Alma Mater Studiorum – Università di Bologna

DOTTORATO DI RICERCA IN

ECONOMIA

Ciclo XXV

**Settore Concorsuale di afferenza: 13/A4 ECONOMIA APPLICATA**

**Settore Scientifico disciplinare: SECS-P/06 ECONOMIA APPLICATA**

TITOLO TESI

# ESSAYS ON THE AXIOMATIC THEORY OF MATCHING

**Presentata da:** **Bora Evci**

**Coordinatore Dottorato** **Relatore**

**Prof. Matteo Cervellati** **Prof. Vincenzo Denicolo**

**Esame finale anno 2013-2014**

**Nota**: *Mantenere le diciture in lingua italiana anche se la tesi è redatta in altra lingua. La traduzione potrà essere effettuata per il solo titolo della tesi.*

# ESSAYS ON THE AXIOMATIC THEORY OF MATCHING

BORA EVCİ

PHD IN ECONOMICS

FACULTY OF ECONOMICS

UNIVERSITY OF BOLOGNA

Under Supervision of

Prof. VINCENZO DENICOLÒ

JUNE, 2014

**Abstract**

The two main frameworks for the college admissions have been proposed so far; one is centralized, like in Turkey, Greece, China and Iran, and the other is decentralized, like in most of the European countries.

In centralized systems, there are clearing houses and those offices execute the placements according to an algorithm. In decentralized markets, the agents match with the agents on the other side themselves.

As Balinski and Sönmez (1999) showed that the algorithm used in Turkey is equivalent to well known Gale and Shapley's stable mechanism. But, because of the restrictions on the market, the outcome matching is unstable.

This dissertation started with the purpose to reduce the inefficiencies of the college admission procedure in Turkey. For this purpose, we propose a mechanism to Turkish college admisson problem. We also introduce a new market structure; as we prefer to call, semi-centralization. A semi-centralized market is the one where the market for one side is centralized, but decentralized for the other. The centralized side, as we call the Restricters, are only supposed to submit their preference orderings before the game starts. Once they submit, their job is done. Then, the other side, as we call the Choosers, play the game.

In chapter 1, we give a brief summary of Matching Theory. We present the first examples in Matching history with the most general papers and mechanisms.

In chapter 2, we propose our mechanism. In real life application, that is in Turkish university placements, the mechanism reduces the inefficiencies of the current system. The success of the mechanism depends on the preference profile. It is easy to show that for some profile the mechanism generates a stable matching.

On the other hand, when we introduce the complete information to the

model, that is the preference profile is publicly known, we get fruitful results. Our mechanism becomes a contribution to the implementation literature. We show that the mechanism implements the full set of stable matchings for a given profile.

We show this result by dividing the full domain of the profiles into two; in one partition, the profiles have one single stable matching and in the other one they have more than one matching. We detect the existence of, as we call, Cyclical Conflicts between the chooser agents for some restricters because of the priority conflicts. We observe that those cyclical conflicts are the reason of such a division. While no chooser experince any cyclical conflict in the profiles from the first division, in the second partition of the domain in all profiles choosers have such conflicts. We prove our main result by using those cyclical conflicts. Depending on the actions of the choosers in those cycles, the game ends up with one of the stable matchings.

In chapter 3, we refine our basic mechanism. The modification on the mechanism has a crucial effect on the results. The new mechanism is, as we call, a middle mechanism. It is middle, because it partitions the full domain into two. In one of the partitions, this mechanism coincides with the original basic mechanism. But, in the other partition, it gives the same results with Gale and Shapley's algorithm. That is, for some profiles, it again implements the full set of stable matchings. But, for the rest of the profiles, it ends up with the chooser-optimal stable matchings.

In chapter 4, we apply our basic mechanism to well known Roommate Problem. We test the success of our mechanism in finding stable matchings of the problem. It is known that there are profiles for this problem where there is no stable solution. Since the roommate problem is in one-sided game patern, firstly we propose an auxiliary function to convert the game semi centralized two-sided game, because our basic mechanism is designed for this framework.

We benefit from a well known scoring rule, the Borda Rule, in a social welfare function form. First we find the Borda scores of each agent and generate a social preference of those agents. The weak Borda ranking order gives us the Restricters order in every stage of the game. Starting from the top, we start the game with one of the top agents being the restricter of our game and the rest of the agents take place in the chooser side. At the end of the first stage, matched agents are deleted from the profile and also from the social preference ranking. Then, we continue with the next top agent among the remaining ones.

We show that this process is mostly succesful in finding a stable matching. Then, we detect the reason why it fails to find any stable matching for some profile in the existence of stability. The reason is the "aggregation fault". As we call the irrelevant alternatives may change the real ordering of some other alternatives. When we "purify" the effects of those externalities, the mechanism becomes successful also in those profiles. So, the basic mechanism successfully finds a stable matching in the existence of stability.

We also show that our mechanism easily and simply tells us if a profile lacks of stability by using purified orderings. Finally, we show a method to find all the stable matching in the existence of multi stability. The method is simply to run the mechanism for all of the top agents in the social preference.

**Acknowledgements**

First of all, I would like to thank my supervisor Prof. Vincenzo Denicolò for his invaluable guidance and positive personal effects on me. His encouraging supervision brought my studies up to this point.

I am very thankful to Prof. Giacomo Calzolari firstly for introducing this field to us. His interesting talks during the lectures and advices in office hours with his absolutely perfect guidance has increased my motivation to take part in the academic life.

I would like to thank Prof. Nadia Burani for her great comments throughout this project.

I would like to thank Prof. Jean Lainé for discussing this project and suggesting further topics.

I am very thankful to the department of Economics in Bologna for the financial support during my phd degree.

The last, but not the least, I am appreciate to my parents for their supports all the. For several months, they have spent endless and sleepless nights with me.

# Contents

# 1 The Literature Review

**Abstract**

In this chapter, we introduce a brief summary of the history of Matching Theory with its most general and milestone papers. Then, we introduce some of the most important properties of the (stable) matchings. And, finally we present some of the well known mechanisms (which are relevant to this dissertation) in Matching Theory.

## 1.1 The History and the Background of the Theory

The oldest issue known in Matching theory literature is American hospital-intern market at mid-twentieth century. For the new medical school graduates, the name of the specific position in the hospitals is called a residency. These positions were an important part of the labor force of the hospitals and also the crucial jobs for the new graduates for their future career.

Between 1900-1945, this market experienced a lot of problems. Given the importance of the market, there was a tough competition for the candidates between the hospitals. To hire the best candidates, the hospitals made the offers two years before the medical shool students' graduation. This was ridicuolus since at the time being, the quality of the candidate could not be observed in a clear way. Another problem was that once a residency program made an offer, they put a very short period of time to respond. This decentralized market suffered much from thickness.

In 1945, to stop these inefficiencies, the medical shools agreed not to announce any information about their students before a specific date. This decision took the control of the time problem of the market. But, another problem appeared. Since, this time, the agreements had to be achieved in a short time, the hospitals offered very high salaries with a very short time to respond for the candidates. This led to the congestion problem.

In 1952, the hospitals, the students and the medical schools agreed to centralize the placement procedure for this malfunctioning market. And, they decided to found a central clearinghouse to coordinate the market. First, the students applied to the residency programs of the hospitals. Then, the hospitals conducted interviews with the students whom had applied to them. After the interviews, both sides were ruqired to submit a preference ordering over the other side to this clearinghouse. That is each agent submited a list of hospitals in a rank order and also each residency program had a preference ordering over the students that they had interviewed. And, then the clearinghouse processed those preference orderings through an *"algorithm"* that they developed and placed the candidates to the hospitals. Today, this clearinghouse is called the National Resident Matching Program (NRMP). This process produced a *"stable"* placement whose meaning will be explained in this chapter soon.

In 1962, David Gale and Lloyd Shapley published a paper, which is regarded as the seminal work of Matching Theory. In their paper, they describe two different problems; one is the college admission problem and the other is the marriage problem.

In the marriage problem, there are two sides, namely the men and the women. Each woman has a preference ordering over men and each man has a preference ordering over women. The problem is to generate the *"marriages"* between these two sides. Basically, we form the couples which consists of one member from both sides. For this reason, the marriage problem is called a *one-to-one* problem. That is each agent on boths sides form a couple with only one agent of the other side.

In the college admission problem, again there are two sides; the colleges and the students. Each student has a preference ordeing over the colleges and each college has rank ordering over the students. The problem here is to place the students to the colleges in a way that each student is placed in only one college but, each college accept many students. So, the college admission problem is known as a *many-to-one* or *one-to-many* problem.

Gale and Shapley showed that the college admission problem is a very simple extension of the marriage problem. The reason is that we can regard the each seat of a school as a school with one seat. By this argument, the seats from the same schools have the same preference orderings over the students. And, so, the students are *indifferent* between the seats of the same school. Then, the college admission problem becomes a one-to-one problem. Therefore, Gale and Shapley modeled their paper on a one-to-one scenario which gives the same results with any other game based on a many-to-many scenario.

They set up the model as the following. The collection of the preference orderings of all agents is called a *preference profile.* The set of the married couples is called a *matching*, $\mu$. The full domain for a marriage problem consist of all the possible combinations of the couples.

As an example, if there are three women Elena, Maria and Silvia, and three men Matteo, Andrea and Mario, then the set of following marriages is a matching: *"Elena and Andrea"*, *"Maria and Matteo"* and *"Silvia and Mario"*. Since there are three members on both sides, the full domain consist of six matchings.

In a matching, if there exist a man and a woman, who are not married to each other, but who prefer each other to their own mate in the matching, then this couple is called a *blocking pair.* For a preference profile and a mathcing, if there exists a blocking pair, then this matching is called *unstable.* Otherwise, it is *stable.*

Gale and Shapley firstly proved that every marriage problem (every preference profile) has at least one stable matching (solution). This result is called *the stability theorem.* Secondly they showed that for both of the sides as a whole group, there exist a *best-optimal matching* for every preference profile. So, every marriage problem, there exist a women-optimal, $\mu_W$, and a men-optimal, $\mu_M$, matching. (In section 1.2 we will describe what men and women optimal matchings mean). And, finally, they proposed an *"algorithm"* to find those optimal matchings.

In 1984, Alvin Roth showed that the algortihms used by NRMP and Gale-Shaple are the same. We will describe the algorithm in a detailed way in section 1.3.

## 1.2 The Properties and the Structure of the Matchings

We know from Gale and Shapley that every marriage problem has a stable solution. What about the upper limit? Do we have a function to determine the number of stable matchings for a given preference profile? Such questions were firstly raised by Donald Knuth in 1976 during his lectures in University of Montréal. This is one of his famous 12 questions he asked in those lecture series.

The answer to question was given by Irving and Leather in 1986 by using the algorithm proposed by McVitie and Wilson (1971) who proposed the algorithm to generate all of the stable matchings for a given profile. Irving and Leather showed that the number of the stable matchings is an exponential function of the number of the agents on both sides.

So, we know that preference profiles, depending on the profile and the number of agents, may have many stable matchings. What about the comparisons of the matchings in view of the agents and the sides as a group?

Let us back to above example. If Matteo prefers Silvia over Maria, then he prefers the matchings where he is matched to Silvia over the matchings in which his mate is Maria. Hence, a matching is men-optimal, $\mu_M$, if for any man this matching is at least as good as any other stable matching. The same argument works for the women side.

Knuth (1976) showed that when all agents have strict preferences, the common preferences of the two sides of the market are opposed on the set of stable matchings. That is let $\mu_i$ and $\mu_j$ be two stable matchings, then all men prefer $\mu_i$ over $\mu_j$ if and only if all women prefer $\mu_j$ over $\mu_i$. This also with the result of Gale and Shapley shows that for a given profile if there is only one stable matching, that matching is men and women optimal at the same time. If there two stable matchings, one of them is men-optimal and the other is women optimal; and all men prefer men-optimal matching to women-optimal one and vise versa for women side. If there are three stable matchings, then we have a strict ordering for both sides. For the men side, there is the men-optimal stable matching, then the middle stable matching and women-optimal one. The preference ordering of the women side over these three stable matchings is the opposite of the one by the men side by Knuth. But, what if there are more than three stable matchings for a given profile?

In 1988, Charles Blair showed that the set of stable matchings for a given profile is a *partial order* for both of the sides, for sure in an opposite way. A partial order is a set with a maximum and a minimum member, but not every subset of it has a maximum and a minimum member. So, for any of the sides, there is a best and a worst member of the set, but the not all the middle members are perfectly comparable. We give more details about the *"incomparable"* stable matchings in *Theorem* 11 and *Example* 12 in section 2.3.1.

The last property we want to give is Pareto Efficieny. For any side, men or women, a matching $\mu$ is *Pareto Efficient* if we cannot improve the mate of an agent without damaging to any other agent on the same side.

## 1.3 The Mechanisms in Matching Theory

In this section, we present some well known mechanisms in Matching Theory. Since, as we have showed, many-to-many is a simple extension of one-to-one, we stick to the marriage problem scenario.

### 1.3.1 Gale and Shapley's Algorithm

As we have stated before, the oldest issue known in Matching Theory is the case of American hospital-intern market. Eventhough it was them who used *"the algorithm"* for the first time, it is known as Gale and Shapley's algorithm (1962). Here how it works;

*We assign one side as the proposer side. Let us assume men are the proposers. In the first stage, each man proposes simultaneously to their first best woman. At that moment, there three types of woman:*

*i) Some women do not receive any proposal,*

*ii) Some woman receive one proposal,*

*iii)Some women receive more than one proposal.*

*A first type of woman moves to the second stage as being single. A second type of woman engages tentatively to the man who has proposed. A third type of woman picks the best man among all proposers, engages tentatively to him and rejects the rest of the proposers.*

*So, at the end of the first stage, there are two types of man:*

*i) Some men tentatively engage,*

*ii) Some men are rejected and they move the second stage as being single.*

*In the second stage, engaged men do not do anything. The rejected, and so single men propose to their second best women. The same scenario in stage one* *works here. But, if an engaged woman receives one or more proposals in this stage, then*

*i) If there is a better man among the new proposers than her tentative husband, she picks this new man and rejects the rest including her tentative husband,*

*ii) If there is no proposer is better her current husband, she rejects all the new proposers and moves to the next stage with her current husband.*

*......................................*

*In stage k, previously rejected men propose to their next best women. A woman always picks the best man and rejects the rest. The process stops at the end of a stage where no man is rejected. And, the currents couples are accepted as the final couples.*

Gale and Shapley showed that this process ends in a finite stage. They showed that as the stages pass through, men get weakly worse off and women get weakly better off. Gale and Shapley proved that their algorithm always finds the proposer-optimal stable matching; that is if the men side is the proposer side, then the outcome of the process is the men-optimal stable matching, $\mu_M$.

