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Asymptotic syzygies of algebraic varieties

Tesi di Laurea Magistrale

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Introduction

The purpose of this thesis is the exposition of some recent results about syzygies of projective varieties. More specifically, consider an algebraically closed field $\mathbf{k}$ of characteristic zero, and a smooth connected projective variety X over $\mathbf{k}$. Let L be a very ample line bundle on X inducing a projectively normal embedding

$$X \hookrightarrow \mathbb{P}(H^0(X,L)) = \mathbb{P}^r$$

and let $S = S^\bullet(H^0(X,L))$ be the homogeneous coordinate ring of the projective space $\mathbb{P}^r$ and $S_X = \bigoplus_{q\in\mathbb{Z}} H^0(X,L^q)$ be the homogeneous coordinate ring of the embedded variety $X \subseteq \mathbb{P}^r$. Then the (extended) minimal free resolution of S_X is the unique shortest possible exact sequence

$$0 \longrightarrow F_s \longrightarrow F_{s-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S_X \longrightarrow 0$$

where every F_p is a finitely generated free graded S-module, called the module of p-th syzygies of X in $\mathbb{P}^r$. We can write

$$F_p = \bigoplus_{q\in\mathbb{Z}} S(-p-q) \otimes_{\mathbf{k}} K_{p,q}(X,L)$$

for certain uniquely determined finite-dimensional vector spaces $K_{p,q}(X,L)$. The numbers

$$k_{p,q}(X,L) \stackrel{\text{def}}{=} \dim_{\mathbf{k}} K_{p,q}(X,L)$$

are called the graded Betti numbers of X and they encode many of the algebraic and geometric properties of the variety.

In this work, we are interested in studying the Betti numbers $k_{p,q}(X,L^{\otimes d})$ as d grows to infinity. The first natural question to ask is about the vanishing of these numbers. From an intuition based on results about smooth curves, one could think that the $k_{p,q}(X,L^{\otimes d})$ would become more sparse as d increases. Instead, in their recent article [EL12], L. Ein and R. Lazarsfeld proved that the Betti numbers $k_{p,q}(X,L^{\otimes d})$ become asymptotically nonzero for almost all possible values of p,q, moreover, they give a precise range of nonvanishings for the case of Veronese embeddings $k_{p,q}(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(d))$. In the thesis, we explain these results, presenting their proof.

The next natural question to ask is about the actual values of the Betti numbers. In particular, inspired by the article [EEL13] of L. Ein, D. Erman and R. Lazarsfeld, we prove that the Betti numbers of smooth curves have an asymptotically normal behavior. More precisely, if X is a smooth curve, then we define a discrete random variable X_d with distribution

$$\mathbb{P}(X_d = p) = \frac{k_{p,1}(X,L^{\otimes d})}{\sum_{h=0}^{+\infty} k_{h,1}(X,L^{\otimes d})} \qquad \text{for all } p \geq 0$$

and then we prove the following result.

Theorem 1. *As $d \to +\infty$ it holds that $\mathbb{E}[X_d] \sim \frac{d}{2}$, $Var[X_d] \sim \frac{d}{4}$ and moreover*

$$\frac{X_d - \mathbb{E}[X_d]}{\sqrt{Var[X_d]}} \longrightarrow \mathcal{N}(0,1)$$

in distribution.

Following a conjecture of Ein, Erman and Lazarsfeld [EEL13], we expect that the above result extends to higher dimensional varieties as well. Then, the next simplest case to consider would be quite naturally that of Betti numbers of plane Veronese embeddings $k_{p,q}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$.

In this case, it is much more difficult to get an hold on the Betti numbers with the methods used for curves, since the result of Ein and Lazarsfeld [EL12] tells us that the Betti table of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ is very non-sparse as $d \to +\infty$. Thus, we need another technique to compute these numbers: the key point is that we can look at $\mathbb{P}^2$ as an $SL(3)$-homogeneous variety, and then the Betti numbers are given as the cohomology of certain $SL(3)$-homogeneous bundles on $\mathbb{P}^2$.

The cohomology of irreducible homogeneous bundles on an homogeneous projective variety $X = G/P$ is described by the classical Bott's Theorem and it is quite simple. In the thesis we present a result, due to G. Ottaviani and E. Rubei [OR06], that extends Bott's Theorem to every homogeneous bundle, exploiting an equivalence between these bundles and representations of a certain quiver.

It is quite difficult to implement directly this method in our situation, since the quiver maps become quickly very complicated, but, under an additional assumption on the $SL(3)$-morphisms that is satisfied in all known cases, the problem is reduced to a more tractable combinatorial statement about representations of $SL(2)$. We do not get to a proof of the asymptotic normality, but through this strategy we are able to give some partial results and we can write an algorithm that computes the Betti numbers $k_{p,q}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ in further cases than existing ones, albeit under the additional hypothesis.

Turning to the contents of the single chapters, in Chapter 1 we give the basic definitions and first results relative to minimal free resolutions and Betti numbers in an algebraic and geometric context, introducing the language of Koszul cohomology.

In Chapter 2 we investigate further some other aspects of Betti numbers. In particular we discuss syzygy bundles, Castelnuovo-Mumford's regularity, duality, a Lefschetz-type theorem, property N_p and we conclude by giving examples of Betti numbers for rational normal curves, elliptic normal curves and Veronese surfaces of low degree.

Chapter 3 is devoted to the exposition of Ein and Lazarsfeld's results, following their article [EL12].

In Chapter 4 we present the proof of asymptotic normality for the Betti numbers of smooth curves. To this end, we generalize the computations done for elliptic normal curves in order to get control on almost every Betti number and then we use some combinatorial computations to get the result.

Chapter 5 presents the technique of Ottaviani and Rubei [OR06] for computing the cohomology of an homogeneous bundles on an Hermitian symmetric variety $X = G/P$. We discuss the connection between Bott's Theorem and quiver representations and we conclude giving the proof of a plethysm decomposition for $SL(2)$ as an application of the technique over $\mathbb{P}^1$.

In Chapter 6 we apply the methods of the previous chapter to compute the Betti numbers $k_{p,q}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ through the cohomology of certain homogeneous bundles. In particular, with an additional hypothesis that is verified in all known cases, we can reduce the problem to a problem in representations of $SL(2)$, that can be attacked explicitly through known rules for plethysm decomposition and tensor product decomposition. This allows to write an algorithm that can compute the Betti numbers in higher degrees than existing ones. In the last part of the chapter we give partial results towards the proof of asymptotic normality, studying the problem over the full flag variety.

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Chapter 1

Minimal free resolutions and Koszul cohomology

In this chapter, we provide the basic definitions and results about minimal free resolutions in the algebraic and in the geometric context. We will work over a fixed algebraically closed field of characteristic 0 denoted by $\mathbf{k}$.

1.1 Preliminary results

We begin by recalling some preliminary results about spectral sequences and Schur functors.

1.1.1 Spectral sequences

We expose briefly some facts about first-quadrant spectral sequences, mainly to clarify the notation (taken from Vakil [Vak]). Let $\mathcal{C}$ be any abelian category, and consider a doubly graded object

$$E = \bigoplus_{p,q\in\mathbb{Z}} E^{p,q}$$

with the $E^{p,q} \in \mathcal{C}$, together with two morphisms

$$d_{\rightarrow}^{p,q} \colon E^{p,q} \longrightarrow E^{p+1,q} \qquad d_{\uparrow}^{p,q} \colon E^{p,q} \longrightarrow E^{p,q+1}$$

such that

$$d_{\rightarrow}^{p+1,q} \circ d_{\rightarrow}^{p,q} = 0, \qquad d_{\uparrow}^{p,q+1} \circ d_{\uparrow}^{p,q} = 0, \qquad d_{\rightarrow}^{p,q+1} \circ d_{\uparrow}^{p,q} = d_{\uparrow}^{p+1,q} \circ d_{\rightarrow}^{p,q}$$

If $E^{p,q} = 0$ for all $p, q < 0$, we call this a **first-quadrant double complex** in the category $\mathcal{C}$.

From this double complex we can define a single complex as follows: we set

$$E^k = \bigoplus_{p+q=k} E^{p,q}$$

and we define a differential $d^k \colon E^k \longrightarrow E^{k+1}$ by $d^k_{|E^{p,q}} = d_{\rightarrow}^{p,q} + (-1)^p d_{\uparrow}^{p,q}$. The cohomology of this complex is called the **cohomology of the double complex** $E^{\bullet,\bullet}$ or the **hypercohomology** of the complex $E^{\bullet,\bullet}$.

Now, for this complex we can build a **spectral sequence with rightward orientation**, that is a sequence of objects ${}_{\rightarrow}E_0^{p,q}, {}_{\rightarrow}E_1^{p,q}, {}_{\rightarrow}E_2^{p,q}, \dots$ such that ${}_{\rightarrow}E_0^{p,q} = E^{p,q}$ together with maps

$${}_{\rightarrow}d_r^{p,q} \colon {}_{\rightarrow}E_r^{p,q} \longrightarrow {}_{\rightarrow}E_r^{p-r+1,q+r}$$

such that ${}_{\rightarrow}d_0^{p,q} = d_{\rightarrow}^{p,q}$, with a differential relation ${}_{\rightarrow}d_r^{p,q} \circ {}_{\rightarrow}d_r^{p+r-1,q-r} = 0$ and an isomorphism

$$\frac{\operatorname{Ker}\, {}_{\rightarrow}d_r^{p,q}}{\operatorname{Im}\, {}_{\rightarrow}d_r^{p+r-1,q-r}} \cong {}_{\rightarrow}E_{r+1}^{p,q}$$

Since we are dealing with a first-quadrant double complex, it is easy to see that we have isomorphisms ${}_{\rightarrow}E_r^{p,q} \cong {}_{\rightarrow}E_{r+1}^{p,q}$ for all $r \gg 0$ and we denote this object by ${}_{\rightarrow}E_\infty^{p,q}$.

We can also build a **spectral sequence with upward orientation**, that is a sequence of objects ${}_{\uparrow}E_0^{p,q}, {}_{\uparrow}E_1^{p,q}, {}_{\uparrow}E_2^{p,q}, \ldots$ together with maps

$${}_{\uparrow}d_r^{p,q} \colon {}_{\uparrow}E_r^{p,q} \longrightarrow {}_{\uparrow}E_r^{p+r,q-r+1}$$

that behave in the same way as the maps ${}_{\rightarrow}d_{\bullet}^{\bullet,\bullet}$.

The fundamental result about spectral sequences is the following:

Theorem 1.1.1. *The cohomology $H^k(E^\bullet)$ of the double complex is filtered by both ${}_{\rightarrow}E_\infty^{k,p-k}$ and ${}_{\uparrow}E_\infty^{k,p-k}$.*

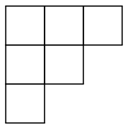
Proof. See [Vak] Section 1.7 . □

1.1.2 Schur functors

Let n be a natural number. A partition of n is a sequence of natural numbers

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \qquad \lambda_1 + \cdots + \lambda_m = n$$

A partition λ can also be represented by its Young diagram: a collection of boxes, with λ_1 boxes in the first row, λ_2 boxes in the second row and so on. For example the following is the Young diagram of the partition $(3, 2, 1)$:



Sometimes we identify a partition with its Young diagram: for example we say that a partition λ has m rows if its Young diagram has m diagram.

For every partition λ one can build as in [FH91] a Schur functor

$$S^\lambda \colon \mathbf{Vec_k} \longrightarrow \mathbf{Vec_k} \qquad V \mapsto S^\lambda(V)$$

where $\mathbf{Vec_k}$ denotes the category of finite-dimensional vector spaces over $\mathbf{k}$. For example, if $\lambda = (n)$ then $S^{(n)}(V)$ is simply the n-th symmetric power $S^n V$, whereas, if $\lambda = (1, 1, \ldots, 1)$ repeated n times, then $S^{(1,\ldots,1)}(V)$ is the n-th alternating power $\wedge^n V$.

Schur functors give a tool to describe the irreducible representations of $SL(V)$ and $GL(V)$.

Theorem 1.1.2. *Let V be a vector space of dimension n. Then every irreducible representation of $SL(V)$ is of the form $S^\lambda(V)$ for an unique partition λ whose Young diagram has at most $n-1$ rows.*

Moreover, every irreducible representation of $GL(V)$ is of the form $S^\lambda(V) \otimes \det^m$, where λ is a partition whose Young diagram has at most at most $n-1$ rows, det *is the determinant representation and $m \in \mathbb{Z}$ is an integer.*

Proof. See [FH91]. □

There are some useful rules for computing with Schur functors:

Proposition 1.1.1. *Let V be a vector space of dimension n and let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition with at most n rows. Then*

$$\dim S^\lambda(V) = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

Proof. See [FH91]. □

Theorem 1.1.3 (Pieri's rule)**.** *Let λ be a partition and let m be a positive integer. Then, as $GL(V)$-representations we have that*

$$S^\lambda(V) \otimes_{\mathbf{k}} S^m(V) = \bigoplus_\nu S^\nu(V)$$

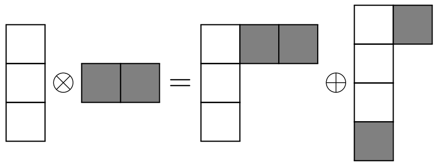
where the sum is over all partitions ν whose Young diagram can be obtained from that of λ by adding m boxes, no two on the same column.

Proof. See [FH91]. □

Example 1.1.1. Let p and q be two positive integers. Then Pieri's rule tells us that

$$\wedge^p V \otimes_{\mathbf{k}} S^q(V) = S^{(q+1,\mathbf{1}^{p-1})}(V) \oplus S^{(q,\mathbf{1}^p)}(V)$$

where by $\mathbf{1}^n$ we mean $(1, 1, \ldots, 1)$ with 1 repeated n times.



Pieri's rule is a special case of the so-called Littlewood-Richardson rule, that tells us how to decompose an arbitrary tensor product $S^\lambda(V) \otimes S^\mu(V)$. Since we won't need to use this more general formula, we will not describe it here, however we will need a further generalization to arbitrary semisimple groups.

Theorem 1.1.4 (Generalized Littlewood-Richardson's rule)**.** *Let G be a simple algebraic group, let λ and μ be two dominant weights for G and let Γ^λ and Γ^μ be their corresponding irreducible representations. Moreover, let $\mu = \mu_1, \ldots, \mu_r$ be all the weights of Γ^μ. Then*

$$\Gamma^\lambda \otimes_{\mathbf{k}} \Gamma^\mu = \bigoplus_{i \in I} \Gamma^{\lambda + \mu_i}$$

as G-representations, for a certain set $I \subseteq \{1, \ldots, r\}$.

Proof. See [Lit90]. □

Now, suppose that the vector space V has dimension 2. Then every irreducible representation of $SL(V)$ is of the form $S^p(V)$ for a certain $p \in \mathbb{N}$. The plethysm problem consist on decomposing into irreducible representations a representation of the form $S^n(S^m(V))$: the Cayley-Sylvester formula is a classic result in Invariant Theory that gives a complete solution to this problem.

Before stating it we need some notation: for every triple of positive integers n, m, e we denote by $p(n, m, e)$ the number of partitions of e in at most n elements bounded by m. More precisely:

$$p(n, m, e) \stackrel{\text{def}}{=} \#\{ (\mu_1, \ldots, \mu_n) \mid m \geq \mu_1 \geq \cdots \geq \mu_n \geq 0,\ \mu_1 + \cdots + \mu_n = e \}$$

Alternatively, we can look at $p(n, m, e)$ as the number of Young diagrams for e that can be fitted into an $n \times m$ rectangle: under this point of view it is easy to see that

$$p(n, m, e) = p(m, n, e) \qquad \text{for all } n, m, e$$

indeed, transposition of rectangles gives the desired bijection. Now we can state the Cayley-Sylvester formula as follows.

Theorem 1.1.5 (Cayley-Sylvester's formula)**.** *Let n, m be two positive integers. Then*

$$S^n(S^m(V)) = \bigoplus_{e \geq 0} S^e(V)^{\oplus N(n,m,e)}$$

where

$$N(n, m, e) \stackrel{\text{def}}{=} p\left(n, m, \frac{nm - e}{2}\right) - p\left(n, m, \frac{nm - e}{2} - 1\right)$$

Proof. See [Dol03]. □

1.2 Algebra

We will denote by V a finite dimensional vector space over $\mathbf{k}$ of dimension $\dim V = r+1$ and by $S = S^\bullet(V)$ the symmetric algebra over V, with the standard grading $S = \bigoplus_{n\geq 0} S^n(V)$. We will denote by $\mathfrak{m} = S_+ = \bigoplus_{n\geq 1} S^n(V)$ its maximal homogeneous ideal. Notice that $\mathbf{k} = S/\mathfrak{m}$ is naturally a graded S-module.

Definition 1.2.1 (Graded resolution)**.** Let M be an S-module. A **graded resolution** of M is a complex of S-modules

$$F_\bullet : \ \ldots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

where the F_i are homogeneous, the maps are homogeneous of degree 0, and the complex is exact everywhere except in degree 0 where it has cohomology M. This means that it can be extended to an exact complex

$$\ldots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

We say that a resolution $F_\bullet$ is **projective** if every module F_i is projective. We say that a resolution $F_\bullet$ is **free** if every module F_i is free.

Definition 1.2.2 (Length of a resolution, Projective dimension)**.** Let M be an S-module and $F_\bullet$ a resolution of M. We define the **length** of the resolution as

$$\sup \{ n \mid F_n \neq 0 \}$$

We define the **projective dimension** of M to be the minimum length of a projective resolution of M and we denote it by $projdim(M)$.

It is easy to see that every graded S-module has a free resolution: if M is such a module, we can just take a set of homogeneous generators of M and get an exact sequence

$$F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

where F_0 is a graded free S-module. Then we can take a set of homogeneous generators of the kernel of ϕ_0 and get another exact sequence

$$F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

It is clear that this process yields a graded free resolution, moreover, if M is finitely generated, the F_i can be taken to be of finite rank. What is nonobvious is the existence of a *finite* graded free resolution. We are going to show that it exists but first we need to give some more definitions.

Lemma 1.2.1 (Graded Nakayama's lemma)**.** *Let M be a finitely generated graded S-module.*

(a) If $\mathfrak{m}M = M$, then $M = 0$.

(b) If $m_1, \ldots, m_r \in M$ are homogeneous elements such that their images $\overline{m_1}, \ldots, \overline{m_r}$ in $M/\mathfrak{m}M$ generate it as a $\mathbf{k}$-module, then they also generate M as an S-module.

Proof. (a) By the usual Nakayama's lemma, there is an element $x \in \mathfrak{m}$ such that $(1+x)M = 0$. We write $x = x_1 + \cdots + x_n$ as a sum of homogeneous elements, where each x_i has degree i or is 0. Then for any homogeneous $m \in M$ we have that

$$m + x_1 \cdot m + x_2 \cdot m + \cdots + x_n \cdot m = 0$$

and since this is a sum of homogeneous elements with different degrees (or that are zero), it follows that $m = 0$. As M is generated by its homogeneous elements, we conclude.

(b) Let N be the submodule of M generated by the m_i. By hypothesis, we have that $M = N + \mathfrak{m}M$, so that $\mathfrak{m}(M/N) = (M/N)$ and from the previous point, it follows that $M = N$. □

Definition 1.2.3 (Minimal free resolution)**.** Let M be a graded S-module. A **minimal free resolution** of M is a graded free resolution

$$F_\bullet\colon \ \dots \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow 0$$

where $\phi_n(F_n) \subseteq \mathfrak{m}F_{n-1}$ for all $n \geq 1$.

Remark 1.2.1*.* We observe that a free resolution

$$F_\bullet\colon \ \dots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

is minimal if and only if, when tensored by $\mathbf{k} \otimes_S$, it gives a trivial complex, i.e. all the maps are just the zero maps. This observation will be very useful in the following.

Remark 1.2.2*.* Let M be a finitely generated graded S-module. We define a set of generators $m_1, \dots, m_r$ to be **minimal** if their images $\overline{m_1}, \dots, \overline{m_r}$ in $\mathbf{k} \otimes_S M$ form a $\mathbf{k}$-basis for this space. Observe that from every set of homogeneous generators we can extract a minimal set of generators: indeed, let $m_1, \dots, m_r$ be any set of generators, then we can suppose that $\overline{m_1}, \dots, \overline{m_s}$ is a basis of $\mathbf{k} \otimes_S M$ as a $\mathbf{k}$-vector space, and from homogeneous Nakayama's lemma we know that $m_1, \dots, m_s$ generate M. This shows that a minimal set of generators coincide with a set of generators of minimal cardinality, which is the dimension of $\mathbf{k} \otimes_S M$ over $\mathbf{k}$.

Thanks to these two observations, we can show that a minimal free resolution is indeed minimal how we would expect it to be.

Proposition 1.2.1. *Let M be a finitely generated graded S-module. Then a graded free resolution*

$$F_\bullet\colon \ \dots \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \longrightarrow \dots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \longrightarrow 0$$

is minimal if and only if each ϕ_i takes a basis of F_i to a minimal set of generators of the image.
In particular, if we write $F_i = \bigoplus_j S(a_{i,j})$, then the restriction $\phi_i\colon S(a_{i,j}) \longrightarrow F_{i-1}$ is injective.

Proof. By Remark 1.2.1, the resolution $F_\bullet$ is minimal if and only if the tensored complex $F_\bullet \otimes_S \mathbf{k}$ is the zero complex: since all the maps $F_{n+1} \otimes_S \mathbf{k} \longrightarrow \operatorname{Im} \phi_{n+1} \otimes_S \mathbf{k}$ are surjective, this is equivalent to saying that the maps $\operatorname{Im} \phi_{n+1} \otimes_S \mathbf{k} \longrightarrow F_n \otimes_S \mathbf{k}$ are zero.
On the other hand, the exact sequence

$$0 \longrightarrow \operatorname{Im} \phi_{n+1} \longrightarrow F_n \longrightarrow \operatorname{Im} \phi_n \longrightarrow 0$$

tensored with $\otimes_S \mathbf{k}$, gives the exact sequence

$$\operatorname{Im} \phi_{n+1} \otimes_S \mathbf{k} \longrightarrow F_n \otimes_S \mathbf{k} \longrightarrow \operatorname{Im} \phi_n \otimes_S \mathbf{k} \longrightarrow 0$$

and from Remark 1.2.1, we see that the map ϕ_n sends a basis of F_n to a minimal system of generators of $\operatorname{Im} \phi_n$ if and only if the map $\operatorname{Im} \phi_{n+1} \otimes_S \mathbf{k} \longrightarrow F_n \otimes_S \mathbf{k}$ is the zero map. □

The following result shows that it makes sense to speak of *the* minimal free resolution.

Theorem 1.2.1. *Let M be a finitely generated graded S-module.*

(a) If $F_\bullet$ and $G_\bullet$ are two minimal free resolutions of M, then they are isomorphic via an unique isomorphism, that induces the identity on M.

(b) The minimal free resolution is contained as a direct summand in any other free resolution of M.

Proof. See [Eis95]. □

The modules that appear in the minimal graded free resolution can be computed by means of the Tor functor.

Proposition 1.2.2. *Take a finitely generated graded free S-module M and consider its minimal free resolution*

$$F_\bullet\colon \dots \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

then we can write every F_i in the form

$$F_i = \bigoplus_{j\in\mathbb{Z}} S(-j) \otimes_{\mathbf{k}} Tor_i^S(M,\mathbf{k})_j$$

Proof. We know that we can write the minimal free resolution as

$$F_i = \bigoplus_{j\in\mathbb{Z}} S(-j) \otimes_{\mathbf{k}} B_{i,j}$$

for certain finite dimensional $\mathbf{k}$-vector spaces $B_{i,j}$. These vector spaces count the multiplicity of $S(-j)$ in F_i. Now the Remark 1.2.1 implies that the maps of the complex

$$F_\bullet \otimes_S \mathbf{k}\colon \dots \longrightarrow F_i \otimes_S \mathbf{k} \longrightarrow F_{i-1} \otimes_S \mathbf{k} \longrightarrow \dots \longrightarrow F_1 \otimes_S \mathbf{k} \longrightarrow F_0 \otimes_S \mathbf{k} \longrightarrow 0$$

are all zero, so that the cohomology in degree i is given precisely by $F_i \otimes_{\mathbf{k}} S$. By definition of Tor this means that

$$\operatorname{Tor}_i^S(M,\mathbf{k}) = \bigoplus_{j\in\mathbb{Z}} \mathbf{k}(-j) \otimes_{\mathbf{k}} B_{i,j}$$

so that we have a natural identification $\operatorname{Tor}_i^S(M,\mathbf{k})_j = B_{i,j}$. □

Definition 1.2.4 (Graded Betti numbers)**.** Let M be a finitely generated graded S-module. Then the **graded Betti numbers** of M are defined as

$$\beta_{i,j} = \beta_{i,j}(M) \stackrel{\text{def}}{=} \dim_{\mathbf{k}} \operatorname{Tor}_i^S(M,\mathbf{k})_j$$

Remark 1.2.3. By definition of Betti numbers, we can write the minimal free resolution of M in the form

$$\dots \longrightarrow \bigoplus_{j\in\mathbb{Z}} S(-j)^{\oplus\beta_{i,j}} \longrightarrow \bigoplus_{j\in\mathbb{Z}} S(-j)^{\oplus\beta_{i-1,j}} \longrightarrow \dots \longrightarrow \bigoplus_{j\in\mathbb{Z}} S(-j)^{\oplus\beta_{1,j}} \longrightarrow \bigoplus_{j\in\mathbb{Z}} S(-j)^{\oplus\beta_{0,j}} \longrightarrow 0$$

Corollary 1.2.1. *Let M be a finitely generated graded S-module. The projective dimension of M coincides with the length of its minimal graded free resolution, that is*

$$projdim(M) = \sup\{\, i \in \mathbb{N} \mid Tor_i^S(M,\mathbf{k}) \neq 0 \,\} = \sup\{\, i \in \mathbb{N} \mid \beta_{i,j}(M) \neq 0 \textit{ for a certain } j \,\}$$

Proof. We just need to show that every projective resolution of M has length at least equal to the length ℓ of the minimal free resolution. Let $P_\bullet$ be a projective resolution of M: then by definition of Tor we know that

$$\operatorname{Tor}_i^S(M,\mathbf{k}) = H^i(P_\bullet \otimes_S \mathbf{k})$$

and by Proposition 1.2.2, we see that $\operatorname{Tor}_\ell^S(M,\mathbf{k})_\bullet \neq 0$ so that $P_\ell \neq 0$. □

1.2.1 The Koszul complex

To compute the Betti numbers of a graded module M we need to compute $\mathrm{Tor}_i^S(M,\mathbf{k})_j$: thanks to the symmetry of Tor, to do this we can either tensor a free resolution of M by $\mathbf{k}$ (this is what we have done before) or tensor a free resolution of $\mathbf{k}$ by M.

Now we are going to find the minimal free resolution of $\mathbf{k}$: notice that for every $p \geq 0$ we have the natural multiplication map

$$\wedge^p V^\vee \otimes_{\mathbf{k}} V^\vee \longrightarrow \wedge^{p+1} V^\vee \qquad \phi_1 \wedge \dots \wedge \phi_p \otimes \phi_{p+1} \mapsto \phi_1 \wedge \dots \wedge \phi_{p+1}$$

That induces by duality a natural $\mathbf{k}$-linear map

$$\wedge^{p+1} V \longrightarrow \wedge^p V \otimes_{\mathbf{k}} V \qquad v_0 \wedge \dots \wedge v_p \mapsto \sum_{i=0}^{p} (-1)^i v_0 \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge v_p \otimes v_i$$

and for every $q \in \mathbb{Z}$ we can consider the composition

$$\wedge^{p+1} V \otimes_{\mathbf{k}} S^q(V) \longrightarrow \wedge^p V \otimes_{\mathbf{k}} V \otimes_{\mathbf{k}} S^q(V) \longrightarrow \wedge^p V \otimes_{\mathbf{k}} S^{q+1}(V)$$

where the first map is obtained by the one above by tensoring with id_M and the second map is obtained by tensoring the natural multiplication map with $\mathrm{id}_{\wedge^p V}$.

This way, we have defined the maps

$$d_{p,q} \colon \wedge^p V \otimes_{\mathbf{k}} S^q(V) \longrightarrow \wedge^{p-1} V \otimes_{\mathbf{k}} S^{q+1}(V) \qquad v_1 \wedge \dots \wedge v_p \otimes f \mapsto \sum_{i=1}^{p} (-1)^{i+1} v_1 \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge v_p \otimes f v_i$$

and taking the direct sum over all q and keeping p fixed, we get homogeneous maps of graded S-modules

$$d_p \colon \wedge^p V \otimes_{\mathbf{k}} S(-1) \longrightarrow \wedge^{p-1} V \otimes_{\mathbf{k}} S$$

Definition 1.2.5 (Koszul complex)**.** The Koszul complex of V is the sequence of graded S-modules

$$K_\bullet(V) \colon 0 \longrightarrow \wedge^{r+1} V \otimes_{\mathbf{k}} S(-r-1) \longrightarrow \wedge^r V \otimes_{\mathbf{k}} S(-r) \longrightarrow \dots \longrightarrow V \otimes_{\mathbf{k}} S(-1) \longrightarrow S \longrightarrow 0$$

We are going to prove that the Koszul complex is the minimal graded free resolution of $\mathbf{k}$. First we recall Euler's formula on homogeneous polynomials.

Lemma 1.2.2 (Euler's formula)**.** *Let $X_0, \dots, X_r$ be a basis of V and let $f \in S$ be a nonzero homogenous element. Then*

$$\sum_{i=0}^{r} X_i \frac{\partial f}{\partial X_i} = (\deg f) f$$

Proof. First we notice that the statement to prove is linear in f so that it is enough to suppose that f is a monomial in the X_i. Now, we notice that the statement to prove is also multiplicative in f: indeed, suppose that the formula holds for two homogeneous elements f, g then

$$\sum_{i=0}^{r} X_i \frac{\partial (fg)}{\partial X_i} = \sum_{i=0}^{r} X_i \left(f \frac{\partial g}{\partial X_i} + g \frac{\partial f}{\partial X_i} \right) = \deg(g) fg + \deg(f) fg = (\deg f + \deg g) fg = \deg(fg) fg$$

Now it is enough to prove the formula in the cases in which f has degree 0 or 1, where it is obvious. □

Theorem 1.2.2. *The Koszul complex $K_\bullet(V)$ is the minimal graded free resolution of* $\mathbf{k}$ *as an S-module.*

Proof. In view of the importance of this result, we are going to give two different proofs of it

- **Standard proof**: first we prove that the Koszul complex is actually a complex. For every $p \geq 1$, $v_1, \dots, v_p \in V$ and $f \in S^q(V)$ we have

$$(d \circ d)(v_1 \wedge \dots \wedge v_p \otimes f) = d\left(\sum_{i=1}^{p} (-1)^{i+1} v_1 \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge v_p \otimes v_i f\right) =$$
$$= \sum_{i=1}^{p} (-1)^{i+1} \sum_{j=1}^{i-1} (-1)^{j+1} v_1 \wedge \dots \wedge \widehat{v_j} \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge v_p \otimes v_j v_i f +$$
$$+ \sum_{i=1}^{p} (-1)^{i+1} \sum_{j=i+1}^{p} (-1)^{j} v_1 \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge \widehat{v_j} \wedge \dots \wedge v_p \otimes v_j v_i f = 0$$

and since the elements $v_1 \wedge \dots \wedge v_p \otimes f$ generate $\wedge^p V \otimes S^q(V)$, it is clear that $d \circ d = 0$. Now we show that this complex gives a free resolution of $\mathbf{k}$: the differential $V \otimes S(-1) \longrightarrow S$ is given by $v \otimes f \mapsto vf$, so that it is clear that the image of this map is $\mathfrak{m}$, and then the cohomology is $S/\mathfrak{m} = \mathbf{k}$. Now we have to prove that the complex is exact in degree $p \geq 1$. In order to do this, fix a basis $X_0, \dots X_r$ of V and consider for every $p \geq 0$ the maps

$$D_p \colon \wedge^p V \otimes_{\mathbf{k}} S^q(V) \longrightarrow \wedge^{p+1} V \otimes_{\mathbf{k}} S^{q-1}(V) \qquad v_1 \wedge \dots \wedge v_p \otimes f \mapsto \sum_{i=0}^{r} X_i \wedge v_1 \wedge \dots \wedge v_p \otimes \frac{\partial f}{\partial X_j}$$

Then, for every $p \geq 1$, every $f \in S^q(V)$ and every choice of $0 \leq i_1 < \dots < i_p \leq r$ we have

$$(d_{p+1} \circ D_p + D_{p-1} \circ d_p)(X_{i_1} \wedge \dots \wedge X_{i_p} \otimes f) =$$
$$= d_{p+1}\left(\sum_{j=0}^{r} X_j \wedge X_{i_1} \wedge \dots \wedge X_{i_p} \otimes \frac{\partial f}{\partial X_j}\right)$$
$$+ D_{p-1}\left(\sum_{h=1}^{p} (-1)^{h+1} X_{i_1} \wedge \dots \wedge \widehat{X_{i_h}} \wedge \dots \wedge X_{i_p} \otimes X_{i_h} f\right) =$$
$$= \sum_{j=0}^{r} X_{i_1} \wedge \dots \wedge X_{i_p} \otimes X_j \frac{\partial f}{\partial X_j} + \sum_{j=0}^{r} \sum_{h=1}^{p} (-1)^h X_j \wedge X_{i_1} \cdots \wedge \widehat{X_{i_h}} \wedge \dots \wedge X_{i_p} \otimes X_{i_h} \frac{\partial f}{\partial X_i} +$$
$$+ \sum_{h=1}^{p} \sum_{j=0}^{r} (-1)^{h+1} X_j \wedge X_{i_1} \cdots \wedge \widehat{X_{i_j}} \wedge \dots X_{i_p} \otimes \frac{\partial (X_{i_h} f)}{\partial X_j}$$

Now, by Euler's formula we have that $\sum_{j=0}^{r} X_{i_1} \wedge \dots \wedge X_{i_p} \otimes X_j \frac{\partial f}{\partial X_j} = q(X_{i_1} \wedge \dots \wedge X_{i_p})$, whereas Leibniz' rule tells us that

$$\frac{\partial (X_{i_h} f)}{\partial X_j} = \delta_{i_h, j} f + X_{i_h} \frac{\partial f}{\partial X_j}$$

Plugging this into the above expression yields

$$(d_{p+1} \circ D_p + D_{p-1} \circ d_p)(X_{i_1} \wedge \dots \wedge X_{i_p} \otimes f) =$$
$$= q(X_{i_1} \wedge \dots \wedge X_{i_p} \otimes f) + \sum_{h=1}^{p} (-1)^{h+1} X_{i_{h+1}} \wedge X_{i_1} \wedge \dots \wedge \hat{X}_{i_h} \wedge \dots \wedge X_{i_p} \otimes f =$$
$$= (p+q)(X_{i_1} \wedge \dots \wedge X_{i_p} \otimes f)$$

Since the elements $X_{i_1} \wedge \dots X_{i_p} \otimes f$ generate $\bigwedge^p V \otimes_{\mathbf{k}} S^q(V)$ it follows that

$$d_{p+1} \circ D_p + D_{p-1} \circ d_p = (p+q)\mathrm{id}$$

and this implies that the complex is exact in every degree $p \geq 1$.

