. 1, article no. 1178, 2016.

[28] F. M. Atici and P. W. Eloe, "Initial value problems in discrete fractional calculus," *Proceedings of the American Mathematical Society*, vol. 137, no. 3, pp. 981–989, 2009.

[29] S. Elaydi, *An Introduction to Difference Equations*, Springer, NY, USA, 2005.

[30] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, Cambridge, UK, 2nd edition, 1959.


Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
The Scientific World Journal
Journal of Probability and Statistics
International Journal of Mathematics and Mathematical Sciences
Hindawi
Submit your manuscripts at
www.hindawi.com
Journal of Optimization
International Journal of Engineering Mathematics
International Journal of Analysis
Journal of Complex Analysis
Advances in Numerical Analysis
Mathematical Problems in Engineering
International Journal of Differential Equations
Discrete Dynamics in Nature and Society
International Journal of Stochastic Analysis
Journal of Mathematics
Journal of Function Spaces
Abstract and Applied Analysis
Advances in Mathematical Physics