### 1.3.2 Multi-Category Serial Dictatorship Algorithm

This algorithm was stated in Balinski and Sönmez (1999). The algorithm described in this paper is used by the central clearinghouse for the college admission procedure in Turkey. Here is the algorithm:

*We assign one side as the non-strategic side (the objects), and the other as the strategic side. The objects do not do anything in the game other than submitting their preferences. Let us assume that men are the objects.*

*In the first stage of the game, independently, we assign each man to their best women. It is possible that more than one man is assigned to the same woman, if she is a favorite woman among the men. If there exist a woman who has more than one man, then we modify her preference ordering in a special* *way. Among the men she is engaged, we find the best man in view of this woman. Then, we delete all the other men less prefered from him in view of the woman from her preference ordering. We apply the same process to the preference orderings of women who are engaged to more than one man. Each time we modify a preference ordering of a woman, we also delete this woman from the preference orderings of those men. Then, at the end of the first stage, we get a new tentative preference profile.*

*In the second stage, we assign men to women by the same argument and if a woman has more than man, then we find the best man engaged in her ordering and we delete less prefered men.*

*...............................................*

*In stage $k$, we apply the process. This procedure stops at the end of a stage where no woman is engagaed to more than one man. Then, these couples are accepted as the final couples.*

Balinkski and Sönmez showed that this process always finds the objects-optimal stable matching. That is if men are the non-strategic players (the objects), then the outcome of the process is the men-optimal stable matching, $\mu_M$.

### 1.3.3 Gale's Top Trading Cycles (TTC) Algorithm

This algorithm was described in Shapley and Scarf (1974). Here is the algorithm;

We assign one side as "the essential" of the game. Let us assume that essentials are women.

*In the first stage, each agent points to their most favorite agent in their preference orderings. In the paper it isproved that there exist at least one cycle if we draw the map. For example, a cycle may consist of 2 or more agents. A 2-agent cycle looks like $m_i \longrightarrow w_j \longrightarrow m_i$. And, a 4-agent cycle looks like $m_i \longrightarrow w_j \longrightarrow m_j \longrightarrow w_i \longrightarrow m_i$. In each cycle, we give the agents that the* *essentials point out. As an example, since we assign women as the essentials, we form the pairs* $(w_j, m_j)$ *and* $(w_i, m_i)$ *from above cycle. Then, we delete these four agents from the preference profile.*

*.................................*

*In stage k, each agent points to their most favorite agent in their preference orderings. We assign the agents that the essentials point out in the cycles. This process stops when either one or no agent remains in the preference profile.*

This pocess finds Pareto efficient matching for the essential side. That is if women are assigned as the essentials, we end with women-Pareto Optimal matching.

This algorithm was proposed as a trade-off with Gale and Shapley's algorithm (1962) by Abdülkadiroğlu and Sönmez (2003). Abdülkadiroğlu and Sönmez claims that if the policy-maker cares about stability, Gale and Shapley's algorithm should be used. But, if Pareto efficiency is the desired property, then Gale's TTC should be applied.

## References


[1] Abdülkadiroğlu, A. & Sönmez, T. (2003), "School Choice: A Mechanism Design Approach", The American Economic Review, 93(3), 729-747.

[2] Balinski, M. & Sönmez, T. (1999), "A Tale of Two Mechanisms: Student Placement", Journal of Economic Theory, 84, 73-94.

[3] Blair, C. (1988), "The Lattice Structure of the Set of Stable Matchings with Multirple Partners", Mathematics of Operations Research, 13(4), 619-628.

[4] Gale, D. & Shapley, L.S. (1962), "College Admissions and the Stability of Marriage", The American Mathematical Monthly, 69: 9-15.

[5] Irving, R.W.&Leather, P. (1986), "The Complexity of Counting Stable Marriages", SIAM Journal on Computing, 15:655-667.

[6] Knuth, D.E. (1976), "Mariages Stables et leurs relations avec d'autres problèmes combinatoires", Les Presses de l'Université Montréal, Montréal.

[7] McVitie, D.G. & Wilson, L.B. (1971), "The Stable Marriage Problem", Comm. ACM 14, 486-490.

[8] Roth, A.E. (1984), "The Evolution of Labor Market for Medical Interns and Residents: A Case Study in Game Theory", Journal of Political Economy, 92(6), 991-1016.

[9] Shapley, L. & Scarf, H. (1974), "On Cores and Indivisibility", Journal of Mathematical Economics, 1, 23-37.

# 2 A New Dynamic Mechanism to the Two-Sided Matching Games

**Abstract**

We know from Gale and Shapley (1962) that every Two-Sided Matching Game has a stable matching. It is also well-known that the number of stable matchings increases with the number of agents on both sides. On the other hand, Gale and Shapley's algorithm selects only the best matchings for either side.

In this paper, we propose a new mechanism to the semi-centralized two-sided matching games. The mechanism ends up with any of the stable matchings for a given profile. Formally, the set of the possible outcomes of the process is the set of the stable matchings for any profile.

## 2.1 Introduction

Gale and Shapley (1962) described the well-known marriage problem. There is a set of men and a set of women, and each man and woman has a strict preference ordering over the agents of the other set. A set of preference orderings, one for each agent, is called a *preference profile*.

We get couples each of which consists of one man and one woman from those sets. We call the set of couples a *matching*. For a given profile, a matching is *unstable* if there exist a man and a woman who are not paired in that matching, but both of them prefer each other to their current mates. The matching is called unstable, because this man and woman do not want to stay in this matching but want to move to another one where they are together. We call such a pair of man and woman a *blocking pair*. For a preference profile and a matching, if there is no blocking pair, then we call such a matching as *stable*. Gale and Shapley showed that there is always a stable matching for every marriage problem.

They also showed that every preference profile, there exist optimal stable matchings for both sides of the market and they distinguish their process to find each of them. We refer to their paper for more details.

One of the famous applications of the two-sided matching games is the college admission problem. This paper mimics the Turkish college admission procedure.

In Turkey, student placements are centralized by a public office. Every year through April-June, high school graduates take several nation-wide exams in all subjects of the high school curriculums. The scores together with their GPAs from their high schools, students get an overall score and so thet are ranked accordingly. Each student, knowing their rank, submits a list of schools to this office and placements are conducted according to an algorithm by processing students' school lists and rankings.

Balinski and Sönmez (1999) showed that the algorithm used by the central college admission authority in Turkey is equivalent to College-Proposing Gale-Shapley algorithm (1962), which had been theoretically known as stable. But, in their paper, they claimed that the algorithm should be converted into Student-Proposing Gale-Shapley for the sake of the students.

Doğan and Yuret (2010) showed that Turkish placement procedure has some inefficiencies. Using the data of a fixed year, they showed that the outcome matching of the placements was not stable and they emprically tried to estimate the ratio of the blocking pairs. They said that the algorithm is equivalent to the one by Gale and Shapley, but since there are restrictions in the application of the algortihm, e.g. in the number of schools allowed to submit and incomplete information between the students, the procedure generates blocking pairs. They claimed that limit for the number of schools should be increased to overcome this problem.

Given that the number of agents is too high, the restrictions of the school lists by the central office is justifiable. Every year nearly two million students take those exams and hundreds of thousands of them are assigned to the universities (the school seats).

This paper started with the aim to decrease, and eliminate if possible, the inefficiencies of this huge market in Turkey. As we have said, the model of the paper is based on the Turkish student placement procedure. We regard the structure of the market as given; that is firstly the schools announce their rankings, and then the students submit their school choices and matching process starts.

Therefore, we proposed a new mechanism for this market. The market is based on the incomplete information (that is the preference profile is not publicly known), so was the mechanism. Eventhough we do not give precise proofs or examples, it is easy to show that this mechanism is successful in reducing the number of blocking pairs, depending on the preference profile.

Introducing the complete information to the model converted the game into an *"implementation problem"*.[1]

We know from Gale and Shapley (1962) that every Two-Sided Matching Game has a stable matching. The question about the number of stable matchings for any profile was raised by Knuth (1976). Irving and Leather (1986) showed that the number of stable matchings is an exponential function of the number of agents.

McVitie and Wilson (1971) described an algorithm to generate all stable matchings starting from either men or women optimal matching found by the algortihm of Gale and Shapley. In section 1.4, we will give a literature review

[1] I would like to thank Vincenzo Denicolò for the suggestion to introduce complete information to the model. His advice has brought this project up to this point.

on the papers published on implementing stable matchings.

We will show that the mechanism we propose in this paper implements the full set of stable matchings for any profile. We propose this mechanism in, as we prefer to say, a semi-centralized market. While for the one side the market is centralized, i.e. the student side, for the other side it is decentralized. The agents on the decentralized part are "the objects" and they are not active players. On the other hand, the agents, who are having a centralized game, are the strategic players and so they play the game.

## 2.2 Basic Definitions and Notations

Let $M = \{m_1, ..., m_k\}$ and $W = \{w_1, ..., w_l\}$ be two non-empty, finite and disjoint sets of men and women.

Each agent has a strict preference ordering $R$ over the agents of the other set; that is $R_{m_i \in M}$ be the preference ordering of $m_i$ over $W$. For any $w_i, w_j \in W$, $w_i R_{m_i} w_j$ means $m_i$ prefers $w_i$ over $w_j$. A *Preference Profile* $\mathbb{R} = (R_i)_{i \in M \cup W}$ is the a set of preferences of all agents in the model. $\mathbb{R}^{W \cup M}$ is the set of all preference profiles for the sets $M$ and $W$.

$r_w(m)$ is the rank of agent $m$ in preference of agent $w$. That is, $r_w(m) = k$ means $m$ is the $k^{th}$ best man of $w$.

A (two-sided) matching $\mu : M \cup W \rightarrow M \cup W$ is an injection. For any $m \in M$ and $w \in W$, $\mu(m) = w$ means $w$ is the match of $m$ and vice versa. $\mu(m) = m$ means $m$ is single in the matching $\mu$. $\Pi^{M \cup W}$ is the set of all matchings between $M$ and $W$.

Let $\mu_i, \mu_j \in \Pi^{M \cup W}$ be two matchings and $m \in M$. If $\mu_i(m) R_m \mu_j(m)$, then we say that for agent $m$, $\mu_i$ *Pareto Dominates* $\mu_j$. If $\mu_i(m) = \mu_j(m)$, then $m$ is *indifferent* between $\mu_i$ and $\mu_j$ and we denote this by $\mu_i I_m \mu_j$. If $\nexists m_i \in M$ such that $\mu_j(m_i) R_{m_i} \mu_i(m_i)$ and $\exists m_j \in M$ such that $\mu_i(m_j) R_{m_j} \mu_j(m_j)$, then we say that for the set of men $M$, $\mu_i$ *Pareto Dominates* $\mu_j$; that is $\mu_i R_M \mu_j$. If $\exists m_i \in M$ such that $\mu_j(m_i) R_{m_i} \mu_i(m_i)$ and $\exists m_j \in M$ such that $\mu_i(m_j) R_{m_j} \mu_j(m_j)$, then we say that for the set of men $M$, $\mu_i$ *and* $\mu_j$ are *incomparable*.

For any $m \in M$ and $w \in W$, $(m, w) \notin \mu$ is called a *blocking pair* for the matching $\mu$, if $w R_m \mu(m)$ and $m R_w \mu(w)$. If there is no blocking pair for $\mu$, then we say $\mu$ is *stable*; otherwise, it is *unstable*.

Gale and Shapley (1962) proved that for any two-sided matching game $\mathbb{R} = (R_i)_{i \in M \cup W}$, there exists a matching $\mu \in \Pi^{M \cup W}$ which is stable for $\mathbb{R}$.

A *Matching Mechanism* $\gamma$ is a procedure to select a matching from every preference profile. Formally

$$\gamma : \mathbb{R}^{W \cup M} \longrightarrow \Pi^{M \cup W}.$$

A Matching Mechanism $\gamma$ is called *stable*, if it always selects a stable matching.

Now, let us consider the following example.

**Example 1** *Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$ be the sets of men and women who have the following preference profile $\mathbb{R}_1$:*

$$\mathbb{R}_1 = \begin{array}{ccccccc} \underline{m_1} & \underline{m_2} & \underline{m_3} & & \underline{w_1} & \underline{w_2} & \underline{w_3} \\ w_1 & w_2 & w_2 & & m_3 & m_1 & m_2 \\ w_2 & w_1 & w_3 & & m_2 & m_2 & m_1 \\ w_3 & w_3 & w_1 & & m_1 & m_3 & m_3 \end{array}$$

*For the sets $M$ and $W$, the set of all possible matchings is $\Pi^{M \cup W} = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$ where*

$$\mu_1 = \{(m_1, w_3), (m_2, w_2), (m_3, w_1)\},$$
$$\mu_2 = \{(m_1, w_2), (m_2, w_3), (m_3, w_1)\},$$
$$\mu_3 = \{(m_1, w_3), (m_2, w_1), (m_3, w_2)\},$$
$$\mu_4 = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\},$$

$$\mu_5 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\},$$
$$\mu_6 = \{(m_1, w_1), (m_2, w_3), (m_3, w_2)\}.$$

*For profile* $\mathbb{R}_1$*, the set of stable matchings is* $\{\mu_2, \mu_4, \mu_5\}$*.*

If we apply Gale and Shapley's algorithm to the profile $\mathbb{R}_1$, we get either $\mu_2$ or $\mu_5$, if we assign women or men as the proposer side, respectively.

There are two major problems with the algorithm by Gale and Shapley. Firstly it is not symetric; if women propose, there is no chance for $\mu_5$ to be chosen and vice versa for $\mu_2$. Secondly, there is no possibility for $\mu_4$ to be chosen in any scenario.

In the next section, we propose a new dynamic mehanism. With that mechanism, any stable matching for any profile could be chosen, e.g. for $\mathbb{R}_1$ the set of the possible outcomes is $\{\mu_2, \mu_4, \mu_5\}$.

## 2.3 The Dynamic Mechanism

For a given matching game $\mathbb{R} = (R_i)_{i \in M \cup W}$, we assign one side as *the Restricter*, and the other side as *the Chooser*. We use the preferences of the restricters as the restrictions or the priorities on the chooser side and the choosers make decisions with their own preferences as their turns come. In that game, *the information is complete*; that is the rule of the game and the preference profile is known by all agents. Here is how the mechanism works.

Without loss of generality, we shall assign $W$ as the restricter and $M$ as the chooser. (Later we will show that the set of the outcomes does not depend on which set is the restricter or the chooser). The preference orderings of any woman is the priorities of the men for those women.

We start with the man/men who are the best in view of women; that is we start with men such that $\{m_i \in M | \exists w_j \in W \text{ such that } r_{w_j}(m_i) = 1\}$. Those men are asked to make a decision; either to say "yes" or "no" to the woman for who they are the best men. Some of them may be the best man for more than one woman. In this case, such a man is asked to choose one of those women. If a man says "yes" to a woman, then they form a pair and both of them are deleted from the profile; if he says "no", he loses that woman/women and waits for his turn for other women.