- **Representation-theoretic proof:** it follows immediately from the construction of the Koszul differentials
$$d_{p,q}\colon \wedge^p V \otimes_{\mathbf{k}} S^q(V) \longrightarrow \wedge^{p-1} V \otimes_{\mathbf{k}} S^{q+1}(V)$$
that they are $GL(V)$-morphisms, with respect to the standard action of $GL(V)$. Now, from Example 1.1.1 we see that we have a decomposition in irreducible $GL(V)$- representations
$$\wedge^p V \otimes_{\mathbf{k}} S^q(V) = S^{(q+1,\mathbf{1}^{p-1})}(V) \oplus S^{(q,\mathbf{1}^p)}(V)$$
where by $\mathbf{1}^n$ we mean $(1,1,\dots,1)$ with 1 repeated n times. The differentials in the Koszul complex have the form
$$\wedge^{p+1} V \otimes_{\mathbf{k}} S^{q-1}(V) \stackrel{d_{p+1,q-1}}{\longrightarrow} \wedge^p V \otimes_{\mathbf{k}} S^q(V) \stackrel{d_{p,q}}{\longrightarrow} \wedge^{p-1} V \otimes_{\mathbf{k}} S^{q+1}(V)$$
and after decomposing into irreducible representation we get a sequence
$$S^{(q,\mathbf{1}^p)}(V) \oplus S^{(q-1,\mathbf{1}^{p+1})}(V) \stackrel{d_{p+1,q-1}}{\longrightarrow} S^{(q+1,\mathbf{1}^{p-1})}(V) \oplus S^{(q,\mathbf{1}^p)}(V) \stackrel{d_{p,q}}{\longrightarrow} S^{(q+2,\mathbf{1}^{p-2})}(V) \oplus S^{(q+1,\mathbf{1}^{p-1})}(V)$$
Now, the composition of the two differentials gives a $GL(V)$-morphism
$$d_{p,q} \circ d_{p+1,q-1}\colon S^{(q,\mathbf{1}^p)}(V) \oplus S^{(q-1,\mathbf{1}^{p+1})}(V) \longrightarrow S^{(q+2,\mathbf{1}^{p-2})}(V) \oplus S^{(q+1,\mathbf{1}^{p-1})}(V)$$
and such a map is forced to be zero by Schur's lemma. This proves that the Koszul complex is an actual complex.

  To show that the Koszul complex gives a resolution of $\mathbf{k}$ we can make use again of Schur's lemma: if $p \geq 1$ the Koszul differential is a $GL(V)$-morphism
$$d_{p,q}\colon S^{(q+1,\mathbf{1}^{p-1})}(V) \oplus S^{(q,\mathbf{1}^p)}(V) \longrightarrow S^{(q+2,\mathbf{1}^{p-2})}(V) \oplus S^{(q+1,\mathbf{1}^{p-1})}(V)$$
and, by Schur's lemma, $d_{p,q}$ can either be the zero map or Ker $(d_{p,q}) = S^{(q,\mathbf{1}^p)}(V)$ and Im $(d_{p,q}) = S^{(q+1,\mathbf{1}^{p-1})}(V)$. Now, it is easy to prove that $d_{p,q}$ is not zero if $q \geq 0$: indeed if $X_0, \dots, X_r$ is a basis of V and $f \in S^q(V)$ is a monomial in the X_i then
$$d_{p,q}(X_0 \wedge \dots \wedge X_p \otimes f) = \sum_{i=0}^{p} (-1)^i X_0 \wedge \dots \wedge \widehat{X_i} \wedge \dots \wedge X_p \otimes e_i f$$
and the latter is different from zero because we know that the elements $X_0 \wedge \dots \wedge \widehat{X_i} \wedge \dots \wedge X_p \otimes X_i f$ are linearly independent in $\bigwedge^{p-1} V \otimes_{\mathbf{k}} S^{q+1}(V)$. This shows that Ker $(d_{p,q}) = S^{(q,\mathbf{1}^p)}(V)$ and Im $(d_{p,q}) = S^{(q+1,\mathbf{1}^{p-1})}(V)$ if $p \geq 1$ and $q \geq 0$, and gives immediately the exactness of the Koszul complex for $p \geq 1$.

  To see what happens in cohomological degree 0 we just notice that Ker $(d_0) = S$ obviously and Im $(d_1) = \oplus_{q\geq 0}$Im $(d_{p,q}) = \bigoplus_{q \geq 0} S^{q+1}(V) = \mathfrak{m}$ by the computation above, so that the cohomology in 0 is precisely $S/\mathfrak{m} = \mathbf{k}$.

These two proofs show that the Koszul complex is a graded free resolution of $\mathbf{k}$. To show that this is the minimal free resolution is then immediate because the Koszul differentials are given by multiplication by homogeneous elements of degree 1. □

As we said before, we can use the Koszul complex to compute $\mathrm{Tor}^S_p(M,\mathbf{k})_q$, for a finitely generated graded S-module M. Indeed, tensoring the Koszul complex with M we obtain the complex:

$$K_\bullet(V) \otimes_S M\colon 0 \longrightarrow \wedge^{n+1} V \otimes_{\mathbf{k}} M(-n-1) \longrightarrow \wedge^n V \otimes_{\mathbf{k}} M(-n) \longrightarrow \dots \longrightarrow V \otimes_{\mathbf{k}} M(-1) \longrightarrow M \longrightarrow 0$$

that in cohomological degree p has the form

$$\dots \longrightarrow \wedge^{p+1}V \otimes_{\mathbf{k}} M(-p-1) \longrightarrow \wedge^{p}V \otimes_{\mathbf{k}} M(-p) \longrightarrow \wedge^{p-1}V \otimes_{\mathbf{k}} M(-p+1) \longrightarrow \dots$$

And taking the part of degree q we obtain the complex

$$\dots \longrightarrow \wedge^{p+1}V \otimes_{\mathbf{k}} M_{q-p-1} \overset{d^M_{p+1,q-p-1}}{\longrightarrow} \wedge^{p}V \otimes_{\mathbf{k}} M_{q-p} \overset{d^M_{p,q-p}}{\longrightarrow} \wedge^{p-1}V \otimes_{\mathbf{k}} M_{q+1-p} \longrightarrow \dots$$

and this tells us that

$$\mathrm{Tor}^S_p(M,\mathbf{k})_q = \frac{\mathrm{Ker}\ d^M_{p,q-p}}{\mathrm{Im}\ d^M_{p+1,q-p-1}}$$

With this in mind we make the following definition:

Definition 1.2.6 (Koszul cohomology of a module)**.** With the notations as above, we define the **Koszul cohomology** of M as

$$K_{p,q}(M,V) \overset{\text{def}}{=} \frac{\mathrm{Ker}\ d^M_{p,q}}{\mathrm{Im}\ d^M_{p+1,q-1}}$$

We also define the notation

$$k_{p,q}(M,V) \overset{\text{def}}{=} \dim_k K_{p,q}(M,V)$$

Remark 1.2.4. By definition, the Koszul cohomology of M is the cohomology at the middle term of the complex

$$\wedge^{p+1}V \otimes_{\mathbf{k}} M_{q-1} \longrightarrow \wedge^{p}V \otimes_{\mathbf{k}} S^q(V) \longrightarrow \wedge^{p-1}V \otimes_{\mathbf{k}} M_{q+1}$$

Proposition 1.2.3. *Let M be a finitely generated graded S-module, then*

$$K_{p,q}(M,V) = Tor^S_p(M,\mathbf{k})_{p+q}$$

Proof. Follows immediately from what we have said before. □

Remark 1.2.5. This shows that the minimal free resolution of a finitely generated graded S-module M can be written as :

$$\dots \longrightarrow \bigoplus_{q\in\mathbb{Z}} S(-p-q) \otimes K_{p,q}(M,V) \longrightarrow \dots \longrightarrow \bigoplus_{q\in\mathbb{Z}} S(-q) \otimes K_{0,q}(M,V) \longrightarrow 0$$

Example 1.2.1 (Koszul cohomology of $\mathbf{k}$). From the Koszul complex, it is immediate to see that the Koszul cohomology of $\mathbf{k}$ is given by

$$K_{p,q}(\mathbf{k},V) = \begin{cases} \wedge^p V & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

Now it is very easy to derive from this an important theorem of Hilbert, that tells us that every finitely generated graded module over S has finite projective dimension, and actually it gives an upper bound to this number:

Theorem 1.2.3 (Hilbert's Syzygy Theorem)**.** *Let M be a finitely generated graded S-module. Then the minimal free resolution of M has length at most $r+1$.*

Proof. It is enough to note that $\bigwedge^p V \otimes_{\mathbf{k}} M_q = 0$ for $p > r+1$ so that $K_{p,q}(M,V) = 0$ for every $p \geq r+1$ and $q \in \mathbb{Z}$. □

1.2.2 Betti tables

We can summarize the informations about the Betti numbers of a module in a convenient way:

Definition 1.2.7 (Betti tables)**.** Let M be a finitely generated graded S-module and let $\beta_{i,j} = \beta_{i,j}(M)$ or $k_{i,j} = k_{i,j}(M, V)$ be its graded Betti numbers. Then the **Betti table** of M is defined as

	0	1	2	$\dots$	i
$\vdots$					
j	$\beta_{0,j}$	$\beta_{1,j+1}$	$\beta_{2,j+2}$	$\dots$	$\beta_{i,j+i}$
$j+1$	$\beta_{0,j+1}$	$\beta_{1,j+2}$	$\beta_{2,j+3}$	$\dots$	$\beta_{i,j+i+1}$
$\vdots$					

or, using Koszul cohomology notations

	0	1	2	$\dots$	i
$\vdots$					
j	$k_{0,j}$	$k_{1,j}$	$k_{2,j}$	$\dots$	$k_{i,j}$
$j+1$	$k_{0,j+1}$	$k_{1,j+1}$	$k_{2,j+1}$	$\dots$	$k_{j,i+j+1}$
$\vdots$					

Example 1.2.2. Suppose that $\dim V = 3$. Then the Betti table of $\mathbf{k}$ as an S-module is

	0	1	2	3	4
0	1	3	3	1	–
1	–	–	–	–	–

where the dashes stay for a 0 and all the elements not shown are 0.

In general, if $\dim V = r+1$, then the Betti table of $\mathbf{k}$ as an S-module is

	0	1	2	3	$\dots$	r	$r+1$	
0	$\binom{r+1}{0}$	$\binom{r+1}{1}$	$\binom{r+1}{2}$	$\binom{r+1}{3}$	$\dots$	$\binom{r+1}{r}$	$\binom{r+1}{r+1}$	–
1	–	–	–	–	$\dots$	–	–	–

The Betti table of a module cannot have an arbitrary shape: for example we have the following result.

Proposition 1.2.4. *Let M be a finitely generated graded S-module with Betti numbers $k_{p,q} = k_{p,q}(M, V)$. Suppose that for a certain $p \geq 0$ it holds that $k_{p,q} = 0$ for all $q \leq \overline{q}$. Then $k_{p,q+1} = 0$ for all $q \leq \overline{q}$ as well.*

This means that if the Betti table of M is zero in the p-th column above the $\overline{q}$-th row, then it is zero also in the $p+1$-th column above the $\overline{q}$-th row.

Proof. Consider the minimal free resolution of M:

$$F_\bullet : \ \dots \longrightarrow F_{p+1} \longrightarrow F_p \longrightarrow \dots$$

then we know from Proposition 1.2.1 that every generator of F_{p+1} must be mapped to a nonzero homogeneous element of $\mathfrak{m}F_p$ of the same degree. By hypothesis we know that

$$F_p = \bigoplus_{q \in \mathbb{Z}} S(-p-q)^{\oplus k_{p,q}} = \bigoplus_{q \geq \overline{q}} S(-p-q)^{\oplus k_{p,q}}$$

so that every generator of F_p has degree greater or equal than $p + \overline{q}$, and this implies that every nonzero homogeneous element of $\mathfrak{m}F_p$ has degree greater or equal than $p + \overline{q} + 1$. By what we have said above, this means that every generator of F_{p+1} has degree greater or equal than $p + \overline{q} + 1$, that means $k_{p+1,q} = 0$ for all $q \leq \overline{q}$. □

Remark 1.2.6. Recently there has been much attention on the possible shapes of graded Betti tables: in particular a new field called Boij-Söderberg theory describes the space of Betti tables up to a rational multiple, and it has been shown by M. Boij, D. Eisenbud, F.-O. Schreyer and J. Soderberg that these Betti table form a rational polyhedral cone whose extremal rays are spanned by so-called pure Betti diagrams. It is very interesting that the proof of this fact relies on a duality with cohomology tables of coherent sheaves on projective space. For a survey of these results, see [ES].

Hilbert function, Hilbert polynomial and Hilbert series

The minimal free resolution is clearly a complete invariant for a module, but even the knowledge of just the Betti table gives us control over many fundamental algebraic properties.

The first one of these is the Hilbert function of a module:

Definition 1.2.8. Let M be a finitely generated graded S-module. Then the **Hilbert function** of M is defined as

$$H_M \colon \mathbb{Z} \longrightarrow \mathbb{N} \qquad d \mapsto \dim_{\mathbf{k}} M_d$$

It is easy to compute the Hilbert function from the Betti table:

Proposition 1.2.5. *Let M be a finitely generated S-module with Betti numbers $k_{p,q} = k_{p,q}(M,V)$ and for every $s \in \mathbb{Z}$ define $K_s \overset{def}{=} \sum_{p\geq 0}(-1)^p k_{p,s-p}$. Then*

$$H_M(d) = \sum_{p=0}^{r+1}(-1)^{p+1}\dim_{\mathbf{k}}(F_p)_d = \sum_{s\geq 0} K_s \binom{d-s+r}{r}$$

Conversely, the K_s can be computed inductively from the Hilbert function by the formula

$$K_s = H_M(s) - \sum_{k<s} K_k \binom{s-k+r}{r}$$

Proof. We have the augmented minimal free resolution of M given by

$$0 \longrightarrow F_{r+1} \longrightarrow \ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where $F_p = \bigoplus_{q\in\mathbb{Z}} S(-p-q)\otimes K_{p,q}(M,V)$. Hence, in degree d we get

$$(F_p)_d = \bigoplus_{q\in\mathbb{Z}} S(-p-q)_d \otimes K_{p,q}(M,V) = \bigoplus_{q\in\mathbb{Z}} S^{d-p-q}(V)\otimes K_{p,q}(M,V)$$

so that

$$H_M(d) = \sum_{p=0}^{r+1}(-1)^{p+1}\dim_{\mathbf{k}}(F_p)_d = \sum_{p=0}^{r+1}\sum_{q\in\mathbb{Z}}(-1)^{p+1}\binom{d-p-q+r}{r} k_{p,q}(M,V)$$

and it is easy to see that this formula can be rearranged to give the one that we want.

For the second statement, observe that $\binom{s-k+r}{r} = 0$ if $s-k<0$ so that

$$H_M(s) = \sum_{k\leq s} K_k \binom{s-k+r}{r} = K_s + \sum_{k<s} K_k \binom{s-k+r}{r}$$

□

Hilbert actually proved his Syzygy Theorem to show that the Hilbert function is eventually polynomial:

Corollary 1.2.2. *Let M be a finitely generated graded S-module and let $H_M(d)$ be its Hilbert function. Then there exists an unique polynomial $P_M(t) \in \mathbb{Q}[t]$ such that $H_M(d) = P_M(d)$ for $d \gg 0$.*

Proof. By Proposition 1.2.5 we know that

$$H_M(d) = \sum_{p=0}^{r+1} (-1)^{p+1} \dim_{\mathbf{k}}(F_p)_d = \sum_{p=0}^{r+1} \sum_{q \in \mathbb{Z}} (-1)^{p+1} \binom{d-p-q+r}{r} k_{p,q}(M,V)$$

and if we define $m = \max\{ p+q \mid k_{p,q}(M,V) \neq 0 \}$, then it is easy to show that for every $d \geq m$ we have that

$$\binom{d-p-q+r}{r} k_{p,q}(M,V) = \frac{(d-p-q+r)(d-p-q+r-1)\dots(d-p-q+1)}{r!} k_{p,q}(M,V)$$

is a polynomial function in d for all p,q. Then it is clear that H_M is polynomial for all $d \geq m$ and this polynomial is unique, because its value is determined on infinite points. □

Definition 1.2.9 (Hilbert polynomial)**.** Let M be a finitely generated graded S-module with Hilbert function H_M. The **Hilbert polynomial** of M is the unique polynomial $P_M(t) \in \mathbb{Q}[t]$ such that $H_M(d) = P_M(d)$ for $d \gg 0$.

Another way to get an hold on the Hilbert function is through its generating function.

Definition 1.2.10 (Hilbert series)**.** Let M be a finitely generated graded S-module with Hilbert function H_M. Then the **Hilbert series** of M is defined as the power series

$$HS_M(z) = \sum_{n \in \mathbb{Z}} H_M(n) z^n$$

Proposition 1.2.6. *Let M be a finitely generated graded S-module. Then its Hilbert series has the form*

$$HS_M(z) = \frac{\Phi_M(z)}{(1-z)^{r+1}}$$

where $\Phi_M(z) \in \mathbb{Z}[z, z^{-1}]$ is a Laurent polynomial with integer coefficients.

Proof. Consider the extended minimal free resolution of M

$$0 \longrightarrow F_{r+1} \longrightarrow \dots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

This is an exact sequence, so that $HS_M(z) = \sum_{p=0}^{r+1} (-1)^p HS_{F_p}(z)$. Now, it is easy to compute the Hilbert function for a free module:

$$HS_{F_p}(z) = \sum_q k_{p,q}(M,V) HS_{S(-p-q)}(z) = \sum_q k_{p,q}(M,V) z^{p+q} HS_S(z) = \frac{\sum_{q \in \mathbb{Z}} k_{p,q}(M,V) z^{p+q}}{(1-z)^{r+1}}$$

and the proof is completed. □

Example: Complete Intersections

We can also use the Betti table to detect regular sequences.

Definition 1.2.11 (Regular sequence)**.** Let M be a finitely generated graded S-module. A sequence $(f_1, \dots, f_s)$ of homogenous elements $f_i \in S$ is said to be an M-**regular sequence** if

1. $M/(f_1, \dots, f_s)M \neq 0$.
2. f_i is not a zerodivisor on $M/(f_1, \dots, f_{i-1})M$ for every i.

Definition 1.2.12 (Complete intersections)**.** A complete intersection is an S-module of the form $S/(f_1, \dots, f_r)$ where the elements $(f_1, \dots, f_r)$ are homogeneous and form a regular sequence in S.

Example 1.2.3 (Principal ideals). The most basic example of complete intersections is clearly that of principal ideals: since S is a domain, any nonzero homogeneous element $f \in S$ gives a regular sequence (f), and a complete instersection $S/(f)$.

If $\deg f = d$, it is quite clear that the minimal free resolution of $S/(f)$ is

$$0 \longrightarrow S(-d) \xrightarrow{\cdot f} S \longrightarrow 0$$

and in particular the Hilbert series of $S/(f)$ is given by

$$HS_{S/(f)}(z) = \frac{1-z^d}{(1-z)^{r+1}}$$

Example 1.2.4 (The residue field). The residue field itself is a complete intersection: indeed if $X_0, \dots, X_r$ is a basis of V, then $\mathbf{k} = S/(X_0, \dots, X_r)$ and $(X_0, \dots, X_r)$ is obviously a regular sequence, whose minimal free resolution is the Koszul complex.

Example 1.2.5. Suppose that $\dim V = 3$ and let $f \in S$ be an irreducible homogeneous polynomial of degree 2 and $g \in S$ an irreducible homogeneous polynomial of degree 3. Then it is clear that (f, g) is a regular sequence: indeed, as S is a domain and $f \neq 0$ it is obvious that f is not a zerodivisor on S. Moreover, take an element $h \in S$ such that $gh \in (f)$: then, as f, g are irreducible and coprime, it must be that $h \in (f)$, so that g is not a zerodivisor on $S/(f)$. Hence, $S/(f,g)$ is a complete intersection.

What is the minimal free resolution of $S/(f,g)$? Clearly, it starts off as

$$S(-2) \oplus S(-3) \xrightarrow{\begin{pmatrix} f & g \end{pmatrix}} S \longrightarrow 0$$

and to continue we have to determine the kernel of the map $(f \ g)$, that is the module

$$K = \{ (h_1, h_2) \in S(-2) \oplus S(-3) \mid h_1 f = -h_2 g \}$$

But since (f, g) is a regular sequence, it is clear that if $(h_1, h_2) \in K$, then we can write $h_1 = gk_1$ and $h_2 = -fk_2$ for certain homogeneous polynomials k_1, k_2. Plugging this into the relation $h_1 f = -h_2 g$ tells us that $k_1 = k_2$. Thus, the next step in the minimal free resolution is given by

$$S(-5) \xrightarrow{\begin{pmatrix} -f \\ g \end{pmatrix}} S(-2) \oplus S(-3) \xrightarrow{\begin{pmatrix} f & g \end{pmatrix}} S \longrightarrow 0$$

and, as this last map is cleary injective, the resolution stops here and we get

$$0 \longrightarrow S(-5) \xrightarrow{\begin{pmatrix} -f \\ g \end{pmatrix}} S(-2) \oplus S(-3) \xrightarrow{\begin{pmatrix} f & g \end{pmatrix}} S \longrightarrow 0$$

In particular, the Hilbert series of $S/(f,g)$ is given by

$$HS_{S/(f,g)}(z) = \frac{1 - z^2 - z^3 + z^5}{(1-z)^{r+1}}$$

Now we want to show that the knowledge of the Betti table of a module allows us to decide whether a certain sequence is regular or not. Actually, this is completely determined by the Hilbert function already:

Proposition 1.2.7. *Let M be a finitely generated graded S-module and let $\mathbf{f} = (f_1, \dots, f_s)$ be a sequence of homogeneous elements of degrees $\deg f_i = d_i$. Then*

$$HS_{M/\mathbf{f}M}(z) \geq \prod_{i=1}^{s} (1 - z^{d_i}) HS_M(z)$$

with equality if and only if $\mathbf{f}$ is an M-regular sequence (the inequality stands for a term-by-term inequality).

Proof. First we prove the statement when $s = 1$. We have the exact sequence

$$0 \longrightarrow K \longrightarrow M(-d_1) \xrightarrow{\cdot f_1} M \longrightarrow M/f_1M \longrightarrow 0$$

so that $HS_{M/f_1M}(z) = (1-z^{d_1})HS_M(z) + HS_K(z)$. Hence, it is clear that $HS_{M/f_1M}(z) \geq (1-z^{d_1})HS_M(z)$, with equality if and only if $HS_K(z) = 0$. But this is equivalent to saying that $K = 0$, i.e. f_1 is M-regular.

In general, we can use repeatedly the above case to get a chain of inequalities

$$HS_{M/(f_1,\dots,f_s)}(z) \geq (1-z^{d_s})HS_{M/(f_1,\dots,f_{s-1})M} \geq \prod_{i=1}^{2}(1-z^{d_{s-i}})HS_{M/(f_1,\dots,f_{s-2})M}(z) \geq \dots \geq \prod_{i=1}^{s}(1-z^{d_i})HS_M(z)$$

and the equality at the two extremes is equivalent to the equality in all the intermediate steps, and again by the case above, this holds if and only if $\mathbf{f}$ is a regular sequence. □

Consider again the example of $S/(f,g)$ above: if we stare for a bit at the minimal free resolution we realize that it looks very much like the Koszul complex, and this is no coincidence, as we are going to see now.

Definition 1.2.13 (Koszul complex of a sequence)**.** Let $\mathbf{f} = (f_1,\dots,f_s)$ be a sequence of homogeneous elements in S and let $W = \mathbf{k}f_1 \oplus \dots \oplus \mathbf{k}f_s$ be the vector space with basis $f_1,\dots,f_s$. Then the Koszul complex of $\mathbf{f}$ is the sequence

$$K(\mathbf{f})_\bullet \colon 0 \longrightarrow \wedge^s W \otimes_{\mathbf{k}} S \longrightarrow \wedge^{s-1}W \otimes_{\mathbf{k}} S \longrightarrow \dots \longrightarrow W \otimes_{\mathbf{k}} S \longrightarrow S \longrightarrow 0$$

where the maps are defined by

$$\wedge^p W \otimes_{\mathbf{k}} S \longrightarrow \wedge^{p-1}W \otimes_{\mathbf{k}} S \qquad w_1 \wedge \dots \wedge w_p \otimes g \mapsto \sum_{i=1}^{p}(-1)^{i+1} w_1 \wedge \dots \wedge \widehat{w_i} \wedge \dots \wedge w_p \otimes w_i g$$

Remark 1.2.7. Of course, we need to change the grading on the various factors $\wedge^p W \otimes_{\mathbf{k}} S$ to get maps of degree zero, but we will not denote this explicitly for reasons of clarity.

Theorem 1.2.4. *Let $\mathbf{f} = (f_1,\dots,f_s)$ be a sequence of homogeneous elements in S. Then the following are equivalent:*

1. *$\mathbf{f}$ is a regular sequence.*
2. *The Koszul complex $K(\mathbf{f})_\bullet$ is a resolution of $S/\mathbf{f}S$.*

Moreover, in this case the Koszul complex is the minimal graded free resolution of $S/\mathbf{f}S$.

Proof. First, we suppose that $\mathbf{f}$ is regular and we prove that $K_\bullet(\mathbf{f})$ is the minimal free resolution. We are going proceed by induction on s, the length of the sequence $\mathbf{f}$. If $s = 0,1$ then the statement is clear. For the inductive step, suppose that $s > 1$ and consider the subsequence $\mathbf{g} = (g_1,\dots,g_{s-1})$. Then we know by inductive hypothesis that the Koszul complex $K(\mathbf{g})_\bullet$ is a free resolution of $S/\mathbf{g}S$. Now, consider the two exact sequences

$$F_\bullet \colon K(f_s)_\bullet \longrightarrow S/f_sS \longrightarrow 0 \qquad G_\bullet \colon K(\mathbf{g}) \longrightarrow S/\mathbf{g}S \longrightarrow 0$$

then if we consider the tensor product of complexes $F_\bullet \otimes_S G_\bullet$, it is easy to show that it corresponds precisely to the sequence $K(\mathbf{f})_\bullet \longrightarrow S/\mathbf{f}S \longrightarrow 0$. By definition, the tensor product $F_\bullet \otimes_S G_\bullet$ is the complex associated to the first-quadrant double complex

$$E^{p,q} = F_{1-q} \otimes_S G_{s+1-p}$$

but now, taking the upward spectral sequence, it is easy to see that ${}_\uparrow E_1^{p,q} = 0$ for all p,q. Indeed, since every free module is flat, it is enough to show that the sequence

$$0 \longrightarrow S/\mathbf{g}S \xrightarrow{\cdot f_s} S/\mathbf{g}S \longrightarrow S/\mathbf{g}S \otimes_S S/f_sS \longrightarrow 0$$

is exact, and this is obvious, as f_s is an $S/\mathbf{g}S$-regular element. Thus, the double complex $E^{p,q}$ has no cohomology, that is, the complex $K(\mathbf{f})_\bullet \longrightarrow S/\mathbf{f}S \longrightarrow 0$ is exact. It is also clear from the definition that the Koszul complex is the minimal free resolution of $S/\mathbf{f}S$.

Suppose now that the Koszul complex $K_\bullet(\mathbf{f})$ is exact. Observe that the Betti numbers of the Koszul complex depend only on the degrees $(d_1,\dots,d_s)$ and not on the fact of whether $\mathbf{f}$ is regular or not. In particular, from what we have proved above and Proposition 1.2.7 we see that

$$\sum_{i=0}^{r}(-1)^i H_{K_\bullet(\mathbf{f})}(z) = \frac{\prod_{i=1}^{r}(1-z^{d_i})}{(1-z)^{r+1}}$$

and since $K_\bullet(\mathbf{f}) \longrightarrow S/\mathbf{f}S \longrightarrow 0$ is exact by hypothesis, it follows that

$$H_{S/\mathbf{f}S}(z) = \frac{\prod_{i=1}^{r}(1-z^{d_i})}{(1-z)^{r+1}}$$

so that $\mathbf{f}$ is a regular sequence by Proposition 1.2.7. □

Cohen-Macaulay modules

One other important property of a module that can be read off its Betti table is the Cohen-Macaulay property.

Definition 1.2.14 (Depth)**.** Let M be a finitely generated graded S-module. The **depth** of M is defined as the maximum length of an M-regular sequence of homogeneous elements in S.

Proposition 1.2.8. *Let M be a finitely generated graded S-module. Then depth $M \leq \dim M$.*

Proof. See [Eis95]. □

Definition 1.2.15 (Cohen-Macaulay)**.** Let M be a finitely generated graded S-module. We say that M is **Cohen-Macaulay** if depth $M = \dim M$, that is, if there exists an M-sequence of length $\dim M$.

Example 1.2.6. Let $X_0,\dots,X_r$ be a basis of V. Then it is clear that $(X_0,\dots,X_r)$ is a regular sequence in S, and since $\dim S = r+1$, this shows that S is Cohen-Macaulay.

The depth of a module is linked to its minimal free resolution by the following fundamental formula:

Theorem 1.2.5 (Auslander-Buchsbaum formula)**.** *Let M be a finitely generated graded S-module. Then*

$$\textit{depth } M + \textit{projdim } M = r+1$$

In particular M is Cohen-Macaulay if and only if

$$\textit{projdim } M = r+1-\dim M$$

Proof. See [Eis95]. □

1.2.3 Koszul cohomology as a functor

First we observe that Koszul cohomology can be defined as a functor. Indeed, fix a vector space V of finite dimension over $\mathbf{k}$ and consider the category $\mathbf{GrMod}_{S^\bullet(V)}$ of finitely generated graded $S^\bullet(V)$-modules (the morphisms in this category are the homogeneous morphisms of degree 0) and the category $Mod_{\mathbf{k}}$ of finite dimensional vector spaces over $\mathbf{k}$. Then it is clear that for every $p,q \in \mathbb{Z}$ we have a functor:

$$K_{p,q}(\cdot,V)\colon \mathbf{GrMod}_{S^\bullet(V)} \longrightarrow \mathbf{Mod}_{\mathbf{k}} \qquad M \mapsto K_{p,q}(M,V)$$

Indeed, for every morphism $f\colon M \longrightarrow N$ of graded $S^\bullet(V)$-modules we have an induced morphism of complexes $M \otimes_S K(V) \longrightarrow N \otimes_S K(V)$ and the corresponding morphism in cohomology gives us what we want.

Observe that we can consider also a global functor

$$K_{p,\bullet}(\cdot, V)\colon \mathbf{GrMod}_{S^\bullet(V)} \longrightarrow \mathbf{GrMod}_k \qquad M \mapsto K_{p,\bullet}(M,V) = \bigoplus_{q\in\mathbb{Z}} K_{p,q}(M,V)$$

and by definition we have $K_{p,\bullet}(\cdot, V) = \mathrm{Tor}_p^{S^\bullet(V)}(\cdot, \mathbf{k})(p)$.

The Koszul cohomology functor inherits the long exact sequence of the *Tor* functor:

Proposition 1.2.9 (Long exact sequence in Koszul cohomology)**.** *Consider a short exact sequence of finitely generated graded S-modules*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

then this induces a long exact sequence in Koszul cohomology

$$\ldots \longrightarrow K_{p,q}(M',V) \longrightarrow K_{p,q}(M,V) \longrightarrow K_{p,q}(M'',V) \longrightarrow K_{p-1,q+1}(M',V) \longrightarrow \ldots$$

for every $p, q \in \mathbb{Z}$.

Proof. This follows immediately from the long exact sequence of the *Tor* functor.