At any step/rank $k$, a man either chooses a woman to marry or refuses and waits for another woman. In that way, we construct our pairs.

First, we shall show that this process produces a matching from any profile. Let $m_i \in M$ be a chooser agent. At any step where he is the best man for any woman, if $m_i$ decides to choose an agent $w_j \in W$, $m_i$ is deleted from the profile and he forms the pair $(m_i, w_j)$. If he never chooses anybody at any step, then he forms the pair $(m_i, m_i)$. As we have said before, any chooser says "no" to wait for his turn for a better restricter. In this model, we explicitely assume that all the agents are acceptable for the agents on the other side, and so they prefer being matched to some agent than being single. If he never says "yes" to any woman, he remains single which contradicts to the rationality assumption. We will analyze when and why a man says "no" in the following sections. If a chooser remains single, it is only because he does not receive any offer. These scenarios are the same for all $m_j \in M$. On the other hand, when any $m_j \in M$ chooses an agent $w_i \in W$, she is deleted from the profile, too. If $w_i$ is not choosen by any $m_j$ (possibly because $l > k$ and $w_i$ is not a favorite woman), then she forms the pair $(w_i, w_i)$. This happens when all of men are matched to some women before she calls for her "best(s)". So, any agent $i \in M \cup W$ could be a member of one pair. Hence, the outcome of this procedure $\gamma$ is a matching $\mu \in \Pi^{M \cup W}$.

Now, we shall demostrate our mechanism with a simple example.

**Example 2** *Let* $M = \{m_1, m_2, m_3\}$ *be the restricter and* $W = \{w_1, w_2, w_3\}$ *be* *the chooser who have the following preference profile* $\mathbb{R}_2$.

$$\mathbb{R}_2 = \begin{array}{ccccccc} \underline{m_1} & \underline{m_2} & \underline{m_3} & & \underline{w_1} & \underline{w_2} & \underline{w_3} \\ w_2 & w_1 & w_2 & & m_1 & m_3 & m_1 \\ w_1 & w_3 & w_3 & & m_3 & m_2 & m_2 \\ w_3 & w_2 & w_1 & & m_2 & m_1 & m_3 \end{array}$$

*In the first round,* $w_1$ *and* $w_2$ *are asked to choose;* $w_1$ *for* $m_2$ *and* $w_2$ *for either* $m_1$ *or* $m_3$. *Since* $r_{w_2}(m_3) = 1$, $w_2$ *says "yes" to* $m_3$. *They construct* $(m_3, w_2)$ *and both of them are deleted from the profile. As the information is complete and so* $w_1$ *knows that* $w_2$ *chooses* $m_3$, *she says "no" to* $m_2$.

*In the second round, since* $r_{w_1}(m_1) = 1$, $w_1$ *says "yes" to* $m_1$. *They construct* $(m_1, w_1)$ *and both of them are deleted from the profile. Eventhough* $r_{w_3}(m_1) = 1$, *since the information is complete and so* $w_3$ *knows that* $w_1$ *chooses* $m_1$, *she says "yes" to* $m_2$ *since* $m_2 R_{w_3} m_3$.

*Hence,using our mechanism* $\gamma$ *we get the matching* $\mu_6 = \{(m_1, w_1), (m_2, w_3), (m_3, w_2)\}$ *which is the only stable matching for* $\mathbb{R}_2$.

### 2.3.1 The Flow of the Mechanism

Our mechanism is based on the "first-come first-served" principle. When a chooser agent is asked to reply an offer, e.g. he is the best man for some restricter(s), he makes his decision by considering the best alternatives better than the current restricter. If all of them have already been taken or regarded as will be taken in the current or the next rounds (thanks to the complete information), then he says "yes" to the offer. In this section, we will examine the game scenarios that the choosers confront.

**Definition 3** *Let* $m \in M$ *be any chooser agent and* $w_i, w_j \in W$ *be any two restricter agents. If* $r_{w_i}(m) > r_{w_j}(m)$ *and* $w_i R_m w_j$ *and at the step* $k = r_{w_j}(m)$ *non of* $w_i$ *and* $w_j$ *have been taken by other choosers yet, then we say the agent* $m$ *experiences a* ***conflict*** *between agents* $w_i$ *and* $w_j$.

The definition says that for a chooser agent if the turn for a worse restricter comes before any better one, given that non of those restricters have not been chosen yet, then the chooser agent experiences a conflict; he *may not* be sure about his decision.

**Definition 4** *If a chooser agent $m \in M$ does not experience any conflict, then we say $m$ has a* ***smooth game****.*

When the information is incomplete, ex. the agents cannot observe the preseferences of any other agents, the chooser agent cannot make a precise decision; but incompleteness is not the topic of this paper. When the preference profile is observable, such an agent estimates what will happen in the current and the successive steps. Hence, he has sufficient information to make a clear decision during those conflicts. So, under complete information, the conflicts turn into smooth games.

In this paper, we pay our attention to the special case of conflicts.

**Definition 5** *Let $\{m_1, ..., m_r\} \subset M$ be a set of choosers and $\{w_1, ..., w_r\} \subset W$ be a set of restricters. If we have such a case;*

- $m_1 R_{w_1} m_2$, $m_2 R_{w_2} m_3$,...,$m_r R_{w_r} m_1$,
- $w_r R_{m_1} w_1$, $w_1 R_{m_2} w_2$,...,$w_{r-1} R_{m_r} w_r$,
- $r_{w_1}(m_1) = r_{w_2}(m_2) = ... = r_{w_r}(m_r) = k$ *(for at least one side),*
- *Each agent of $\{w_1, ..., w_r\}$ and $\{m_1, ..., m_r\}$ is present at step $k$.*

*Then, we say that agents in $\{m_1, ..., m_r\}$ experience a* ***cyclical conflict*** *with each other at step $k$.*

Therefore, eventhough the preference profile is publicly known, if some group of choosers experience a cyclical conflict, they cannot have a precise decisions, since the actions are not observable for the current step. Hence, when a chooser has a smooth game, the process is a *squential game* for that agent. On the other hand, when he experiences a cyclical conflict, it is a *simultaneous game* for him (including other agents in the cycle).

**Claim 6** *Any chooser agent could be a member of at most one cyclical conflict at any step $k$.*

***Proof.*** *The proof is straight forward. Let $m \in M$ be any chooser being a member of more than one cyclical conflict at step $k$. Let $w_i, w_j \in W$ be any two restricters where $r_{w_i}(m) = r_{w_j}(m) = k$ and each of these restricters is from different cyclical conflicts. As an assumption, all the agents have strict preferences. So, we have either $w_i R_m w_j$ or $w_j R_m w_i$. In any case, the agent $m$ does not consider the worse agent. Hence, any chooser $m$ experiences only one single cyclical conflict at any single step.* ■

Now, we will focus on the affects of such cycles on the relationship between any preference profile and the set of the stable matchings for that profile; ex. the number of stable matchings for a given profile.

**Theorem 7** *For a given preference profile $\mathbb{R} = (R_i)_{i \in M \cup W}$, there exists only one single stable matching if and only if there exists no cyclical conflict for the choosers.*

**Proof.** ($\Longleftarrow$). Suppose that we do not have any cyclical conflict. We shall assume multiple stable matchings for a preference profile and prove that this leads to a contradiction. Then, from Gale and Shapley there exist optimum stable matchings $\mu_M$ and $\mu_W$ for men and women, respectively, with $\mu_M R_M \mu_W$ and $\mu_W R_W \mu_M$. Since the preferences are strict, then $\exists m_k, m_l \in M$ and $\exists w_i, w_j \in W$, such that

1. $\mu_W(w_i) = m_k$, $\mu_W(w_j) = m_l$, $\mu_M(w_i) = m_l$, $\mu_M(w^*) = m_k$ and $\mu_M(w_j) = m^*$ for some $w^* \in W$, $m^* \in M$ with

2. $w^* R_{m_k} w_i$, $w_i R_{m_l} w_j$, $m_k R_{w_i} m_l$, $m_l R_{w_j} m^*$.

In that case, we may have two scenarios:

**Scenario 1:** Let $A = \{\overline{w} \in W | \overline{w} R_{m_l} w_i\}$ be the set of the agents who are better than agent $w_i$ according to agent $m_l$. Since $\mu_M(w_i) = m_l$, either all of $\overline{w} \in A$ have been taken by some other $\acute{m} \neq m_l$ before the round where $m_l$ chooses $w_i$ or $m_l$ regards each $\overline{w}$ as will be taken. Hence, any $\overline{w} \in A$ is not achievable for $m_l$. On the other hand, we have $\mu_W(w_j) = m_l$. In that case, the fact $w_i R_{m_l} w_j$ contradicts the rationality of agent $m_l$; while he could choose $w_i$, he did not. Then this leads us to $w_i = w_j$ which gives $\mu_M = \mu_W = \mu$.

**Scenario 2:** Maintaining the assumptions on the rationality of the agents and existence of multiple stable matchings, we have the following scenario. If we have stable matchings $\mu_M$ and $\mu_W$, using the information in 2 above, we may have either of the followings;

1. $m^* = m_k$ and $w^* = w_j$, that is the sets $\{m_l, m_k\}$ and $\{w_i, w_j\}$ had a cyclical conflict so that we have such two stable matchings, or

2. $m^* = m'$ and $w^* = w'$, that is there is a bigger cycle including $m_l, m_k, w_i$ and $w_j$; by iterative construction, there may be cycle including all the agents.

Both of them contradicts the fact that there is no cycle. Hence, there is only one single stable matchings.

($\Longrightarrow$). For any profile $\mathbb{R} = (R_i)_{i \in M \cup W}$, there exists only one stable matching $\mu \in \Pi^{M \cup W}$. And, let us assume there exists a cyclical conflict between the agents of $M' = \{m_1, ..., m_r\} \subset M$ for the agents $W' = \{w_1, ..., w_r\} \subset W$ at step $k$.

Since each $m_i \in M'$ is in the cycle, any better restricters than the ones in the cycle have either been taken or regarded as taken in the current or next steps. For that reason, in the matching $\mu$ we cannot have any pair such that $(m_i, \hat{w})$ where $m_i \in M'$ and $\hat{w} \notin W'$; because $(m_i, w_j)$ would block the matching $\mu$, where $w_j \in W'$. For that reason, in the matching $\mu$, $\forall m \in M'$ and $\forall w \in W'$, $\mu(m) \in W'$ and $\mu(w) \in M'$.

We shall assume that all the choosers say "yes" at step $k$. In that case, we may have two scenarios:

**Scenario 1:** $M' = M$ and $W' = W$. In such case, the matching $\mu$ would be stable since no pair blocks it; that is $\forall w \in W'$ get better choosers in their own cycle, so no woman admires any man in the cycle. But, in that case, another matching $\mu'$ would be stable where no agent $m \in M'$ says "yes" at step $k$; in that situation $\forall m \in M'$ $\mu'(m) R_m \mu(m)$. Hence, in $\mu'$ $\forall m \in M'$ gets their better restricters in their own cycle, so no man admires admires any woman in the cycle. Hence, we have two stable matching which contradicts to the single stable matching.

**Scenario 2:** $M' \subset M$ and $W' \subset W$ are proper subsets. $\forall m \in M/M'$ and $\forall w \in W/W'$ do not confront any cycle. Hence, each of them experince either smooth or (simple) conflict games. By using the argument in **Scenario 1** of ($\Longleftarrow$), we end up with a unique set of pairs for $m \in M/M'$ and $w \in W/W'$, and $\forall m \in M/M'$ $\mu(m) \in W/W'$ and $\forall w \in W/W'$ $\mu(w) \in M/M'$. The remaining part is same with scenario 1. ■

With Theorem 7, we have showed that when there is a unique stable matching for a preference profile, we do not have any cyclical conflicts for the choosers, and vice versa. And, our proof also showed that when there is a single stable matching, our mechanism gives us that matching. As we have a two-sided implication in the theorem, we get the symmetry between the sides: if one side has no cyclical conflict, then there is a single stable matching and if so, other side has no cyclical conflict. Then, in any scenario, we get the same matching.

Now, we know that the reason for multiple stable matchings is the existence of cyclical conflicts. The following example shows that when there are multiple stable matchings, the set of the agents on both sides having cyclical conflicts need not be the same.

**Example 8** *Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$ be the sets of men and woman who have the following preference profile $\mathbb{R}_3$:*

$$\mathbb{R}_3 = \begin{array}{ccccccc} \underline{m_1} & \underline{m_2} & \underline{m_3} & & \underline{w_1} & \underline{w_2} & \underline{w_3} \\ w_1 & w_2 & w_2 & & m_3 & m_1 & m_3 \\ w_3 & w_1 & w_3 & & m_2 & m_2 & m_1 \\ w_2 & w_3 & w_1 & & m_1 & m_3 & m_2 \end{array}$$

*The set of the stable matchings for $\mathbb{R}_3$ is $\{\mu_4, \mu_5\}$. When $W$ is the chooser, the sets of agents in the cycle are $\{m_1, m_2\}$ and $\{w_1, w_2\}$. On the other hand, when $M$ is the chooser, the sets of agents in the cycle are $\{m_1, m_3\}$ and $\{w_2, w_3\}$.*

$$\begin{array}{ccccccc} \multicolumn{7}{c}{\underline{\textit{W-Chooser case}}} \\ \underline{m_1} & \underline{m_2} & \underline{m_3} & & \underline{w_1} & \underline{w_2} & \underline{w_3} \\ \mathbf{w}_1 & \mathbf{w}_2 & w_2 & & m_3 & \mathbf{m}_1 & m_3 \\ w_3 & \mathbf{w}_1 & w_3 & & \mathbf{m}_2 & \mathbf{m}_2 & m_1 \\ \mathbf{w}_2 & w_3 & w_1 & & \mathbf{m}_1 & m_3 & m_2 \end{array} \qquad \begin{array}{ccccccc} \multicolumn{7}{c}{\underline{\textit{M-Chooser case}}} \\ \underline{m_1} & \underline{m_2} & \underline{m_3} & & \underline{w_1} & \underline{w_2} & \underline{w_3} \\ w_1 & w_2 & \mathbf{w}_2 & & m_3 & \mathbf{m}_1 & \mathbf{m}_3 \\ \mathbf{w}_3 & w_1 & \mathbf{w}_3 & & m_2 & m_2 & \mathbf{m}_1 \\ \mathbf{w}_2 & w_3 & w_1 & & m_1 & \mathbf{m}_3 & m_2 \end{array}$$

We have seen that existence of a cyclical conflict generates two stable matching; one of them is constructed if all the choosers in the cycle say "yes" in the first step and the second is created if all say "no". But, we cannot conclude that this is always the case.