More concretely, since the Koszul complex is made up of free modules, we have a short exact sequence of complexes

$$0 \longrightarrow K_\bullet(V) \otimes_S M' \longrightarrow K_\bullet(V) \otimes_S M \longrightarrow K_\bullet(V) \otimes_S M'' \longrightarrow 0$$

and taking the corresponding long exact sequence in cohomology we get an exact sequence

$$\ldots \longrightarrow \mathrm{Tor}_p^S(M',V) \longrightarrow \mathrm{Tor}_p^S(M,V) \longrightarrow \mathrm{Tor}_p^S(M',V) \longrightarrow \mathrm{Tor}_{p+1}^S(M,V) \longrightarrow \ldots$$

that in degree $p+q$ yelds the desired exact sequence. □

Koszul cohomology behaves well with respect to direct sums:

Lemma 1.2.3. *Let M, N be two finitely generated S-modules, with minimal free resolutions $F_\bullet$ and $G_\bullet$ repectively. Then $F_\bullet \oplus G_\bullet$ is the minimal free resolution of $M \oplus N$. In particular*

$$K_{p,q}(M \oplus N, V) = K_{p,q}(M,V) \oplus K_{p,q}(N,V)$$

Proof. It is clear that $F_\bullet \oplus G_\bullet$ is the minimal free resolution of $M \oplus N$, and now the statement about Koszul cohomology groups is obvious. □

1.3 Geometry

Now we want to explain the concepts of Koszul cohomology in a geometric context. Let V be as before a vector space over $\mathbf{k}$ of dimension $r+1$, and let $S = S^\bullet(V)$ be its symmetric algebra. Then we denote with $\mathbb{P}(V)$ the projective space of quotients, i.e.

$$\mathbb{P}(V) = \mathrm{Proj}\, S^\bullet(V)$$

Observe that we have a canonical identification

$$H^0(\mathbb{P}(V), \mathcal{O}(1)) = V$$

that in turn induces identifications

$$H^0(\mathbb{P}(V), \mathcal{O}(k)) = S^k(V) \qquad \forall k \in \mathbb{Z}$$

so that we can consider

$$S^\bullet(V) = \bigoplus_{k\in\mathbb{Z}} H^0(\mathbb{P}(V), \mathcal{O}(k))$$

1.3.1 Minimal free resolutions of coherent sheaves

Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}(V)$. Then we define the graded group

$$R(\mathcal{F}, V) = \bigoplus_{k \in \mathbb{Z}} H^0(\mathbb{P}(V), \mathcal{F}(k))$$

and we note that this has a natural structure of an $S^\bullet(V)$-module, thanks to the multiplication maps

$$H^0(\mathbb{P}(V), \mathcal{O}(k)) \otimes H^0(\mathbb{P}(V), \mathcal{F}(m)) \longrightarrow H^0(\mathbb{P}(V), \mathcal{F}(m+k))$$

Lemma 1.3.1. *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}(V)$ such that there are no closed points in $Ass(\mathcal{F})$, then $R(\mathcal{F}, V)$ is a finitely generated graded $S^\bullet(V)$-module. Conversely, if there is a closed point in $Ass(\mathcal{F})$, then $R(\mathcal{F}, V)$ is not finitely generated.*

Proof. See [Eis05]. □

Definition 1.3.1 (Minimal free resolution of a sheaf)**.** Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}(V)$ with no closed points in its support and let $R(\mathcal{F}, V)$ be its associated S-module. Then the minimal free resolution of $\mathcal{F}$ is defined as the sheafification of the minimal free resolution of $R(\mathcal{F}, V)$ as an S-module.

Definition 1.3.2 (Koszul cohomology of a coherent sheaf)**.** Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}(V)$. Then the Koszul cohomology of $\mathcal{F}$ is defined as

$$K_{p,q}(\mathcal{F}, V) \overset{\text{def}}{=} K_{p,q}(R(\mathcal{F}, V), V)$$

More concretely, it is defined as the cohomology at the middle term of the complex

$$\wedge^{p+1} V \otimes_{\mathbf{k}} H^0(\mathbb{P}(V), \mathcal{F}(q-1)) \longrightarrow \wedge^p V \otimes_{\mathbf{k}} H^0(\mathbb{P}(V), \mathcal{F}(q)) \longrightarrow \wedge^{p-1} V \otimes_{\mathbf{k}} H^0(\mathbb{P}(V), \mathcal{F}(q+1))$$

Remark 1.3.1. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}(V)$: then the minimal free resolutions of $R(\mathcal{F})$ and $\mathcal{F}$ respectively are given by

$$0 \longrightarrow F_r \longrightarrow F_{r-1} \longrightarrow \ldots \longrightarrow F_0 \longrightarrow F \longrightarrow 0$$
$$0 \longrightarrow \mathcal{F}_r \longrightarrow \mathcal{F}_{r-1} \longrightarrow \ldots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where

$$F_p = \bigoplus_{q \in \mathbb{Z}} S(-p-q) \otimes K_{p,q}(\mathcal{F}, V) \qquad \mathcal{F}_p = \bigoplus_{q \in \mathbb{Z}} \mathcal{O}(-p-q) \otimes K_{p,q}(\mathcal{F}, V)$$

and in particular we see that $H^0(\mathbb{P}(V), \mathcal{F}_p(k))$ is precisely the degree k part of F_p. By exactness of the minimal free resolution at the level of moduli, we see that the sequence

$$0 \longrightarrow H^0(\mathbb{P}(V), \mathcal{F}_r(k)) \longrightarrow H^0(\mathbb{P}(V), \mathcal{F}_{r-1}(k)) \longrightarrow \ldots \longrightarrow H^0(\mathbb{P}(V), \mathcal{F}_0(k)) \longrightarrow H^0(\mathbb{P}(V), \mathcal{F}(k)) \longrightarrow 0$$

is exact. Moreover, taking the direct sum of these sequences for all $k \in \mathbb{Z}$ we recover the minimal free resolution of $R(\mathcal{F})$.

Example 1.3.1. Let X be the complete intersection of an irreducible conic and an irreducible cubic in $\mathbb{P}^3$. Then the minimal free resolution of $\mathcal{O}_X$ as a sheaf on $\mathbb{P}^3$ has the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-5) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

1.3.2 Koszul cohomology of projective varieties

Let X be a projective variety, L a line bundle on X and $V \subseteq H^0(X,L)$ a subspace. Then we can consider the graded ring $S = S(X,L,V) = S^\bullet(V)$ and for every coherent sheaf $\mathcal{F}$ on X we have the graded $S^\bullet(V)$-module

$$R(\mathcal{F},L) = \bigoplus_{k\in\mathbb{Z}} H^0(X,\mathcal{F}\otimes L^{\otimes k})$$

Definition 1.3.3 (Koszul cohomology of a projective variety)**.** Let notations be a s before. Then we define the Koszul cohomology of $\mathcal{F}$ with respect to V as

$$K_{p,q}(X,L,V;\mathcal{F}) \stackrel{\text{def}}{=} K_{p,q}(R(\mathcal{F},L),S^\bullet(V))$$

In particular, if $V = H^0(X,L)$ we use the notation $K_{p,q}(X,L;\mathcal{F}) \stackrel{\text{def}}{=} K_{p,q}(X,L,V;\mathcal{F})$ whereas, if $\mathcal{F} = \mathcal{O}_X$ we use the notation $K_{p,q}(X,L,V) \stackrel{\text{def}}{=} K_{p,q}(X,L,V;\mathcal{O}_X)$.

Remark 1.3.2*.* More concretely, the Koszul cohomology $K_{p,q}(X,L,V;\mathcal{F})$ is the cohomology at the middle term of the complex

$$\wedge^{p+1}V\otimes H^0(X,\mathcal{F}\otimes L^{\otimes(q-1)}) \longrightarrow \wedge^{p}V\otimes H^0(X,\mathcal{F}\otimes L^{\otimes q}) \longrightarrow \wedge^{p-1}V\otimes H^0(X,\mathcal{F}\otimes L^{\otimes(q+1)})$$

Remark 1.3.3*.* If $X = \mathbb{P}(V)$ and $L = \mathcal{O}_{\mathbb{P}(V)}(1)$ then the Koszul cohomology $K_{p,q}(\mathbb{P}(V),\mathcal{O}_{\mathbb{P}(V)}(1);\mathcal{F})$ corresponds to the Koszul cohomology $K_{p,q}(\mathcal{F},V)$ that we have defined before.

Remark 1.3.4*.* Let X be a projective variety and suppose that L is a very ample line bundle that defines a closed embedding $\phi_L\colon X \hookrightarrow \mathbb{P}(H^0(X,L))$. Then for every coherent sheaf $\mathcal{F}$ we see that

$$K_{p,q}(X,L;\mathcal{F}) = K_{p,q}(\phi_{L*}\mathcal{F},V)$$

Chapter 2

Techniques of Koszul cohomology

In this chapter we are going to present some properties of Koszul cohomology and some examples of Betti tables.

2.1 Syzygy bundles

Let X be a projective variety over $\mathbf{k}$ and let L be a line bundle on it. Suppose that L is globally generated: then it defines a morphism

$$\phi_L \colon X \longrightarrow \mathbb{P}(H^0(X, L))$$

On the projective space $\mathbb{P} = \mathbb{P}(H^0(X, L))$ we have the Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}}(1) \longrightarrow H^0(X, L) \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(1) \longrightarrow 0$$

and pulling back this sequence on X via ϕ_L we obtain the exact sequence

$$0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \mathcal{O}_X \xrightarrow{ev} L \longrightarrow 0$$

where $ev \colon H^0(X, L) \otimes \mathcal{O}_X \longrightarrow L$ is the evaluation map and $M_L \overset{\text{def}}{=} \phi_L^*(\Omega_{\mathbb{P}(H^0(X,L))}(1))$ is a vector bundle of rank $h^0(X, L) - 1$.

Definition 2.1.1 (Syzygy bundle)**.** Let X be a projective variety and L a globally generated line bundle over X. Then, in the above notation, M_L is called the **syzygy bundle** on X relative to L.

The syzygy bundle is strictly related to the Koszul cohomology of X and L: indeed, from the exact sequence

$$0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \longrightarrow L \longrightarrow 0$$

for every $p \geq 1$ we have exact sequences

$$0 \longrightarrow \wedge^p M_L \longrightarrow \wedge^p H^0(X, L) \otimes \mathcal{O}_X \longrightarrow \wedge^{p-1} M_L \otimes L \longrightarrow 0$$

and twisting this sequence by $L^q \overset{\text{def}}{=} L^{\otimes q}$ gives the exact sequences

$$0 \longrightarrow \wedge^p M_L \otimes L^q \longrightarrow \wedge^p H^0(X, L) \otimes L^q \longrightarrow \wedge^{p-1} M_L \otimes L^{q+1} \longrightarrow 0$$

so that, taking cohomology, we have the long exact sequence

$$0 \longrightarrow H^0(X, \wedge^p M_L \otimes L^q) \longrightarrow \wedge^p H^0(X, L) \otimes H^0(X, L^q) \longrightarrow H^0(X, \wedge^{p-1} M_L \otimes L^{q+1}) \longrightarrow \dots$$

Now, consider the commutative diagram:

$$
\begin{array}{ccccc}
\wedge^{p+1} H^0(X,L) \otimes H^0(X,L^{q-1}) & \xrightarrow{d_{p+1,q-1}} & & & \\
\downarrow & \searrow & & & \\
H^0(X, \wedge^p M_L \otimes L^q) & \hookrightarrow & \wedge^p H^0(X,L) \otimes H^0(X,L^q) & \longrightarrow & H^0(X, \wedge^{p-1} M_L \otimes L^{q+1}) \\
 & & & \searrow^{d_{p,q}} & \downarrow \\
 & & & & \wedge^{p-1} H^0(X,L) \otimes H^0(X,L^{q+1})
\end{array}
$$

Where the diagonal maps are the differentials in the Koszul complex and the row is exact. Then we have:

Proposition 2.1.1. *Suppose that L is a globally generated line bundle on X. Then there are canonical isomorphisms*

$$
\begin{aligned}
K_{p,q}(X,L) &\cong \mathit{Coker}\ (\wedge^{p+1} H^0(X,L) \otimes H^0(X,L^{q-1}) \longrightarrow H^0(X, \wedge^p M_L \otimes L^q)) \\
&\cong \mathit{Ker}\ (H^1(X, \wedge^{p+1} M_L \otimes L^{q-1}) \longrightarrow \wedge^{p+1} H^0(X,L) \otimes H^1(X, L^{q-1}))
\end{aligned}
$$

Proof. For the first isomorphism, consider the above commutative diagram: as the map

$$H^0(X, \wedge^{p-1} M_L \otimes L^{q+1}) \hookrightarrow \wedge^{p-1} H^0(X,L) \otimes H^0(X, L^{q+1})$$

is injective, we see that

$$\mathit{Ker}(d_{p,q}) = \mathrm{Ker}\ (\wedge^p H^0(X,L) \otimes H^0(X,L^q) \longrightarrow H^0(X, \wedge^{p-1} M_L \otimes L^{q+1})) = H^0(X, \wedge^p M_L \otimes L^q)$$

and then it is clear how to conclude.

For the second isomorphism, consider the exact sequence in cohomology given by

$$\wedge^{p+1} H^0(X,L) \otimes H^0(X,L^{q-1}) \longrightarrow H^0(X, \wedge^p M_L \otimes L^q) \longrightarrow H^1(X, \wedge^{p+1} M_L \otimes L^{q-1}) \longrightarrow \wedge^{p+1} H^0(X,L) \otimes H^1(X, L^q)$$

□

Corollary 2.1.1. *Suppose that L is a globally generated line bundle on X and suppose that for a certain $h > 0$ we have*

$$
\begin{aligned}
H^i(X, L^{q-i}) &= 0 \qquad \text{for all } i = 1, \ldots, h \\
H^i(X, L^{q-1-i}) &= 0 \qquad \text{for all } i = 1, \ldots, h-1
\end{aligned}
$$

Then

$$K_{p,q}(X,L) \cong H^h(X, \wedge^{p+h} M_L \otimes L^{q-h})$$

Proof. The proof is by induction on h, starting from $h = 1$: we know from the Proposition 2.1.1 that

$$K_{p,q}(X,L) = \mathrm{Ker}\ (H^1(X, \wedge^{p+1} M_L \otimes L^{q-1}) \longrightarrow \wedge^{p+1} H^0(X,L) \otimes H^1(X, L^{q-1}))$$

so that, if $H^1(X, L^{q-1}) = 0$, then it is obvious that $K_{p,q}(X,L) = H^1(X, \wedge^{p+1} M_L \otimes L^{q-1})$.

Now, suppose that the thesis has been proved for $h-1$. Then by inductive hypothesis we know that $K_{p,q}(X,L) \cong H^{h-1}(X, \wedge^{p+h-1} M_L \otimes L^{q-h+1})$, and using the long exact sequence in cohomology associated to the short exact sequence

$$0 \longrightarrow \wedge^{p+h} M_L \otimes L^{q-h} \longrightarrow \wedge^{p+h} H^0(X,L) \otimes L^{q-h} \longrightarrow \wedge^{p+h-1} M_L \otimes L^{q-h+1} \longrightarrow 0$$

it is easy to see that we have

$$H^{h-1}(X, \wedge^{p+h-1} M_L \otimes L^{q-h+1}) \cong H^h(X, \wedge^{p+h} M_L \otimes L^{q-h})$$

□

Corollary 2.1.2. *Suppose that L is a globally generated line bundle on X such that $\dim_{\mathbf{k}} H^0(X,L) = r+1$, and suppose that for a certain $p \geq 0$ it holds that*

$$H^i(X, L^{q-i}) = 0 \qquad \text{for all } i = 1,\dots,r-p+1$$
$$H^i(X, L^{q-1-i}) = 0 \qquad \text{for all } i = 1,\dots,r-p$$

then $K_{p,q}(X,L) = 0$.

Proof. From Corollary 2.1.1, we know that in these hypotheses

$$K_{p,q}(X,L) = H^{r-p+1}(X, \wedge^{r+1} M_L \otimes L^{q-h})$$

but $\wedge^{r+1} M_L = 0$ since M_L is a vector bundle of rank r. □

Corollary 2.1.3. *Suppose that L is a globally generated line bundle on X such that*

$$H^0(X, L^q) = 0 \qquad \text{for all } q < 0$$

then

$$K_{p,0}(X,L) = H^0(X, \wedge^p M_L) \qquad \text{and} \qquad K_{p,q}(X,L) = 0 \qquad \text{for all } q < 0$$

Proof. We know from Proposition 2.1.1 that

$$K_{p,q}(X,L) = \text{Coker}\left(\wedge^{p+1} H^0(X,L) \otimes H^0(X, L^{q-1}) \longrightarrow H^0(X, \wedge^p M_L \otimes L^q)\right)$$

and in our hypotheses $H^0(X, L^{q-1}) = 0$ for all $q \leq 0$, so that

$$K_{p,q}(X,L) = H^0(X, \wedge^p M_L \otimes L^q) \qquad \text{for all } q \leq 0$$

Now, since we have an inclusion $H^0(X, \wedge^p M_L \otimes L^q) \subseteq \wedge^p H^0(X,L) \otimes H^0(X, L^q)$, it is clear that

$$K_{p,q}(X,L) = 0 \qquad \text{for all } q < 0$$

□

Example 2.1.1 (Veronese embeddings). Let V be a vector space of dimension $r+1$ over $\mathbf{k}$ and let $\mathbb{P}(V)$ be the projective space of quotients. Consider the line bundle $L = \mathcal{O}_{\mathbb{P}(V)}(d)$ for a certain positive $d > 0$ and let $M_d = M_L$ be the corresponding syzygy bundle: then it is easy to see from the Corollary 2.1.1 that

$$K_{p,q}(X,L) = H^q(\mathbb{P}(V), \wedge^{p+q} M_d)$$

Indeed, if $q < r$ we know that line bundles on projective spaces do not have any intermediate cohomology, that is $H^i(X, L^s) = 0$ for every $s \in \mathbb{Z}$ and every $i = 1,\dots,r-1$. Then we only need to check the case $q \geq r$, but this reduces to the condition $H^r(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}) = 0$, that is clearly true.

In particular, we see that $K_{p,q}(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d)) = 0$ if $q > r$.

Remark 2.1.1. The above results can be generalized as follows: let L be a globally generated line bundle on a projective variety X and let $V \subseteq H^0(X,L)$ be a subspace that generates L. Then we have an exact sequence

$$0 \longrightarrow M_V \longrightarrow V \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0$$

where M_V is a vector bundle, called the syzygy bundle of V. Now we can rephrase our previous results as follows:

Proposition 2.1.2. *Suppose that X is a projective variety and that L is a line bundle generated by a subspace $V \subseteq H^0(X,L)$. Then, for every $\mathcal{F}$ coherent sheaf on X we have that*

$$\begin{aligned} K_{p,q}(X,L,V;\mathcal{F}) &\cong \text{Coker}\left(\wedge^{p+1} V \otimes H^0(X, \mathcal{F} \otimes L^{q-1}) \longrightarrow H^0(X, \mathcal{F} \otimes \wedge^p M_V \otimes L^q)\right) \\ &\cong \text{Ker}\left(H^1(X, \mathcal{F} \otimes \wedge^{p+1} M_V \otimes L^{q-1}) \longrightarrow \wedge^{p+1} V \otimes H^0(X, \mathcal{F} \otimes L^{q-1})\right) \end{aligned}$$

Proof. The proof is analogous to that of Proposition 2.1.1. □

Corollary 2.1.4. *Let X be a projective variety, L a line bundle on X generated by a subspace $V \subseteq H^0(X, L)$ and $\mathcal{F}$ a coherent sheaf on X such that for a certain $h > 0$ we have the vanishings*

$$H^i(X, \mathcal{F} \otimes L^{q-i}) = 0 \qquad \text{for all } i = 1, \ldots, h$$
$$H^i(X, \mathcal{F} \otimes L^{q-1-i}) = 0 \qquad \text{for all } i = 1, \ldots, h-1$$

then

$$K_{p,q}(X, L, V; \mathcal{F}) \cong H^h(X, \mathcal{F} \otimes \wedge^{p+h} M_L \otimes L^{q-h})$$

Proof. The proof is analogous to that of Corollary 2.1.1. □

In particular, this result tells us that we can recover the cohomology of a sheaf from its Koszul cohomology wih respect to an appropriate line bundle.

Proposition 2.1.3. *Let X be a connected projective variety, L a line bundle on X generated by a linear subspace $V \subseteq H^0(X, L)$ of dimension $\dim V = r + 1$ and $\mathcal{F}$ a coherent sheaf on X. Suppose that for a certain $q > 0$ we have*

$$H^i(X, \mathcal{F} \otimes L^{q-i}) = 0 \qquad \text{for all } i = 1, \ldots, q-1$$
$$H^i(X, \mathcal{F} \otimes L^{q+1-i}) = 0 \qquad \text{for all } i = 1, \ldots, q$$

then

$$H^q(X, \mathcal{F}) \cong K_{r-q,q+1}(X, L, V; \mathcal{F})$$

Proof. In these hypotheses, Corollary 2.1.4 tells us that

$$K_{r-q,q+1}(X, L, V; \mathcal{F}) \cong H^q(X, \mathcal{F} \otimes \wedge^r M_V \otimes L)$$

but from the exact sequence

$$0 \longrightarrow M_V \longrightarrow V \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0$$

we see that $\wedge^r M_V = \det M_V \cong L^{-1}$ and this yields the thesis. □

Remark 2.1.2. In the hypotheses of Proposition 2.1.3, we see that the Koszul cohomology spaces $K_{r-q,q+1}(X, L, V; \mathcal{F})$ do not depend on V.

Corollary 2.1.5. *Let X be a projective variety, L a line bundle on X generated by a subspace $V \subseteq H^0(X, L)$ of dimension $\dim V = r + 1$. Suppose that $\mathcal{F}$ is a coherent sheaf on X such that*

$$H^i(X, \mathcal{F} \otimes L^{q-i}) = 0 \qquad \text{for all } i = 1, \ldots, r-p+1$$
$$H^i(X, \mathcal{F} \otimes L^{q-1-i}) = 0 \qquad \text{for all } i = 1, \ldots, r-p$$

then $K_{p,q}(X, L, V; \mathcal{F}) = 0$.

Proof. The proof is analogous to that of Corollary 2.1.2. □

Corollary 2.1.6. *Let X be a projective variety, L a line bundle on X generated by a subspace $V \subseteq H^0(X, L)$ and $\mathcal{F}$ a coherent sheaf on X such that $H^0(X, \mathcal{F} \otimes L^q) = 0$ for each $q < 0$. Then*

$$K_{p,0}(X, L, V; \mathcal{F}) = H^0(X, \mathcal{F} \otimes \wedge^p M_L), \qquad K_{p,q}(X, L, V; \mathcal{F}) = 0 \qquad \text{for all } q < 0$$

Proof. The proof is analogous to that of Corollary 2.1.3. □

2.2 Lefschetz theorem

We want to study what happens to Koszul cohomology upon taking hyperplane sections. Take a vector space V of dimension $\dim V = r+1$ and let $S^\bullet(V)$ be its symmetric algebra, with the standard grading. Take an homogeneous element of degree one $\ell \in V$: we denote by $\langle \ell \rangle$ the vector space in V spanned by ℓ. Notice that $S^\bullet(V/\langle \ell \rangle) = S^\bullet(V)/(\ell)$.

Lemma 2.2.1. *Let M be a finitely generated graded $S^\bullet(V)$-module such that ℓ is M-regular. Then*

$$K_{p,q}(M,V) \cong K_{p,q}(M/\ell M, V/\ell) \qquad \textit{for all } p,q$$

Proof. Since ℓ is an M-regular element, we have an exact sequence of graded $S^\bullet(V)$-modules

$$0 \longrightarrow M(-1) \xrightarrow{\cdot \ell} M \longrightarrow M/\ell M \longrightarrow 0$$

that induces by Proposition 1.2.3 a long exact sequence in Koszul cohomology

$$\ldots \longrightarrow K_{p,q-1}(M,V) \xrightarrow{\cdot \ell} K_{p,q}(M,V) \longrightarrow K_{p,q}(M/\ell M, V) \longrightarrow K_{p-1,q}(M,V) \xrightarrow{\cdot \ell} \ldots$$

We need to study the map $K_{p,q-1}(M,V) \xrightarrow{\cdot \ell} K_{p,q}(M,V)$: observe that the multiplication map $\cdot\ell\colon M(-1) \longrightarrow M$ factors as

$$M(-1) \cong \langle \ell \rangle \otimes M(-1) \longrightarrow V \otimes M(-1) \xrightarrow{d} M$$

where the last map d is the differential in the Koszul complex. Now, it is obvious that the differential d induces the zero map in Koszul cohomology, so that the map $K_{p,q-1}(M,V) \xrightarrow{\cdot \ell} K_{p,q}(M,V)$ is zero as well. This shows that for every p,q we have exact sequences

$$0 \longrightarrow K_{p,q}(M,V) \longrightarrow K_{p,q}(M/\ell M, V) \longrightarrow K_{p-1,q}(M,V) \longrightarrow 0 \tag{2.1}$$

Now, by definition $K_{p,q}(M/\ell M, V)$ is the cohomology at the middle term of

$$0 \longrightarrow \wedge^{p+1} V \otimes (M/\ell M)_{q-1} \longrightarrow \wedge^p V \otimes (M/\ell M)_q \longrightarrow \wedge^{p-1} V \otimes (M/\ell M)_{q+1}$$

but thanks to the splitting $V = \langle \ell \rangle \oplus V/\langle \ell \rangle$ we see that we have

$$K_{p,q}(M/\ell M, V) = K_{p,q}(M/\ell M, V/\langle \ell \rangle) \oplus [\langle \ell \rangle \otimes K_{p-1,q}(M/\ell M, V/\langle \ell \rangle)]$$

Putting this in the exact sequence 2.1, we can prove by induction on p (the case $p=0$ being obvious) that $K_{p,q}(M,V) = K_{p,q}(M/\ell M, V/\langle \ell \rangle)$. □

Remark 2.2.1. Either with the same ideas of the above proof or by applying repeatedly the above result, one can show that if $\ell = (\ell_1, \ldots, \ell_s)$ is an M-regular sequence of homogeneous elements $\ell_i \in V$ of degree 1 , then $K_{p,q}(M,V) \cong K_{p,q}(M/\ell M, V/\ell V)$ for each p,q.

Theorem 2.2.1 (Lefschetz theorem for Koszul cohomology)**.** *Let X be an irreducible projective variety, L a line bundle on X and $Y \in |L|$ a connected divisor. If $H^1(X, L^q) = 0$ for all $q \geq 0$, then*

$$K_{p,q}(X,L) \cong K_{p,q}(Y, L_{|Y}) \qquad \textit{for all } p,q$$

Proof. First we observe that in these hypotheses we have $H^0(X, L^q) = H^0(Y, L^q_{|Y}) = 0$ for all $q < 0$. Now, let $\sigma \in H^0(X,L)$ be a global section such that $Y = \operatorname{div}(\sigma)$ and take the exact sequence relative to the divisor Y:

$$0 \longrightarrow L^{-1} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

tensoring this sequence by L we find the exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\cdot \sigma} L \longrightarrow L_{|Y} \longrightarrow 0$$

and since $H^1(X, \mathcal{O}_X) = 0$ we have the exact sequence in cohomology

$$0 \longrightarrow \mathbf{k} \xrightarrow{\cdot\sigma} H^0(X, L) \longrightarrow H^0(Y, L_{|Y}) \longrightarrow 0$$

and this shows that $H^0(Y, L_{|Y}) \cong H^0(X, L)/ \langle\sigma\rangle$.

In the same way, we see that for every $q > 0$ we have the exact sequence

$$0 \longrightarrow H^0(X, L^{q-1}) \xrightarrow{\cdot\sigma} H^0(X, L^q) \longrightarrow H^0(Y, L^q_{|Y}) \longrightarrow 0$$

and adding all of these we find the exact sequence of graded $S^\bullet(H^0(X, L))$-modules

$$0 \longrightarrow R(X, L)(-1) \xrightarrow{\cdot\sigma} R(X, L) \longrightarrow R(Y, L_{|Y}) \longrightarrow 0$$

This means precisely that σ is an $R(X, L)$-regular element, so that we conclude invoking Lemma 2.2.1. □

2.3 Duality

We now discuss briefly duality in Koszul cohomology: first we need a technical lemma.

Lemma 2.3.1. *Let X be a projective variety over $\mathbf{k}$. Let*

$$0 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}_1 \longrightarrow \ldots \longrightarrow \mathcal{F}_s \longrightarrow \mathcal{F}_{s+1} \longrightarrow 0$$

be an exact sequence of sheaves on X such that

$$H^{s-i}(X, \mathcal{F}_i) = H^{s-i}(X, \mathcal{F}_{i+1}) = 0 \qquad \text{for all } i = 1, \ldots, s-1$$

then

$$\text{Ker } (H^s(X, \mathcal{F}_0) \longrightarrow H^s(X, \mathcal{F}_1)) \cong \text{Coker } (H^0(X, \mathcal{F}_s) \longrightarrow H^0(X, \mathcal{F}_{s+1}))$$

Proof. Let $\mathcal{U}$ be a finite affine open cover of X, and consider the first-quadrant double complex

$$E^{p,q} \stackrel{\text{def}}{=} C^q(X, \mathcal{F}_p, \mathcal{U})$$

where the horizontal maps are induced by the exact sequence and the vertical map are the maps of the Cech complex. Since the rows of the complex are exact, the cohomology of the double complex is zero. Now, considering the spectral sequence starting with the vertical maps we get that

$${}_{\uparrow}E_1^{p,q} = H^q(X, \mathcal{F}_p)$$

and now using the hypothesis it is easy to conclude.

Another way to prove this proposition is by splitting the long exact sequence into short exact sequences. □

The following result is due to Green [Gre84a].

Theorem 2.3.1 (Duality Theorem)**.** *Let X be smooth projective variety of dimension* $\dim X = n$ *and let L be a globally generated line bundle on X. Set $r = h^0(X, L) - 1$. If*

$$H^i(X, L^{q-i}) = H^i(X, L^{q+1-i}) = 0 \qquad \text{for all } i = 1, \ldots, n-1$$

then it follows that

$$K_{p,q}(X, L) \cong K_{r-n-p,n+1-q}(X, L; K_X)^\vee$$

Proof. Set $V = H^0(X,L)$. Then we know from Proposition 2.1.1 that

$$K_{p,q}(X,L) \cong \text{Coker } (\wedge^{p+1}V \otimes H^0(X,L^{q-1}) \longrightarrow H^0(X,\wedge^p M_L \otimes L^q))$$

so that

$$K_{p,q}(X,L)^\vee \cong \text{Ker } (H^0(X,\wedge^p M_L \otimes L^q)^\vee \longrightarrow \wedge^{p+1}V^\vee \otimes H^0(X,L^{q-1})^\vee)$$

and using Serre's duality we see that

$$\begin{aligned} K_{p,q}(X,L)^\vee &\cong \text{Ker } (H^n(X,K_X \otimes \wedge^p M_L^\vee \otimes L^{-q}) \longrightarrow \wedge^{p+1}V^\vee \otimes H^n(X,K_X \otimes L^{1-q})) \\ &\cong \text{Ker } (H^n(X,K_X \otimes \wedge^{r-p} M_L \otimes L^{1-q}) \longrightarrow \wedge^{r-p}V \otimes H^n(X,K_X \otimes L^{1-q})) \end{aligned}$$

where for the second isomorphism we have used the fact that $\det M_L \cong L^{-1}$ and $\det V \cong \mathbf{k}$.

From the exact sequence

$$0 \longrightarrow M_L \longrightarrow V \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0$$

we get (taking wedge powers and gluing) the exact sequence

$$0 \longrightarrow \wedge^{r-p}V \otimes K_X \otimes L^{1-q} \longrightarrow \dots \longrightarrow \wedge^{r-p-n+1}V \otimes K_X \otimes L^{n-q} \longrightarrow \wedge^{r-p-n}M_L \otimes K_X \otimes L^{n+1-q} \longrightarrow 0$$

Now it is enough to use Serre duality to show that the hypotheses of the Lemma 2.3.1 hold, so that we get

$$\begin{aligned} &\text{Ker } (H^n(X,K_X \otimes \wedge^{r-p} M_L \otimes L^{1-q}) \longrightarrow \wedge^{r-p}V \otimes H^n(X,K_X \otimes L^{1-q})) \cong \\ &\cong \text{Coker } (\wedge^{n-p-n+1}V \otimes H^0(X,K_X \otimes L^{n-q}) \longrightarrow H^0(\wedge^{r-n-p}M_L \otimes K_X \otimes L)) \end{aligned}$$

and from this we can conclude thanks to Proposition 2.1.1. □

Remark 2.3.1. Notice that the conditions of the theorem are automatically satisfied if X is a smooth curve or if $X = \mathbb{P}(W)$ is a projective space. Indeed, in both cases the intermediate cohomology of any line bundle is zero.

Remark 2.3.2. In the previous Theorem, suppose that $X = \mathbb{P}(W)$. Then the proof of the theorem gives an isomorphism of $SL(W)$-representations

$$K_{p,q}(X,L) \cong K_{r-n-p,n+1-q}(X,L;K_X)^\vee$$

and not merely of vector spaces. Indeed, it is enough to note that the isomorphisms $\det V = \mathbf{k}$ and $\det M_L = L^{-1}$ hold as $SL(W)$-isomorphisms, and this follows from the fact that for every finite-dimensional vector space U, we have

$$\det U \cong \mathbf{k}$$

as $SL(U)$-representation, where $\mathbf{k}$ is the trivial representation (notice that this does not hold for $GL(U)$).

Remark 2.3.3. With a similar argument, one can prove the following more general result: let X be a smooth projective variety of dimension n, L a line bundle on X, $V \subseteq H^0(X,L)$ a base-point-free subspace of dimension $r+1$ and E a vector bundle on X such that

$$H^i(X,E \otimes L^{q-i}) = H^i(X,E \otimes L^{q-i-1}) = 0 \qquad \text{for all } i = 1,\dots,n-1$$

then we have an isomorphism

$$K_{p,q}(X,L,V;E)^\vee \cong K_{r-n-p,n+1-q}(X,L,V;K_X \otimes E^\vee)$$

2.4 Castelnuovo-Mumford regularity

Koszul cohomology is deeply connected with Castelnuovo-Mumford regularity, which is a numerical measure of the cohomological complexity of a coherent sheaf.

Definition 2.4.1 (Regularity on projective spaces)**.** Let $\mathcal{F}$ be a coherent sheaf on a projective space $\mathbb{P}(V)$ and let $m \in \mathbb{Z}$ be an integer. We say that $\mathcal{F}$ is m-**regular** if

$$H^i(\mathbb{P}(V), \mathcal{F}(m-i)) = 0 \qquad \text{for all } i > 0$$

Remark 2.4.1. It is clear that $\mathcal{F}$ is m-regular if and only if $\mathcal{F}(m)$ is 0-regular.

Theorem 2.4.1 (Mumford's theorem on regularity - I)**.** *Let $\mathcal{F}$ be a m-regular coherent sheaf on a projective space $\mathbb{P} = \mathbb{P}(V)$, then for every $h \geq 0$ one has that*

1. *$\mathcal{F}$ is $(m+h)$-regular.*
2. *$\mathcal{F}(m+h)$ is generated by global sections.*
3. *The multiplication map $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(h)) \otimes H^0(\mathbb{P}(V), \mathcal{F}(m)) \longrightarrow H^0(\mathbb{P}(V), \mathcal{F}(m+h))$ is surjective.*

Proof. Substituting $\mathcal{F}$ with $\mathcal{F}(m)$ we can suppose that $m = 0$.

First, suppose that we have already proved (1) and (3). Then to prove (2) it is sufficient to consider the case $h = 0$: we know that $\mathcal{F}(H)$ is globally generated for a certain $H > 0$, so that the map $H^0(\mathbb{P}(V), \mathcal{F}(H)) \longrightarrow \mathcal{F}(H)$ is surjective. Now, using (3) we find that the morphism $H^0(\mathbb{P}(V), \mathcal{F}) \otimes H^0(P(V), \mathcal{O}_{\mathbb{P}}(H)) \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{F}(H)$ is surjective, but this factors through

$$H^0(\mathbb{P}(V), \mathcal{F}) \otimes \mathcal{O}_{\mathbb{P}}(H) \longrightarrow \mathcal{F}(H)$$

and tensoring this sequence with $\mathcal{O}_{\mathbb{P}}(-H)$ shows that $\mathcal{F}$ is generated by global sections.

Now, to prove (1) and (3) we proceed by induction on h, starting with $h = 1$ (for $h = 0$ there is nothing to prove): set $\dim V = r+1$ and consider the Koszul complex on $\mathbb{P}(V)$ given by

$$0 \longrightarrow \wedge^{r+1} V \otimes \mathcal{O}_{\mathbb{P}}(-r-1) \longrightarrow \wedge^r V \otimes \mathcal{O}_{\mathbb{P}}(-r) \longrightarrow \ldots \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0$$

this is exact, since it is the sheafification of the augmented Koszul complex

$$K_{\bullet}(V) \longrightarrow \mathbf{k} \longrightarrow 0$$

of $S^{\bullet}(V)$-modules (notice that the sheafification of $\mathbf{k}$ is 0).

Now, for every $i \geq 0$, tensoring the above complex by $\mathcal{F}(1-i)$ we get an exact sequence

$$0 \longrightarrow \wedge^{r+1} V \otimes \mathcal{F}(-r-i) \longrightarrow \wedge^r V \otimes \mathcal{F}(-r+1-i) \longrightarrow \ldots \longrightarrow V \otimes \mathcal{F}(-i) \longrightarrow \mathcal{F}(1-i) \longrightarrow 0$$

Now, fix a finite open affine cover $\mathcal{U}$ of $\mathbb{P}(V)$ and consider the first-quadrant double complex

$$E^{p,q} \stackrel{\text{def}}{=} C^q(\wedge^{r+1-p} V \otimes \mathcal{F}(-r-i+p), \mathcal{U})$$

where the horizontal maps are induced by the exact sequence above and the vertical maps are the differentials in the Cech complex. Since that the rows of this complex are exact, the spectral sequence that starts with the vertical differentials abuts to zero.

At the first step of this spectral sequence we find ${}_{\uparrow}E_1^{p,q} = \wedge^{r+1-p} V \otimes H^q(\mathbb{P}(V), \mathcal{F}(-r-i+p))$ and by hypothesis we have that ${}_{\uparrow}E_1^{p,r+i-p} = 0$ for all $p < r+i$.

Now, taking $i = 0$, this implies that ${}_{\uparrow}E_2^{0,r+1} = {}_{\uparrow}E_{\infty}^{0,r+1} = 0$ but by definition we see that

$${}_{\uparrow}E_2^{0,r+1} = \text{Coker}\,(V \otimes H^0(\mathbb{P}(V), \mathcal{F}) \longrightarrow H^0(\mathbb{P}(V), \mathcal{F}(1)))$$

and this proves point (3).

Instead, taking $i > 0$ we see that this implies ${}_{\uparrow}E_1^{r+1,i} = {}_{\uparrow}E_\infty^{r+1,i} = 0$, that is $H^i(\mathbb{P}(V), \mathcal{F}(1-i)) = 0$, and this proves point (1).

For the inductive step, take $h > 1$ and suppose that the thesis is true for every $0 \leq s < h$. Then $\mathcal{F}$ is $h-1$-regular, that is $\mathcal{F}(h-1)$ is 0-regular, and by what we have proved before we know that $\mathcal{F}(h-1)$ is 1-regular, that is $\mathcal{F}$ is h-regular.

For the surjectivity of the multiplication map, observe that by inductive hypothesis we know that the map

$$H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(h-1)) \otimes H^0(\mathbb{P}(V), \mathcal{F}) \longrightarrow H^0(\mathbb{P}(V), \mathcal{F}(h-1))$$

is surjective, so that the tensored map

$$H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(1)) \otimes H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(h-1)) \otimes H^0(\mathbb{P}(V), \mathcal{F}) \longrightarrow H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(1)) \otimes H^0(\mathbb{P}(V), \mathcal{F}(h))$$

is surjective as well. Now, we know that $\mathcal{F}(h)$ is 0-regular by inductive hypothesis, and from what we have proved before it follows that the map $H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(1)) \otimes H^0(\mathbb{P}(V), \mathcal{F}(h)) \longrightarrow H^0(\mathbb{P}(V), \mathcal{F}(h+1))$ is surjective. To conclude it is enough to observe that the resulting surjective map

$$H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(1)) \otimes H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(h-1)) \otimes H^0(\mathbb{P}(V), \mathcal{F}) \longrightarrow H^0(\mathbb{P}(V), \mathcal{F}(h))$$

factors through

$$H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(1)) \otimes H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(h-1)) \otimes H^0(\mathbb{P}(V), \mathcal{F}) \longrightarrow H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(h)) \otimes H^0(\mathbb{P}(V), \mathcal{F}) \longrightarrow H^0(\mathbb{P}(V), \mathcal{F}(h))$$

so that the last map is surjective. □

Thanks to this theorem, we can give a meaningful definition of the regularity of a sheaf:

Definition 2.4.2 (Castelnuovo-Mumford's regularity)**.** Let $\mathcal{F}$ be a coherent sheaf on a projective space $\mathbb{P}(V)$. Then we define the **Castelnuovo-Mumford regularity** of $\mathcal{F}$ as

$$\operatorname{reg}(\mathcal{F}) \stackrel{\text{def}}{=} \min \{ m \in \mathbb{Z} \mid \mathcal{F} \text{ is } m\text{-regular } \}$$

Example 2.4.1. Let V be a vector space over $\mathbf{k}$ of dimension $r+1$ and let $d \geq 1$ be a positive integer. We want to compute the regularity of the line bundle $\mathcal{O}_{\mathbb{P}}(d)$: we need to find the smallest $m \in \mathbb{Z}$ such that

$$H^i(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(d+m-i)) = 0 \qquad \text{for all } i > 0$$

and since line bundles on projective spaces do not have any intermediate cohomology, this is equivalent to

$$H^r(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(d+m-r)) = H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(-d-m-1))^\vee = 0$$

and now it is easy to see that $\operatorname{reg} \mathcal{O}_{\mathbb{P}}(d) = -d$.

Example 2.4.2. Let V be as before. We want to study the Castelnuovo-Mumford regularity of the syzygy bundle corresponding to $\mathcal{O}_{\mathbb{P}}(d)$, that is, the vector bundle M_d defined by the exact sequence

$$0 \longrightarrow M_d \longrightarrow S^d V \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(d) \longrightarrow 0$$

First, we note that M_d is not 0-regular: indeed, tensoring the above exact sequence by $\mathcal{O}_{\mathbb{P}}(-1)$ and taking the induced exact sequence in cohomology we get an exact sequence

$$0 \longrightarrow S^{d-1}V \longrightarrow H^1(\mathbb{P}(V), M_d(-1)) \longrightarrow S^d V \otimes H^1(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(-1))$$

Now we observe that $H^1(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(-1))$ is always zero: this is clear if $r > 1$, as line bundles on projective space do not have any intermediate cohomology, and for $r = 1$ we see that $H^1(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(-1)) = H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(-1))^\vee = 0$. Hence, this gives $H^1(\mathbb{P}(V), M_d(-1)) \cong S^{d-1}V$ and this is always different from zero.

Let's check whether M_d is 1-regular: for every $i > 0$ we have the exact sequence

$$0 \longrightarrow M_d(1-i) \longrightarrow S^d V \otimes \mathcal{O}_{\mathbb{P}}(1-i) \longrightarrow \mathcal{O}_{\mathbb{P}}(d+1-i) \longrightarrow 0$$

that induces the exact sequence in cohomology

$$H^{i-1}(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(d+1-i)) \longrightarrow H^i(\mathbb{P}(V), M_d(1-i)) \longrightarrow S^d V \otimes H^i(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(1-i)) \longrightarrow 0$$

and now it is easy to check that $H^{i-1}(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(d+1-i)) = H^i(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(1-i)) = 0$ so that $H^i(\mathbb{P}(V), M_d(1-i)) = 0$ as well. This shows that $\operatorname{reg} M_d = 1$.

The definition of regularity and the above theorem can be generalized as follows:

Definition 2.4.3 (Regularity w.r.t. an ample line bundle)**.** Let X be a projective variety, L an ample globally generated line bundle on X and $\mathcal{F}$ a coherent sheaf on X. For every fixed integer $m \in \mathbb{Z}$ we say that $\mathcal{F}$ is m-regular with respect to L if

$$H^i(X, \mathcal{F} \otimes L^{m-i}) = 0 \qquad \text{for all } i > 0$$

Remark 2.4.2. Take $X = \mathbb{P}(V)$ and let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}(V)$. Then saying that $\mathcal{F}$ is m-regular means precisely that $\mathcal{F}$ is m-regular w.r.t the line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$.

In order to generalize the above result we need a small lemma:

Lemma 2.4.1. *Let X be a projective variety of finite type over* $\mathbf{k}$*, L a globally generated line bundle on X and $\phi_L \colon X \longrightarrow \mathbb{P}(H^0(X, L))$ the morphism associated to the complete linear system of L. Then L is ample if and only if ϕ_L is finite.*

Proof. Recall that if $f \colon X \longrightarrow Y$ is a finite morphism of projective varieties and if M is an ample line bundle on Y, then f^*M is an ample line bundle on X (see for example [Har77]). In particular, if ϕ_L is finite, it is clear that L is ample, since $L = \phi_L^* \mathcal{O}_{\mathbb{P}}(1)$.

Conversely, suppose that ϕ_L is not finite, then there must be a closed subscheme $Z \subseteq X$ of positive dimension that is contracted to a point by ϕ_L (since a projective morphism is finite if and only if it has finite fibers). But then the line bundle $L_{|Z}$ is trivial, and since the closed embedding $Z \subseteq X$ is finite, we see that L cannot be ample. □

Remark 2.4.3. Let X be a projective variety, L an ample globally generated line bundle on X and $\mathcal{F}$ a coherent sheaf on X. Then $\mathcal{F}$ is m-regular w.r.t. L if and only if $\phi_{L*}\mathcal{F}$ is m-regular on $\mathbb{P} = \mathbb{P}(H^0(X, L))$.

Indeed, since the map $\phi_L \colon X \longrightarrow \mathbb{P}$ is finite, hence affine, we see that

$$H^i(\mathbb{P}, \phi_{L*}\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}}(m-i)) = H^i(\mathbb{P}, \phi_{L*}(\mathcal{F} \otimes \phi_L^* \mathcal{O}_{\mathbb{P}}(m-i))) = H^i(X, \mathcal{F} \otimes L^{m-i})$$

Theorem 2.4.2 (Mumford's theorem on regularity - II)**.** *Let X be a projective variety, L an ample globally generated line bundle on X and $\mathcal{F}$ a coherent sheaf on X. Suppose that $\mathcal{F}$ is m-regular w.r.t. L. Then for every $h \geq 0$ one has that*

1. *$\mathcal{F}$ is $(m+h)$-regular w.r.t. L.*
2. *$\mathcal{F} \otimes L^{m+h}$ is generated by global sections.*
3. *The multiplication map $H^0(\mathbb{P}(V), L^h) \otimes H^0(\mathbb{P}(V), \mathcal{F} \otimes L^m) \longrightarrow H^0(\mathbb{P}(V), \mathcal{F} \otimes L^{m+h})$ is surjective.*

Proof. It is enough to apply the above remark and Theorem 2.4.1 to the sheaf $\phi_{L*}\mathcal{F}$ on $\mathbb{P}(H^0(X, L))$. □

Definition 2.4.4. Let X be a projective variety, L an ample globally generated line bundle on X and $\mathcal{F}$ a coherent sheaf on X. We define the **Castelnuovo-Mumford regularity** of $\mathcal{F}$ w.r.t L as

$$\operatorname{reg}_L(\mathcal{F}) = \min \{ m \mid \mathcal{F} \text{ is } m-\text{regular w.r.t. } L \}$$

Now, the proof of Theorem 2.4.1 shows that the regularity of a sheaf is strongly linked to the Koszul complex. In fact, the regularity can be read off directly from the Koszul cohomology of $\mathcal{F}$.

Theorem 2.4.3. *Let X be a projective variety, L an ample globally generated line bundle on X and $\mathcal{F}$ be a coherent sheaf on X. Suppose that $\mathcal{F}$ is $(m+1)$-regular w.r.t. L, then it is m-regular w.r.t. L if and only if $K_{p,m+1}(X, L; \mathcal{F}) = 0$ for all p. In particular*

$$reg_L(\mathcal{F}) = \max\{\, m \mid K_{p,m}(X, L; \mathcal{F}) \neq 0 \text{ for a certain } p \,\}$$

Proof. Suppose first that $\mathcal{F}$ is m-regular w.r.t L. Then $K_{p,m+1}(X, L; \mathcal{F}) = 0$ for all p thanks to Corollary 2.1.5.

Conversely, suppose that $\mathcal{F}$ is $(m+1)$-regular w.r.t L and that $K_{p,m+1}(X, L; \mathcal{F}) = 0$ for all p. Then observe that, thanks to what we have already proved and Theorem 2.4.2, we have $K_{p,q}(X, L; \mathcal{F}) = 0$ for all $q \geq m+1$.

Now, set $V = H^0(X, L)$, then we know that we have an exact sequence of sheaves on $\mathbb{P}(V)$

$$0 \longrightarrow L_{r+1} \longrightarrow L_r \longrightarrow \ldots \longrightarrow L_0 \longrightarrow L_{-1} \longrightarrow 0$$

where

$$L_p = \bigoplus_{q\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}(V)}(-p-q) \otimes K_{p,q}(X, L; \mathcal{F}) \qquad \text{for all } p \geq 0, \qquad L_{-1} = \phi_{L*}\mathcal{F}$$

and $r+1 = \dim V$. Now, for every $i > 0$ we tensor this sequence by $\mathcal{O}_{\mathbb{P}(V)}(m-i)$ so that we get the exact sequence

$$0 \longrightarrow L_{r+1}(m-i) \longrightarrow L_r(m-i) \longrightarrow \ldots \longrightarrow L_0(m-i) \longrightarrow L_{-1}(m-i) \longrightarrow 0$$

Take a finite affine open cover $\mathcal{U}$ of $\mathbb{P}(V)$ and consider the first-quadrant double complex

$$E^{p,q} = C^q(L_{r+1-p}(m-i), \mathcal{U})$$

where the horizontal maps are induced by the previous exact sequence and the vertical maps are the differentials in the Cech complex. This complex has exact rows, so that the spectral sequence of the vertical maps abuts to zero. Computing the sequence at the first step gives

$${}_{\uparrow}E_1^{p,q} = H^q(\mathbb{P}(V), L_{r+1-p}(m-i))$$

In particular ${}_{\uparrow}E_1^{r+2,q} = H^q(\mathbb{P}(V), \phi_{L*}\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(V)}(m-i)) = H^q(X, \mathcal{F} \otimes L^{m-i})$.

Now, to show that $H^i(X, \mathcal{F} \otimes L^{m-i}) = 0$ it is enough to prove that

$${}_{\uparrow}E_1^{r+1+i-q,q} = H^q(\mathbb{P}(V), L_{q-i}(m-i)) = 0 \qquad \text{for all } q \geq i$$

but since line bundles on projective spaces do not have any intermediate cohomology, it is enough to prove that $H^r(\mathbb{P}(V), L_{r-i}(m-i)) = 0$. We see that

$$\begin{aligned} H^r(\mathbb{P}(V), L_{r-i}(m-i)) &= \bigoplus_{q\in\mathbb{Z}} H^r(\mathbb{P}(V), \mathcal{O}(m-r-q)) \otimes K_{r-i,q}(X, L; \mathcal{F}) \\ &\cong \bigoplus_{q\in\mathbb{Z}} H^0(\mathbb{P}(V), \mathcal{O}(q-m-1))^\vee \otimes K_{r-i,q}(X, L; \mathcal{F}) \end{aligned}$$

and since $K_{p,q}(X, L; \mathcal{F}) = 0$ for all $q \geq m+1$ and for all p, we get the desired vanishing. □

Remark 2.4.4. This theorem says that the regularity of a sheaf $\mathcal{F}$ is the highest index of a non zero row in the Betti table of $\mathcal{F}$.

As an application of the above result we can study the regularity of a tensor product:

Proposition 2.4.1. *Let X be a projective variety, L an ample globally generated line bundle, $\mathcal{F}$ a coherent sheaf on X and E a vector bundle on X. Suppose that $\mathcal{F}$ is m-regular w.r.t. L and E is n-regular w.r.t. L, then $\mathcal{F} \otimes E$ is $(n+m)$-regular w.r.t L.*

Proof. Substituting $\mathcal{F}$ by $\mathcal{F} \otimes L^m$ and E by $E \otimes L^n$ we can suppose that $m = n = 0$.

Exploiting the fact that the map $\phi_L \colon X \longrightarrow \mathbb{P}(H^0(X, L))$ is finite, we can suppose that $X = \mathbb{P}(V)$ for a certain vector space of dimension $r+1$ and that $L = \mathcal{O}_{\mathbb{P}(V)}(1)$. Then the augmented minimal free resolution of E gives us an exact sequence

$$0 \longrightarrow E_{r+1} \longrightarrow E_r \longrightarrow \dots \longrightarrow E_0 \longrightarrow E_{-1} = E \longrightarrow 0$$

where $E_p = \oplus_q \mathcal{O}_{\mathbb{P}(V)}(-p-q) \otimes K_{p,q}(\mathbb{P}(V), E)$. Then, for every $i > 0$ we can tensor this exact sequence by $\mathcal{F}(-i)$ to get an exact sequence

$$0 \longrightarrow \mathcal{F} \otimes E_{r+1}(-i) \longrightarrow \dots \longrightarrow \mathcal{F} \otimes E_0(-i) \longrightarrow \mathcal{F} \otimes E_{-1}(-i) \longrightarrow 0$$

Now, take an affine and finite open cover $\mathcal{U}$ of $\mathbb{P}(V)$ and consider the first-quadrant double complex

$$E^{p,q} = C^q(\mathcal{F} \otimes E_{r+1-p}, \mathcal{U})$$

where the horizontal differentials are induced by the above exact sequence and the vertical differentials are the differential in the Cech complex. Since the rows are exact, the spectral sequence starting with the vertical maps abuts to 0. Computing this sequence at the first step gives

$${}_{\uparrow}E_1^{p,q} = H^q(\mathbb{P}(V), \mathcal{F} \otimes E_{r+1-p}(-i))$$

in particular, we see that ${}_{\uparrow}E_1^{r+2,i} = H^i(\mathbb{P}(V), \mathcal{F} \otimes E(-i))$ and then, to show that this is zero it is enough to prove that ${}_{\uparrow}E_1^{r+1,i} = 0$ and ${}_{\uparrow}E_1^{r+2-j,j} = 0$ for all $j \geq i+1$.

For the first vanishing observe that

$${}_{\uparrow}E_1^{r+1,i} = H^i(\mathbb{P}(V), \mathcal{F} \otimes E_0(-i)) = \bigoplus_{q \in \mathbb{Z}} H^i(\mathbb{P}(V), \mathcal{F}(-q-i)) \otimes K_{0,q}(E, V)$$

and now $K_{0,q}(E, V) = 0$ for every $q > 0$ since E is 0-regular (cfr. Theorem 2.4.3), whereas $H^i(\mathbb{P}(V), \mathcal{F}(-q-i)) = 0$ for every $q \leq 0$ since $\mathcal{F}$ is 0-regular. For the other vanishings we see that for every $j \geq i+1$ we have

$${}_{\uparrow}E_1^{r+2-j,j} = H^j(\mathbb{P}(V), \mathcal{F} \otimes E_{j-1}(-i)) = \bigoplus_{q \in \mathbb{Z}} H^i(\mathbb{P}(V), \mathcal{F}(1-j-i-q)) \otimes K_{j-1,q}(E, V)$$

and we can use the same reasoning as before. □

Corollary 2.4.1. *Let X be a projective variety, L an ample globally generated line bundle on X and E a vector bundle on X that is m-regular w.r.t. L. Then $E^{\otimes p}, \wedge^p E, S^p E$ are pm-regular w.r.t. L.*

Proof. Applying repeatedly Proposition 2.4.1 yields that $E^{\otimes p}$ is pm-regular. Now, we just need to observe that we can regard $\wedge^p E$ and $S^p E$ as direct summands of $E^{\otimes p}$ and the thesis follows. □

2.5 Vanishing theorems

As for standard sheaf cohomology, it is of great interest to know whether a certain Koszul cohomology group is zero or nonzero. In this section we want to present two useful vanishing statements for Koszul cohomology: the first one is quite generic and due to Green [Gre].

Theorem 2.5.1. *Let X be an irreducible projective variety, L a line bundle on X, $V \subseteq H^0(X, L)$ a subspace and E a vector bundle on X such that $h^0(X, E \otimes L^q) \leq p$. Then*

$$Ker\ (\wedge^p V \otimes H^0(X, E \otimes L^q) \xrightarrow{d} \wedge^{p-1} V \otimes H^0(X, E \otimes L^{q+1})) = 0$$

and in particular

$$K_{p,q}(X, L, V; E) = 0$$

Proof. Substituting E with $E \otimes L^q$ we can suppose that $q = 0$. Suppose $\dim_{\mathbf{k}} V = r+1$ and take $r+1$ general points $P_0, \dots, P_r \in X$. Choose a basis $v_0, \dots, v_r$ of V such that $v_i(P_j) = 0$ for every $i \neq j$ and $v_i(P_i) \neq 0$. Then every element $\alpha \in \wedge^p V \otimes H^0(X, E)$ can be written in the form

$$\alpha = \sum_{0 \leq i_1 < \dots < i_p \leq r} v_{i_1} \wedge \dots \wedge v_{i_p} \otimes \alpha_{i_1 \dots i_p}$$

where $\alpha_{i_1,\dots,i_p} \in H^0(X, E)$. Now, suppose that $\alpha \in Ker(\wedge^p V \otimes H^0(X, E) \xrightarrow{d} \wedge^{p-1} V \otimes H^0(X, E \otimes L))$, then

$$0 = d(\alpha) = \sum_{0 \leq i_1 < \dots < i_p \leq r} \left[\sum_{j=1}^{p} (-1)^{j+1} v_{i_1} \wedge \dots \wedge \widehat{v_{i_j}} \wedge \dots \wedge v_{i_p} \otimes v_{i_j} \alpha_{i_1 \dots i_p} \right]$$

$$= \sum_{0 \leq h_1 < \dots < h_{p-1} \leq r} v_{h_1} \wedge \dots \wedge v_{h_{p-1}} \otimes \left[\sum_{0 \leq j < h_1} v_j \alpha_{j h_1 \dots h_{p-1}} + \dots + (-1)^{p-1} \sum_{h_{p-1} < j \leq r} v_j \alpha_{h_1 \dots h_{p-1} j} \right]$$

For every sequence $(i_1, \dots, i_p) \in \{0, \dots, r\}^p$ we define $\alpha_{i_1,\dots,i_p} = 0$ if $i_h = i_k$ for certain $h \neq k$, and $\alpha_{i_1,\dots,i_p} = \operatorname{sgn}(\sigma) \alpha_{i_{\sigma(1)} \dots i_{\sigma(p)}}$ where $\sigma \colon \{1, \dots, p\} \longrightarrow \{1, \dots, p\}$ is the unique permutation such that $i_{\sigma(1)} < \dots < i_{\sigma(p)}$. With this notation, we can rephrase the condition $d(\alpha) = 0$ as

$$\sum_{j=0}^{r} v_j \alpha_{j h_1 \dots h_{p-1}} = 0 \qquad \text{for all } 0 \leq h_1 < \dots < h_{p-1} \leq r$$

and evaluating this at the point P_j tells us that $\alpha_{j h_1 \dots h_{p-1}}(P_j) = 0$ for every $j = 0, \dots r$, that can also be written as

$$\alpha_{i_1 \dots i_p}(P_{i_j}) = 0 \qquad \text{for all } j = 1, \dots, p$$

But, now, since the points P_j were chosen generally, and $h^0(X, E) \leq p$ by hypothesis, we know that the evaluation map

$$H^0(X, E) \longrightarrow \bigoplus_{j=0}^{r} E \otimes \kappa(P_j)$$

is injective, so that $\alpha_{i_1 \dots i_p} = 0$ and $\alpha = 0$ as desired. □

The second statement, again due to Green [Gre84b], is about projective spaces.

Theorem 2.5.2. *Let V be a vector space over $\mathbf{k}$ of dimension $n+1$, let $d > 0, k \geq 0$ be two integers and consider a base-point free subspace $W \subseteq S^d V = H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(d))$ of codimension c. Then*

$$K_{p,q}(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(d), W; \mathcal{O}_{\mathbb{P}}(k)) = 0 \qquad \textit{for all } k + (q-1)d \geq p + c$$

Proof. Thanks to Corollary 2.1.4, we see that

$$K_{p,q}(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(d), W; \mathcal{O}_{\mathbb{P}}(k)) \cong H^1(\mathbb{P}(V), \wedge^{p+1} M_W \otimes \mathcal{O}_{\mathbb{P}}((q-1)d + k))$$

Indeed, thanks to the fact that line bundles on projective space do not have any intermediate cohomology, it is enough to check that, if $n = 1$ then $H^1(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}((q-1)d + k)) = 0$, but

$$H^1(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}((q-1)d + k)) \cong H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(-(q-1)d - k - 2))^{\vee} = 0$$

since $(q-1)d + k \geq p + c \geq 0$.

Now, to conclude, it is enough to show that $\wedge^{p+1} M_W$ is $((q-1)d + k + 1)$-regular on $\mathbb{P}(V)$. To this end, choose a flag of linear subspaces

$$W = W_c \subseteq W_{c-1} \subseteq \dots \subseteq W_1 \subseteq W_0 = S^d V$$

such that $\dim W_{i-1}/W_i = 1$, and for every $i = 0, \dots, c$ let M_i be defined by the exact sequence

$$0 \longrightarrow M_i \longrightarrow W_i \otimes \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{O}_{\mathbb{P}}(d) \longrightarrow 0$$

we are going to show by induction that $\wedge^{p+1} M_i$ is $(p+1+i)$-regular, for all $p \geq 0, i = 0, \dots, c$ and this will give us the thesis.

For $i = 0$, we already know from Example 2.4.2 that M_0 is 1-regular, so that $\wedge^{p+1} M_0$ is $(p+1)$-regular thanks to Lemma 2.4.1.

For the inductive step, suppose that $i > 0$: then we have an exact sequence

$$0 \longrightarrow M_i \longrightarrow M_{i-1} \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow 0$$

that yields another exact sequence

$$0 \longrightarrow \wedge^{p+2} M_i \longrightarrow \wedge^{p+2} M_{i-1} \longrightarrow \wedge^{p+1} M_i \longrightarrow 0$$

Now we prove by descending induction on p that $\wedge^{p+1} M_i$ is $(p+i+1)$-regular. If $p+1 > \operatorname{rk} M_i$ this is obvious, and for the inductive step, we observe that the above exact sequence gives the exact sequence in cohomology

$$H^i(\mathbb{P}(V), \wedge^{p+2} M_{i-1}(p+1)) \longrightarrow H^i(\mathbb{P}(V), \wedge^{p+1} M_i(p+1)) \longrightarrow H^{i+1}(\mathbb{P}(V), \wedge^{p+2} M_i(p+1))$$

Now, $\wedge^{p+2} M_i$ is $(p+2+i)$-regular thanks to the inductive hypothesis on i, so that $H^i(\mathbb{P}(V), \wedge^{p+2} M_{i-1}(p+1)) = 0$, whereas $\wedge^{p+2} M_{i-1}$ is $(p+2+i)$-regular, thanks to the inductive hypothesis on p, so that $H^{i+1}(\mathbb{P}(V), \wedge^{p+2} M_i(p+1)) = 0$. □

2.6 Property N_p

Now we are going to introduce a measure of "niceness" for syzygies, called property N_p. In general, let us consider a connected projective variety X with an embedding $X \hookrightarrow \mathbb{P} = \mathbb{P}(H^0(X, L))$ determined by a very ample line bundle L. What is the best possible minimal free resolution of (X, L), that is, the minimal free resolution of the $S = S^\bullet(H^0(X, L))$ graded module $R(X, L) = \oplus_{q \in \mathbb{Z}} H^0(X, L^q)$?

First, we would like $R(X, L)$ to coincide with the coordinate ring of X in $\mathbb{P}(H^0(X, L))$, that is $S_X \stackrel{\text{def}}{=} S/I_X$, where I_X is the homogeneous ideal of $X \subseteq \mathbb{P}$. Taking global sections in the exact sequences

$$0 \longrightarrow \mathcal{I}_X(q) \longrightarrow \mathcal{O}_{\mathbb{P}}(q) \longrightarrow \mathcal{O}_X(q) \longrightarrow 0$$

we get an exact sequence

$$0 \longrightarrow I_X \longrightarrow S \longrightarrow R(X, L)$$

so that $R(X, L) = S_X$ if and only if all the maps $S^q(H^0(X, L)) \longrightarrow H^0(X, L^q)$ are surjective. This property has a name:

Definition 2.6.1 (Projective normality)**.** Let X be a projective variety and L an ample line bundle on X. Then we say that (X, L) is **projectively normal** if the natural multiplication maps $S^q(H^0(X, L)) \longrightarrow H^0(X, L^q)$ are surjective for all $q \geq 0$.

Now, the condition of projective normality on (X, L) can be also rephrased by saying that the multiplication maps

$$H^0(X, L) \otimes H^0(X, L^{q-1}) \longrightarrow H^0(X, L^q)$$

are surjective for all $q \geq 1$, but by definition of Koszul cohomology this means that $K_{0,q}(X, L) = 0$ for all $q > 0$. Since X is connected, it follows that $K_{0,1}(X, L) = 0$ and this brings us to the following definition.

Definition 2.6.2 (Property N_0)**.** Let X be a connected projective variety and let L be an ample line bundle on X. Then we say that L satisfies property N_0 if $K_{0,q}(X, L) = 0$ for all $q \geq 2$.

Remark *2.6.1.* Let X be a connected projective variety and L an ample line bundle on X. Then (X, L) is projectively normal if and only if L satisfies property N_0. An equivalent characterization is that the augmented minimal free resolution of $R(X, L)$ starts as

$$S \longrightarrow R(X, L) \longrightarrow 0$$

Indeed, since X is connected it follows that $K_{0,0}(X, L) = \mathbf{k}$, whereas, since L is ample, it follows that $K_{0,q}(X, L) = 0$ for all $q < 0$.

Now, suppose that (X, L) satisfies property N_0. Then the next step in the minimal free resolution tells us about the generators of the ideal I_X of X. Observe that there cannot be any elements of degree 1 in I_X, as any embedding determined by a complete linear system is nondegenerate. Thus, the simplest possible case is that I_X is generated by elements of degree 2, that is, X is an intersection of quadrics. In this case, the augmented minimal free resolution of $R(X, L)$ continues as

$$S(-2) \otimes K_{1,1}(X, L) \longrightarrow S \longrightarrow R(X, L) \longrightarrow 0$$

Then we make the following definition:

Definition 2.6.3 (Property N_1)**.** Let X be a connected projective variety and let L be an ample line bundle on X. We say that L has property N_1 if it has property N_0 and moreover $K_{1,q}(X, L) = 0$ for all $q \geq 2$.