**Definition 9** *Let $M$ and $W$ be the sets of choosers and restricters, respectively. Let $M_1, M_2 \subset M$ be the set of the agents of two cycles. If $M_1 \cap M_2 = \emptyset$, we say* *the cycles are* ***independent****. Otherwise, they are* ***(sequentially) dependent*** *cycles.*

From *Claim* 6, we know that the any chooser could be a member of at most cyclical conflict at a single step; but he could be a member of another cycle in any consecutive steps. This is why we call such cycles as sequential.

**Proposition 10** *If there exist two dependent cycles in the profile, they generate three stable matchings.*

**Proof.** The proof is simple. Let $M_1, M_2 \subset M$ and $W_1, W_2 \subset W$ with $M_1$ and $W_1$ be the agents of the cycle at step $k$ and $M_2$ and $W_2$ be the agents of the cycle at step $l$. Let us assume that for $M_1$ if all say "yes" at step $k$, $\mu_1$ is generates; if nobody says "yes", $\mu_2$ is generated. And, also we shall assume that for $M_2$ if all say "yes" at step $l$, $\mu_3$ is generates; if nobody says "yes", $\mu_4$ is generated. Let $m \subset M_1 \cap M_2$ be any chooser member of the both of the cycles.

In such a case, we may have two scenarios; either $k = l$ or $k \neq l$ and we examine here each one of them. The first scenario is trivial. Let $w_i \in W_1$ and $w_j \in W_2$ such that $r_{w_i}(m) = k$ and $r_{w_j}(m) = l$ with (wolg) $k < l$. If $m$ (like other agents in $M_1$) says "no", they come to the consecutive step to $l$ (here $l - k \geq 1$). If $m$ says "yes" (like other agents in $M_1$) at step $l$ which is also the consecutive step of $k$, $\mu_2$ is generated where all the agents without cycles construct unique couples as we have proved in *Thm* 7. But, at the same time, if $m$ says "yes" (like other agents in $M_2$), then $\mu_3$ is generated with same uniwue couples by the same argument. Then $\mu_2 = \mu_3$.

The same argument works for the case $k = l$ from which we end up with $\mu_1 = \mu_3$. Hence, two sequentially cycles generate three stable matchings. ■

From *Proposition* 10, the idea saying that "each cycle produces two stable matchings" fails. We refer to the profile $\mathbb{R}_1$ of *Example 1* above where $k \neq l$. For example, we may have four stable matchings from either two independent cycles or three sequentially dependent cycles. Hence, unfortunately we cannot have a relationship between the number of cycles and the number of stable matchings for a given profile $\mathbb{R}$.

We have one more property of the cyclical conflicts related to the stable matchings.

**Theorem 11** *There exist (independent) cyclical conflicts which occur at the same step $k$ if and only if we have* ***incomparable*** *stable matchings.*

**Proof.** ($\Longrightarrow$).Let $M$ and $W$ be the sets of choosers and restricters, respectively, and let $M_1, M_2 \subset M$ and $W_1, W_2 \subset W$ with $M_1$ and $W_1$ be the agents of the one of the cycles and $M_2$ and $W_2$ be the agents of the other cycle.

We shall focus on the scenarios of those two cycles. Let us assume that at step $k$, if the agents of $M_1$ say "yes", they construct the couples $C1$ and nobody says "yes", they construct the couples $C2$. Same argument works for the agents of $M_2$ and they construct the couples $C3$ from "yes" at step $k$ and $C4$ from "no" at step $k$. Since the cycles are independent, then the couples are different constructed by the two cyclec. Hence, the combinations of those cycles give us four matchings: $C1 \cup C3 \in \mu_1$, $C1 \cup C4 \in \mu_2$, $C2 \cup C3 \in \mu_3$ and $C2 \cup C4 \in \mu_4$. And, the agents out of those cycles construct the same couples in all those matchings from the proof of $Thm$ 7. From the definition of a cycle, we have

1. $C2 R_{M_1} C1$ and $C4 R_{M_2} C3$,

2. $C1 R_{W_1} C2$ and $C3 R_{W_2} C4$.

Hence, from (1) we have $\mu_1 R_W \mu_4$ and $\mu_4 R_M \mu_1$. And, from (2), we have $\mu_2$ and $\mu_3$ are incomparable for both $M$ and $W$.

($\Longleftarrow$). Let $\mu_i$ and $\mu_j$ be any two incomparable stable mathcings for the profile $\mathbb{R} = (R_i)_{i \in M \cup W}$ with $M$ be the chooser and $W$ be the restricter. And, let us assume we do not have any cyclical conlicts that occur at the same step $k$. We may have three scenarios.

**Scenario 1:** We do not have any cycles. But, in that case since we have one single stable matching and so there is nothing to compare, our theorem becomes true.

**Scenario 2:** We have independent cycles that occur consecutively at different steps of the game. In that case, each cycle produces two stable matchings. From the defition of cyclical conflicts, we have for the matchings $\mu_k, \mu_l$ that are created from a single cycle such that $\mu_k$ from "yeses" in the first step and $\mu_l$ from "yeses" in the second that $\mu_k R_W \mu_l$ and $\mu_l R_M \mu_k$ and this is the case for all such matchings. Contradiction to the existence for incopmrable stable matchings.

**Scenario 3:** We have independent cycles and/or sequentially dependent cycles. As stated before, dependent cycles produce such common cycles which do not effect the comparison. The same argument works with the one in *Scenario 2*. Contradiction. $\blacksquare$

Now we shall give an example of incomparable stable matchings.

**Example 12** *(Roth and Sotomayor, Example 2.17, page 37). Let* $M = \{m_1, m_2, m_3, m_4\}$ *and* $W = \{w_1, w_2, w_3, w_4\}$ *be the sets of men as the chooser and woman as the restricter who have the following preference profile* $\mathbb{R}_4$*:*

$$\mathbb{R}_4 = \begin{array}{cccccccc} \underline{w_1} & \underline{w_2} & \underline{w_3} & \underline{w_4} & \underline{m_1} & \underline{m_2} & \underline{m_3} & \underline{m_4} \\ \widehat{m_4} & \widehat{m_3} & \widetilde{m_2} & \widetilde{m_1} & w_1 & w_2 & w_3 & w_4 \\ \widehat{m_3} & \widehat{m_4} & \widetilde{m_1} & \widetilde{m_2} & w_2 & w_1 & w_4 & w_3 \\ m_2 & m_1 & m_4 & m_3 & \widetilde{w_3} & \widetilde{w_4} & \widehat{w_1} & \widehat{w_2} \\ m_1 & m_2 & m_3 & m_4 & \widetilde{w_4} & \widetilde{w_3} & \widehat{w_2} & \widehat{w_1} \end{array}$$

$\Pi^{M \cup W}$ *has 24 matchings 10 of which are stable for* $\mathbb{R}_4$*. Two of them are* $\mu_8 =$

*$\{(m_1, w_3), (m_2, w_4), (m_3, w_2), (m_4, w_1)\}$ and $\mu_9 = \{(m_1, w_4), (m_2, w_3), (m_3, w_1), (m_4, w_2)\}$. $\mu_8$ is generated if the agents of $\{m_4, m_3\}$ and $\{m_2, m_1\}$ say "yes" and "no", respectively, and the opposite for $\mu_9$. And, both for $M$ and $W$, $\mu_8$ and $\mu_9$ are incomparable.*

Before we state and prove our main theorem, let us examine the following trivial example.

**Example 13** *Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$ be the sets of men as the chooser and woman as the restricter who have the following preference profile $\mathbb{R}_5$:*

$$\mathbb{R}_5 = \begin{array}{cccc} \underline{m_1} & \underline{m_2} & \underline{w_1} & \underline{w_2} \\ w_2 & w_1 & m_1 & m_2 \\ w_1 & w_2 & m_2 & m_1 \end{array}$$

*For the sets $M$ and $W$, the set of all the possible matchings between them is $\Pi^{M \cup W} = \{\mu_1, \mu_2\}$ where*

$$\mu_1 = \{(m_1, w_1), (m_2, w_2)\},$$
$$\mu_2 = \{(m_1, w_2), (m_2, w_1)\}.$$

*For the profile $\mathbb{R}_5$, the set of the stable matchings is $\{\mu_1, \mu_2\}$. If $w_2$ says "yes" to $m_1$, then the best response of $w_1$ would be "yes" to $m_2$ (otherwise, we get the couples $\mu = \{(m_1, w_2), (m_2, m_2), (w_1, w_1)\}$) and vice versa. And, if $w_2$ says "no" to $m_1$, then the best response of $w_1$ would be "no" to $m_2$; otherwise $w_1$ would miss the chance to construct $(w_1, m_1)$ which she prefers over $(w_1, m_2)$. Hence, the Nash Equilibrium (NE) of this game is $NE(w_1, w_2) = \{(yes, yes), (no, no)\}$. And, it is easy to show that the argument is same for any cyclical conflicts.*

**Theorem 14** *If Nash Equilibria of the cycles are chosen, the outcome of our mechanism is the set of the stable matchings for any preference profile. In other words, we always end up with one of the stable matchings for any profile.*

**Proof.** The proof is straight forward. Let $\mathbb{R} = (R_i)_{i \in M \cup W}$ be any profile with $M$ be the chooser and $W$ be the restricter. If there is no cyclical conflict, then there is only one stable matching and our mechanism finds it as we have proved in *Thm* 7.

Hence, let us assume there are some cyclical conflicts for that profile, and so we have multiple stable matchings. Let $\Pi^{M \cup W} = \{\mu_1, ..., \mu_r\}$ be the set of all stable matchings for $\mathbb{R}$. So, let us assume $\exists \mu \in \Pi^{M \cup W}$, but our mechanism does not find it in any scenario.

From *Thm* 7 and *Proposition* 10, we know that $\forall \mu_i \in \Pi^{M \cup W}$ are generated by some cyclical conflicts. Any cycle that has $n$ choosers and $n$ restricters generates $n!$ matchings; two of them are stable and $(n! - 2)$ are unstable. The game ends when all agents construct a pair. With NE assumption, either all the agents in any cycle say "yes" or all say "no". As they say "no", we prooceed from top to the bottom on the preferences of $W$. With the assumption on NE solutions, we ommit $(n! - 2)$ unstable matchings for each cycle which means our mechanism always ends up with a stable matching. So, for each cycle, one of two stable matchings is chosen.

Hence, if there exists such a matching $\mu$, then either it was not generated by any cycle or it is an unstable matching. If $\mu$ was not generated by any cycle, then from *Thm* 7, it is the unique stable matching for the profile $\mathbb{R}$ which is a contradiction to the existence of multiple stable matchings. If $\mu$ is unstable, then we are done. Hence, any stable matching could be chosen by our mechanism and there is no possibility to end up with unstable matching. ■

### 2.3.2 Strategy-Proofness of the Mechanism

Now, we will investigate whether our mechanism is vulnerable to the strategic manipulation or not.

**Theorem 15** *Truth telling is weakly dominant in our mechanism.*

**Proof.** Let us assume that our mechanism is manipulable. Let $\mathbb{R} = (R_i)_{i \in M \cup W}$ be any profile based on true preferences with $M$ be the chooser and $W$ be the restricter. And, let $\mathbb{R}^* = (R_i)_{i \in M \cup W}$ be any other profile with $R_i = R^*_{i \in M \cup W/\{m\}}$and $R_m \neq R^*_m$ for a chooser agent $m \in M$. $\mathbb{R}^*$ is the preference profile based on mispresented preferences and $m$ is the manipulator agent with $\gamma(\mathbb{R}^*) R_m \gamma(\mathbb{R})$. We may have two scenarios:

**Scenario 1:** For $\mathbb{R}$ we have one single stable matching; that is there is no cyclical conflict. So let $\gamma(\mathbb{R}) = \mu$ and $\exists \mu^* \in \gamma(\mathbb{R}^*)$. Let $w_i, w_j \in W$ with $\mu^*(m) = w_i R_m w_j = \mu(m)$. As we have proved, our mechanism $\gamma$ is stable, and so is $\mu^*$. To satisfy stability for $\mu^*$, $\mu^*(w_j) R_{w_j} m$ and $m R_{w_i} \mu(w_i)$. If we have $\mu^*(w) = \mu(w)$ for $\forall w \in W/\{w_i, w_j\}$, then we get $\mu^*(w_j) = \mu(w_i) = m^*$ such that the sets $\{m, m^*\}$ and $\{w_i, w_j\}$ construct a cyclical conflict. If not, to keep the stability of $\mu^*$ we should have a pair $(\hat{m}, w_j)$ with $\hat{m} R_{w_j} m$, and so on. In every step, we should assign a better mate to every agents which iteratively leads us to the *full cyclical conflict*. This is a contradiction to fact that there is no cyclical conflict.

**Scenario 2:** For $\mathbb{R}$ we have one single cyclical conflict. If $m$ is not a member of the cycle, then the above argument works. Let $w_i, w_j \in W$ be the agents that $m$ experiences the conflict with $w_i R_m w_j$. Let $\mu_M$ and $\mu_W$ be the stable matchings from Gale and Shapley such that $\mu_M = w_i$ and $\mu_W = w_j$. From the definition of a cycle, any agent in the set $\{w | w R_m w_i, w \in W\}$ is not achievable for $m$. Any matching $\mu^*$ such that $w_j R_m \mu^*(m)$ would be unstable. Hence, the question becomes *"Can m guarantee to make $\mu_M = w_i$ chosen?"*.

Firstly, we shall examine the trvial example, ex. *Example 12* above, where $\mu_2 = \mu_M$ and $\mu_1 = \mu_W$. Let $m = m_1$ be our manipulator. A change by $m$ breaks the cycle and $\gamma(\mathbb{R}^*) = \mu_1 = \mu_W$. Hence, m damages himself by abolishing the possiblity for $\mu_2 = \mu_M$ to be chosen.

Now, we shall consider any cycle which has three agents on each side with $\exists m, m_i, m_j \in M$ and $w_i, w_j, w_k \in W$ such that $w_k R_m w_i$, $w_i R_{m_i} w_j$ and $w_j R_{m_j} w_k$. If $m$ changes his ordering, he breaks the cycle and saying "yes" to the offers is a dominant-rational strategy. And, we end up with $\mu_W$. By the same argument, iteratively for any number of agents, breaking the cycle damages to the manipulator. The argument works for any number of cycles for the choosers.

Hence, we conclude that truth telling is weakly dominant for the choosers.

■

## 2.4 The Related Literature

In the related literature, there are papers, which have different model and pattern, on implementing the stable matchings. The main difference of those papers are that some of them are modeled on the centralized market and the others are on decentralized markets.

While in the centralized markets, there is a social planner who collects the preferences of all agents and constructs the matching, in the decentralized ones, the agents on both sides match with other themselves.

Among the centralized based paper in the literature, the closest one to our paper is Alcalde (1996). Alcalde proposed a deferred acceptance algorithm similar to Gale-Shapley, but in a now-or-never scenario, that is if an agent receives an offer, she can never receive an offer in the subsequent stages. Alcalde showed that in undomainated Nash equilibria, the mechanism ends up with the full set of stable matchings.