Remark *2.6.2.* By the above discussion, an ample line bundle X has property N_1 if and only if the minimal free resolution of $R(X, L)$ starts as

$$S(-2) \otimes K_{1,1}(X, L) \longrightarrow S \longrightarrow 0$$

that is, (X, L) is projectively normal and the ideal of X is generated by quadrics.

Now, suppose that L satisfies property N_1. Then the simplest possible case for the next step in the minimal free resolution of $R(X, L)$ is

$$S(-3) \otimes K_{2,1}(X, L) \longrightarrow S(-2) \otimes K_{1,1}(X, L) \longrightarrow S \longrightarrow 0$$

this means that all the relations between the generators of the ideal I_X are generated by linear relations. In terms of Koszul cohomology we can write this as $K_{2,q}(X, L) = 0$ for all $q \geq 2$, and in this case we say that L has property N_2.

Now it is obvious how to proceed:

Definition 2.6.4 (Property N_p)**.** Let X be a connected projective variety and L an ample line bundle on X. Then for any positive integer $p \geq 0$, we say that L has **property** N_p if

$$K_{h,q}(X, L) = 0 \qquad \text{for all } 0 \leq h \leq p, q \geq 2$$

Remark *2.6.3.* L has property N_p if and only if the minimal free resolution of $R(X, L)$ starts as

$$\ldots \longrightarrow S(-p-1) \otimes K_{p,1}(X, L) \longrightarrow S(-p) \otimes K_{p-1,1}(X, L) \longrightarrow \ldots \longrightarrow S(-2) \otimes K_{1,1}(X, L) \longrightarrow S \longrightarrow 0$$

This can be rephrased by saying that the Betti table of $R(X, L)$ has shape

	0	1	2	3	...	$p-1$	p	$p+1$	...
0	1	−	−	−	...	−	−	−	...
1	−	∗	∗	∗	...	∗	∗	∗	...
2	−	−	−	−	...	−	−	∗	...

and by definition, this could be also expressed in terms of the minimal free resolution of the sheaf $\mathcal{O}_X$ on $\mathbb{P}(H^0(X, L))$.

2.6.1 Arithmetically Cohen-Macaulay embeddings

The concept of projective normality can be used to study the Cohen-Macaulay property for coordinate rings of projective varieties.

Definition 2.6.5 (Arithmetically Cohen-Macaulay line bundles)**.** Let X be a projective variety and let L be a line bundle on X. We say that L is an **arithmetically Cohen-Macaulay** line bundle if $R(X,L)$ is a Cohen-Macaulay $S^\bullet(H^0(X,L))$-module.

Proposition 2.6.1. *Let X be an irreducible projective variety of dimension $\dim X = n$ and let L be a projectively normal very ample line bundle on X. Suppose that*

$$H^i(X,L^q) = 0 \qquad \text{for all } 0 < i < n \text{ and } q \in \mathbb{Z}$$

then L is arithmetically Cohen-Macaulay.

Proof. Since L is projectively normal and very ample, we see that $\dim R(X,L) = \dim S^\bullet(H^0(X,L))/I_X = n+1$. Hence, using the Auslander-Buchsbaum formula 1.2.5, we see that L is arithmetically Cohen-Macaulay if and only if

$$\operatorname{projdim} R(X,L) \leq h^0(X,L) - n - 1$$

that is, if and only if

$$K_{p,\bullet}(X,L) = 0 \qquad \text{for all } p \geq h^0(X,L) - n$$

Now, from Corollary 2.1.1, we see that for any $p \geq 0$ and $q \in \mathbb{Z}$ we have

$$K_{p,q}(X,L) = H^{n-1}(X, \wedge^{n-1+p} M_L \otimes L^{q-n+1})$$

Observe now that, $\operatorname{rk} M_L = h^0(X,L) - 1$, so that, if $p = h^0(X,L) - n$, then $H^{n-1}(X, \wedge^{n+1-p} M_L \otimes L^{q-n+1}) = H^{n-1}(X, L^{q-n}) = 0$,whereas, if $p > h^0(X,L) - n$ then $\wedge^{n-1+p} M_L = 0$. □

2.6.2 Property N_p for curves of high degree

As a first example, we study property N_p for smooth curves.

Let X be a connected smooth projective curve of genus g, and let L be a line bundle on X: it is a standard principle that the line bundle L gets "nicer" as its degree grows:

Proposition 2.6.2. *Let X be a smooth curve of genus g and let L be a line bundle on X of degree d. Then*

1. *If $d \geq 2g-1$ then $h^0(X,L) = d+1-g$ and $h^1(X,L) = 0$.*
2. *If $d \geq 2g$ then L is base-point-free.*
3. *If $d \geq 2g+1$ then L is very ample.*

Proof.
1. By Serre's Duality we know that $h^1(X,L) = h^0(X, \omega_X \otimes L^\vee)$ but this is zero since $\deg(\omega_X \otimes L^\vee) = 2g-2-d < 0$. Now, by Riemann-Roch we see that $h^0(X,L) = \chi(X,L) = d+1-g$.
2. We need to show that $h^0(X, L(-p)) = h^0(X,L) - 1$ for every closed point $p \in X$, but this follows at once from point 1.
3. We need to show that $h^0(X, L(-p-q)) = h^0(X,L) - 2$ for every two closed points $p,q \in X$, but this follows at once from point 1.

□

Then, it is quite natural to ask ourselves whether the line bundle L gets "nicer" also from a syzygy point of view. It turns out that this is precisely the case:

Theorem 2.6.1. *Let X be a smooth curve of genus g, let $k \geq 0$ be a nonnegative integer and let L be a line bundle on X of degree $\deg L = 2g+1+k$. Then L has property N_k.*

Proof. By definition, we need to check that $K_{p,q}(X,L) = 0$ for all $0 \le p \le k$ and $q \ge 2$.

Observe that by Riemann-Roch we have

$$h^0(X,L) = \deg(L) + 1 - g = 2g + 2 + k - g = g + 2 + k$$

and then Duality Theorem 2.3.1 tells us that

$$K_{p,q}(X,L) \cong K_{g+k-p,2-q}(X,L;K_X)^{\vee}$$

Now, by Vanishing Theorem 2.5.1, we know that $K_{g-1+k-p,2-q}(X,L;K_X) = 0$ as soon as

$$h^0(X, K_X \otimes L^{2-q}) \le g + k - p$$

In particular, we see that $\deg K_X \otimes L^{2-q} = (2g-2) + (2-q)(2g+k+1)$ so that $h^0(X, K_X \otimes L^{2-q}) = 0$ as soon as $q > 2$. The only case remaining is $q = 2$, but we see that $h^0(X, K_X) = g \le g + k - p$ for every $k \ge p$. □

Remark 2.6.4. The above result was proved in the case $k = 0$ by Castelnuovo [Cas93], Mattuck [Mat61] and Mumford [Mum]. The case $k = 1$ is due to Fujita [Fuj] and Saint-Donat [SD72]. Then Green understood that the property N_p was the right generalizations of these results and proved the above theorem in [Gre84a].

Remark 2.6.5. In the hypotheses of Theorem 2.6.1 , we see from Proposition 2.6.1 that the embedded curve $X \subseteq \mathbb{P}(H^0(X,L))$ is arithmetically Cohen-Macaulay, since the embedding is projectively normal and there is no intermediate cohomology on a curve. In particular, we know that

$$\operatorname{projdim}(R(X,L)) = h^0(X,L) - \dim R(X,L) = g + 2 + k - 2 = g + k$$

and by the above Theorem we can see the the Betti table of (X,L) has the shape

	0	1	2	3	...	$k-1$	k	$k+1$	...	$k+g$
0	1	–	–	–	...	–	–	–	...	–
1	–	$*$	$*$	$*$	...	$*$	$*$	?	...	?
2	–	–	–	–	...	–	–	?	...	?

where the asterisks stand for a nonzero element and the question marks stand for an element that could be either zero or nonzero.

2.6.3 Property N_p for Veronese embeddings

Another interesting case for studying property N_p is provided by Veronese embeddings.

Proposition 2.6.3. *Let V be a vector space over $\mathbf{k}$ of dimension $r+1$ and consider the line bundle $\mathcal{O}_{\mathbb{P}}(d)$ for $d \ge 2$. Then $(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d))$ has property N_d.*

Proof. By Theorem 2.5.2, we see that

$$K_{p,q}(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(d)) = 0$$

if $p \le (q-1)d$, and this is true for every $0 \le p \le d$ and $q \ge 2$. □

Remark 2.6.6. The above result is due to M. Green ([Gre84b]). In the paper [Rub04] E. Rubei showed that the Veronese embedding $\mathcal{O}_{\mathbb{P}^n}(3)$ satisfies property N_4 and this result has been improved by W. Bruns, A. Conca and T. Römer that in their paper [BCR11] proved that the Veronese embedding given by $\mathcal{O}_{\mathbb{P}}(d)$ satisfies property N_{d+1}.

We will see later that the minimal free resolution of the Veronese embeddings of $\mathbb{P}^1$ is explicitly known, and from this it can be seen that they satisfy property N_p for every p.

In the case of $\mathbb{P}^n$ with $n \geq 2$ this is not true anymore. To see this, we need a preliminary result about the structure of syzygies. Specifically, we consider the functor

$$\mathbf{K}_{p,q}(d) \colon \mathrm{Vec}_{\mathbf{k}} \longrightarrow \mathrm{Vec}_{\mathbf{k}} \qquad V \mapsto K_{p,q}(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(d))$$

then, from [Rub04] or [Sno13] one can see that

$$\mathbf{K}_{p,q}(d) = \bigoplus_{\lambda \vdash (p+q)d} M_\lambda \otimes_{\mathbf{k}} \mathbf{S}^\lambda$$

where S^λ is the schur functor associated to the partition λ and $M_\lambda = M_\lambda(p,q;d)$ is a vector space that counts multiplicities. In particular we see that

Corollary 2.6.1. *If $K_{p,q}(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d)) \neq 0$ then $K_{p,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \neq 0$ for all $n \geq m$.*

Proof. Immediate from the above remark. □

Now we can prove that we have an upper bound for property N_p, due to G. Ottaviani and R. Paoletti [OP01]:

Proposition 2.6.4. *For every positive integer $d \geq 1$ set $r_d = \dim \mathbb{P}(S^d(\mathbf{k}^3)) = \dim S^d(\mathbf{k}^3) - 1$. Then for every vector space V of dimension greater or equal than 3 we have*

$$K_{p,2}(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(d)) \neq 0 \qquad \textit{for all } 3d-2 \leq p \leq r_d - 2$$

Proof. Thanks to Corollary 2.6.1, we can suppose that $\dim V = 3$. First we observe that there is nothing to prove for $d = 1,2$, hence we can suppose that $d \geq 3$. By Duality Theorem 2.3.1, we have that

$$K_{p,2}(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(d)) \cong K_{r_d-2-p,1}(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(d); \mathcal{O}_{\mathbb{P}}(-3))^\vee$$

and by definition this is the cohomology at the middle term of the complex

$$0 \longrightarrow \wedge^{r_d-2-p}(S^dV) \otimes S^{d-3}V \xrightarrow{d} \wedge^{r_d-3-p}(S^dV) \otimes S^{2d-3}V$$

Now, suppose that $p \leq r_d - 2$ (otherwise $\wedge^{r_d-2-p}(S^dV) = 0$) and consider an element $\alpha \in \wedge^{r_d-2-p}(S^dV) \otimes S^{d-3}V$ of the form

$$\alpha = \sum_{i=0}^{r_d-2-p} (-1)^i f_0 \wedge \cdots \wedge \widehat{f_i} \wedge \cdots \wedge f_{r_d-2-p} \otimes g_j$$

for certain linearly independent elements $f_i \in S^dV$ and nonzero elements $g_j \in S^{d-3}V$. Then we see that $\alpha \neq 0$ and if $f_ig_j = f_jg_i$ for all i,j, then $d(\alpha) = 0$. In particular, this happens when $f_i = sg_i$ for a certain $s \in S^3V$. Thus, to get a nonzero element in $K_{r_d-2-p,1}(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}}(d); \mathcal{O}_{\mathbb{P}}(-3))$ it suffices to get $r_d - 1 - p$ linearly independent elements in $S^{d-3}V$ and this is possible as soon as $p \geq 3d-2$. □

Remark 2.6.7. After this result, Ottaviani and Paoletti conjectured that $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ always satisfies property N_{3d-3}.

2.7 Examples of Betti tables

We want to present some examples of Betti tables and minimal free resolutions in geometrical situations.

2.7.1 Rational normal curves

Let V be a vector space of dimension 2 over $\mathbf{k}$ and let $\mathbb{P}^1 = \mathbb{P}(V)$. We want to study the Betti numbers $K_{p,q}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ and actually this is one of the few cases in which these are known explicitly. Since it is a basic example, we are going to give several different methods that can be used to compute this Betti table.

Computation by syzygy bundles

Consider for each $d \geq 0$ the syzygy bundle M_d defined by the exact sequence on $\mathbb{P}^1$:

$$0 \longrightarrow M_d \longrightarrow S^dV \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d) \longrightarrow 0$$

then we know from Example 2.1.1 that

$$K_{p,q}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \cong H^q(\mathbb{P}^1, \wedge^{p+q} M_d)$$

Now, the by the Grothendieck-Segre Theorem, we know that every vector bundle on $\mathbb{P}^1$ splits as the direct sum of line bundles: to determine this splitting for M_d it is enough to consider the exact sequence in cohomology associated to the above exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^1, M_d) \longrightarrow S^dV \xrightarrow{\text{id}} S^dV \longrightarrow H^1(\mathbb{P}^1, M_d) \longrightarrow 0$$

This shows that $H^0(\mathbb{P}^1, M_d) = H^1(\mathbb{P}^1, M_d) = 0$ and then it is clear that $M_d \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus d}$.

We can also determine completely M_d as an homogeneous bundle: indeed, twisting the above exact sequence by $\mathcal{O}_{\mathbb{P}^1}(1)$ and taking global sections we find the exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^1, M_d(1)) \longrightarrow S^dV \otimes V \longrightarrow S^{d+1}V$$

and a simple application of Pieri's rule shows that $H^0(\mathbb{P}^1, M_d(1)) \cong S^{d-1}V$ as an $SL(V)$-module. Hence

$$M_d = S^{d-1}V \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \qquad \wedge^{p+q} M_d = \wedge^{p+q}(S^{d-1}V) \otimes \mathcal{O}_{\mathbb{P}^1}(-p-q)$$

Now it is easy to compute the Betti numbers: indeed we have

- $K_{p,0}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = \wedge^p(S^{d-1}V) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-p)) = \begin{cases} \mathbf{k} & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 \end{cases}$
- $K_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = \wedge^{p+1}(S^{d-1}V) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-p-1)) = \begin{cases} \wedge^{p+1}(S^{d-1}V) \otimes S^{p-1}V & \text{if } 1 \leq p \leq d \\ 0 & \text{otherwise} \end{cases}$
- $K_{p,q}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = 0$ for all $q \geq 2, p \geq 0$.

And the Betti table of $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ is the following

	0	1	2	...	p	$p+1$	...	$d-1$
0	1	–	–	...	–	–	–	
1	–	$\binom{d}{2}$	$2\binom{d}{3}$	...	$p\binom{d}{p+1}$	$(p+1)\binom{d}{p+2}$	...	$d-1$

Computation from the shape of the Betti table

Another strategy is to compute the shape of the Betti table and then infer the values of the Betti numbers. To compute the shape of the Betti table, observe that $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ satisfies property N_{d-1} by Theorem 2.6.1 and, moreover we know that it is arithmetically Cohen-Macaulay since there is no intermediate cohomology on a curve. Hence, the projective dimension of $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ is precisely $d-1$ and then we know that the Betti table has the following shape

	0	1	2	...	p	$p+1$	...	$d-1$
0	1	–	–	...	–	–	–	
1	–	*	*	...	*	*	...	*

Now, using again Example 2.1.1, we see that $K_{p,q}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) \cong H^q(\mathbb{P}^1, \wedge^{p+q} M_d)$ and in particular

$$\chi(\mathbb{P}^1, \wedge^{p+1} M_d) = k_{p+1,0}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) - k_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$$

Now, the shape of the Betti table tells us that $k_{p+1,0}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = 0$ for all $p \geq 1$ so that

$$k_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = -\chi(\mathbb{P}^1, \wedge^{p+1} M_d) \qquad \text{for all } p \geq 1$$

and the Euler characteristic is easy to compute:

Lemma 2.7.1. *Let X be a smooth projective curve of genus g and let L be a globally generated line bundle on X with syzygy bundle M_L. Let $h^0(X, L) = r+1$ and $\deg L = d$. then*

$$\chi(X, \wedge^h M_L \otimes L^t) = \binom{r}{h}(td + 1 - g) - \binom{r-1}{h-1} d$$

Proof. By Riemann-Roch we know that

$$\begin{aligned}
\chi(X, \wedge^h M_L \otimes L^t) &= c_1(\wedge^h M_L \otimes L^t) + \operatorname{rank}(\wedge^h M_L \otimes L^t)\chi(\mathcal{O}_X) \\
&= c_1(\wedge^h M_L) + \operatorname{rank}(\wedge^h M_L) c_1(L^t) + \operatorname{rank}(\wedge^h M_L)\chi(\mathcal{O}_X) \\
&= \binom{r-1}{h-1} c_1(M_L) + \operatorname{rank}(\wedge^h M_L)(td + 1 - g)
\end{aligned}$$

and from the exact sequence

$$0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0$$

we see that $c_1(M_L) = -d$ and $\operatorname{rank}(M_L) = r$ and we conclude. □

In our case, we see that

$$k_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = -\binom{d}{p+1} + \binom{d-1}{p} d = \binom{d}{p+1} p$$

as above.

Computation from the Hilbert function

Suppose that we already know that the Betti table of $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(d))$ has the shape

	0	1	2	...	p	$p+1$	...	$d-1$
0	1	–	–	...	–	...	–	
1	–	*	*	...	*	*	...	*

To compute explicitly the Betti numbers, consider the module $R = R(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = \bigoplus_{q \geq 0} S^{qd}(V)$. It is easy to see that the Hilbert function of R is given by

$$H_R(t) = \begin{cases} td + 1, & \text{if } d \geq 0 \\ 0, & \text{if } d < 0 \end{cases}$$

Now, using the notations of Proposition 1.2.5, we see from the Betti table that

$$k_{p,1} = (-1)^p K_{p+1} \qquad \text{for every } p \geq 1$$

and the Proposition 1.2.5 tells us how to compute the K_s from the Hilbert function by the formula

$$K_s = H_R(s) - \sum_{k<s} K_k \binom{s-k+d}{d}$$

***Remark* 2.7.1.** Actually, for $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ one can give an explicit description of the whole minimal free resolution using the Eagon-Northcott complex, which is a generalization of the Koszul complex. We are not going to describe this construction here, but the interested reader can consult [Eis05].

2.7.2 Elliptic normal curve

Another example in which the Betti numbers are known explicitly is that of elliptic normal curves: let X be a smooth connected projective curve of genus 1 and let L be a line bundle on X of degree $\deg L = d$. Observe that since $K_X = \mathcal{O}_X$ the Riemann-Roch Theorem tells us that

$$h^0(X, L) - h^0(X, L^\vee) = d$$

and in particular $h^0(X, L) = d$ if $d > 0$ and $h^0(X, L) = 0$ if $d < 0$.

Now, Proposition 2.6.2 tells us that L is very ample as soon as $d \geq 3$ and it is clear that this is not true if $d = 1, 2$. Then, we want to study the Betti table of (X, L) for $d \geq 3$. Theorem 2.6.1 tells us that (X, L) satisfies property N_{d-3} and in particular it is always projectively normal. Since there is no intermediate cohomology on a curve, it is also arithmetically Cohen-Macaulay from Proposition 2.6.1 so that

$$\text{projdim } R(X, L) = h^0(X, L) - 2 = d - 2$$

Moreover, Duality Theorem 2.3.1, tells us that

$$K_{p,q}(X, L) \cong K_{d-2-p,2-q}(X, L)^\vee$$

and in particular

$$K_{d-2,q}(X, L) \cong K_{0,2-q}(X, L)^\vee \cong \begin{cases} \mathbf{k}, & \text{if } q = 2 \\ 0, & \text{if } q \neq 2 \end{cases}$$

Hence, the Betti table of (X, L) has the following shape

	0	1	2	...	p	$p+1$	...	$d-3$	$d-2$
0	1	–	–	...	–	–	...	–	–
1	–	*	*	...	*	*	...	*	–
2	–	–	–	...	–	–	...	–	1

To compute the Betti numbers explicitly, we can use a reasoning similar to the one that we used for the rational normal curve: fix a $1 \leq p \leq d - 3$, then from Proposition 2.1.1, we know that

$$K_{p,1}(X, L) \cong \text{Coker } (\wedge^{p+1} H^0(X, L) \longrightarrow H^0(X, \wedge^p M_L \otimes L))$$

and from the exact sequence

$$0 \longrightarrow \wedge^{p+1} M_L \longrightarrow \wedge^{p+1} H^0(X, L) \otimes \mathcal{O}_X \longrightarrow \wedge^p M_L \otimes L \longrightarrow 0$$

we get an exact sequence

$$0 \longrightarrow H^0(X, \wedge^{p+1} M_L) \longrightarrow \wedge^{p+1} H^0(X, L) \longrightarrow H^0(X, \wedge^p M_L \otimes L) \longrightarrow K_{p,1}(X, L) \longrightarrow 0$$

so that

$$k_{p,1}(X, L) = h^0(X, \wedge^p M_L \otimes L) - \binom{d}{p+1} + h^0(X, \wedge^{p+1} M_L)$$

Now, from the Betti table and Corollary 2.1.3 we see that $h^0(X, \wedge^{p+1} M_L) = 0$, whereas from Corollary 2.1.1 we see that $H^1(X, \wedge^p M_L \otimes L) \cong K_{p-1,2}(X, L) = 0$. Then, from Lemma 2.7.1, we see that

$$h^0(X, \wedge^p M_L \otimes L) = \chi(X, \wedge^p M_L \otimes L) = -\binom{d-2}{p-1} d + \binom{d-1}{p} d$$

and putting everything together we see that

$$k_{p,1}(X,L) = -\binom{d-2}{p-1}d + \binom{d-1}{p}d - \binom{d}{p+1} = \frac{d(d-p-2)}{p+1}\binom{d-2}{p-1}$$

Remark 2.7.2. Once we know that the Betti table of a normal elliptic curve has the above shape, we can compute the values of the Betti numbers also through the Hilbert function, as for the rational normal curves.

We can also compute explicitly the whole minimal free resolution through an Eagon-Northcott complex: for details see [Eis05].

2.7.3 Veronese surface

Let V be a vector space over $\mathbf{k}$ of dimension 3 and let $\mathbb{P}^2 = \mathbb{P}(V)$. We want to study the Betti tables of the Veronese embeddings $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$, for $d \geq 2$.

Quadratic Veronese surface

The first case is $d = 2$: observe that $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(2q)) = 0$ for every $q \geq 0$, so that we can use the Lefschetz Theorem 2.2. We can take as a connected divisor $Y \in |\mathcal{O}_{\mathbb{P}^2}(2)|$ a smooth conic $Y \cong \mathbb{P}^1$ and then $\mathcal{O}_{\mathbb{P}^2}(2)_{|Y} \cong \mathcal{O}_{\mathbb{P}^1}(4)$.

Thus, the Betti table of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ corresponds to the Betti table of $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$ that we already know:

	0	1	2	3
0	1	–	–	–
1	–	6	8	3

For instance, this tells us that the ideal of $X_2 = \phi_{\mathcal{O}_{\mathbb{P}^2}(2)}(\mathbb{P}^2) \subseteq \mathbb{P}^5$ is generated by 6 quadratic polynomials. Indeed, the Veronese surface can be characterized as the locus of points $[x_0, x_1, x_2, x_3, x_4, x_5] \in \mathbb{P}^5$ such that

$$\operatorname{rk}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix} \leq 1$$

and this condition is equivalent to the vanishing of the upper-triangular 2×2 minors, so that

$$I_{X_2} = (x_0x_3 - x_1^2,\ x_0x_4 - x_1x_2,\ x_0x_5 - x_2^2,\ x_1x_4 - x_2x_3,\ x_1x_5 - x_2x_4,\ x_3x_5 - x_4^2)$$

Cubic Veronese surface

The next case is $d = 3$: we can use the Lefschetz Theorem 2.2 as before, and as a connected divisor $Y \in |\mathcal{O}_{\mathbb{P}^2}(3)|$ we can take a smooth plane cubic, and now $L = \mathcal{O}_{\mathbb{P}^2}(3)_{|Y}$ is a line bundle of degree 9 on Y. Hence, the Betti table of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ corresponds of the Betti table of (Y, L), where Y is a smooth connected curve of genus 1 and L is a line bundle on Y of degree 9: from our previous computations, we see that this Betti table is given by

	0	1	2	3	4	5	6	7
0	1	–	–	–	–	–	–	–
1	–	27	105	189	189	105	27	–
2	–	–	–	–	–	–	–	1

Quartic Veronese surface

Now we proceed to the case $d = 4$: we can use again the Lefschetz Theorem 2.2, taking a smooth plane quartic $Y \in |\mathcal{O}_{\mathbb{P}^2}(4)|$ and then $L = \mathcal{O}_{\mathbb{P}^2}(4)_{|Y}$ is a line bundle of degree 16 on Y.

Observe that Y is a smooth curve of genus 3 and from Theorem 2.6.1, we see that (Y, L) satisfies property N_9. In particular, from Proposition 2.6.1, we know that (Y, L) is arithmetically Cohen-Macaulay, so that its minimal free resolution has length $h^0(Y, L) - 2 = 14 - 2 = 12$. Proposition 2.6.4 tells us that $K_{p,2}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) \neq 0$ for $10 \leq p \leq 12$ so that the Betti table of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$ has the shape

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	–	–	–	–	–	–	–	–	–	–	–	–
1	–	*	*	*	*	*	*	*	*	*	?	?	?
2	–	–	–	–	–	–	–	–	–	–	*	*	*

where the asterisk * denote a nonzero element and the question marks ? denote an element that can be either zero or nonzero.

Now, we study the group $K_{11,1}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$: by Duality Theorem 2.3.1, we see that $K_{11,1}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) \cong K_{1,2}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4); \mathcal{O}_{\mathbb{P}^2}(-3))^\vee$, and from Theorem 2.5.2 we see that $K_{1,2}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4); \mathcal{O}_{\mathbb{P}^2}(-3)) = 0$. Now, using Proposition 1.2.4, we see that $K_{12,1}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) = 0$ as well, so that the Betti table is

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	–	–	–	–	–	–	–	–	–	–	–	–
1	–	*	*	*	*	*	*	*	*	*	?	–	–
2	–	–	–	–	–	–	–	–	–	–	*	*	*

To conclude, we can observe that, by Example 2.1.1, we have that $K_{p,q}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) \cong H^q(\mathbb{P}^2, \wedge^{p+q} M_4)$ for every p, q. Then, for every $s \in \mathbb{Z}$ we see that

$$k_{s,0}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) - k_{s-1,1}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) + k_{s-2,2}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4)) = \chi(\mathbb{P}^2, \wedge^s M_4)$$

and since on every diagonal $p + q = s$ of the Betti table there is at most one Betti number that is nonzero, this must be equal to the absolute value $|\chi(\mathbb{P}^2, \wedge^s M_4)|$. This Euler characteristic can be computed inductively

Lemma 2.7.2. *Let X be an irreducible projective variety and let L be a globally generated line bundle on X with syzygy bundle M_L. Set $h^0(X, L) = r + 1$, then for every $s \geq 0$ and $t \in \mathbb{Z}$*

$$\chi(\wedge^s M_L \otimes L^t) = \sum_{h=0}^{s} (-1)^h \binom{r+1}{s-h} \chi(L^{t+h})$$

Proof. The proof is by induction on s: if $s = 0$ the identity is obvious, and if $s > 0$ it is enough to observe that the exact sequence

$$0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0$$

yields the exact sequence

$$0 \longrightarrow \wedge^s M_L \otimes L^t \longrightarrow \wedge^s H^0(X, L) \otimes L^t \longrightarrow \wedge^{s-1} M_L \otimes L^{t+1} \longrightarrow 0$$

and now the thesis follows immediately by induction. □

In our case, this lemma yields

$$\chi(\wedge^s M_4) = \sum_{h=0}^{s} (-1)^h \binom{15}{s-h} \chi(\mathcal{O}_{\mathbb{P}^2}(4h)) = \sum_{h=0}^{s} (-1)^h \binom{15}{s-h} \binom{4h+2}{2}$$

and after some computations we see that the Betti table of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(4))$ is given by

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	–	–	–	–	–	–	–	–	–	–	–	–
1	–	75	536	1947	4488	7095	7920	6237	3344	1089	120	–	–
2	–	–	–	–	–	–	–	–	–	–	55	24	3

2.7.4 Higher degree Veronese surfaces

What can we say in general about the Betti table of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$? Using as before the Lefschetz Theorem 2.2, we see that this Betti table coincides with that of (Y, L), where Y is a smooth plane curve of degree d and genus $g = \frac{(d-1)(d-2)}{2}$ and L is a line bundle on Y of degree d^2. In particular, Theorem 2.6.1, tells us that (Y, L) satisfies property N_{3d-3}.

Moreover, setting $r_d = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) - 1$, we know that $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ is arithmetically Cohen-Macaulay, so that projdim $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) - 3 = r_d - 2$, and Proposition 2.6.4 tells us that $K_{p,2}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \neq 0$ for every $3d - 2 \leq p \leq r_d - 2$.

Then, as before we can observe that $K_{p,1}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \cong K_{r_d-2-p,2}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d); \mathcal{O}_{\mathbb{P}^2}(-3))$ and now we can use Theorem 2.5.2, to show that $K_{p,1}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) = 0$ for $p \geq r_d - d + 1$.

We can sum up what we have said saying that the Betti table of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ has the shape

	0	1	2	...	$3d-3$	$3d-2$	$3d-1$	...	$r_d - d$	$r_d - d + 1$	...	$r_d - 3$	$r_d - 2$
0	1	–	–	–	–	–	–	–	–	–	–	–	–
1	–	*	*	...	*	?	?	...	?	–	...	–	–
2	–	–	–	...	–	*	*	...	*	*	...	*	*

and the mistery of the question marks will be unveiled in the next chapter.

Chapter 3

Asymptotic syzygies of algebraic varieties

We have seen in Remark 2.6.5 that if X is a smooth curve of genus g and if L is a line bundle on X of degree $d \gg 0$ then the Betti table of (X, L) has the following shape:

	0	1	2	3	...	$d-2g-2$	$d-2g-1$	$d-2g$	...	$d-g-1$
0	1	–	–	–	...	–	–	–	...	–
1	–	*	*	*	...	*	*	?	...	?
2	–	–	–	–	...	–	–	?	...	?

In particular, we note that the length of the segment in which there can be two nonzero elements in the same column has constant length g, thus as $d \to +\infty$, it becomes negligible w.r.t. the length of the Betti table. Thus, the Betti table becomes nicer, in the sense of more sparse, as $d \to +\infty$.

Hence, we ask ourselves whether this facts generalizes to higher dimensions: more precisely, we pose the following question.

Question 1. *Let X be a smooth irreducible variety of dimension $n \geq 2$ and let L be a very ample line bundle on X. Is it true that the Betti table of (X, L^d) becomes more sparse as d grows ?*

The most basic example of the above situation is that of Veronese embeddings: in the case of $\mathbb{P}^2$ we know that the Betti table of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ has the shape

	0	1	2	...	$3d-3$	$3d-2$	$3d-1$	...	r_d-d	r_d-d+1	...	r_d-3	r_d-2
0	1	–	–	–	–	–	–	–	–	–	–	–	–
1	–	*	*	...	*	?	?	...	?	–	...	–	–
2	–	–	–	...	–	*	*	...	*	*	...	*	*

but with what we know up to now we have control only over the segments $0 \leq p \leq 3d-3$ and $r_d - d + 1 \leq p \leq r_d - 2$, that become negligible as $d \to +\infty$, because their length grows linearly with d, whereas the length of the Betti table grows quadratically with d.

Then, to answer our question we need to study further the asymptotic behaviour of Betti numbers.

3.1 Asymptotic Betti tables

We are going to work in the following slightly more general setting: we consider a smooth irreducible projective variety X of positive dimension $\dim X = n$, an ample line bundle A on X, two arbitrary divisors P, B and for each d we set

$$L_d = A \otimes \mathcal{O}_X(P) \qquad r_d = h^0(X, L_d) - 1$$

Observe that, thanks to Riemann-Roch, $h^0(X, L_d)$ and r_d are given by polynomials of degree n in d, for $d \gg 0$.

To begin with, we set up upper and lower bounds for the rows of the Betti table.

Proposition 3.1.1. *For $d \gg 0$ we have that*

$$K_{p,q}(X, L_d, \mathcal{O}_X(B)) = 0 \qquad \text{for all } p \geq 0,\ q < 0 \text{ and } q > n+1$$

Proof. Take d large enough so that $H^0(X, \mathcal{O}_X(B) \otimes L_d^q) = 0$ for all $q < 0$, then from Corollary 2.1.6 it follows that

$$K_{p,q}(X, L_d; \mathcal{O}_X(B)) = 0 \qquad \text{for all } p \geq 0, q < 0$$

To prove the other vanishings, thanks to Theorem 2.4.3, it is enough to show that $\mathcal{O}_X(B)$ is $(n+1)$-regular w.r.t. L_d if d is large enough. But this is clearly true because by definition of Castelnuovo-Mumford's regularity, it means that

$$H^i(X, \mathcal{O}_X(B) \otimes L_d^{n+1-i}) = 0 \qquad \text{for } i = 1, \ldots, n$$

and we can achieve this if $d \gg 0$. □

Now we study the groups $K_{p,0}$.

Proposition 3.1.2. *For $d \gg 0$ we have that*

$$K_{p,0}(X, L_d; \mathcal{O}_X(B)) \neq 0 \iff 0 \leq p \leq h^0(X, \mathcal{O}_X(B)) - 1$$

Proof. First, from Theorem 2.5.1, we see that $K_{p,0}(X, L_d; \mathcal{O}_X(B)) = 0$ for all $p \geq h^0(X, \mathcal{O}_X(B))$. Note that this is true independently of d.

For the other implication, we can proceed as in the proof of Proposition 2.6.4. More precisely, we begin by taking d large enough so that $H^0(X, L_d^{-1} \otimes \mathcal{O}_X(B)) = 0$ and then, by definition, the group $K_{p,0}(X, L_d; \mathcal{O}_X(B))$ is the cohomology at the middle term of the complex

$$0 \longrightarrow \wedge^p H^0(X, L_d) \otimes H^0(X, \mathcal{O}_X(B)) \xrightarrow{d_{p,0}} \wedge^{p-1} H^0(X, L_d) \otimes H^0(X, L_d \otimes \mathcal{O}_X(B))$$

that is

$$K_{p,0}(X, L_d; \mathcal{O}_X(B)) = \operatorname{Ker} d_{p,0}$$

Then, suppose that $p < h^0(X, \mathcal{O}_X(B))$ and take linearly independent elements $f_0, \ldots, f_p \in H^0(X, \mathcal{O}_X(B))$ together with a nonzero element $s \in H^0(X, L_d \otimes \mathcal{O}_X(-B))$ (we can take d large enough so that this last group is nonzero). Consider now the element

$$\alpha \overset{\text{def}}{=} \sum_{j=0}^{p} (-1)^j (f_0 s \wedge \cdots \wedge \widehat{f_j s} \wedge \cdots \wedge f_p s) \otimes f_j \in \wedge^p H^0(X, L_d) \otimes H^0(X, \mathcal{O}_X(B))$$

it is then clear that $\alpha \neq 0$ and $d_{p,0}(\alpha) = 0$ and in this way we have constructed a nonzero element in $\operatorname{Ker} d_{p,0}$. □

This result, coupled with the Duality Theorem 2.3.1, gives us control over the groups $K_{p,n+1}$ as well.

Corollary 3.1.1. *For $d \gg 0$ we have that*

$$K_{p,n+1}(X, L_d; \mathcal{O}_X(B)) \neq 0 \iff r_d - n - h^0(X, \mathcal{O}_X(K_X - B)) + 1 \leq p \leq r_d - n$$

Proof. Suppose that d is large enough so that

$$H^i(X, \mathcal{O}_X(B) \otimes L^{n+1-q}) = H^i(X, \mathcal{O}_X(B) \otimes L^{n-i}) = 0 \qquad \text{for all } i = 1, \ldots, n-1$$

then we can use the Duality Theorem in the form of Remark 2.3.3 to see that

$$K_{p,n+1}(X, L_d; \mathcal{O}_X(B)) \cong K_{r_d-n-p,0}(X, L, \mathcal{O}_X(K_X - B))^\vee$$

and then the result follows applying Proposition 3.1.2 to the right-hand term. □

The above results tell us that for $d \gg 0$ the Betti table of (X, L_d) is zero above the 0-th row and under the $(n+1)$-th row, and moreover, on these two rows the segments with nonzero elements have constant length.

In particular, all the action is concentrated between the 1-st and the n-th row. What we want to present now is a recent result due to L. Ein and R. Lazarsfeld [EL12] that gives the asymptotic behaviour of these rows.

Theorem 3.1.1. *For every $1 \leq q \leq n$ there exist constants $C_1, C_2 > 0$ such that, if $d \gg 0$, then $K_{p,q}(X, L_d) \neq 0$ for every*

$$C_1 \cdot d^{q-1} \leq p \leq r_d - C_2 \cdot d^{n-1}$$

If moreover $H^i(X, \mathcal{O}_X) = 0$ for all $0 < i < n$ then $K_{p,q}(X, L_d) \neq 0$ for every

$$C_1 \cdot d^{q-1} \leq p \leq r_d - C_2 \cdot d^{n-q}$$

In particular, this result tells us that our previous question has a negative answer. The situation, indeed, is quite the opposite: imagine to construct a normalized Betti table for every d by rescaling the horizontal length so that every Betti diagram stays into a rectangle of fixed length and $n+2$ rows. Then the nonzero Betti numbers fill up completely the rows from the 1-st one to the n-th one for $d \to +\infty$, whereas the 0-th row and the $n+1$-th row become asymptotically zero.

In the same paper [EL12], Ein and Lazarsfeld also give an explicit result for the case of Veronese embeddings:

Theorem 3.1.2. *For every $1 \leq q \leq n$, if $d \gg 0$ then*

$$K_{p,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \neq 0$$

for every

$$\binom{d+q}{q} - \binom{d-1}{q} - q \leq p \leq \binom{d+n}{n} - \binom{d+n-q}{n-q} + \binom{n}{n-q} - q - 1$$

***Remark** 3.1.1.* In particular, Theorem 3.1.2 tells us that the question marks in the Betti table of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ are actually asterisks, so that this Betti table has the form

	0	1	2	...	$3d-3$	$3d-2$	$3d-1$	...	r_d-d	r_d-d+1	...	r_d-3	r_d-2
0	1	–	–	–	–	–	–	–	–	–	–	–	–
1	–	*	*	...	*	*	*	...	*	–	...	–	–
2	–	–	–	...	–	*	*	...	*	*	...	*	*

***Remark** 3.1.2.* Zhou [Zho14] gives a simplified proof of Theorem 3.1.1 when B is an adjoint divisor of type $B = K_X + bA$. Moreover, his result specializes to Theorem 3.1.2 in the case of projective spaces.

3.1.1 The case of $K_{p,1}$

We begin by giving the proof of Theorem 3.1.1 in the case of $q = 1$. We keep the same notation of before and, moreover we choose a very ample divisor H on X, a general divisor $\overline{X} \in |H|$ and we set

$$V_d \stackrel{\text{def}}{=} H^0(X, L_d), \qquad V'_d \stackrel{\text{def}}{=} H^0(X, \mathcal{I}_{\overline{X}/X} \otimes L_d), \qquad v'_d = \dim V'_d$$

Observe that from the exact sequence

$$0 \longrightarrow \mathcal{I}_{X/\overline{X}} \otimes L_d \longrightarrow L_d \longrightarrow \mathcal{O}_{\overline{X}} \otimes L_d \longrightarrow 0$$

and Riemann-Roch we see that

$$v'_d = r_d - O(d^{n-1})$$

for $d \gg 0$.

Then we have the following result that, in particular, proves Theorem 3.1.1 for $q = 1$.

Proposition 3.1.3. *If $d \gg 0$ then $K_{p,1}(X, L_d; \mathcal{O}_X(B)) \neq 0$ for all*

$$h^0(X, \mathcal{O}_X(B+H)) \leq p \leq v'_d - 1$$

Proof. We denote by $\overline{B}, \overline{L_d}, \overline{H}$ the restrictions of the corresponding objects to $\overline{X}$. Then consider the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow \wedge^{p+1} V_d \otimes H^0(X, \mathcal{O}_X(B)) & \longrightarrow & \wedge^{p+1} V_d \otimes H^0(X, \mathcal{O}_X(B+H)) & \xrightarrow{r} & \wedge^{p+1} V_d \otimes H^0(\overline{X}, \mathcal{O}_{\overline{X}}(\overline{B}+\overline{H})) \\ \downarrow d & & \downarrow d & & \downarrow \overline{d} \\ 0 \to \wedge^{p+1} V_d \otimes H^0(X, \mathcal{O}_X(B) \otimes L_d) & \longrightarrow & \wedge^{p+1} V_d \otimes H^0(X, \mathcal{O}_X(B+H) \otimes L_d) & \to & \wedge^{p+1} V_d \otimes H^0(\overline{X}, \mathcal{O}_{\overline{X}}(\overline{B}+\overline{H}) \otimes \overline{L}_d) \end{array}$$

where the columns are Koszul differentials. Suppose that $p \geq h^0(X, \mathcal{O}_X(B+H))$: then by Theorem 2.5.1 we see that the two vertical maps on the left are injective. Now, a little bit of diagram chasing shows that, in order to conclude, we just need to find an element $\alpha \in \wedge^{p+1} V_d \otimes H^0(X, \mathcal{O}_X(B+H))$ such that

$$\alpha \neq 0, \qquad r(\alpha) = 0, \qquad \overline{d}(r(\alpha)) = 0$$

In particular, any nonzero element in $\wedge^{p+1} V'_d \otimes H^0(X, \mathcal{O}_X(B+H))$ will do, and this group is nonzero precisely when $p \leq v'_d - 1$. □

The rest of the chapter is devoted to complete the proof of Theorem 3.1.1: the basic idea of the proof is to use secant hyperplanes to prove the statements inductively on the dimension. After that, we will also give the proof of Theorem 3.1.2. The exposition here follows faithfully that of [EL12].

3.2 Secant constructions

We consider an irreducible projective variety X of dimension $\dim X = n$ and a very ample line bundle L on X. We also fix a basepoint-free subspace $V \subseteq H^0(X, L)$ of dimension $\dim V = v$ that defines an embedding $X \subseteq \mathbb{P}(V)$. We suppose also to have a divisor a divisor B on X such that

$$H^i(X, \mathcal{O}_X(B) \otimes L^m) = 0 \qquad \text{for all } i > 0, \ m > 0 \tag{3.1}$$

so that, from Corollary 2.1.4 we have

$$K_{p,q}(X, L, V; \mathcal{O}_X(B)) \cong H^{q-1}(X, \mathcal{O}_X(B) \otimes \wedge^{p+q-1} M_V \otimes L) \qquad \text{for } q \geq 2 \tag{3.2}$$

and in particular

$$K_{p,n+1}(X, L, V; \mathcal{O}_X(B)) \cong H^n(X, \mathcal{O}_X(B) \otimes \wedge^{p+n-1} M_V \otimes L)$$

and $K_{p,q}(X; L, V; \mathcal{O}_X(B)) = 0$ if $q > n+1$.

In this section, we expose a technique to prove inductively on the dimension the non-vanishing of certain cohomology groups $K_{p,q}(X, L, V; \mathcal{O}_X(B))$, using secant planes.

Turning to details, we fix a quotient

$$\pi \colon V \longrightarrow W$$

of dimension $\dim W = w < v$, that defines a linear subspace $\mathbb{P}(W) \subseteq \mathbb{P}(V)$ and we set

$$Z \stackrel{\text{def}}{=} X \cap \mathbb{P}(W)$$

as the scheme-theoretic intersection. If we set $J = \operatorname{Ker} \pi$, then the twisted ideal sheaf $\mathcal{I}_{Z/X} \otimes L$ is generated by J through the natural evaluation map

$$V \otimes \mathcal{O}_X \longrightarrow L$$

We suppose in what follows that $Z \neq \emptyset$. Now, we have a commutative diagram of sheaves on X

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_V & \longrightarrow & V \otimes \mathcal{O}_X & \xrightarrow{ev_W} & L & \longrightarrow & 0 \\ & & \downarrow \rho & & \downarrow \pi & & \downarrow & & \\ 0 & \longrightarrow & \Sigma_V & \xrightarrow{i} & W \otimes \mathcal{O}_X & \xrightarrow{ev_V} & \mathcal{O}_Z \otimes L & \longrightarrow & 0 \end{array} \tag{3.3}$$

where the rows are exact, the columns surjective and Σ_W is defined by the diagram itself. Observe that Σ_W is torsion-free of rank w.

Lemma 3.2.1. *Let $p \in X$ be a closed point, then there are isomorphisms*

$$\Sigma_{W,p} \cong \mathcal{I}_{Z/X,p} \oplus \mathcal{O}_{X,p}^{\oplus w-1}$$
$$W \otimes \mathcal{O}_{X,p} \cong \mathcal{O}_{X,p} \oplus \mathcal{O}_{X,p}^{\oplus w-1}$$

compatible with the inclusion $\Sigma_W \subseteq W \otimes \mathcal{O}_X$, meaning that at the level of stalks this map induces the identity between the two components on the right.

Proof. If $p \notin Z$, then $\mathcal{I}_{Z/X,p} = \mathcal{O}_{X,p}$ and we are done. Then suppose that $p \in Z$ and fix an isomorphism $\mathcal{O}_{Z,p} \cong (\mathcal{O}_Z \otimes L)_p$. Then, since the map ev_W is surjective, there is a basis $e_1, \dots, e_w \in W \otimes \mathcal{O}_{X,p}$ such that $ev_W(e_1) = 1 \in \mathcal{O}_{Z,p}$ is the standard generator, whereas $ev_W(e_i) = 0$ if $i \geq 0$. The assertion follows. □

Note that the above isomorphism is non-canonical, but it has a canonical consequence

Corollary 3.2.1. *There is a canonical surjective map $\varepsilon \colon \wedge^w \Sigma_W \longrightarrow \mathcal{I}_{Z/X}$ that makes the following diagram commutative*

$$\begin{array}{ccc} \wedge^w \Sigma_W & \xrightarrow{\wedge^w i} & \wedge^w W \otimes \mathcal{O}_X \\ \downarrow \varepsilon & & \downarrow \wr \\ \mathcal{I}_{Z/X} & \longrightarrow & \mathcal{O}_X \end{array}$$

where the bottom map is the natural inclusion.

Proof. It follows from Lemma 3.2.1. □

Now, if we take the composition

$$\wedge^w M_V \xrightarrow{\wedge^w \rho} \wedge^w \Sigma_W \xrightarrow{\varepsilon} \mathcal{I}_{Z/X}$$

we get a surjective map $\sigma \colon \wedge^w M_V \longrightarrow \mathcal{I}_{Z/X}$. This map will be our main tool to show the nonvanishing of the Koszul cohomology group:

Definition 3.2.1. Fix an integer $q > 2$. Then we say that W **carries weight q syzygies** of B (with respect to V) if the map

$$H^{q-1}(X, \mathcal{O}_X(B) \otimes \wedge^w M_V \otimes L) \longrightarrow H^{q-1}(X, \mathcal{O}_X(B) \otimes \mathcal{I}_{Z/X} \otimes L)$$

induced by σ is surjective.

In particular, if we manage to find a W that carries weight q syzygies and such that $H^{q-1}(X, \mathcal{O}_X(B) \otimes \mathcal{I}_{Z/X} \otimes L) \neq 0$ then $H^{q-1}(X, \mathcal{O}_X(B) \otimes \wedge^w M_V \otimes L) \neq 0$ as well, so that

$$K_{w+1-q,q}(X, L, V; \mathcal{O}_X(B)) \neq 0$$

thanks to the isomorphism 3.2.

Now we suppose that $n \geq 2$ and we study the behaviour of the previous construction with respect to hyperplane sections. We fix a very ample divisor H on X and we take a general divisor $\overline{X} \in |H|$. We can choose this divisor such that $\overline{X}$ is again irreducible and such that the commutative diagram 3.3 has still exact rows when tensored by $\mathcal{O}_{\overline{X}}$.

Now we set

$$V' = V \cap H^0(X, \mathcal{I}_{\overline{X}/X} \otimes L) \subseteq H^0(X, L)$$

and $W' = \pi(V') \subseteq W$. If we write $\overline{V} = V/V'$ and $\overline{W} = W/W'$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & \overline{V} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \overline{\pi}} & & \\
0 & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & \overline{W} & \longrightarrow & 0
\end{array}$$

and setting $v' = \dim V', w' = \dim W', \overline{v} = \dim \overline{V}, \overline{w} = \dim \overline{W}$, we have that

$$v = v' + \overline{v}, \qquad w = w' + \overline{w}$$

Now, taking

$$\overline{L} = L_{|\overline{X}}, \qquad \overline{Z} = Z \cap \overline{X}, \qquad \overline{B} = B_{|\overline{X}}, \overline{H} = H_{|\overline{X}}$$

we can apply the same constructions of before on $\overline{X}$ and get an exact commutative diagram of sheaves on $\overline{X}$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{\overline{V}} & \longrightarrow & \overline{V} \otimes \mathcal{O}_{\overline{X}} & \xrightarrow{\overline{ev}_{\overline{W}}} & \overline{L} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \overline{\rho}} & & \downarrow{\scriptstyle \overline{\pi}} & & \downarrow & & \\
0 & \longrightarrow & \Sigma_{\overline{W}} & \xrightarrow{\overline{i}} & \overline{W} \otimes \mathcal{O}_{\overline{X}} & \xrightarrow{\overline{ev}_{\overline{V}}} & \mathcal{O}_{\overline{Z}} \otimes \overline{L} & \longrightarrow & 0
\end{array}$$

with exact rows and surjective columns (the sheaves $M_{\overline{V}}$ and $\Sigma_{\overline{W}}$ are defined by the diagram itself). As before, we have a surjective map

$$\overline{\sigma} \colon \wedge^{w} M_{\overline{V}} \longrightarrow \mathcal{I}_{\overline{Z}/\overline{X}}$$

Observe that if $\dim Z = 0$, then $\overline{Z} = \emptyset$ and $\Sigma_{\overline{W}} = \overline{W}_{\overline{X}}$: in this case, if moreover $W' = W$, we take $\overline{\sigma}$ to be the identity map $\mathcal{O}_{\overline{X}} \longrightarrow \mathcal{O}_{\overline{X}}$.

Lemma 3.2.2. *There are isomorphisms*

$$M_V \otimes \mathcal{O}_{\overline{X}} \xrightarrow{\sim} (V' \otimes \mathcal{O}_{\overline{X}}) \oplus M_{\overline{V}}$$
$$\Sigma_W \otimes \mathcal{O}_{\overline{X}} \xrightarrow{\sim} W'_{\overline{X}} \oplus \Sigma_{\overline{W}}$$

under which the quotient map $\rho \otimes id \colon M_V \otimes \mathcal{O}_{\overline{X}} \longrightarrow \Sigma_W \otimes \mathcal{O}_{\overline{X}}$ is identified with the direct sum of the two maps

$$\pi' \colon V' \otimes \mathcal{O}_{\overline{X}} \longrightarrow W' \otimes \mathcal{O}_{\overline{X}}, \qquad \overline{\rho} \colon M_{\overline{V}} \longrightarrow \Sigma_{\overline{W}}$$

Proof. First we observe that by construction

$$V' = \operatorname{Ker}\left(H^0(\overline{X}, V \otimes \mathcal{O}_{\overline{X}}) \longrightarrow H^0(\overline{X}, \mathcal{O}_{\overline{X}} \otimes \overline{L})\right)$$

and in the same way the sections in W' vanish on $\mathcal{O}_{\overline{Z}} \otimes \overline{L}$. Then we get a commutative diagram of sheaves on $\overline{X}$ with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V'\otimes\mathcal{O}_{\overline{X}} & \longrightarrow & M_V\otimes\mathcal{O}_{\overline{X}} & \longrightarrow & M_{\overline{V}} & \longrightarrow & 0 \\ & & \downarrow \pi' & & \downarrow \rho\otimes\mathrm{id} & & \downarrow \overline{\rho} & & \\ 0 & \longrightarrow & W'_{\overline{X}} & \longrightarrow & \Sigma_W\otimes\mathcal{O}_{\overline{X}} & \longrightarrow & \Sigma_{\overline{W}} & \longrightarrow & 0 \end{array}$$

if we can show that there are compatible splittings of the two rows, then we are done.

Now, from diagram 3.2, we can choose a section $\overline{V} \longrightarrow V$ that maps Ker $(\overline{\pi})$ into Ker (π): then this section induces a compatible splitting of the two rows of 3.2 and consequently the left-square diagram of 3.3 restricts to

$$\begin{array}{ccc} M_V\otimes\mathcal{O}_{\overline{X}} & \longrightarrow & V'\otimes\mathcal{O}_{\overline{X}}\oplus(\overline{V}\otimes\mathcal{O}_{\overline{X}}) \\ \downarrow \rho\otimes\mathrm{id} & & \downarrow \pi'\oplus\overline{\pi} \\ \Sigma_W\otimes\mathcal{O}_{\overline{X}} & \longrightarrow & W'_{\overline{X}}\oplus\overline{W}_{\overline{X}} \end{array}$$

and this gives the required splitting. □

Now we can prove the first main technical result, that is a criterion for verifying inductively when W carries weight q syzygies.

Theorem 3.2.1. *Fix $q \geq 2$ and suppose that $H^{q-1}(X, \mathcal{I}_{Z/X}\otimes\mathcal{O}_X(B+H)\otimes L) = 0$ and that $\overline{W}$ carries weight $q-1$ syzygies of $\overline{B}+\overline{H}$ on $\overline{X}$. Then W carries weight q syzygies of B on X.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \wedge^w M_V(B+H) & \longrightarrow & \wedge^w M_V(B+H)\otimes L & \longrightarrow & \wedge^w M_V\otimes\mathcal{O}_{\overline{X}}(\overline{B}+\overline{H})\otimes_{\mathcal{O}_{\overline{X}}}\overline{L} & \longrightarrow & 0 \\ & & \downarrow\sigma & & \downarrow\sigma & & \downarrow\overline{\sigma} & & \\ 0 & \longrightarrow & \mathcal{I}_{Z/X}(B)\otimes L & \longrightarrow & \mathcal{I}_{Z/X}(B+H)\otimes L & \longrightarrow & \mathcal{I}_{\overline{Z}/\overline{X}}\otimes\mathcal{O}_{\overline{X}}(\overline{B}+\overline{H})\otimes_{\mathcal{O}_{\overline{X}}}\overline{L} & \longrightarrow & 0 \end{array}$$

from this we get the commutative diagram with exact rows

$$\begin{array}{ccccc} H^{q-2}(X, \wedge^w M_V\otimes\mathcal{O}_{\overline{X}}(\overline{B}+\overline{H})\otimes_{\mathcal{O}_{\overline{X}}}\overline{L}) & \longrightarrow & H^{q-1}(X, \wedge^w M_V(B)\otimes L) & \rightarrow & H^{q-1}(X, \wedge^w M_V(B+H)\otimes L \\ \downarrow & & \downarrow & & \downarrow \\ H^{q-2}(\mathcal{I}_{Z/X}\otimes\mathcal{O}_{\overline{X}}(\overline{B}+\overline{H})\otimes_{\mathcal{O}_{\overline{X}}}\overline{L}) & \longrightarrow & H^{q-1}(X, \mathcal{I}_{Z/X}(B)\otimes L & \longrightarrow & H^{q-1}(\mathcal{I}_{Z/X}(B+H)\otimes L) \end{array}$$

whose middle column is the map in Definition 3.2.1, that we need to prove to be surjective. To do this, since $H^{q-1}(X, \mathcal{I}_{Z/X}\otimes\mathcal{O}_X(B+H)\otimes L) = 0$, it is enough to prove that the leftmost map in the diagram (that is induced by the restriction of σ to $\overline{X}$) is surjective. Then, consider the restriction to $\overline{X}$ of the composition $\varepsilon\circ\wedge^w\rho = \sigma$

$$\wedge^w(M_V\otimes\mathcal{O}_{\overline{X}}) \longrightarrow \wedge^w(\Sigma_W\otimes\mathcal{O}_{\overline{X}}) \longrightarrow \mathcal{I}_{Z/X}\otimes\mathcal{O}_{\overline{X}}$$

using the identification of Lemma 3.2.2, we see that this is precisely the bottom row of the commutative diagram

$$\begin{array}{ccccc} \wedge^{w'}(V'\otimes\mathcal{O}_{\overline{X}})\otimes\wedge^{\overline{w}}M_{\overline{V}} & \xrightarrow{\wedge^{w'}\pi'\otimes\wedge^{\overline{w}}\overline{\rho}} & \wedge^{w'}(W'\otimes\mathcal{O}_{\overline{X}})\otimes\wedge^{\overline{w}}\Sigma_{\overline{W}} & \xrightarrow{\mathrm{id}\otimes\overline{\varepsilon}} & \wedge^{w'}W'_{\overline{X}}\otimes\mathcal{I}_{\overline{Z}/\overline{X}} \\ \downarrow & & \downarrow & & \downarrow\wr \\ \wedge^w(V'\otimes\mathcal{O}_{\overline{X}}\oplus M_{\overline{V}}) & \longrightarrow & \wedge^w(W'_{\overline{X}}\oplus\Sigma_{\overline{W}}) & \xrightarrow{\varepsilon_{|H}} & \mathcal{I}_{Z/X}\otimes\mathcal{O}_{\overline{X}} \end{array}$$

then to get the surjectivity of

$$H^{q-2}(X,\wedge^{w}M_V\otimes\mathcal{O}_{\overline{X}}(\overline{B}+\overline{H})\otimes_{\mathcal{O}_{\overline{X}}}\overline{L})\longrightarrow H^{q-2}(X,\mathcal{I}_{Z/X}\otimes\mathcal{O}_{\overline{X}}(\overline{B}+\overline{H})\otimes_{\mathcal{O}_{\overline{X}}}\overline{L})$$

it is enough to prove the surjectivity of the map

$$H^{q-2}(\overline{X},\wedge^{w'}(V'\otimes\mathcal{O}_{\overline{X}})\otimes\wedge^{\overline{w}}M_{\overline{V}}(\overline{B}+\overline{H})\otimes\overline{L})\longrightarrow H^{q-2}(\overline{X},\wedge^{w'}(W'\otimes\mathcal{O}_{\overline{X}})\otimes\mathcal{I}_{\overline{Z}/H}(\overline{B}+\overline{H})\otimes\overline{L})$$

induced by the top row of the last commutative diagram. But now we see that we can identify this map with

$$\wedge^{w'}V'\otimes H^{q-2}(\overline{X},\wedge^{\overline{w}}M_{\overline{V}}(\overline{B}+\overline{H})\otimes\overline{L})\overset{\wedge^{w'}\pi'\otimes\overline{\sigma}}{\longrightarrow}\wedge^{w'}W'\otimes H^{q-2}(\overline{X},\mathcal{I}_{\overline{Z}/\overline{X}}(\overline{B}+\overline{H})\otimes\overline{L})$$

that is surjective by the hypothesis that $\overline{W}$ carries weight $q-1$ syzygies. □

Remark 3.2.1. Observe that when $q=2$ the hypothesis of the theorem is that the map

$$H^0(\overline{X},\wedge^{\overline{w}}M_{\overline{V}}\otimes\mathcal{O}_{\overline{X}}(\overline{B}+\overline{H})\otimes\overline{L})\longrightarrow H^0(\overline{X},\mathcal{I}_{\overline{Z}/\overline{X}}(\overline{B}+\overline{H})\otimes\overline{L})$$

determined by $\overline{\sigma}$ is surjective, and this is automatically verified if $\overline{w}=0$.

We now give the second main technical result

Theorem 3.2.2. *Suppose that the hypotheses of Theorem 3.2.1 are verified and that, moreover, $Z\subseteq X$ is a local complete intersection, that $\overline{v}-\overline{w}>n$ and that $H^{q-1}(X,\mathcal{I}_{Z/X}(B)\otimes L)\neq 0$. Then*

$$K_{p,q}(X,L,V;\mathcal{O}_X(B))\neq 0 \qquad \text{for all } w+1\leq p+q\leq v'+\overline{w}+1$$

Proof. Suppose that U is a quotient of V such that π factors as

$$V\overset{\pi_1}{\longrightarrow}U\overset{\pi_2}{\longrightarrow}W$$

then we have a chain of linear subspaces $\mathbb{P}(W)\subseteq\mathbb{P}(U)\subseteq\mathbb{P}(V)$. We set $U'=\pi_1(V'),\overline{U}=U/U'$ and we write $u=\dim U, u'=\dim U',\overline{u}=\dim\overline{U}$, in particular $u=u'+\overline{u}$. Suppose now that we can choose U such that

$$\text{The natural map } \overline{U}\longrightarrow\overline{W} \text{ is an isomorphism} \tag{3.4}$$

$$\mathbb{P}(U)\cap X=Z. \tag{3.5}$$

Then, applying the constructions of before to U, we get maps

$$\varepsilon^{\#}\colon\ \wedge^{u}\Sigma_U\longrightarrow\mathcal{I}_{Z/X},\qquad \sigma^{\#}\colon\ \wedge^{u}M_U\longrightarrow\mathcal{I}_{Z/X}$$

and thanks to property 3.4, the map

$$\overline{\sigma}^{\#}\colon\ \wedge^{\overline{u}}M_{\overline{V}}\longrightarrow\mathcal{I}_{Z/X}\otimes\mathcal{O}_{\overline{X}}$$

coincides with the map

$$\overline{\sigma}\colon\ \wedge^{\overline{w}}M_{\overline{V}}\longrightarrow\mathcal{I}_{\overline{Z}/\overline{X}}$$

determined by $\overline{W}$. Thus, from Theorem 3.2.1 we see that U carries weight q syzygies of B, and then from the hypothesis $H^{q-1}(X,\mathcal{I}_{Z/X}(B)\otimes L)\neq 0$ it follows that $K_{u+1-q,q}(X,L,V;\mathcal{O}_X(B))\neq 0$. Hence we just need to build one such U for every $w\leq u\leq\overline{w}+1$.

To this end, we consider a subspace $\Lambda\subseteq V$ and we ask ourselves when does the quotient $U\overset{\text{def}}{=}V/\Lambda$ meet the necessary conditions. First, to get a factorization of π we need to have

$$\Lambda\subseteq\operatorname{Ker}\pi$$

then, condition 3.4 is equivalent to

$$V'+\Lambda=V'+\operatorname{Ker}\pi$$

and condition 3.5 is equivalent to the fact that $\mathcal{I}_{Z/X}\otimes L$ is generated by Λ.

Now, we see that $\dim(V'+\operatorname{Ker}\pi)=v-\overline{w}$, so that the condition $V'+\Lambda=V'+\operatorname{Ker}\pi$ holds for a general subspace $\Lambda\subseteq\operatorname{Ker}\pi$ such that $\dim\Lambda\geq\overline{v}-\overline{w}$. Now, since Z is a local complete intersection and $\operatorname{Ker}\pi$ generates $\mathcal{I}_{Z/X}$, any general subspace $\Lambda\subseteq\operatorname{Ker}\pi$ of dimension $\dim\Lambda\geq n+1$ will generate $\mathcal{I}_{Z/X}$ as well. Since $\overline{v}-\overline{w}>n$ we can then find a subspace that satisfies all these conditions for every $w\leq u\leq\overline{w}+1$. □

3.3 The asymptotic non-vanishing theorem

Now we want to give a proof of Theorem 3.1.1. The notation is as follows: we fix an irreducible smooth projective variety X of dimension $n \geq 2$, an ample line bundle A on X, an arbitrary divisor P and for every $d \geq 0$ we set

$$L_d = A \otimes \mathcal{O}_X(P) \qquad V_d \stackrel{\text{def}}{=} H^0(X, L_d), \qquad v_d \stackrel{\text{def}}{=} h^0(X, L_d)$$

We are interested in the behaviour of L_d for $d \gg 0$ so that we can suppose that L_d is very ample and that it defines an embedding

$$X \subseteq \mathbb{P}(V_d) = \mathbb{P}^{r_d}$$

with $r_d = v_d - 1$. From Riemann-Roch we know that v_d and r_d are polynomials of degree n in d, for $d \gg 0$.

The main step in the proof of Theorem 3.1.1 is the following result.

Theorem 3.3.1. *Fix an index $2 \leq q \leq n$ and consider a divisor B on X. Then there are positive constants $C_1, C_2 > 0$ such that, if $d \gg 0$ then*

$$K_{p,q}(X, L_d; \mathcal{O}_X(B)) \neq 0 \qquad \textit{for all} \;\; C_1 \cdot d^{q-1} \leq p \leq r_d - C_2 \cdot d^{n-1}$$

If moreover $H^1(X, \mathcal{O}_X(B)) = 0$ for all $0 < i < n$ then we have

$$K_{p,q}(X, L_d; \mathcal{O}_X(B)) \neq 0 \qquad \textit{for all} \;\; C_1 \cdot d^{q-1} \leq p \leq r_d - C_2 \cdot d^{n-q}$$

The strategy is to prove this theorem inductively on n using Theorem 3.2.1 and Theorem 3.2.2, but first we need to perform some auxiliary constructions.