The related papers to ours have been published for the decentralized markets. Blum, Roth and Rothblum (1997) poposed an defer-acceptance process. They assume there is uncertainty; each proposer only knows to who she proposes and each receiver-replier knows only his offer. Also, the order of the proposers to make offers is randomized. They analyzed the Nash equilibria. They show that since the order of the proposers are randomized, whether the mechanism ends up with proposer-optimal matching or not depends on the initial point of the game.

Alcalde et al. (1998) proposed a one-stage game. In the first stage the offers are made, and in the second the candidates-receivers accept at most one proposal. The poposers simultaneously make the offer all the agents they want on the other side. Then, then the proposals are accepted or rejected and the game ends. They show that this implements the full set of stable matchings in subgame perfect nash equilibrium.

Alcalde and Romero-Medina (2000) proposed a many-to-one sequential one-stage mechanism similar to Gale and Shapley. In the first stage, the students simultaneously send a letter to at most one college and in the second stage the colleges select the set of best students among their candidates. They show that this mechanism implements the full set of stable matchings in subgame Nash equilibrium.

Peleg (1997) proposed a one-stage one-to-one model for the marriage problem. The agents on both sides propose to at most one agent on the other side. If a man and a woman propose each other, then they form a pair. Peleg showed that his mechanism implements the full set of stable matchings by strong Nash equilibria. He also showed that an extensive form game finds the same set in subgame Nash equilibrium.

Roth and Xiaolin (1997) proposed a deferred acceptance algortihm for the market for clinical psychologists. When the agents on one side of the market make the offers, the other side can hold the offer for a while. They show that the results coincide with Gale-Shapley.

Haeringer and Wooders (2011) proposed a sequential mechanism and they studied the mechanism for four different scenarios. In their game, the firms propose and the workers accept or reject the offers. Their scenarios are based on whether firms and workers acts simultaneously or non-simultaneously. They show that regarless of the firms, if the workers act simultaneously the outcome includes the full set of stable matchings, but also includes unstable ones. If they act non-simultaneously, the result is worker-optimal stable matching.

Romero-Medina and Triossi (2013) proposed an extension of the model by Alcalde and Romero-Medina (2000). Precisely, they extended the serial dictatorship. The students simultaneously propose to the colleges. And, then, the colleges in a fixed ordering are allowed to accept their offers in one single queue. They show that this extended-serial dictatorship mechanism implements the full set of stable matchings in subgame perfect Nash equilibrium.

Among those papers, our work is based on a semi-centralized work with a multi-stage game for one side. In this perspective, it is similar to Romero-Medina and Triossi (2013), but we allow multi-ordering; not restricted to one queue. There is a similarity to the paper by Haeringer and Wooders (2011) in the sense that there are multi-stages for non-proposers. But, we fix the the preferences of one side which makes them non-strategic players, namely *"objects"* in the game. Moreover, their looks like a chess game; the sides of the market play after the other side. The game consists of multi one-stage games. But, in our paper for one side (restricters) it is one stage game and for the other side (choosers) it is a multi stage game and they play the game with each other; not with the restricter agents on the other side. Since our models are different, we observe the differences in the outcomes; our mechanims never ends up with a best stable matching for any side, unless there is only one single matching of the game. Besides, our mechanism does not choose any unstable matching in subgame Nash equilibrium.

## 2.5 A Simple Extension of the Mechanism into Many-to-Many Case

Our basic mechanism $\gamma$ is defined for one-to-one matching games. And, we have showed that it implements the full set of stable matchings.

We can simply extent the mechanism into many-to-one games for which the college admission problem is a well-known example. First, we should convert the game into one-to-one. We can do this conversion by regarding each "seat" of a school as a "school with one seat". In this way, we seperate the restricters the schools into the seats. And, the seats of the same schools have the preference orderings over the set of the students. Tehn, it is easy to show that our previous results hold.

More interesting extension of one-to-one games is many-to-many games for which many of the properties of one-to-one models do not extend. The main reason for that is in this wider class of two-sided matching games, object comparison is introduced different from one-to-one games.

The usual example for this class is the match of the workers and the firms. There are two disjoint sets of the workers and the firms and each agent has a preference ordering over the agents of the other set. The main difference of many-to-many games from other scenarios is that any agent may have more than one mate. That is any worker may be matched to one than one firm and also the opposite. Formally,

**Definition 16** *Let $W = \{w_1, ..., w_k\}$ and $F = \{f_1, ..., f_l\}$ be two non-empty, finite and disjoint sets of workers and firms, with the quotas $Q^W = \{w_1^q, ..., w_k^q\}$ and $Q^F = \{f_1^q, ..., f_l^q\}$. Each agent has a strict preference ordering $R$ over the agents of the other set; that is $R_{w_i \in W}$ be the preference ordering of $w_i$ over*

*$F$. For any $w_i, w_j \in W$, $w_i R_{f_i} w_j$ means $f_i$ prefers $w_i$ over $w_j$. A Preference Profile $\mathbb{R} = (R_i)_{i \in F \cup W}$ is the a set of preferences of all agents in the model. $\mathbb{R}^{W \cup F}$ is the set of all preference profiles for the sets $F$ and $W$.*

**Definition 17** *$r_w(f)$ is the rank of agent $f$ in preference of agent $w$. That is, $r_w(f) = k$ means $f$ is the $k^{th}$ best firm of $w$.*

**Definition 18** *A matching $\mu : P(F)/\emptyset \rightarrow P(W)/\emptyset$ is a mapping, where $P(.)$ is the power set. For any $f \in F$ and $\overline{W} \subseteq W$, $\mu(f) = \overline{W}$ means that the set $\overline{W}$ is the match of $f$. $\mu(f) = f$ means $f$ is single in the matching if $\overline{W} = \emptyset$. $\mu$. $\Pi^{(P(F)/\emptyset)\cup(P(W)/\emptyset)}$ is the set of all matchings between $(P(F)/\emptyset)$ and $(P(W)/\emptyset)$.*

In this many-to-many model, we study the Pairwise Stability concept. But, *"object comparison"* is not the topic of this paper. So, we drop the object comparison part from the usual definition of pairwise stability.

**Definition 19** *Let $W$ and $F$ be the sets of the workers and the firms. Let $f, \overline{f} \in F$ and $w, \overline{w} \in W$. Let $\mu : P(F)/\emptyset \rightarrow P(W)/\emptyset$ be a matching. Let $\overline{w} \in \mu(f)$ and $\overline{f} \in \mu(w)$ with $\overline{f} R_{\overline{w}} f$ and $\overline{w} R_{\overline{f}} w$. Then, we say the matching $\mu$ is pairwise blocked by the pair $(\overline{f}, \overline{w})$. Then, we call $\mu$ **pairwise unstable**. If there is no pairwise blocking, then $\mu$ is **pairwise stable**.*

Then we define our mechanism $\gamma$ as,

$$\gamma : \mathbb{R}^{W \cup F} \longrightarrow \Pi^{(P(F)/\emptyset)\cup(P(W)/\emptyset)}$$

We shall apply our mechanism $\gamma$ to this many-to-many game.

### 2.5.1 The Flow of the Mechanism

Without loss of generality, we shall assign the firms as the restricters and the workers as the choosers; previously we have showed that order of the game does not change the result for one-to-one case and this many-to-many model is not an execption.

If a chooser agent accepts the offer, both of the agents fill one of their quotas. Any agent is deleted from the game, when he fills all of his positions.

Like in section 1.3.1, we have the concepts of a "smooth game" and a "conflict" for the chooser side. But, since the choosers stay in the game longer in this model, we need to modify our definitions.

**Definition 20** *Let $w \in W$ be any chooser agent and $f_i, f_j \in F$ be any two restricter agents. If $r_{f_i}(w) > r_{f_j}(w)$ and $f_i R_w f_j$ and at the step $k = r_{f_j}(w)$ non of $f_i$ and $f_j$ have been deleted from the game yet, then we say the agent $w$ experiences a* ***conflict*** *between agents $f_i$ and $f_j$ if the current rank of $f_j$ in $R_w^*$, $r_{f_j}^*(w)$, is bigger than the current quota of $w$, that is $r_{f_j}^*(w) > w^{q^*}$, in the current subgame.*

The definition says that for a chooser agent if the turn for a worse restricter comes before any better one, given that non of those restricters have not been deleted from the game yet, and if this restricter is not one of the favorite agents for the remaining positions, then the chooser agent experiences a conflict; he *may not* be sure about his decision.

The above definition is the key factor of this section. Since, the choosers may match to multi partners, we only observe conflicts when the restricter of the issue is not a top candidate for the quotas of the chooser. If the numerical values of both of the agents' ranks are less than their capacity, the agent directly accepts the offer.

**Definition 21** *If a chooser agent $w \in W$ does not experience any conflict, then we say $w$ has a* ***smooth game****.*

The definition of a cycle is same as the one in section 1.3.1.

**Definition 22** *Let $wi, wj \in W$ be the chooser agents who experince conflicts on the same restricter agents $fi, fj \in F$, in an opposite way. Then, we say that the set of the choosers $\{wi, wj\}$ experince a cyclical conflict for the set of the restricters $\{fi, fj\}$.*

According to above set up and definitions, all the results of the section 1.3 also hold for this many-to-many game.

**Theorem 23** *If Nash Equilibria of the cycles are chosen, the outcome of our mechanism is the set of the pairwise stable matchings for any preference profile. In other words, the mechanism $\gamma$ implements the full set of the pariwise stable matchings.*

**Proof.** The proof is based on the same arguments with Theorem 14. ■

It is easy to show that truth telling is weakly dominant for the choosers by Theorem 15.

## 2.6 The Conclusion

In this paper, we have proposed a new dynamic mechanism for the semi-centralized two-sided mathcing games. The model mimics the college admission procedures where the number of agents is too high in the market, like Turkey, Greece, Iran and China, where the admissions to the universities are centralized.

The mechanism is definedon a market where the preferences of one side are fixed (the schools) and we the other side (the students) play the game simultaneously. Which matching to be found is determined by the actions of the students at the decision steps.

The mechanism is an improvement under incomplete information in the sense that it partially or fully eliminates the blocking pairs depending on the preference profile. Under complete information, it ends up with one of the stable matchings. Precisely, the set of the possible outcomes of the procedure is the set of the stable matchings for a given market.

We also shoewd that a simple extension of the mechanism into many-to-many games, generates the full set of pairwise stable matchings, where we drop the object comparison part.

## References


[1] Alcalde, J. (1996), "Implementation of Stable Solutions to Marriage Problems", Journal of Economic Theory, 69, 240-254.

[2] Alcalde, J., & Romero-Medina, A., (2000), "Simple Mechanisms to Implement the Core of the College Admission Problems", Games and Economic Behaviour, 31, 294-302.

[3] Alcalde, J., Pérez-Castrillo, D. & Romero-Medina, A., (1998), "Hiring Procedures to Implement of Stable Allocations", Journal of Economics Theory, 82, 469-480.

[4] Balinski, M. & Sönmez, T. (1999), "A Tale of Two Mechanisms: Student Placement", Journal of Economic Theory, 84, 73-94.

[5] Blum, Y., Roth, A.E., & Rothblum, U.G., (1997), "Vacancy Chains and Equilibration in Senior-level Labour Markets", Journal of Economic Theory, 76, 362-411.

[6] Doğan, K. & Yuret, D. (2010), "Üniversitelere Öğrenci Yerleştirme Sisteminde Tercih Bildirimindeki Kısıtlamanın Etkileri", Ankara Üniversitesi SBF Dergisi, vol 65, 59-88 (in Turkish).

[7] Gale, D. & Shapley, L.S. (1962), "College Admissions and the Stability of Marriage", The American Mathematical Monthly, 69: 9-15.

[8] Haeinger, G. & Wooders, M. (2011), "Decentralized Job Matching", International Journal of Game Theory, vol 40, 1-28.

[9] Irving, R.W.&Leather, P. (1986), "The Complexity of Counting Stable Marriages", SIAM Journal on Computing, 15:655-667.

[10] Knuth, D.E. (1976), "Mariages Stables et leurs relations avec d'autres problèmes combinatoires", Les Presses de l'Université Montréal, Montréal.

[11] McVitie, D.G. & Wilson, L.B. (1971), "The Stable Marriage Problem", Comm. ACM 14, 486-490.

[12] Peleg, B., (1997), "Implementation the core of a marriage problem", Discussion Paper #132, Hebrew University of Jerusalem.

[13] Romero-Medina, A., & Triossi, M., (2013), "Non-revelation mechanisms in many-to-one markets", Games and Economic Behaviour, http://dx.doi.org/10.1016/j.geb.2013.08.005

[14] Roth, A., E., & Xing, X., (1997), "Turnaround time and bottlenecks in market clearing: Decentralized matching in the market for clinical psychologists", Journal of Political Economy, 105, 284-329.

[15] Roth,A.E.&Sotomayor,M. (1990), "Two-Sided Matching: A Study in Game-Theoretic Modelling and Analysis", Econometric Society Monograph Series, Cambridge Univ. Press, New York.

# 3 A Partially Biased Mechanism for the College Admission Problem

**Abstract**

Evci (2014) showed that their mechanism implements the full set of stable matchings for a semi-centralized market. In this paper, we propose a new mechanism to the same semi-centralized two-sided matching games.

We show that the mechanism generates a bias for the strategic player; that is our mechanism improves the outcome for the centralized side. The mechanism partitions the full domain; for the profiles in one partition, our mechanism coincides with the mechanism by Evci (2014) and for the other partition it end up with the algortihm by Chooser-Optimal Gale - Shapley (1962).

## 3.1 Introduction

The seminal work by Gale and Shapley (1962) showed that every two-sided matching game has a stable matching, which is known as the *"stability theorem"*. They also proved that there exist matchings which are best for either of the side in profile. Depending on the proposer side, their algorithm ends up with the stable matching which is the best for the prposers.

Evci (2014) proposed a mechanims for the semi-centralized two-sided markets. They propose the concept semi-centralized to the huge markets where the number of agents is too high. In such markets, either applying an centralized algorithm is not efficient, or it is possible with some restrictions on the procedure, which brings some inefficiencies as Doğan and Yuret (2010) have showed. Their mechanism in this specifically modified market implements the full set of stable matchings.

Under the same conditions, that is in a huge market where centralization is not possible in an efficient way, we propose a mechanism to improve the outcome for the strategic players, namely for the students in college admission problem. We show that the mechanism is partially successful in achieving this goal.

Moreover, the mechanism we propose here is in fact a refinement of the one by Evci (2014). We apply a little change to their mechanism and we improve the result for the choosers, as they call in their paper.

Since, we refine their mechanism, we directly adopt their notation. In the next section, we propose our refinement and the analysis of the game with the characterization the results.

## 3.2 The Mechanism

In this section, we propose a refinement of the mechanism by Evci (2014) and analyze the effects of this little modification[2]. Since they call their mechanism $\gamma$, we shall use the letter $\alpha$ for ours.

The difference of $\alpha$ from $\gamma$ is that when a chooser agent refuses the offer, he is re-placed to the end of the queue of the same restricter agent instead of loosing her forever as in $\gamma$. Now, we shall start with the most trivial example to analyze the equilibrium.

**Example 24** *We will focus on $\mathbb{R}_5$ in Example 13.*

$$\mathbb{R}_5 = \begin{array}{cccc} \underline{m_1} & \underline{m_2} & \underline{w_1} & \underline{w_2} \\ w_2 & w_1 & m_1 & m_2 \\ w_1 & w_2 & m_2 & m_1 \end{array}$$

*For the sets $M$ and $W$, the set of all the possible matchings between them is $\Pi^{M \cup W} = \{\mu_1, \mu_2\}$ where*

$$\mu_1 = \{(m_1, w_1), (m_2, w_2)\},$$

$$\mu_2 = \{(m_1, w_2), (m_2, w_1)\}.$$

[2] I would like to thank Giacomo Calzolari for suggesting this refinement.

*For the profile $\mathbb{R}_5$, the matchings $\{\mu_1, \mu_2\}$ are both stable and for the mechanism $\gamma$ the Nash Equilibrium (NE) of this game is $NE(w_1, w_2) = \{(yes, yes), (no, no)\}$. Under the mechanism $\alpha$, the story changes.*

*First, let us assume that $w_1$ says "no" and she is replaced to the end of the queue of the agent $m_2$. At the same step, if $w_2$ says "no", then in the second stage of the game, the tentative preference profile will look like,*

$$\mathbb{R}_5^* = \begin{array}{cccc} \underline{m_1} & \underline{m_2} & \underline{w_1} & \underline{w_2} \\ w_1 & w_2 & m_1 & m_2 \\ w_2 & w_1 & m_2 & m_1 \end{array}$$

*Then, in this step both of the choosers say "yes" and we end up with the matching $\mu_1$ which is chooser-optimal.*

*On the other hand, at the first step if $w_2$ says "yes", then she forms the pair $(m_1, w_2)$ and both of them are deleted from the profile. In the second stage of the game, the tentative preference profile will look like*

$$\mathbb{R}_5^* = \begin{array}{c} \underline{m_2} \\ w_1 \end{array}$$

*Then, $w_1$ forms the pair $(m_2, w_1)$ and we end up with the matching $\mu_2$ which is restricter-optimal.*

*Secondly, let us assume that $w_1$ says "yes", then she forms the pair $(m_2, w_1)$ and both of them are deleted from the profile. At the first step, whatever $w_2$ says, we end up with the matching $\mu_2$.*

*Hence, at the first step, if $w_1$ says "yes", the game ends up with $\{\mu_2\}$ and if $w_1$ says "no", the game ends up with one of $\{\mu_1, \mu_2\}$. So, regardless of which action $w_2$ takes, for $w_1$ rejecting the offer is* ***weakly dominant****. The arguments are the same for $w_2$.*

*Therefore, for the mechanism $\alpha$ the Nash Equilibrium (NE) of this game is $NE(w_1, w_2) = \{(no, no)\}$, which ends up with $\mu_1$, the chooser-optimal matching.*

We have showed that for the profile $\mathbb{R}_5$, $\alpha$ is an improvement for the choosers;

of course the case is opposite for the restricters. We continue with another example.

**Example 25** *We will study $\mathbb{R}_3$ in Example 8.*

$$\mathbb{R}_3 = \begin{array}{ccc} \underline{m_1} & \underline{m_2} & \underline{m_3} \\ w_1 & w_2 & w_2 \\ w_3 & w_1 & w_3 \\ w_2 & w_3 & w_1 \end{array} \qquad \begin{array}{ccc} \underline{w_1} & \underline{w_2} & \underline{w_3} \\ m_3 & m_1 & m_3 \\ m_2 & m_2 & m_1 \\ m_1 & m_3 & m_2 \end{array}$$

*The set of the stable matchings for $\mathbb{R}_3$ is $\{\mu_4, \mu_5\}$ according to the list of the matchings in Example 1. Here, the strategic players are $w_1$ and $w_2$.*

*The same arguments with those in Example 16 work. For both of $w_1$ and $w_2$, rejecting the offeres at step 1 is weakly dominant. Therefore, for the mechanism $\alpha$ the Nash Equilibrium (NE) of this game is $NE(w_1, w_2) = \{(no, no)\}$, which ends up with $\mu_5$, the chooser-optimal matching.*

Example 25 shows that, eventhough it is not a trivial example, for $\mathbb{R}_3$, which has two stable matchings, $\alpha$ is an improvement for the choosers.

**Claim 26** *From Example 24 and 25, can we conclude that for all profiles with two stable matchings (one cyclical conflict), $\alpha$ is an improvement for the choosers?*

The following example shows that the answer is negative.

**Example 27** *Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$ be the sets of men and women who have the following preference profile $\mathbb{R}_6$:*

$$\mathbb{R}_6 = \begin{array}{ccc} \underline{m_1} & \underline{m_2} & \underline{m_3} \\ w_1 & w_2 & w_1 \\ w_3 & w_1 & w_3 \\ w_2 & w_3 & w_2 \end{array} \qquad \begin{array}{ccc} \underline{w_1} & \underline{w_2} & \underline{w_3} \\ m_2 & m_1 & m_1 \\ m_1 & m_3 & m_2 \\ m_3 & m_2 & m_3 \end{array}$$

*For profile $\mathbb{R}_6$, the set of stable matchings is $\{\mu_3, \mu_5\}$. Now, we will analyze the possible scenarios.*

*First, let us assume that $w_1$ accepts the offer and $w_2$ does not at step 1, then we end up with $\mu_6$, which $w_1$ prefers less than $\mu_3$, chooser-optimal matching. So, if $w_2$ rejects the offer, so does $w_1$.*

*Second, let us assume that $w_1$ rejects the offer and $w_2$ accepts at step 1, then we end up with $\mu_1$, which $w_1$ prefers less than $\mu_5$, restricter-optimal matching. Therefore, if $w_2$ accepts the offer, so does $w_1$.*

*Hence, there is no dominant strategy at step 1. We conclude that $\alpha$ is not an improvement for the choosers for $\mathbb{R}_6$.*

Next example will be on a profile with three stable matchings.

**Example 28** *Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$ be the sets of men and women who have the following preference profile $\mathbb{R}_7$:*

$$\mathbb{R}_7 = \begin{array}{cccccc} \underline{m_1} & \underline{m_2} & \underline{m_3} & \underline{w_1} & \underline{w_2} & \underline{w_3} \\ w_1 & w_2 & w_3 & m_3 & m_1 & m_2 \\ w_2 & w_1 & w_2 & m_2 & m_3 & m_1 \\ w_3 & w_3 & w_1 & m_1 & m_2 & m_3 \end{array}$$

*For profile $\mathbb{R}_7$, the set of stable matchings is $\{\mu_2, \mu_4, \mu_5\}$. Now, we will analyze the possible scenarios for $w_1$. The table below shows all possible decisions at step 1 and corresponding mate that $w_1$ matches from the game.*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $w_1$ | $Yes$ | $No$ | $Yes$ | $No$ | $Yes$ | $No$ | $Yes$ | $No$ |
| $w_2$ | $Yes$ | $Yes$ | $Yes$ | $Yes$ | $No$ | $No$ | $No$ | $No$ |
| $w_3$ | $Yes$ | $Yes$ | $No$ | $No$ | $Yes$ | $Yes$ | $No$ | $No$ |
| $Mate$ | $m_1$ | $m_1$ | $m_1$ | $m_3$ | $m_1$ | $m_2$ | $m_1$ | $m_3$ |

*So, we conclude that for $w_1$ rejecting the offer at step 1 is weakly dominant. Same analysis for $w_2$ and $w_3$ shows that also for those agents it is a dominant strategy to refuse. Therefore, for the mechanism $\alpha$ the Nash Equilibrium (NE) of this game is $NE(w_1, w_2, w_3) = \{(no, no, no)\}$, which ends up with $\mu_2$, the chooser-optimal matching.*

Example 28 shows that, for $\mathbb{R}_7$, which has three stable matchings (inter-dependent cycles), $\alpha$ is an improvement for the choosers.

**Claim 29** *From Example 28, can we conclude that for all profiles with three stable matchings (inter-dependent cycles), $\alpha$ is an improvement for the choosers?*

The following example shows that the answer is negative.

**Example 30** *We will study $\mathbb{R}_1$ in Example 1.*

$$\mathbb{R}_1 = \begin{array}{ccc c ccc} \underline{m_1} & \underline{m_2} & \underline{m_3} & \quad & \underline{w_1} & \underline{w_2} & \underline{w_3} \\ w_1 & w_2 & w_2 & & m_3 & m_1 & m_2 \\ w_2 & w_1 & w_3 & & m_2 & m_2 & m_1 \\ w_3 & w_3 & w_1 & & m_1 & m_3 & m_3 \end{array}$$

*For profile $\mathbb{R}_1$, the set of stable matchings is $\{\mu_2, \mu_4, \mu_5\}$. Now, we will analyze the possible scenarios.*

*First, let us assume that $w_1$ accepts the offer and $w_2$ does not at step 1, then we end up with $\mu_6$, which $w_2$ prefers less than $\mu_5$, men-optimal matching. So, if $w_1$ accepts the offer, so does $w_2$.*

*Second, let us assume that $w_1$ rejects the offer and $w_2$ accepts at step 1, then we end up with $\mu_1$, which $w_2$ prefers less than $\mu_4$ or $\mu_2$ (women-optimal matching). Hence, $w_2$ is bounded by the action of $w_1$ at step 1.*

*Finally, let us assume that both of $w_1$ and $w_2$ reject the offers at step 1, then we will have the following tentative profile,*

$$\mathbb{R}_1^* = \begin{array}{ccc c ccc} \underline{m_1} & \underline{m_2} & \underline{m_3} & \quad & \underline{w_1} & \underline{w_2} & \underline{w_3} \\ w_2 & w_1 & w_3 & & m_3 & m_1 & m_2 \\ w_3 & w_3 & w_1 & & m_2 & m_2 & m_1 \\ w_1 & w_2 & w_2 & & m_1 & m_3 & m_3 \end{array}$$

*For profile $\mathbb{R}_1^*$, the set of stable matchings is $\{\mu_2, \mu_4\}$. And, it is easy to show that at the second step of $\mathbb{R}_1^*$ for $w_1$ and $w_3$, rejecting the offers is weakly dominant which leads us to $\mu_2$, the women-optimal matching.*

*Since there is no dominant strategy at the first step of* $\mathbb{R}_1$*, we cannot conclude that the outcome set is* $\{\mu_2, \mu_5\}$*. Even if we ended up with the set* $\{\mu_2, \mu_5\}$ *by dominant strategies, we could not say that this set is better than the set* $\{\mu_2, \mu_4, \mu_5\}$*;* ***object comparison*** *is not the topic of this paper.*

So far we have showed that for the profiles with independent or interdependent cyclical conflicts $\alpha$ may or may not be an improvement for the choosers compared to $\gamma$. The bigger profiles with more than three stable matchings (including incomparable matchings) consist of both independent and interdependent cycles. Hence, the same arguments and similar examples, like above, work also for those profiles.

**Proposition 31** *Let* $M = \{m_1, ..., m_k\}$ *be the restricter and* $W = \{w_1, ..., w_l\}$ *be the chooser side, with* $l > k$*.* $\nexists \mathbb{R} = (R_i)_{i \in M \cup W} \in \mathbb{R}^{W \cup M}$*,* $\alpha$ *is an improvement for the choosers. In other words,* $\forall \mathbb{R} = (R_i)_{i \in M \cup W} \in \mathbb{R}^{W \cup M}$*,* $\gamma$ *and* $\alpha$ *coincide.*

**Proof.** The proof is straight forward. For simplicity let us assume $k = n$ and $l = n + 1$. Let $\mathbb{R} \in \mathbb{R}^{W \cup M}$ be any profile.

If $\mathbb{R}$ has only one stable matching, then both of $\gamma$ and $\alpha$ finds it. Hence, they coincide.

Let us assume $\mathbb{R}$ has more than one stable matching. Let $\Pi^{M \cup W} = \{\mu_1, ..., \mu_r\}$ be the set of all stable matchings for $\mathbb{R}$. From Roth and Sotomayor (1990), we know that for a profile $\mathbb{R}$, the set of the agents that are matched is the same for all stable matchings. Therefore, the set of $n$ women matched to $n$ men are same for all $\mu \in \Pi^{M \cup W}$.

Let $w^* \in W$ be the agent who remains single for all stable matchings. Let $\overline{w} \in W$ be one the agents in stable matchings. We only need to show that there exists a cycle where $\overline{w}$ does not have a weakly dominant strategy.

Without loss of generalirt, let us assume $\forall w \in W/\{\overline{w}\}$ accept the offers in the first cycle they confront and $\overline{w}$ rejects the offer by $\overline{m} \in M$ and she is re-placed to the end of the same queue. Since $\forall m \in M/\{\overline{m}\}$ have been taken and deleted from the profile, in the next step $w^*$ accepts the offer by $\overline{m} \in M$ and forms the pair $(\overline{m}, w^*)$. $\overline{w}$ remains single which she prefers less than being matched to $\overline{m}$. Hence, $\overline{w}$ is bounded by the actions of the other agents, like in $\gamma$. The same argument works for any other agent that is matched in stable matchings.

We conclude that $\alpha$ is not an improvement and it coincides with $\gamma$. ■

Finally, we state our most general result.

**Theorem 32** *Let $M = \{m_1, ..., m_k\}$ be the restricter and $W = \{w_1, ..., w_l\}$ be the chooser side. Let $\mathbb{R}^{W \cup M}$ be the set of all profiles.*

*Let $\mathbb{R}^*, \mathbb{R}^{**}, \mathbb{R}^{***} \in \mathbb{R}^{W \cup M}$ be disjoint sub-domains, that is $\mathbb{R}^* \cup \mathbb{R}^{**} \cup \mathbb{R}^{***} = \mathbb{R}^{W \cup M}$ and $\mathbb{R}^* \cap \mathbb{R}^{**} = \emptyset$, $\mathbb{R}^* \cap \mathbb{R}^{***} = \emptyset$, $\mathbb{R}^{**} \cap \mathbb{R}^{***} = \emptyset$, where we have $k \geq l$ for $\mathbb{R}^*$ and $\mathbb{R}^{**}$ and $k < l$ for $\mathbb{R}^{***}$.*

*Let $GS^{Chooser}$ denote the mechanism by Gale and Shapley where, as we call, the chooser side propose. Then, we have*

$$\alpha = \begin{cases} GS^{Chooser}, & \text{if } \mathbb{R} \in \mathbb{R}^* \\ \gamma, & \text{if } \mathbb{R} \in \mathbb{R}^{**} \\ \gamma, & \text{if } \mathbb{R} \in \mathbb{R}^{***} \end{cases}$$

**Proof.** *Examples* $24 - 30$ and *Proposition* 31 proves the theorem. ■

## 3.3 The Conclusion

In this paper, we have proposed a new mechanism to generate a bias for the chooser side in semi-centralized two-sided matching game as described in Evci (2014). We have showed that we are partially successful for this purpose; the refined mechanism is an improvement in a subdomain.