3.3.1 Preliminary constructions

Let B be a fixed divisor on X. Then we choose a very ample divisor H on X such that

$$H + B - K_X \text{ is very ample} \tag{3.6}$$

$$H^i(X, \mathcal{O}_X(mH + B)) = 0 \text{ for all } i > 0, \;\; m \geq 1 \tag{3.7}$$

$$H^i(X, \mathcal{O}_X(mH + K_X)) = 0 \text{ for all } i > 0, \;\; m \geq 1. \tag{3.8}$$

and we fix smooth irreducible divisors $\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n \in |H|$ that meet transversely. For the next step, we set $c = n + 2 - q$ and we assume that d is sufficiently large so that $L_d \otimes \mathcal{O}_X((c-1)H)$ is very ample. Then we choose divisors

$$D_1, D_2, \ldots, D_{c-2} \in |H|, \qquad D_{c-1} \in |H + B - K_X|, \qquad D_c \in |L_d - (c-1)H| \tag{3.9}$$

in such a way that $\sum \overline{X_i} + \sum D_i$ has simple normal crossing and set

$$Z = Z_d = D_1 \cap \cdots \cap D_c \tag{3.10}$$

as the complete intersection of the D_i. Then Z is smooth and $\dim Z = n - c = q - 2$. By construction, we have that

$$\mathcal{O}_X(D_1 + \cdots + D_c) \cong L_d \otimes \mathcal{O}_X(B - K_X) \tag{3.11}$$

so that only D_c involves L_d. If $d \gg 0$, then every $L_d \otimes \mathcal{O}_X(-D_i)$ is base-point-free and we will always suppose so in the future. In particular, this implies that $\mathcal{I}_{Z_d/X} \otimes L_d$ is globally generated.

Now, we define a quotient $V_d \longrightarrow W_d$ such that $X \cap \mathbb{P}(W_d) = Z_d$. Set

$$J_{0,d} \stackrel{\text{def}}{=} H^0(X, \mathcal{I}_{Z_d/X} \otimes L_d), \qquad W_{0,d} \stackrel{\text{def}}{=} V_d / J_{0,d}$$

then we see that

$$W_{0,d} = \mathrm{Im}\,(H^0(X, L_d) \longrightarrow H^0(Z_d, L_d \otimes \mathcal{O}_{Z_d}))$$

and since $\mathcal{I}_{Z_d/X}$ is globally generated, it follows that $\mathbb{P}(W_{0,d}) \cap X = Z_d$. We could try to apply now the constructions in the previous section to this quotient, but it will actually convenient to work with a modification of it.

Suppose that for any $d \gg 0$ one has a subspace $J_d \subseteq J_{0,d}$ such that

$$\mathcal{I}_{Z_d/X} \otimes L_d \text{ is generated by } J_d \qquad \dim J_{0,d}/J_d \leq a \tag{3.12}$$

for a certain a independent of d. Then set $W_d = V_d/J_d$, $w_d = \dim W_d$ and let $\pi_d \colon V_d \longrightarrow W_d$ be the natural map. Observe that the first condition ensures that $\mathbb{P}(W_d) \cap X = Z_d$ in $\mathbb{P}(V_d)$.

Definition 3.3.1. With notations as before, we say that H, X_j, Z_d and W_d are **adapted to** B if

1. H satisfies the conditions 3.6,3.7,3.8.
2. Z_d is constructed as in 3.9 and 3.10, for $d \gg 0$.
3. W_d arise from a subspace J_d as in 3.12 for $d \gg 0$.

In what follows, we will always suppose H, X_j, Z_d and W_d to be adapted to B.

Lemma 3.3.1. *It holds that*

1. $H^{q-1}(X, \mathcal{I}_{Z_d/X} \otimes \mathcal{O}_X(B) \otimes L_d) \neq 0$.
2. $H^{q-1}(X, \mathcal{I}_{Z_d/X} \otimes \mathcal{O}_X(B+H) \otimes L_d) = 0$.
3. *The dimensions* $\dim H^i(X, \mathcal{I}_{Z_d/X} \otimes L_d)$ *and* $\dim H^i(X, \mathcal{I}_{Z_d/X} \otimes L_d(-H))$ *are bounded from above independently from d.*

Proof. Set $E = \bigoplus_{i=1}^{c} \mathcal{O}_X(-D_i)$. Then it can be shown that we have an exact complex

$$0 \longrightarrow \wedge^c E \longrightarrow \wedge^{c-1} E \longrightarrow \ldots \longrightarrow E \longrightarrow \mathcal{I}_{Z_d/X} \longrightarrow 0$$

given by the Koszul complex of a complete intersection. Now, using the double complex arising to the above resolution and the Cech complex, it can be shown that to prove the first point we just need to show that

$$H^n(X, \wedge^c E \otimes \mathcal{O}_X(B+H)) \neq 0 \text{ and } H^i(X, \wedge^j E \otimes \mathcal{O}_X(B+L_d)) = 0 \text{ for } i > 0,\ j < c$$

Thanks to 3.11 we see that $\wedge^c E \otimes \mathcal{O}_X(B) \otimes L_d \cong \mathcal{O}_X(X_X)$, that implies the first condition. For the second one, we note that for $j < c$ the bundle $\wedge^j E$ is the direct sum of twists of $\mathcal{O}_X(K_X)$ by line bundles of the form $\mathcal{O}_X(mH)$, for $m \geq 1$ as well as possibly one or both of $\mathcal{O}_X(H+B-K_X)$ and $L_d \otimes \mathcal{O}_X(-(c-1)H)$. Then we see that the summands with L_d have vanishing cohomology for $d \gg 0$, whereas the other terms are covered by 3.7 and 3.8. The second point can be proved in a similar way.

For the last point, we see that the line bundle summands that appear in $\wedge^j E \otimes \mathcal{O}_X(L_d)$ and $\wedge^j E \otimes \mathcal{O}_X(L_d - H)$ either involve only B, H and K_X or else have vanishing cohomology if $d \gg 0$. This shows that for $i > 0$ the dimensions of $H^i(X, \wedge^j E \otimes L_d)$ and $H^i(X, \wedge^j E \otimes L_d \otimes \mathcal{O}_X(-H))$ are independent from d if $d \gg 0$. Then the conclusion that we want follows as before considering the double complex arising from the resolution and the Cech complex. □

Lemma 3.3.2. *There exists a polynomial* $Q(d)$ *of degree* $q-1$ *such that the difference*

$$|\dim W_d - Q(d)|$$

is bounded.

Proof. Thanks to condition 3.12, it is enough to construct a polynomial $P(d)$ of degree $q-1$ such that $|\dim W_{0,d} - P(d)|$ is bounded. To do this we take $Y = D_1 \cap D_2 \cap \cdots \cap D_{c-1}$: this is is a smooth variety of dimension $n-(c-1) = q-1$ and $Z_d \subseteq Y$ is a divisor corresponding to the line bundle $L_d \otimes \mathcal{O}_Y(-(c-1)H)$, so that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_Y((c-1)H) \longrightarrow \mathcal{O}_Y \otimes L_d \longrightarrow \mathcal{O}_{Z_d} \otimes L_d \longrightarrow 0$$

of sheaves on Y. Taking cohomology we get the exact sequence

$$0 \longrightarrow H^0(Y, \mathcal{O}_Y((c-1)H)) \longrightarrow H^0(Y, \mathcal{O}_Y \otimes L_d) \longrightarrow H^0(Z_d, \mathcal{O}_{Z_d} \otimes L_d) \longrightarrow H^1(Y, \mathcal{O}_Y((c-1)H)) \longrightarrow C_d \longrightarrow$$

which tells us that

$$|h^0(Y, \mathcal{O}_Y \otimes L_d) - h^0(Z_d, \mathcal{O}_{Z_d} \otimes L_d)| \leq h^0(Y, \mathcal{O}_Y((c-1)H)) + h^1(Y, \mathcal{O}_Y((c-1)H)) + \dim C_d$$

and the right-hand term is bounded independently from d, as $\dim C_d \leq h^1(Y, \mathcal{O}_Y((c-1)H))$. Moreover, we know from Riemann-Roch that $h^0(Y, \mathcal{O}_Y \otimes L_d)$ is a polynomial function in d of degree $\dim Y = q-1$ for $d \gg 0$.

To conclude, it is enough to observe that, if $d \gg 0$ then $H^1(X, L_d) = 0$ so that, taking cohomology in the exact sequence

$$0 \longrightarrow \mathcal{I}_{Z_d/X} \otimes L_d \longrightarrow L_d \longrightarrow \mathcal{O}_{Z_d} \otimes L_d \longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow J_{0,d} \longrightarrow V_d \longrightarrow H^0(Z_d, \mathcal{O}_{Z_d} \otimes L_d) \longrightarrow H^1(X, \mathcal{I}_{Z_d/X} \otimes L_d) \longrightarrow 0$$

Now, this last sequence tells us that

$$h^0(Z_d, \mathcal{O}_{Z_d} \otimes L_d) - \dim W_{0,d} = h^1(X, \mathcal{I}_{Z_d/X} \otimes L_d)$$

and the right-hand term is bounded thanks to Lemma 3.3.1. □

We also need to consider the kernel bundle M_{L_d}:

Lemma 3.3.3. *The vector bundle $M_{L_d} \otimes \mathcal{O}_X(H)$ is globally generated if $d \gg 0$. In particular, M_{L_d} is a quotient of $\mathcal{O}_X(-H)^{\oplus N}$ for a certain N.*

Proof. We denote by M_H the kernel bundles corresponding to the line bundle $\mathcal{O}_X(H)$. We observe that the fibers at a point $p \in X$ of M_H and M_{L_d} are canonically identified with $H^0(X, \mathcal{I}_p \otimes \mathcal{O}_X(H))$ and $H^0(X, \mathcal{I}_p \otimes L_d)$ respectively and we have a natural multiplication map

$$H^0(X, L_d \otimes \mathcal{O}_X(-H)) \otimes H^0(X, L \otimes \mathcal{O}_X(H)) \longrightarrow H^0(X, \mathcal{I}_p \otimes L_d)$$

It can be seen that the above map globalizes to morphism of vector bundles

$$H^0(X, L_d \otimes \mathcal{O}_X(-H)) \otimes_{\mathbf{k}} M_H \longrightarrow M_{L_d}$$

and moreover this morphism is surjective for $d \gg 0$, because it is surjective at the level of stalks. To conclude, it is enough to observe that $M_H \otimes \mathcal{O}_X(H)$ is globally generated □

Proposition 3.3.1. *Suppose that for every $d \gg 0$ we have a quotient*

$$M_{L_d} \xrightarrow{s_d} T_d \otimes_{\mathbf{k}} \mathcal{O}_X \longrightarrow 0$$

for a certain vector space T_d of dimension $t_d \leq c$, where c is a positive integer independent of d. Then for any fixed divisor C the map

$$H^0(X, \wedge^{t_d} M_{L_d} \otimes L_d \otimes \mathcal{O}_X(C)) \longrightarrow H^0(X, \wedge^{t_d} T_d \otimes L_d \otimes \mathcal{O}_X(C)) = H^0(X, L_d \otimes \mathcal{O}_X(C))$$

determined by $\wedge^{t_d} s_d$ is surjective for $d \gg 0$.

Proof. By Lemma 3.3.3, we have a surjective map

$$\mathcal{O}_X(-H)^{\oplus N} \longrightarrow M_{L_d}$$

for a certain $N = N_d$. Then, we get another surjective map

$$u_d \colon \mathcal{O}_X(-H)^{\oplus N} \longrightarrow T_d \otimes_{\mathbf{k}} \mathcal{O}_X$$

Set $\phi_d \stackrel{\text{def}}{=} \wedge^{t_d} u_d$: then it is enough to show that the map

$$H^0(X, \wedge^{t_d}(\mathcal{O}_X(-H)^{\oplus N} \otimes L_d \otimes \mathcal{O}_X(C))) \longrightarrow H^0(X, L_d \otimes \mathcal{O}_X(C))$$

induced by ϕ_d is surjective if $d \gg 0$.

Now, ϕ_d is resolved by an Eagon-Northcott complex (see [Eis05]) that is of the form

$$\dots \longrightarrow \mathcal{O}_X(-(t_d+2)H) \otimes_{\mathbf{k}} S_2 \longrightarrow \mathcal{O}_X(-(t_d+1)H) \otimes_{\mathbf{k}} S_1 \longrightarrow \wedge^{t_d}(\mathcal{O}_X^{\oplus N})(-t_d H) \xrightarrow{\phi_d} \mathcal{O}_X \longrightarrow 0$$

where the s_i are certain vector spaces with dimension depending on t_d. Now, to get the surjectivity result that we need it is enough to prove that

$$H^i(X, L_d \otimes \mathcal{O}_X(C - (t_d+i)H)) = 0 \qquad \text{for all } i > 0$$

but since t_d is bounded independently by d, we can make this happen taking $d \gg 0$. □

3.3.2 The proof

Now we proceed to prove Theorem 3.3.1. We begin by setting notation: assume that we have data H, Z_d, W_d adapted to B: then we set $\overline{X} \stackrel{\text{def}}{=} X_n \in |H|$ and for every $1 \le j \le n-1$ we denote by $\overline{\overline{X}}_j$ the restriction of $\overline{X}_j$ to $\overline{X}$. We also denote by $\overline{Z}_d, \overline{L}_d, \overline{B}, \overline{H}, \overline{D}_i$ the restrictions of the corresponding objects to $\overline{X}$; we point out that $\overline{Z}_d = \overline{D}_1 \cap \dots \cap \overline{D}_c$.

Now set

$$V_d' \stackrel{\text{def}}{=} H^0(X, L_d \otimes \mathcal{I}_{\overline{X}/X}) = H^0(X, L_d \otimes \mathcal{O}_X(-H))$$

and define $\overline{V}_d \stackrel{\text{def}}{=} V_d/V_d'$, $v_d' \stackrel{\text{def}}{=} \dim V_d'$. We observe that $\overline{V}_d = H^0(\overline{X}, \overline{L}_d)$ for $d \gg 0$ and that $v_d' = v_d - O(d^{n-1})$.

Similarly, we set $W_d' \stackrel{\text{def}}{=} \pi_d(W_d), \overline{W}_d \stackrel{\text{def}}{=} W_d/W_d', \overline{w}_d \stackrel{\text{def}}{=} \dim \overline{W}_d$ and we denote as $\overline{J}_d \subseteq H^0(\overline{X}, \mathcal{I}_{\overline{Z}_d/\overline{X}} \otimes \overline{L}_d)$ the image of J_d on $\overline{X}$, so that $\overline{W}_d = \overline{V}_d/\overline{J}_d$.

Lemma 3.3.4. *If the data $H, \overline{X}_j, Z_d$ and W_d are adapted to B on X, then the restrictions $\overline{H}, \overline{\overline{X}}_j, \overline{Z}_d$ and $\overline{W}_d$ are adapted to $\overline{B} + \overline{H}$ on $\overline{X}$.*

Proof. First we remark that

$$(B - K_X)_{|\overline{X}} \equiv_{lin} \overline{B} + \overline{H} - K_{\overline{X}}$$

thanks to the adjunction formula. Then, the fact that $\overline{H}$ and $\overline{Z}_d$ satisfy the required conditions can be shown using the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\overline{X}} \longrightarrow 0$$

of sheaves on X.

Regarding the conditions for $\overline{W}_d$, it is easy to show that the sections in $\overline{J}_d$ generate the sheaf $\mathcal{I}_{\overline{Z}_d/\overline{X}} \otimes \overline{L}_d$, and it remains to show that

$$\dim \frac{H^0(\overline{X}, \mathcal{I}_{\overline{Z}_d/\overline{X}} \otimes \overline{L}_d)}{\overline{J}_d} \le a$$

for a certain $\overline{a}$ independent of d. In order to do this we consider

$$\overline{I}_d = \mathrm{Im}\,(H^0(X, \mathcal{I}_{Z_d/X} \otimes L_d) \longrightarrow H^0(\overline{X}, \mathcal{I}_{\overline{Z}_d/\overline{X}} \otimes \overline{L}_d))$$

as the image of the restriction map. Then the condition 3.12 tells us that $\dim \overline{I}_d / \overline{J}_d \leq a$, but there is also a natural embedding

$$\frac{H^0(X, \mathcal{I}_{\overline{Z}_d/\overline{X}} \otimes \overline{L}_d)}{\overline{I}_d} \subseteq H^1(X, \mathcal{I}_{Z_d/X} \otimes L_d \otimes \mathcal{O}_X(-H))$$

and then we can conclude thanks to Lemma 3.3.1. □

Lemma 3.3.5. *Suppose that H, X_j, Z_d and W_d are adapted to a divisor B.*

1. *If $q \geq 2$, then W_d carries weight q syzygies of B with respect to L_d, for $d \gg 0$.*
2. *If $q \geq 3$, then $\overline{W}_d$ carries weight $q-1$ syzygies of $\overline{B}+\overline{H}$ with respect to $\overline{L}_d$ for $d \gg 0$.*

Proof. We want to apply Theorem 3.2.1. In order to do this, consider the exact sequence

$$0 \longrightarrow \Sigma_{\overline{W}_d} \longrightarrow \overline{W}_d \otimes \mathcal{O}_{\overline{X}} \longrightarrow \mathcal{O}_{\overline{Z}_d} \otimes \overline{L}_d \longrightarrow 0$$

of sheaves on $\overline{X}$ and the quotient $M_{\overline{L}_d} \longrightarrow \Sigma_{\overline{W}_d}$ that induces the map $\sigma_d \colon \wedge^{\overline{w}_d} M_{\overline{L}_d} \longrightarrow \mathcal{I}_{\overline{Z}_d/\overline{X}}$.

Now we fix the codimension $c = n+2-q$ and we proceed to prove the result by induction on q. If $q=2$, we see that $\dim Z_d = 0$ so that $\overline{Z}_d = \emptyset$ and then the above exact sequence shows that $\Sigma_{\overline{W}_d} = \overline{W}_d \otimes_{\mathbf{k}} \mathcal{O}_{\overline{X}}$. Moreover, we also claim that the map

$$H^0(\overline{X}, \wedge^{\overline{w}_d} M_{\overline{L}_d} \otimes \overline{L}_d \otimes \mathcal{O}_{\overline{X}}(\overline{B}+\overline{H})) \longrightarrow H^0(\overline{X}, \overline{L}_d \otimes \mathcal{O}_{\overline{X}}(\overline{B}+\overline{H})) \tag{3.13}$$

induced by $\overline{\sigma}_d$ is surjective for $d \gg 0$. Indeed, we observe that $\overline{\sigma}_d$ is an exterior power of the quotient

$$M_{\overline{L}_d} \longrightarrow \Sigma_{\overline{W}_d} = \overline{W}_d \otimes \mathcal{O}_{\overline{X}}$$

and then the fact that $\overline{W}_d$ is adapted to $\overline{B}+\overline{H}$ (cf Lemma 3.3.4) tells us that $\overline{w}_d$ is bounded independently of d. Then the claim on surjectivity follows from Proposition 3.3.1.

Now, if $d \gg 0$, the hypotheses of Theorem 3.2.1 are satisfied thanks to Lemma 3.3.1, and then from the surjectivity of 3.13 we see that the statement (1) of the Lemma holds when $q=2$. But now, thanks to Lemma 3.3.4, we can apply this statement to the divisor $\overline{B}+\overline{H}$ on $\overline{X}$ and get the statement (2) of the Lemma in the case $q=3$. Now, using once again Theorem 3.2.1, we prove the statement (1) in the case $q=3$ and we continue like this. □

Finally, we prove Theorem 3.3.1:

Proof of Theorem 3.3.1. We just make use of Theorem 3.2.2. Indeed, using the surjectivity of 3.13 (for the case $q=2$), the Lemma 3.3.5 (for the case $q \geq 3$), as well as the Lemma 3.3.1 and the observation that $\overline{v}_d - \overline{w}_d = O(d^{n-1})$, we see that the hypotheses of Theorem 3.2.2 are satisfied, so that, if d is large enough, then

$$K_{p,q}(X, L_d; \mathcal{O}_X(B)) \neq 0 \qquad \text{for all } w_d + 1 - q \leq p \leq v'_d + \overline{w}_d + 1 - q$$

Now, the first part of the Theorem follows from Lemma 3.3.2 and the observation that $v'_d - v_d = O(d^{n-1})$.

To get the second part, it is enough to apply (for $d \gg 0$) the Duality Theorem 2.3.3, the part of the Theorem already proved and the statement for the groups $K_{p,1}$ given by Proposition 3.1.3. □

3.4 The asymptotic non-vanishing theorem for Veronese varieties

We now aim at proving Theorem 3.1.2: we already know that it is true in the case of $\mathbb{P}^1$ so that now we address the higher dimensional cases.

We begin with the extremal cases of $K_{p,0}$ and $K_{p,n}$.

Lemma 3.4.1. *Let $d \geq 0$ and $b \geq 0$ be two integers. Then*

1. *If $d \geq b+1$ then $K_{p,0}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d); \mathcal{O}_{\mathbb{P}^n}(b)) \neq 0$ if and only if*
$$0 \leq p \leq \binom{b+n}{n} - 1$$
2. *If $d \geq b+n+1$ then $K_{p,n}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}; \mathcal{O}_{\mathbb{P}^n}(d)) \neq 0$ if and only if*
$$\binom{d+n}{n} - \binom{d-b-1}{n} - n \leq p \leq \binom{d+n}{n} - n - 1$$

Proof. The point (1) follows from the proof of Proposition 3.1.2. The point (2) follows from point (1) by Duality Theorem in Remark 2.3.3. □

Next, we address the case of $K_{p,1}$.

Proposition 3.4.1. *Suppose $n \geq 2$ and take integers $b \geq 0$ and $d \geq b+2$. Then $K_{p,1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d); \mathcal{O}_{\mathbb{P}^n}(b)) \neq 0$ for*
$$b+1 \leq p \leq \binom{d+n-1}{n} - 1$$

Proof. We denote by M_d the syzygy bundle associated to $\mathcal{O}_{\mathbb{P}^n}(d)$. Then we know from Corollary 2.1.4 that
$$K_{p,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d); \mathcal{O}_{\mathbb{P}^n}(b)) \cong H^1(\mathbb{P}^n, \wedge^{p+1} M_d \otimes \mathcal{O}_{\mathbb{P}^n}(b))$$
Now, take a subset $Z \subseteq \mathbb{P}^n$ of $b+2$ collinear points, so that
$$H^1(\mathbb{P}^n, \mathcal{I}_{Z/\mathbb{P}^n}(b)) \neq 0, \qquad H^1(\mathbb{P}^n, \mathcal{I}_{Z/\mathbb{P}^n}(b+1)) = 0$$
Then we apply the constructions of Section 3.2 with
$$W = \operatorname{Im}\left(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \xrightarrow{\pi} H^0(Z, \mathcal{O}_Z(d))\right) = H^0(Z, \mathcal{O}_Z(d))$$
and in this way we get a mapping
$$\sigma\colon H^1(\mathbb{P}^n, \wedge^w M_d \otimes \mathcal{O}_{\mathbb{P}^n}(b)) \longrightarrow H^1(\mathbb{P}^n, \mathcal{I}_{Z/\mathbb{P}^n}(b))$$
with $w = \dim W = b+2$ and we want to prove that this is surjective. In order to do this, we observe that, using the same notations as in Section 3.2, we see that $W' = W$, $\overline{W} = 0$ and then the homomorphism $\overline{\sigma}$ of Theorem 3.2.1 is automatically surjective, so that we can apply the same reasoning as in the proof of that theorem.

This proves that $K_{p,1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d); \mathcal{O}_{\mathbb{P}^n}(b)) \neq 0$ for $p = b+1$. But now, as in the proof of Theorem 3.2.2, we can substitute W with a bigger quotient $U = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))/\Lambda$ with the same properties and such that $\dim U$ takes any value between $b+1$ and $\binom{n+d-1}{n}$. The conclusion follows. □

To conclude, we study the case $2 \leq q \leq n-1$.

Proposition 3.4.2. *Consider integers $2 \leq q \leq n-1$, $b \geq 0$ and $d \geq b+q+1$. Then $K_{p,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(b); \mathcal{O}_{\mathbb{P}^n}(d)) \neq 0$ for*
$$\binom{d+q}{q} - \binom{d-b-1}{q} - q \leq p \leq \binom{d+n-1}{n} + \binom{d+q-1}{q-1} - \binom{d-b-2}{q-1} - q$$

Proof. We begin by adapting some constructions of Section 3.2 to the case of projective space $\mathbb{P}^m$. Fix an integer $2 \le c \le m$, and integer $a \ge 0$ and set $s = m - c + 2$; assume, moreover, that $d \ge a + s + 1$. Now we choose general divisors

$$D_1, D_2, \dots, D_{c-2} \in |\mathcal{O}_{\mathbb{P}^m}(1)|, \qquad D_{c-1} \in |\mathcal{O}_{\mathbb{P}^m}(a+s+1)|, \qquad D_c \in |\mathcal{O}_{\mathbb{P}^m}(d)|$$

and we denote their complete intersection by

$$Z_{d,m,a} \stackrel{\text{def}}{=} D_1 \cap \dots \cap D_c$$

We see that $Z_{d,m,a}$ is a smooth variety of dimension $n - c = s - 2$ such that $\mathcal{I}_{Z_{d,m,a}/\mathbb{P}^m}(d)$ is generated by global sections and

$$H^{s-1}(\mathbb{P}^m, \mathcal{I}_{Z_{d,m,a}/\mathbb{P}^m}(d+a)) \neq 0, \qquad H^{s-1}(\mathbb{P}^m, \mathcal{I}_{Z_{d,m,a}/\mathbb{P}^m}(d+a+1)) = 0$$

Now we set

$$W_{d,m,a} = \operatorname{Im}\left(H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d)) \longrightarrow H^0(Z_{d,m,a}, \mathcal{O}_{Z_{d,m,a}}(d))\right)$$

then, if $\dim Z > 0$, one can show that $W_{d,m,a}$ is surjective and that

$$w_{d,m,a} \stackrel{\text{def}}{=} \dim W_{d,m,a} \stackrel{\text{def}}{=} \binom{d+s}{s} - \binom{d-a-1}{s} - 1$$

for every $s \ge 2$. Now, if we denote by $M_{m,d}$ the syzygy bundle on $\mathbb{P}^m$ associated to $\mathcal{O}_{\mathbb{P}^m}(d)$, we can get as in Section 3.2 a map

$$\sigma_{m,d,a} \colon \wedge^{w_{m,d,a}} M_{d,m} \longrightarrow \mathcal{I}_{Z_{d,m,a}/\mathbb{P}^m}$$

If we take a general hyperplane $\mathbb{P}^{m-1} \subseteq \mathbb{P}^m$, we see that the intersection

$$\overline{Z}_{d,m,a} = Z_{d,m,a} \cap \mathbb{P}^{m-1}$$

can be identified with $Z_{d,m-1,a+1}$ inside $\mathbb{P}^{m-1}$. In particular we see that c is unchanged and that s is decreased by 1. Using the same notations as in Section 3.2 we have an identification $\overline{W}_{d,m,a} = W_{d,m-1,a+1}$ if $c < m$. Instead, if $c = m$ we get $\dim Z_{m,d,a} = 0$ and

$$\overline{W}_{d,m,a} = \operatorname{Ker}\left(H^1(\mathbb{P}^m, \mathcal{I}_{Z_{d,m,a}/\mathbb{P}^m}(d-1)) \longrightarrow H^1(\mathbb{P}^m, \mathcal{I}_{Z_{d,m,a}})\right)$$

It can be shown that in this case

$$\overline{w}_{d,m,a} \stackrel{\text{def}}{=} \dim \overline{W}_{d,m,a} = a + 2$$

To prove our Proposition, we begin by applying this constructions with $m = n$, $a = b$ and $c = n + 2 - q$, for a fixed $2 \le q \le n$. We want to show that $W_{n,d,a}$ satisfies the hypotheses of Theorem 3.2.1 and Theorem 3.2.2. The strategy is to apply repeatedly Theorem 3.2.2 and use descending induction on $0 \le i \le 2$ to prove that the homomorphisms

$$H^{q-1-i}(\mathbb{P}^{n-i}, \wedge^{w_{d,n-i,b+1}} M_{d,n-i}(b+d-i)) \longrightarrow H^{q-1-i}(\mathbb{P}^{n-i}, \mathcal{I}_{Z_{d,n-i,b+i}/\mathbb{P}^n}(d+b+i))$$

determined by $\sigma_{d,n-i,b+i}$ are surjective. Thanks to the discussion above, the only issue is with the base of induction with $i = q - 2$. In this case, we need to prove the surjectivity of the map

$$H^0(\mathbb{P}^{n+1-q}, \wedge^{\overline{w}_{d,n+2-q,b+q-2}} M_{d,n+1-q}(b+d+q-1)) \longrightarrow H^0(\mathbb{P}^{n+1-s}, \mathcal{O}_{\mathbb{P}^{n+1-q}}(b+d+q-1))$$

induced by

$$\overline{\rho}_{d,n+1-q,b+q-2} \colon M_{d,n+1-q} \longrightarrow \overline{W}_{d,n+2-q,b+q-2} \otimes_{\mathbf{k}} \mathcal{O}_{\mathbb{P}^{n+1-q}}$$

but this follows using the Eagon-Northcott complex as in the proof of Proposition 3.3.1 and the fact that $M_{d,n+1-q}(1)$ is globally generated (by Castelnuovo-Mumford regularity for example).

Now, we can apply Theorem 3.2.2 that in this case tells us that $K_{p,q}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d); \mathcal{O}_{\mathbb{P}^n}(b)) \neq 0$ for

$$w_{d,n,b} + 1 - q \le p \le h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)) + \overline{w}_{d,n,b} + 1 - q$$

and we conclude observing that

$$\overline{w}_{d,n,b} = w_{d,n-1,b+1} = \binom{d-q-1}{q-1} - \binom{d-b-2}{q-1} - 1$$

□

Chapter 4

Asymptotic normality of Betti numbers

In the previous chapter we have seen results about the asymptotic shape of the Betti tables. It is then natural to ask ourselves what we can say about the asymptotic values of the Betti numbers.

In this chapter, we want to present some results about the asymptotic distribution of these values for curves and give a conjecture for higher-dimensional varieties. The questions addressed here are inspired by the paper [EEL13], of L. Ein, D. Erman and R. Lazarsfeld.

4.1 Asymptotic Betti numbers of rational normal curves

To begin with, we study the rational normal curves $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$: indeed, they are a perfect example to examine, since we know explicitly from Subsection 2.7.1 their Betti table:

	0	1	2	...	p	$p+1$	...	$d-1$
0	1	–	–	...	–	–	–	
1	–	$\binom{d}{2}$	$2\binom{d}{3}$	...	$p\binom{d}{p+1}$	$(p+1)\binom{d}{p+2}$	...	$d-1$

In particular, we see that the Betti numbers $k_{p,0}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ are uninteresting, and we concentrate on the Betti numbers $k_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. It is enlightening to plot this Betti numbers for various values of d:

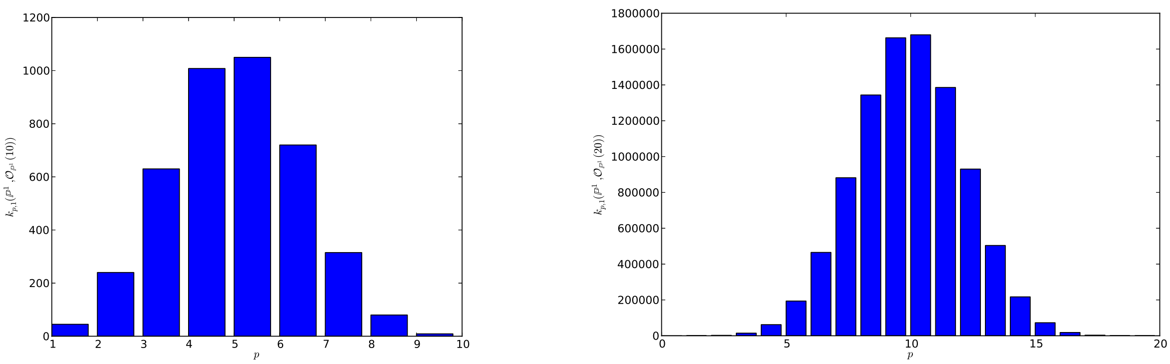


Figure 4.1: The plot on the left shows the Betti numbers $k_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(10))$ and the plot on the right shows the Betti numbers $k_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(20)$.

It is then quite natural from these figures to conjecture that the Betti numbers $k_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ tend to a normal distribution as $d \longrightarrow +\infty$.

More precisely, for every $d \geq 0$ we fix a random variable X_d with natural values such that

$$\mathbb{P}(X_d = p) = \frac{k_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))}{\sum_{p \in \mathbb{Z}} k_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))} \qquad \text{for all } p \in \mathbb{N}$$

then we are going to prove the following result.

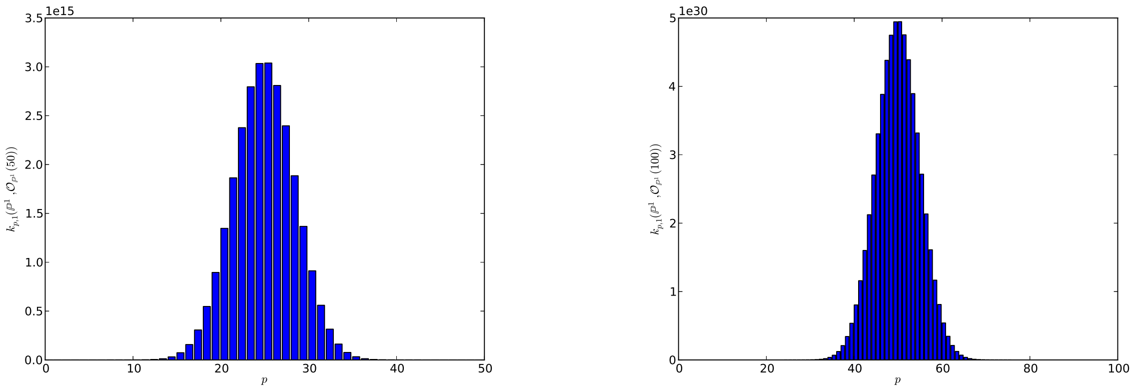


Figure 4.2: The plot on the left shows the Betti numbers $k_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(50))$ and the plot on the right shows the Betti numbers $k_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(100)$.