Basically the mechanism $\alpha$ partitions the full domain into two. In one of them, it coincides with $GS^{Chooser}$, that is it ends up with chooser-optimal stable matching, and in the other it coincides with $\gamma$, that is it ends up with any of the stable matchings. Then, for the second case we partition this sub-domain into two as $\mathbb{R}^{**}$ and $\mathbb{R}^{***}$.

Unfortunately, for now we cannot know further about the distinction between $\mathbb{R}^{*}$ and $\mathbb{R}^{**}$. This is because the improvement for the case $k \geq l$ is profile based and we do not have any extra relation for the profiles within $\mathbb{R}^{*}$ or $\mathbb{R}^{**}$.

# References


[1] Doğan, K. & Yuret, D. (2010), "Üniversitelere Öğrenci Yerleştirme Sisteminde Tercih Bildirimindeki Kısıtlamanın Etkileri", Ankara Üniversitesi SBF Dergisi, vol 65, 59-88 (in Turkish).

[2] Evci, B. (2014), "A New Mechanism to the Two-Sided Matching Games", Unpublished manuscript.

[3] Gale, D. & Shapley, L.S. (1962), "College Admissions and the Stability of Marriage", The American Mathematical Monthly, 69: 9-15.

[4] Roth,A.E.&Sotomayor,M. (1990), "Two-Sided Matching: A Study in Game-Theoretic Modelling and Analysis", Econometric Society Monograph Series, Cambridge Univ. Press, New York.

# 4 A Simple Mechanism for the Roommate Problem

**Abstract**

Gale and Shapley (1962) proposed that there is a similar game to marriage problem called "the roommate problem". And, they showed that unlike the marriage problem, the roommate problem may have unstable solutions. In other words, the stability theorem fails for the roommate problem.

In this paper, we propose a new mechanism to the roommate problem. Firstly, the mechanism is successful in determining the reason of instability. Hence, it detects whether any given profile has a stable matching or not. If the profile has a stable solution, the mechanism finds that matching. We also show how we can end up with any stable matchings in the existence of multi stability using our mechanism.

## 4.1 Introduction

Gale and Shapley (1962) described the well known roommate problem in their seminal paper. In the problem, there are even number of boys and rooms for paired boys. Each boy has a preference ordering over the other boys. The objective is to allocate these boys to the rooms. They showed that this problem does not hold the stability theorem and they describe a counter-example in their paper.

Knuth (1976) showed that multiple solutions could exist, like the marriage problem. In his 12 famous questions he raised in these lecture notes he asked for an efficient algorithm to find a stable solution for the roommate problem.

Irving (1985) proposed a deferred acceptance algorithm for the roommate problem. The algorithm tells whether a given profile has a stable solution or not and if there exists the algorithm finds it.

In this paper, we propose a simple mechanism to this problem. We benefit from the mechanism proposed in Evci (2014). The problem here is that their mechanism is for semi-centralized two-sided matching games. But, the roommate problem is a one-sided matching game. Hence, firstly we convert the model of the roommate problem into a semi-centralized two-sided game by using auxiliary functions, and then we will apply the mechanism by Evci (2014) to this modified market. And also, we give our solution to multi stability case using our mechanism.

In the next section, we describe the roommate problem with an example. Later, we propose our mechanism and model.

## 4.2 Basics and Examples

"The Roommate Problem" is one of the most interesting examples of matching theory.[3] In the game we match the agents, but there is only side.

The problem was proposed firstly by Gale and Shapley (1962). In the roommate problem, as one-sided game, we have an even-number cardinal set of agents. There are $2n$ number boys and $n$ rooms. Each boy has a preference ordering over the other $(2n-1)$ boys. The objective is to allocate those boys to $n$ rooms in pairs.

In their paper, Gale and Shapley give a counter example which shows that stability theorem, which holds for the marriage problem, fails for the roommate problem. They say *"...consider boys $\alpha$, $\beta$, $\gamma$ and $\delta$, where $\alpha$ ranks $\beta$ first, $\beta$ ranks $\gamma$ first, $\gamma$ ranks $\alpha$ first, and $\alpha$, $\beta$ and $\gamma$ all rank $\delta$ last. Then regarless of $\delta$'s preferences..."*. We shall demonstrate their example with the following preference profile $\mathbb{R}_8$,

[3] I would like to thank Jean Lainé for the suggestion to test the mechanism for the roommate problem.

$$\mathbb{R}_8 = \begin{array}{cccc} \underline{a} & \underline{b} & \underline{c} & \underline{d} \\ b & c & a & c \\ c & a & b & b \\ d & d & d & a \end{array}$$

where $N = \{a, b, c, d\}$ be the set of the agents and $\Pi^N = \{\mu_1, \mu_2, \mu_3\}$ be the set of all possible matchings, where

$$\mu_1 = \{(a, b), (c, d)\},$$
$$\mu_2 = \{(a, c), (b, d)\},$$
$$\mu_3 = \{(a, d), (b, c)\}.$$

None of those matchings is stable; $\mu_1$ is blocked by $(b, c)$, $\mu_2$ is blocked by $(a, b)$ and $\mu_3$ is blocked by $(a, c)$. So, in this one-sided game, we observe unstable matching games as well as the stable ones.

## 4.3 The Mechanism and The Model

In this section we refine the mechanism $\gamma$ of Evci (2014) to the roommate problem. The mechanism $\gamma$ is designed for two-sided matching games. Therefore, we should modify either the mechanism or the roommate problem. To stick with the mechanism and its structure, we shall modify the game. So, we need to convert this game into two-sided case. For this purpose, we benefit from a well-known social welfare function (SWF).

**Definition 33** *Let $A$ be a set of alternatives, with $Card(A) = m$, and $N$ be a set of objects, with $Card(n)$. $\forall i \in N$, $R_i$ be the (strict) preference ordering of $i$ over the set of alternatives $A$. $\mathbb{R}$ be the set of all orderings and $\mathbb{R}^N$ be a preference profile.*

*A Social Welfare Function (SWF) $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ gives the social preference of the society $N$ over the alternative set $A$.*

This is the usual definition of a social welfare function in Social Choice Theory. Basically, we aggregate the orderings of all agents. Next, we give the definition of a famous SWF, which is one of the Scoring Rules.

**Definition 34** *In a profile, the Borda Score $BS(x)$ of an alternative $x$ is $BS(x) = \sum_{i \in N} [(m+1) - r_i(x)]$. In a voting system, the Borda Rule as a SWF, ranks the alternatives according to their Borda Scores. We allow weak orders in social preference.*

And, this is the usual the Borda Rule definition. In the roommate problem, since there is no alternative set, we modify the definiton of the Borda Rule to this game. Now, we shall show this modification in an example.

**Example 35** *Let $N = \{a, b, c, d, e, f\}$ be the set boys with a preference profile $\mathbb{R}_9$,*

$$\mathbb{R}_9 = \begin{array}{cccccc} \underline{a} & \underline{b} & \underline{c} & \underline{d} & \underline{e} & \underline{f} \\ b & c & d & a & a & b \\ c & d & a & b & b & a \\ d & a & b & c & c & c \\ e & e & f & f & d & d \\ f & f & e & e & f & e \end{array}$$

*Now, we shall compute the Borda scores of the agents.*

$$B(a) = 0 + 3 + 4 + 5 + 5 + 4 = 21$$
$$B(b) = 5 + 0 + 3 + 4 + 4 + 5 = 21$$
$$B(c) = 4 + 5 + 0 + 3 + 3 + 3 = 18$$
$$B(d) = 3 + 4 + 5 + 0 + 2 + 2 = 16$$
$$B(e) = 2 + 2 + 1 + 1 + 0 + 1 = 7$$
$$B(f) = 1 + 1 + 2 + 2 + 1 + 0 = 7$$

*From these scores, we get the profile to be used for our mechanism,*

$$\mathbb{R}_9^* = \begin{array}{cccccccc} \underline{a} & \underline{b} & \underline{c} & \underline{d} & \underline{e} & \underline{f} & & \underline{B(R)} \\ b & c & d & a & a & b & & ab \\ c & d & a & b & b & a & & c \\ d & a & b & c & c & c & & d \\ e & e & f & f & d & d & & ef \\ f & f & e & e & f & e & & \end{array}$$

Since the mechanism in Evci (2014) is called $\gamma$, we shall denote ours by $\beta$. $\beta$ is defined over the preference profile $\mathbb{R}^N$ and its Borda ranking $B(\mathbb{R}^N)$ into the set of matchings $\Pi^N$. Formally,

$$\beta : (\mathbb{R}^N, B(\mathbb{R}^N)) \longrightarrow \Pi^N.$$

Now, we describe how our mechanism works here. We use the Borda ranking of preference profile to split the game into two-sided case. The Borda ranking gives the order of the agents that will be the restricter in all successive steps.

We assign the first agent in Borda ranking as the restricter. Then, all the other agents take place in the chooser side. If there is more than one agent at the top, we randomly choose one of them and assign him as the restricter. Then, we run the mechanism $\gamma$.

**Claim 36** *At the end of the first stage, we get a pair which consists of the restricter and one of the choosers.*

**Proof.** The proof is easy. Since the restricter is (one of) the top agent(s) in the Borda ranking, he is a favorite agent. If there exists a chooser agent whose the best agent is the restricter, then trivially they form a pair.

So, let us assume that there is no agent whose the best agent is this restricter. This is possible under the Borda rule. If all the choosers reject, then we get an unstable matching (in the existence of stable matchings) which is against the rationality of the agents. Since there is only one ordering, the choosers do not confront any conflict or cyclical conflict as they do in the games for $\gamma$, it is easy to show that in subgame perfect Nash equilibrium, since there is only one ranking (that is serial dictatorship), there exists a chooser that accepts the offer because the other better alternatives are not achievable. ■

Then, we delete the agents of the pair from the preference profile and the Borda ranking. For the second stage of the game, we assign the best agent in the Borda ranking as the restricter. Then, we run our mechanism. And, so on.

Now, we demonstrate the mechanism $\beta$ with an example.

**Example 37** *We will study $\mathbb{R}_9$ in Example 35.*

$$\mathbb{R}_9^* = \begin{array}{cccccccc} \underline{a} & \underline{b} & \underline{c} & \underline{d} & \underline{e} & \underline{f} & & \underline{B(R)} \\ b & c & d & a & a & b & & ab \\ c & d & a & b & b & a & & c \\ d & a & b & c & c & c & & d \\ e & e & f & f & d & d & & ef \\ f & f & e & e & f & e & & \end{array}$$

*Since there is tie between $a$ and $b$, we randomly choose one of them.*

*Firstly, let us pick $a$ as the restricter. Firstly, $b$ is called to make a choice. If he accepts the offer, then he forms the pair $(a, b)$. In the second stage $c$ becomes the restricter. $d$ is called for an offer and $d$ accepts the offer since $a$ and $b$ are deleted from the profile and so there is no better mate remained. Then, he forms the pair $(c, d)$. The final pair $(e, f)$ is automatically formed. Now, let us assume that $b$ rejects the offer in the first stage. Then, $a$ offers to $c$. If $c$ rejects the offer, then $d$ will be called and surely he will accept the offer which means $c$ will loose his chance for both of $a$ and $d$. So, $c$ accepts the offer and forms the pair $(a, c)$. In the second stage, $b$ will be the restricter and he offers to $d$. $d$ will* *definitely accepts and forms the pair $(b, d)$. And, the pair is $(e, f)$. Now, let us back to the beginning of the first stage. If $b$ accepts the offer from $a$, he forms the pair $(a, b)$. If he rejects $a$'s offer, then he forms the pair $(b, d)$. Since, he prefers $d$ over $a$, then in the equilibrium, he rejects at the first stage and we end up with matching $\mu = \{(a, c), (b, d), (e, f)\}$, which is the only stable matching for $\mathbb{R}_9$.*

*Secondly, let us pick $b$ as the restricter of the first stage. It is easy to show that for $c$ rejecting $b$'s offer is dominant and we end up with the same matching $\mu$.*

Example 37 showed that the mechanism finds a stable matching for $\mathbb{R}_9$. From Theorem 14 in section 2.3.1, this result is not unexpected. $\mathbb{R}_9$ has a stable matching and our mechanism $\beta$ finds it.

But, what about $\mathbb{R}_8$, the example by Gale and Shapley? What do we observe if we apply our mechanism to some profile that does not have any stable matching?

**Example 38** *We will study $\mathbb{R}_8$.*

$$\mathbb{R}_8^* = \begin{array}{cccccc} \underline{a} & \underline{b} & \underline{c} & \underline{d} & & \underline{B(R)} \\ b & c & a & c & & c \\ c & a & b & b & & b \\ d & d & d & a & & a \\ & & & & & d \end{array}$$

*In the first stage, $c$ will be the restricter. $a$ is called to make a decision. If $a$ accepts the offer, he forms the pair $(a, c)$. Then, the other pair will be $(b, d)$. If $a$ rejects, then $b$ will be called. Definitely $b$ accpets the offer and forms $(b, c)$. Then, the other pair will be automatically $(a, d)$. So, at the beginning of the stage, if $a$ accepts the offer, then he matches to $c$. If he rejects, then his mate* *will be $d$. Since he prefers $c$ over $d$, $a$ accepts the offer by $c$. Then, we end up with matching $\mu_2 = \{(a, c), (b, d)\}$.*

We applied the mechanism $\beta$ to $\mathbb{R}_8$ and we end up with an unstable matching. This result is not a surprise; we knew that there is no stable matching for this profile. The unexpected point is the behaviour of our mechanism. In section 2, we showed that $\gamma$ is a stable mechanism; it always finds a stable matching. The surprise part is that as if there is stable matching for $\mathbb{R}_8$, the procedure is very smooth. But, in the end it gives an ustable matching. Then, what is the mistery for $\mathbb{R}_8$?

Next example will give us some clue about the perspective that we should have to evaluate $\mathbb{R}_8$ or any other profile for which we do not end up with any stable matching.

**Example 39** *Let $N = \{a, b, c, d, e, f\}$ be the set boys with a preference profile $\mathbb{R}_{10}$,*

$$\mathbb{R}^*_{10} = \begin{array}{cccccccc} \underline{a} & \underline{b} & \underline{c} & \underline{d} & \underline{e} & \underline{f} & & \underline{B(R)} \\ c & c & d & a & a & b & & a \\ b & d & a & c & b & a & & c \\ d & a & b & b & c & c & & b \\ e & e & f & f & d & d & & d \\ f & f & e & e & f & e & & ef \end{array}$$

*In the first stage, $a$ will be the restricter. $c$ is called to make a decision. If $c$ accepts the offer, he forms the pair $(a, c)$. In the second stage $b$ becomes the restricter. $d$ is called for an offer and $d$ accepts the offer since $a$ is deleted from the profile and so there is no better mate remained. Then, he forms the pair $(b, d)$. The final pair $(e, f)$ is automatically formed. And, we end up with matching $\mu_i = \{(a, c), (b, d), (e, f)\}$.*

*Now, let us assume that c rejects the offer in the first stage. Then, a offers to b. If b accepts the offer, he forms the pair $(a, b)$. In the second stage c becomes the restricter. Then, c offers to d and d accepts the offer since a is deleted from the profile. Then, he forms the pair $(c, d)$. The final pair $(e, f)$ is automatically formed. Then, we end up with matching $\mu_j = \{(a, b), (c, d), (e, f)\}$.*

*If b rejects the offer in the first stage, then d will be called and surely he will definitely accept the offer and forms the pair $(a, d)$. In the second stage c becomes the restricter. Then, c offers to b and b definitely accepts the offer. Then, he forms the pair $(b, c)$. The final pair $(e, f)$ is automatically formed. Then, we end up with matching $\mu_k = \{(a, d), (b, c), (e, f)\}$.*

*In the first stage (after c rejects a's offer), if b accepts a's offer, we end up with $\mu_j$. If he rejects a's offer, we end up with $\mu_k$. Since $\mu_k R_b \mu_j$, b rejects the offer by a.*

*In the first stage, if c accepts a's offer, we end up with $\mu_i$. If c rejects a's offer, we end up with $\mu_k$. Since $\mu_i R_c \mu_k$, c accepts the offer by a.*

*Hence, we end up with matching $\mu_i$, which is unstable for $\mathbb{R}_{10}$.*

We applied our mechanism $\beta$ to $\mathbb{R}_{10}$ and we find an unstable matching. Is $\mathbb{R}_{10}$ one of the profiles which do not have any stable matching?

The answer is "No!". $\mathbb{R}_{10}$ has absolutely and only one stable matching and it is $\mu_j$.

As we have stated and proved, the mechanism $\gamma$ is stable. $\beta$ is stronger than $\gamma$, since there is only one queue and the chooser agents never experience any conflict. Then, why cannot $\beta$ end up with a/the stable matching while there exist some?

The definition below will help us to figure out the reason.

**Definition 40** *Let $N$ be a set of agents. Let $R$ be the preference profile and $B(R)$ is the corresponding Borda ranking. Let $M \subset N$ be a proper subset of $N$.*

*The preference profile $\hat{R}$ of $M$ is constructed by deleting the agents $i \in N/M$ in $R$. Namely, $\hat{R}$ is the profile **Purified from Irrelevant Alternatives (PIA)** of $M$ and $B(\hat{R})$ is the corresponding Borda ranking.*

In the next example, we will examine $\mathbb{R}_{10}$ with purified Borda ranking.

**Example 41** $\hat{\mathbb{R}}_{10} =$

$$\begin{array}{cccccccc} \underline{a} & \underline{b} & \underline{c} & \underline{d} & \underline{e} & \underline{f} & \underline{B(R)} & \underline{B(\hat{R})} \\ c & c & d & a & a & b & a & c \\ b & d & a & c & b & a & c & ad \\ d & a & b & b & c & c & b & b \\ e & e & f & f & d & d & d & ef \\ f & f & e & e & f & e & ef & \end{array}$$

*In $\mathbb{R}_{10}$, the agents $e$ and $f$ are the worst alternatives for the rest of the society and for each other they are same. If we purify $\mathbb{R}_{10}$ by excluding $e$ and $f$, we get the relationships of $\{a, b, c, d\}$ with each other, as seen in profile $\mathbb{R}_{10}^{\{a,b,c,d\}}$ below,*

$$\mathbb{R}_{10}^{\{a,b,c,d\}} = \begin{array}{ccccc} \underline{a} & \underline{b} & \underline{c} & \underline{d} & \underline{B(\hat{R})} \\ c & c & d & a & c \\ b & d & a & c & ad \\ d & a & b & b & b \end{array}$$

*The comparison of $B(R)$ and $B(\hat{R})$ tells us that eventhough $a$ is the most favorite member of the set $\{a, b, c, d\}$, the support from an "irrelevant" set $\{e, f\}$ makes him the best, which falsifies the result of the game. The fake position of $a$ makes him get a better mate which leads to an unstable matching.*

*If we start the game with $a$, then we get $\mu_i = \{(a, c), (b, d), (e, f)\}$. On the other hand, strating with $c$ gives $\mu_j = \{(a, b), (c, d), (e, f)\}$ and we have $\mu_i R_a \mu_j$.*

Before we find out why the same scenario does not happen for $\mathbb{R}_9$, we have a corollary to understand the story in $\mathbb{R}_9$.

**Corollary 42** *Let $M \subset N$ be an even-cardinal subset whose members form a (top) cycle in purified Borda ranking. There is only one stable matching and in that matching agents form pairs with their mid-rank agents.*

**Proof.** If $Card(M) = 2$, then it is trivial; for both of the agents, the other is the best for him and they form the pair.

Let $Card(M) > 2$. Mid-ranks give the stable matching which is easy to check. Let $i, j \in M$ be two agents in pair in the stable case. Now, we shall assume there is another stable matching where we have the pair $(i, k)$. If we give, wlog, $k \in M$ to $i$ which $i$ prefers more than $j$, because of the structure of a cycle, $k$ gets worse off. If $Card(M) = 4$, then the other agent $l \in M$ forms the pair with $j$, $(j, l)$. Using the structure of a cycle, we find that $j$ becomes better off and $l$ becomes worse off. Then, $(k, l)$ blocks the matching. If $Card(M) > 4$, we keep giving better agents to those who has become worse off and we end with a conflict that there is no cycle among the members of $M$. ■

Now, we check the case for $\mathbb{R}_9$.

**Example 43** $\hat{\mathbb{R}}_9 =$

| $\underline{a}$ | $\underline{b}$ | $\underline{c}$ | $\underline{d}$ | $\underline{e}$ | $\underline{f}$ | $\underline{B(R)}$ | $\underline{B(\hat{R})}$ |
|---|---|---|---|---|---|---|---|
| $b$ | $c$ | $d$ | $a$ | $a$ | $b$ | $ab$ | $abcd$ |
| $c$ | $d$ | $a$ | $b$ | $b$ | $a$ | $c$ | $ef$ |
| $d$ | $a$ | $b$ | $c$ | $c$ | $c$ | $d$ | |
| $e$ | $e$ | $f$ | $f$ | $d$ | $d$ | $ef$ | |
| $f$ | $f$ | $e$ | $e$ | $f$ | $e$ | | |

*For $\mathbb{R}_9$ with unpurified ranking, since $\{a, b\}$ is the top cycle of $B(R)$, we run our mechanism for both of $a$ and $b$ and we showed that starting with either $a$ or $b$ gives the same stable matching.*

*As we have showed in Corollary 42, there is only one stable matching for the set $\{a, b, c, d\}$ among each other. Since our mechanism $\beta$ is stable, whoever we start with, the procedure ends up with the same matching.*

*Since $B(R)$'s top cycle $\{a, b\}$ is included by the one of $B(\hat{R})$, $\mathbb{R}_9$ with unpurified Borda ranking gives coincidetanlly the stable matching. c and d would give the same matching from the corollary.*

Now, we will examine the profile $\mathbb{R}_8$, the example of Gale and Shapley, with purified orderings.

**Example 44** $\hat{\mathbb{R}}_8 =$

| $\underline{a}$ | $\underline{b}$ | $\underline{c}$ | $\underline{d}$ | $\underline{B(R)}$ | $\underline{B(\hat{R})}$ |
|---|---|---|---|---|---|
| $b$ | $c$ | $a$ | $c$ | $c$ | $abc$ |
| $c$ | $a$ | $b$ | $b$ | $b$ | $d$ |
| $d$ | $d$ | $d$ | $a$ | $a$ | |
| | | | | $d$ | |

*Since there is a cycle between $\{a, b, c\}$, we randomly pick one of the agents and assign him as the restricter. If we start with a, we end up with $\mu_1 = \{(a, b), (c, d)\}$. Starting with b gives $\mu_3 = \{(a, d), (b, c)\}$. And, finally if c is the restricter of the first stage, the game reaches at $\mu_2 = \{(a, c), (b, d)\}$. As we have already said, none of them is stable.*

From above example, the following question arises; what is the stability condition (in terms of $\beta$)?

Following proposition generalizes the example of Gale and Shapley.

**Proposition 45** *Let $N$ be a society and $\hat{R}$ be their preference profile with purified orderings. The profile $R$ does not have any stable matching if and only if in the game for $\hat{R}$, the mechanism $\beta$ confronts a cycle with odd number of agents in a subgame.*

**Proof.** ($\Longleftarrow$). Let $M \subset N$ be a set of agents with odd number of cardinality. From Corollary 42, we know that the agents of $M$ match with each other. So, one of them has to form a pair with an agent from the bottom set. The existence of an agent in a bottom set is guaranteed by the number of agents in the set $N$. Let $i \in M$ be that agent and $j \in N/M$ be his mate. From the definition of a cycle, it is clear that $xR_ij$ where $x \in M/\{i\}$. And, again from Corollary 42, there is an agent $k \in M$ such that $r_k(i) = 1$. So, the pair $(k, i)$ blocks the matching.

($\Longrightarrow$). We suppose that R does not have any stable matching and there is no cycle with odd number of agents in game of $\beta$. We will show that this leads to a contradiction.

Firstly, let us assume that there is no cycle at all. So, $B(\hat{R})$ is a sequence of agents. Let $(i, j)$ be a blocking pair. Wlog, let us assume $i$ has a higher raking than $j$ does in $B(\hat{R})$. Since $(i, j)$ blocks the matching, $i$ has a mate $k$ such that $jR_ik$ and also $j$ has a mate $l$ such that $iR_jl$. This means that until $j$'s turn, $i$ has not been taken. Since, we have $(j, l)$, $j$ has not chosen any agent until his turn for $i$, because better agents are not achievable for him.. Antd, finally, when it is his turn, he does not choose $i$ and in a later stage he chooses $l$. Eventhough he has a chance, he does not choose $i$, which contradicts to the rationality axiom. This contradicts to instability.

Secondly, let us assume that the game consist(s) of cycle(s) with even number of agents. From Corollary 42, we know that agents end up one and only one matching where they form pairs with each other in the cycle. This contradicts to instability.

Finally, if the game is a combination of two above cases, same arguments work. ■

The final topic of this paper is multi stability. Like for the marriage problem, in the roommate problem some profiles have more than one stable matching. The reason of multi stability is, not surprisingly, the existence of the cycles of a set of agents for another disjoint set. Since we already did an exhaustive analysis of such cycles, we will not do it again in this section.

The next example is on a profile with multi stability and the outcome of the mechainsm $\beta$.

**Example 46** *Let $N = \{a, b, c, d\}$ be any set of boys with the profile $\mathbb{R}_{11}$,*

$$\hat{\mathbb{R}}_{11} = \begin{array}{cccc} \underline{a} & \underline{b} & \underline{c} & \underline{d} \\ c & d & b & a \\ d & a & a & b \\ b & c & d & c \end{array} \qquad \begin{array}{llll} \underline{B(R)} & \underline{B(R^{\{a,b,c\}})} & \underline{B(R^{\{a,b,d\}})} & \underline{B(R^{\{b,c,d\}})} \\ a & abc & d & b \\ bd & & a & d \\ c & & b & c \end{array}$$

*$\Pi^N = \{\mu_1, \mu_2, \mu_3\}$ be the set of all possible matchings, where*

$$\mu_1 = \{(a, b), (c, d)\},$$
$$\mu_2 = \{(a, c), (b, d)\},$$
$$\mu_3 = \{(a, d), (b, c)\}.$$

*For profile $\mathbb{R}_{11}$, the set of stable matchings is $\{\mu_2, \mu_3\}$. As we see, there is no one purified ordering for the whole set. So, any agent could be the restricter of the first stage. This is because of the cycle between the sets $\{a, b\}$ and $\{c, d\}$.*

*It is easy to that if we start the game with a or b, we end up with matching $\mu_3$. On the other hand, starting with c or d gives us matching $\mu_2$.*

*Hence, in the existence of multi stability, we need to run all purified orderings in order to find all of the stable matchings of the profile.*

Now, we state the most general result.

**Theorem 47** *Let $N$ be a set agents and $R$ be their preference profile. Let $\hat{R}$ be the purified ordering (or one of the purified orderings when multiple). The mechanism $\beta$ defined over $\hat{R}$, formally*

$$\beta : (\mathbb{R}^N, B(\hat{\mathbb{R}}^N)) \longrightarrow \Pi^N$$

*gives a stable matching, when there exists some, and gives a Pareto Efficient matching in the absence of stability.*

**Proof.** The stability part has been proved by the examples, corollaries, claims and propositions so far.

Pareto efficiency is proved from the definition of a cycle with odd number of agents. There always exists an agent who matches to his most favorite choice. Increasing the "payoffs" of the blocking pair damages to this agent. Hence, the matching from the procedure is Pareto efficient. ■

## 4.4 The Conclusion

In this paper, we have proposed a simple mechanism to the roommate problem. The mechanism is a refinement of the mechanism by Evci (2014). While applying their mechanism to this problem, we have benefitted from the Borda rule in a scial welfare function form. Then, we analyze the effect of this SWF in two scenarios by simply seperating the raw and purified orderings.

First of all, as we have showed, the mechanism $\beta$ is quite successful under the purified orderings in determining the stability of any given profile. The success of $\beta$ for the raw orderings depends on whether it coincide with the purified orderings or not. As long as the top set of the raw orderings concide with those of purified orderings, we end up with a/the stable matchings.

We have showed that in the absence of stability, the mechanism $\beta$ ends up with a Pareto efficient matching.

And finally, we showed that $\beta$ is also an easy and strong mechanism to find all the stable matchings for a given profile in the existence of multiple case.

# References

[1] Evci, B. (2014), "A New Mechanism to the Two-Sided Matching Games", Unpublished manuscript.

[2] Gale, D. & Shapley, L.S. (1962), "College Admissions and the Stability of Marriage", The American Mathematical Monthly, 69: 9-15.

[3] Irving, R.W. (1985), "An Efficient Algorithm for the Stable Roommates Prolem", Journal of Algorithms 6, 577-595.

[4] Knuth, D.E. (1976), "Mariages Stables et leurs relations avec d'autres problèmes combinatoires", Les Presses de l'Université Montréal, Montréal.