Proposition 4.1.1. *We have that*

$$\frac{X_d - \mathbb{E}[X_d]}{\sqrt{Var[X_d]}} \longrightarrow \mathcal{N}(0,1)$$

in distribution as $d \to +\infty$.

Before attacking this problem, we need some preliminaries from probability.

4.1.1 Discrete random variables and generating functions

The law of a discrete random variable is completely determined by its probability generating function.

Definition 4.1.1 (Probability generating function)**.** Let X be a random variable with values in $\mathbb{N}$. Then we define the **probability generating function** (or **PGF**) of X as the power series

$$G_X(z) = \mathbb{E}[z^X] = \sum_{n \geq 0} \mathbb{P}(X = n) z^n$$

Definition 4.1.2 (Moment generating function)**.** Let X be a random variable such that $\mathbb{E}[X^n] < +\infty$ for every $n \geq 0$. Then we define the **moment generating function** (or **MGF**) of X as the power series

$$\psi_X(z) = \mathbb{E}[e^{zX}] = \sum_{n \geq 0} \frac{\mathbb{E}[X^n]}{n!} z^n$$

Lemma 4.1.1. *Let X be a random variable that takes a finite number of values in $\mathbb{N}$. Then*

1. *G_X is a polynomial.*
2. *$\psi_X(t) = \mathbb{E}[e^{tX}] = G_X(e^t)$.*
3. *$\mathbb{E}[X] = G'_X(1)$ and $\mathbb{E}[X^2] = G''_X(1) + G'_x(1)$.*
4. *$Var[X] = G''_X(1) + G'_X(1) - G'_X(1)^2$.*

Proof.
1. This is clear.
2. This is also clear.
3. By definition, we see that $\mathbb{E}[X^n] = \psi_X^{(n)}(0)$ so that the thesis follows from point (2).
4. This follows from point (3) and the formula $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$.

□

The moment generating function gives us a criterion for convergence to a normal distribution.

Lemma 4.1.2. *Let X_n be a sequence of random variables that take finite values in $\mathbb{N}$. If*

$$\psi_{X_n}(t) \longrightarrow e^{\frac{t^2}{2}} \qquad \text{for all } t \in \mathbb{R}$$

then $X_n \longrightarrow \mathcal{N}(0,1)$ in distribution.

Proof. See [Fel71]. □

Corollary 4.1.1. *Let X_n be a sequence of random variables that take finite values in $\mathbb{N}$. Set $m_n \overset{def}{=} \mathbb{E}[X_n]$ and $\sigma_d \overset{def}{=} \sqrt{Var[X_n]}$. If*

$$e^{-\frac{m_n}{\sigma_n}t} G_{X_n}(e^{\frac{t}{\sigma_n}}) \longrightarrow e^{\frac{t^2}{2}} \qquad \text{for all } t \in \mathbb{R}$$

then

$$\frac{X_n - m_n}{\sigma_n} \longrightarrow \mathcal{N}(0,1)$$

in distribution.

Proof. Thanks to Lemma 4.1.2, it is enough to observe that

$$\psi_{\frac{X_n - m_n}{\sigma_n}}(t) = \mathbb{E}[e^{t\frac{X_n - m_n}{\sigma_n}}] = e^{-\frac{m_n}{\sigma_n}t}\mathbb{E}[e^{\frac{t}{\sigma_n}X_n}] = e^{-\frac{m_n}{\sigma_n}t} G_{X_n}(e^{\frac{t}{\sigma_n}})$$

□

Random variables from power series

We have seen before that to a discrete random variable we can associate a power series with positive coefficients. Conversely, given a power series with positive coefficients, we can associate to it a random variable, or better a distribution on $\mathbb{N}$:

Definition 4.1.3 (Random variable associated to a power series)**.** Consider a power series

$$F(z) = \sum_{n \geq 0} p_n z^n$$

with nonnegative conefficients not all zero, which converges on the whole of $\mathbb{C}$. Then a random variable X with values in $\mathbb{N}$ is said to be **associated to** F if its law is given by

$$\mathbb{P}(X = n) = \frac{p_n}{F(1)}$$

Lemma 4.1.3. *Let X be a random variable associated to the power series F, as above. Then*

1. $G_X(z) = \frac{F(z)}{F(1)}$.
2. $\mathbb{E}[X] = \frac{F'(1)}{F(1)}$.
3. $Var[X] = \frac{F''(1)}{F(1)} + \frac{F'(1)}{F(1)} - \left(\frac{F'(1)}{F(1)}\right)^2$.

Proof. The first point is clear from the definition of a random variable associated to a power series. The other points follow from the first point and Lemma 4.1.1. □

4.1.2 Asymptotic normality for Betti numbers of rational normal curves

With the language that we have just introduced, Proposition 4.1.1 can be formulated as follows: for every $d \geq 1$ we define the polynomial

$$F_d(z) \stackrel{\text{def}}{=} \sum_{p\geq 0} k_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))z^p = \sum_{p=1}^{d-1} p\binom{d}{p+1}z^p$$

we also define X_d to be a random variable associated to F_d and we set

$$m_d \stackrel{\text{def}}{=} \mathbb{E}[X_d], \qquad \sigma_d = \sqrt{\text{Var}[X_d]}$$

Then we want to prove that

$$\frac{X_d - m_d}{\sigma_d} \longrightarrow \mathcal{N}(0,1)$$

in distribution, as $d \to +\infty$.

Lemma 4.1.4. *With notations as before we have*

1. $F_d(z) = d(1+z)^{d-1} - \frac{(1+z)^d}{z} + 1.$
2. $F_d(1) = \frac{d}{2}2^d + 1.$
3. $m_d \sim \frac{d}{2}$ *as* $d \to +\infty$.
4. $\sigma_d \sim \frac{\sqrt{d}}{2}$ *as* $d \to +\infty$.

Proof. 1. This follows from a simple computation:

$$\begin{aligned} F_d(z) &= \sum_{p=1}^{d-1} p\binom{d}{p+1}z^p = z\sum_{p=1}^{d-1} p\binom{d}{p+1}z^{p-1} = z\cdot\frac{d}{dz}\left[\sum_{p=1}^{d-1}\binom{d}{p+1}z^p\right] = z\cdot\frac{d}{dz}\left[\sum_{p=2}^{d}\binom{d}{p}z^{p-1}\right] \\ &= z\cdot\frac{d}{dz}\left[\frac{1}{z}\sum_{p=2}^{d}\binom{d}{p}z^p\right] = z\cdot\frac{d}{dz}\left[\frac{(1+z)^d-(1+dz)}{z}\right] \\ &= z\left[\frac{(d(1+z)^{d-1}-d)z-((1+z)^d-(1+dz))}{z^2}\right] = d(1+z)^{d-1}-d-\frac{(1+z)^d}{z}+\frac{1+dz}{z} \\ &= d(1+z)^{d-1}-\frac{(1+z)^d}{z}+1 \end{aligned}$$

2. From point (1) we see that

$$F_d(1) = d2^{d-1} - 2^d + 1 = (d-2)2^{d-1} + 1$$

and then the asymptotic estimate is obvious.

3. After a small computation, we see that the derivative of $F_d(z)$ is given by

$$F_d'(z) = d(d-1)(1+z)^{d-2} - d\frac{(1+z)^{d-1}}{z} + \frac{(1+z)^d}{z^2}$$

so that

$$F_d'(1) = \frac{d^2-3d+4}{4}2^d$$

and then we obtain

$$m_d = \frac{F_d'(1)}{F_d(1)} = \frac{1}{2}\left((d-1)+\frac{2}{d-2}\right) \sim \frac{d}{2}$$

for $d \to +\infty$.

4. Computing the second derivative of $F_d(z)$ we get

$$F_d''(z) = d(d-1)(d-2)(1+z)^{d-3} - d(d-1)\frac{(1+z)^{d-2}}{z} + 2d\frac{(1+z)^{d-1}}{z^2} - 2\frac{z(1+z)^d}{z^4}$$

so that

$$F_d''(1) = \frac{d^3 - 5d^2 + 12d - 16}{8} 2^d$$

and

$$\frac{F_d''(1)}{F_d(1)} = \frac{1}{4}\left(d^2 - 3d + 6 - \frac{4}{d-2}\right)$$

To conclude, we just need to use the formula for the variance of Lemma 4.1.3.

$$\sigma_d^2 = \frac{F_d''(1)}{F_d(1)} + \frac{F_d'(1)}{F_d(1)} - \left(\frac{F_d'(1)}{F_d(1)}\right)^2 \sim \frac{1}{4}(d^2 - 3d + 6) + \frac{1}{2}(d-1) - \frac{1}{4}(d^2 - 2d + 1)$$
$$\sim \frac{1}{4}d^2 - \frac{3}{4}d + \frac{1}{2}d - \frac{1}{4}d^2 + \frac{1}{2}d = \frac{d}{4}$$

□

Now we can prove Proposition 4.1.1:

Proof of Proposition 4.1.1. Thanks to Lemma 4.1.1 and Lemma 4.1.3, it is enough to prove that for every fixed $t \in \mathbb{R}$ we have

$$\lim_{d\to+\infty} e^{-\frac{m_d}{\sigma_d}t} \frac{F_d\left(e^{\frac{t}{\sigma_d}}\right)}{F_d(1)} = e^{\frac{t^2}{2}}$$

Thanks to the estimates of Lemma 4.1.4, the limit that we have to compute is

$$\lim_{d\to+\infty} \frac{2e^{-\sqrt{d}t}}{d2^d} F_d\left(e^{\frac{2t}{\sqrt{d}}}\right) = \lim_{d\to+\infty} \frac{2e^{-\sqrt{d}t}}{d2^d}\left[d\left(1+e^{\frac{2t}{\sqrt{d}}}\right)^{d-1} - \frac{\left(1+e^{\frac{2t}{\sqrt{d}}}\right)^d}{e^{\frac{2t}{\sqrt{d}}}} + 1\right]$$

Now, we observe that, as $d \to +\infty$, we have $e^{\frac{2t}{\sqrt{d}}} \to 1$ and $\frac{2e^{-\sqrt{d}t}}{d2^d} \to 0$, so that the limit reduces to

$$\lim_{d\to+\infty} \frac{2e^{-\sqrt{d}t}}{d2^d}\left[d\left(1+e^{\frac{2t}{\sqrt{d}}}\right)^{d-1} - \left(1+e^{\frac{2t}{\sqrt{d}}}\right)^d\right] = \lim_{d\to+\infty} \frac{2e^{-\sqrt{d}t}}{d2^d}\left(1+e^{\frac{2t}{\sqrt{d}}}\right)^d\left[\frac{d}{\left(1+e^{\frac{2t}{\sqrt{d}}}\right)} - 1\right]$$
$$= \lim_{d\to+\infty} \frac{2e^{-\sqrt{d}t}}{2^d}\left(1+e^{\frac{2t}{\sqrt{d}}}\right)^d\left[\frac{1}{\left(1+e^{\frac{2t}{\sqrt{d}}}\right)} - \frac{1}{d}\right] = \lim_{d\to+\infty} \frac{e^{-\sqrt{d}t}}{2^d}\left(1+e^{\frac{2t}{\sqrt{d}}}\right)^d$$

We can write

$$\frac{e^{-\sqrt{d}t}}{2^d}\left(1+e^{\frac{2t}{\sqrt{d}}}\right)^d = \frac{\left(e^{-\frac{t}{\sqrt{d}}}\right)^d}{2^d}\left(1+e^{\frac{2t}{\sqrt{d}}}\right)^d = \left(\frac{e^{\frac{t}{\sqrt{d}}} + e^{-\frac{t}{\sqrt{d}}}}{2}\right)^d$$

but we observe that

$$e^{\frac{t}{\sqrt{d}}} = 1 + \frac{t}{\sqrt{d}} + \frac{t^2}{2d} + o\left(\frac{1}{d}\right)$$

as $d \to +\infty$, so that

$$e^{\frac{t}{\sqrt{d}}} + e^{-\frac{t}{\sqrt{d}}} = 2\left(1 + \frac{t^2}{2d}\right) + o\left(\frac{1}{d}\right)$$

and then

$$\lim_{d\to+\infty}\left(\frac{e^{\frac{t}{\sqrt{d}}} + e^{-\frac{t}{\sqrt{d}}}}{2}\right)^d = e^{\frac{t^2}{2}}$$

as we wanted. □

4.2 Asymptotic normality for Betti numbers of smooth curves

Now we want to prove a generalization of the above result to curves of arbitrary genus. More precisely, let C be smooth, connected projective curve of genus g, and, for every $d \gg 0$, let L_d be a line bundle on C of degree d.

Then for every d we consider the random variable X_d with distribution

$$\mathbb{P}(X_d = p) = \frac{k_{p,1}(C, L_d)}{\sum_{n\geq 0} k_{n,1}(C, L_d)} \qquad \text{for all } p \in \mathbb{N}$$

and we want to prove the following.

Theorem 4.2.1. *As $d \to +\infty$ it holds that $\mathbb{E}[X_d] \sim \frac{d}{2}$, $Var[X_d] \sim \frac{d}{4}$ and moreover*

$$\frac{X_d - \mathbb{E}[X_d]}{\sqrt{Var[X_d]}} \longrightarrow \mathcal{N}(0,1)$$

in distribution.

Remark 4.2.1. We recall from Remark 2.6.5 that the Betti table of (C, L_d) is given by

	0	1	2	3	...	$d-2g-2$	$d-2g-1$	$d-2g$	...	$d-g-1$
0	1	–	–	–	...	–	–	–	...	–
1	–	$*$	$*$	$*$	...	$*$	$*$	?	...	?
2	–	–	–	–	...	–	–	?	...	?

In particular, as we have already observed, the segment in which $k_{p,2}(C, L)$ could be different from zero has constant lenght, so that it becomes negligible with respect to the lenght of the table. This is the reason why we restrict our attention to the Betti numbers $k_{p,1}$.

To prove the above proposition, we need to study the Betti numbers $k_{p,1}$ for a curve of arbitrary genus.

Proposition 4.2.1. *Let C be a smooth connected projective curve of genus g and let L_d be a line bundle of degree d on C. Then, if $d \gg 0$ we have that*

1. $k_{p,1}(C, L_d) = \binom{d-g}{p}(d+1-g) - \binom{d-g-1}{p-1}d - \binom{d+1-g}{p+1}$ *for* $1 \leq p \leq d - 2g - 1$.
2. $k_{p,1}(C, L_d) \leq (d+1-g)\binom{d+1-g}{p}$ *for all* p.

Proof. By Riemann-Roch, we know that $h^0(C, L_d) = d + 1 - g$ and now the second point is immediate, as $K_{p,1}(C, L_d)$ is by definition a quotient of a subspace of $\wedge^p H^0(X, L_d) \otimes H^0(X, L_d)$.

For the first point, we can proceed as we have done for the elliptic normal curves, in Subsection 2.7.2. We know from Proposition 2.1.1 that we have an exact sequence

$$0 \longrightarrow H^0(C, \wedge^{p+1} M_{L_d}) \longrightarrow \wedge^{p+1} H^0(C, L_d) \longrightarrow H^0(C, \wedge^p M_{L_d} \otimes L_d) \longrightarrow K_{p,1}(C, L_d) \longrightarrow 0$$

so that

$$k_{p,1}(C, L_d) = h^0(C, \wedge^p M_{L_d} \otimes L_d) - \binom{d+1-g}{p+1} + h^0(C, \wedge^{p+1} M_{L_d})$$

Now suppose that $1 \leq p \leq d - 2g - 1$, then from the Betti table of (C, L_d) and Corollary 2.1.3 we see that $h^0(C, \wedge^{p+1} M_{L_d}) = 0$, whereas from Corollary 2.1.1 we see that $H^1(X, \wedge^p M_L \otimes L) \cong K_{p-1,2}(X, L) = 0$. Then

$$k_{p,1}(C, L_d) = \chi(C, \wedge^p M_{L_d} \otimes L_d) - \binom{d+1-g}{p+1}$$

and we can conclude thanks to Lemma 2.7.1. □

Now, for every $d \gg 0$ we define the polynomial

$$F_d(z) \stackrel{\text{def}}{=} \sum_{p=0}^{+\infty} k_{p,1}(C, L_{d+g}) z^p$$

and if we set $Y_d \stackrel{\text{def}}{=} X_{d+g}$ we see that the random variable Y_d is associated to F_d by construction.

We also define the polynomials

$$A_d(z) \stackrel{\text{def}}{=} -(d+1)\left(1 + \sum_{p=d-g}^{d} \binom{d}{p} z^p\right), \qquad B_d(z) \stackrel{\text{def}}{=} (d+g)z\left(\sum_{p=d-g-1}^{d-1} \binom{d-1}{p} z^p\right)$$

$$C_d(z) \stackrel{\text{def}}{=} \frac{1}{z}\left[1 + (1+d)z + \sum_{p=d-g+1}^{d+1} \binom{d+1}{p} z^p\right], \quad D_d(z) \stackrel{\text{def}}{=} \sum_{p=d-g}^{d-1} k_{p,1}(C, L_{d+g}) z^p$$

and we observe that these polynomials have always the same number of terms, indipendently of d.

As the last ingredient, we define another polynomial

$$H_d(z) \stackrel{\text{def}}{=} (d+1)(1+z)^d - (d+g)z(1+z)^{d-1} - \frac{(1+z)^{d+1}}{z}$$

Lemma 4.2.1. *1. If $d \gg 0$ then*

$$F_d(z) = H_d(z) + A_d(z) + B_d(z) + C_d(z) + D_d(z)$$

2. *$F_d(1) \sim \frac{d}{2} 2^d$ as $d \to +\infty$.*
3. *$\mathbb{E}[Y_d] \sim \frac{d}{2}$ as $d \to +\infty$*
4. *$Var[Y_d] \sim \frac{d}{4}$ as $d \to +\infty$.*

Proof. 1. If $d \gg 0$ we see that $k_{0,1}(C, L_{d+g}) = 0$ from the Betti table, so that

$$F_d(z) = \sum_{p=1}^{d-g-1} k_{p,1}(C, L_{d+g}) z^p + D_d(z)$$

by definition of $D_d(z)$. Now, from Proposition 4.2.1, we see that

$$\sum_{p=1}^{d-g-1} k_{p,1}(C, L_{d+g}) z^p = (d+1) \sum_{p=1}^{d-g-1} \binom{d}{p} (d+1) z^p - (d+g) \sum_{p=1}^{d-g-1} \binom{d-1}{p-1} z^p - \sum_{p=1}^{d-g-1} \binom{d+1}{p+1} z^p$$

and we proceed to analyze the three summands separately. The first one is

$$(d+1) \sum_{p=1}^{d-g-1} \binom{d}{p} (d+1) z^p = (d+1)\left[(1+z)^d - \left(1 + \sum_{p=1}^{d-g-1} \binom{d}{p} z^p\right)\right] = (d+1)(1+z)^d + A_d(z)$$

the second one is

$$(d+g) \sum_{p=1}^{d-g-1} \binom{d-1}{p-1} z^p = (d+g) \sum_{p=0}^{d-g-2} \binom{d-1}{p} z^{p+1} = (d+g)z[(1+z)^d - B_d(z)]$$

the third one is

$$\sum_{p=1}^{d-g-1} \binom{d+1}{p+1} z^p = \sum_{p=2}^{d-g} \binom{d+1}{p} z^{p-1} = \frac{(1+z)^{d+1}}{z} - C_d(z)$$

and we conclude.

2. We begin by computing
$$H_d(1) = \frac{d-2-g}{2}2^d$$
and then we observe that $A_d(1), B_d(1), C_d(1)$ and $D_d(1)$ are at most polynomial in d: this follows from the fact that they always have the same number of terms, indipendently of d and (for D_d) from the estimate of Proposition 4.2.1. Then, as $d \to +\infty$ it follows that $F_d(1) \sim H_d(1) \sim \frac{d}{2}2^d$.

3. To get the mean $\mathbb{E}[Y_d]$ we have to estimate the ratio $\frac{F_d'(1)}{F_d(1)}$. However, we see that $A_d'(1), B_d'(1), C_d'(1)$ and $D_d'(1)$ are at most polynomial in d for the same reason as before, whereas $H_d(1) \sim d2^{d-1}$. Hence, as far as only asymptotics is concerned, we can replace F_d with H_d, that is $\frac{F_{d+g}'(1)}{F_{d+g}(1)} \sim \frac{H_d'(1)}{H_d(1)}$ as $d \to +\infty$. Now, it is easy to compute that
$$H_d'(z) = d(d+1)(1+z)^{d-1} - (d+g)(1+z)^{d-1} - (d-1)(d+g)z(1+z)^{d-2} - (d+1)\frac{(1+z)^d}{z} + \frac{(1+z)^{d+1}}{z^2}$$
and then we see that
$$H_d'(1) = \frac{d^2-(3+g)d+(4+g)}{4}2^d$$
and consequently, as $d \to +\infty$ we have
$$\mathbb{E}[Y_d] \sim \frac{H_d'(1)}{H_d(1)} = \frac{d-1}{2} + \frac{1}{d-2-g}$$

4. We have to estimate
$$\mathrm{Var}[Y_d] = \frac{F_d''(1)}{F_d(1)} + \frac{F_d'(1)}{F_d(1)} - \left(\frac{F_d'(1)}{F_d(1)}\right)^2$$
but reasoning as in the previous point, we can substitute F_d with H_d, and after some computations we get the desired result.

□

Proof of Theorem 4.2.1. Since $Y_d = X_{d+g}$, we can just work with the Y_d. The estimates for the mean and the variance follow from the previous lemma. Now, we have to prove that
$$\frac{Y_d - \mathbb{E}[Y_d]}{\sqrt{\mathrm{Var}[Y_d]}} \longrightarrow \mathcal{N}(0,1)$$
and to this end it is sufficient to prove that
$$\lim_{d\to+\infty} e^{-\frac{\mathbb{E}[Y_d]t}{\sqrt{\mathrm{Var}[Y_d]}}} \frac{F_d(e^{\frac{t}{\sqrt{\mathrm{Var}[Y_d]}}})}{F_d(1)} = e^{\frac{t^2}{2}}$$
for every fixed $t \in \mathbb{R}$. Thanks to the estimates in the previous lemma, this limit is the same as
$$\lim_{d\to+\infty} \frac{2e^{-\sqrt{d}t}}{d2^d} F_d\left(e^{\frac{t}{\sqrt{d}}}\right)$$
We claim that
$$\lim_{d\to+\infty} \frac{2e^{-\sqrt{d}t}}{d2^d} A_d\left(e^{\frac{t}{\sqrt{d}}}\right) = 0$$

and indeed

$$\lim_{d\to+\infty}\frac{e^{-\sqrt{d}t}}{d2^d}A_d\left(e^{\frac{t}{\sqrt{d}}}\right) = -\lim_{d\to+\infty}\frac{e^{-\sqrt{d}t}}{2^d}\frac{d+1}{d}\left(1+\sum_{p=d-g}^{d}\binom{d}{p}e^{\frac{pt}{\sqrt{d}}}\right)$$

$$= \lim_{d\to+\infty}\left(\frac{e^{-\sqrt{d}t}}{2^d}+\frac{e^{\frac{(d-g)t}{\sqrt{d}}-\sqrt{d}t}}{2^d}\sum_{p=0}^{g}\binom{d}{p}e^{\frac{pt}{\sqrt{d}}}\right)$$

$$= \lim_{d\to+\infty}\frac{e^{\frac{(d-g)t}{\sqrt{d}}-\sqrt{d}t}}{2^d}P(d)$$

where $P(d)$ is a polynomial in d. And then one can easily see that this limit is zero.

In the same way, we can show that

$$\lim_{d\to+\infty}\frac{2e^{-\sqrt{d}t}}{d2^d}B_d\left(e^{\frac{t}{\sqrt{d}}}\right) = \lim_{d\to+\infty}\frac{2e^{-\sqrt{d}t}}{d2^d}C_d\left(e^{\frac{t}{\sqrt{d}}}\right) = \lim_{d\to+\infty}\frac{2e^{-\sqrt{d}t}}{d2^d}D_d\left(e^{\frac{t}{\sqrt{d}}}\right) = 0$$

and then what remains to be proved is that

$$\lim_{d\to+\infty}\frac{2e^{-\sqrt{d}t}}{d2^d}H_d\left(e^{\frac{t}{\sqrt{d}}}\right) = e^{\frac{t^2}{2}}$$

but it is easy to show that

$$\lim_{d\to+\infty}\frac{2e^{-\sqrt{d}t}}{d2^d}H_d\left(e^{\frac{t}{\sqrt{d}}}\right) = \lim_{d\to+\infty}\frac{2(d-g-2)}{2d}\frac{e^{-\sqrt{d}t}}{2^d}\left(1+e^{\frac{t}{\sqrt{d}}}\right)^d = \lim_{d\to+\infty}\frac{e^{-\sqrt{d}t}}{2^d}\left(1+e^{\frac{t}{\sqrt{d}}}\right)^d$$

and we have already seen in the proof of Proposition 4.1.1 that this last limit is precisely $e^{\frac{t^2}{2}}$. □

Ein, Erman and Lazarsfeld [EEL13] have conjectured that this asymptotic normality remains true for higher dimensions as well.

Conjecture 1. *Let X be a smooth connected projective variety of dimension n. Let A be an ample line bundle on X ans for every $d \geq 0$ set $L_d = A^{\otimes d}$. Then fix an index $1 \leq q \leq n$ and consider for every d a random variable Y_d with distribution*

$$\mathbb{P}(Y_d = p) = \frac{k_{p,q}(X, L_d)}{\sum_{h=0}^{+\infty} k_{h,q}(X, L_d)} \qquad \textit{for all } p \geq 0$$

Then, as $d \to +\infty$

$$\frac{Y_d - \mathbb{E}[Y_d]}{\sqrt{Var\left[Y_d\right]}} \longrightarrow \mathcal{N}(0,1)$$

in distribution.

We remark that the techniques that we have used for curves were numerical: we were able to prove our results because we had an explicit knowledge of the Betti numbers. In higher dimensions, we do not have this knowledge, and the problem appears to be much more difficult. In the next two chapters, we are going to expose some techniques that could be used to attack it in the case of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$.

Chapter 5

Cohomology of homogeneous vector bundles

In this chapter we are going to present a method to compute the cohomology of homogeneous vector bundles on homogeneous varieties, following [OR06]. The running example for this chapter will be that of the projective space $\mathbb{P}(V)$.

5.1 Notations and preliminaries

Throughout this chapter we fix a semisimple, connected linear algebraic group G, a maximal torus $T < G$ and a Borel subgroup $B < G$ such that $T < B$.

We denote by $X^*(T)$ the group of characters of T, by $W = N_G(T)/Z_G(T)$ the Weyl group of T, by $\Lambda_{\mathbb{R}}(T) = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ the real vector space spanned by the characters and we fix on $\Lambda_{\mathbb{R}}(T)$ a W-invariant positive-definite scalar product $(\cdot, \cdot)$.

Example 5.1.1 (The special linear group - I). Let n be a positive integer and consider the groups $\widetilde{G} = GL(n+1)$ and $G = SL(n+1)$. We choose the maximal tori $\widetilde{T} < \widetilde{G}$ and $T < G$ given by diagonal matrices

$$\widetilde{T} = \left\{ \left. \begin{pmatrix} t_0 & & & \\ & t_1 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \right| t_i \in \mathbf{k}^* \right\} \qquad T = \widetilde{T} \cap G = \left\{ \left. \begin{pmatrix} t_0 & & & \\ & t_1 & & \\ & & \ddots & \\ & & & t_n \end{pmatrix} \right| t_i \in \mathbf{k}^*,\ t_0 \dots t_n = 1 \right\}$$

and the Borel subgroups $\widetilde{B} < \widetilde{G}$ and $B < G$ given by lower-triangular matrices

$$\widetilde{B} = \left\{ \left. \begin{pmatrix} t_0 & & & \\ * & t_1 & & \\ \vdots & \vdots & \ddots & \\ * & * & * & t_n \end{pmatrix} \right| t_i \in \mathbf{k}^* \right\} \qquad B = \widetilde{B} \cap G = \left\{ \left. \begin{pmatrix} t_0 & & & \\ * & t_1 & & \\ \vdots & \vdots & \ddots & \\ * & * & * & t_n \end{pmatrix} \right| t_i \in \mathbf{k}^*,\ t_0 t_1 \dots t_n = 1 \right\}$$

For every $i = 0, \dots, n$ we denote by $\widetilde{\chi}_i$ and χ_i the characters of $\widetilde{T}$ and T respectively given by

$$\widetilde{\chi}_i \colon \widetilde{T} \longrightarrow \mathbb{G}_m \qquad t \mapsto t_i \qquad\qquad \chi_i = \widetilde{\chi}_{i|T} \colon T \longrightarrow \mathbb{G}_m \qquad t \mapsto t_i$$

Then we see easily that $X^*(\widetilde{T}) = \mathbb{Z}\widetilde{\chi_0} \oplus \dots \oplus \mathbb{Z}\widetilde{\chi_n}$ and that $X^*(T)$ is generated by the χ_i, so that we have a surjective homomorphism

$$X^*(\widetilde{T}) \longrightarrow X^*(T) \qquad \widetilde{\chi}_i \mapsto \chi_i$$

that gives an isomorphism

$$X^*(T) \cong \mathbb{Z}\widetilde{\chi_0} \oplus \dots \oplus \mathbb{Z}\widetilde{\chi_n} / \langle \widetilde{\chi}_0 + \dots + \widetilde{\chi}_n \rangle$$

In particular, tensoring with $\cdot \otimes_{\mathbb{Z}} \mathbb{R}$ we see that

$$\Lambda_{\mathbb{R}}(\widetilde{T}) = \mathbb{R}\widetilde{\chi_0} \oplus \ldots \oplus \mathbb{R}\widetilde{\chi_n} \qquad \Lambda_{\mathbb{R}}(T) \cong \mathbb{R}\widetilde{\chi_0} \oplus \ldots \oplus \mathbb{R}\widetilde{\chi_n} / \langle \widetilde{\chi}_0 + \cdots + \widetilde{\chi}_n \rangle$$

Now, it is a simple exercise to show that the Weyl group of both $\widetilde{T}$ and T is the symmetric group $W = S_{n+1}$ that acts by permutations of the diagonal elements. In particular, we see that the standard euclidean product on $\Lambda_{\mathbb{R}}(\widetilde{T})$ is W-invariant, and since the action of W leaves the subspace $\langle \widetilde{\chi}_0 + \cdots + \widetilde{\chi}_n \rangle$ fixed, we get a W-equivariant isomorphism $\Lambda_{\mathbb{R}}(T) \cong \langle \widetilde{\chi}_0 + \cdots + \widetilde{\chi}_n \rangle^{\perp}$, that induces a W-invariant scalar product on $\Lambda_{\mathbb{R}}(T)$. In particular, taking as a basis of $\Lambda_{\mathbb{R}}(T)$ the elements

$$\alpha_i = \chi_i - \chi_{i-1} \qquad i = 1, \ldots, n$$

we see that under this scalar product we have

$$(\alpha_i, \alpha_j) = \begin{cases} 2, & \text{if } i = j \\ -1, & \text{if } j = i \pm 1, \\ 0, & \text{otherwise} \end{cases}$$

Continuing with the general case, the torus T acts on the Lie algebra $\mathfrak{g}$ and induces a decomposition into root spaces

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where for every character $\alpha \in X^*(T)$ we define

$$\mathfrak{g}_\alpha = \{\, v \in \mathfrak{g} \mid t \cdot v = \alpha(t) v, \text{ for all } t \in T \,\}$$

and then $\mathfrak{t} = \mathfrak{g}_0 = Lie(T)$ whereas $\Phi = \{\, \alpha \in X^*(T) \mid \mathfrak{g}_\alpha \neq 0 \,\}$ is the set of roots of T. As G is semisimple, the roots span $\Lambda_{\mathbb{R}}(T)$.

The choice of a Borel subgroup defines a subset of the roots

$$\Phi^+ = \Phi(B) = \{\, \alpha \in \Phi \mid \mathfrak{g}_\alpha \subseteq Lie(B) \,\}$$

and then we know that $\Phi(B)$ is a positive subset of roots and that

$$Lie(B) = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \qquad \mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$$

The choice of the positive subset Φ^+ defines an unique basis of simple roots $\Delta = (\alpha_i)_{i \in I} \subseteq \Phi^+$, that is a subset of linearly independent roots, such that every positive root is a linear combination with nonnegative coefficients of the α_i.

For every root $\alpha \in \Phi$ we denote by $r_\alpha \in W$ the reflection with respect to α, that is the isometry that fixes pointwise the hyperplane orthogonal to α. It can be proved (see for example [Hum81]) that the Weyl group W is generated by the reflections r_{α_i} with respect to the simple roots $\alpha_i \in \Delta$. For every $w \in W$ we define its length $\ell(w)$ as the minimum number of reflection r_{α_i} with respect to simple roots needed to generate w. Equivalently, $\ell(w)$ coincides with the number of positive roots sent into negative roots by w.

We denote by $(\lambda_i)_{i \in I}$ the fundamental weights corresponding to Δ, that is, the elements of $\Lambda_{\mathbb{R}}$ defined by

$$\frac{2(\lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \qquad \text{for all } i \in I$$

In particular, if we define the Cartan matrix of T as the matrix C such that

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

we see that the columns of C are the coordinates of the (α_i) with respect to the basis (λ_i), or that the columns of C^{-1} are the coordinates of the (λ_i) with respect to the basis (α_i).

The weights of G are the linear combinations with integer coefficients of the fundamental weights and the dominant weights are the linear combinations with nonnegative integer coefficients. The fundamental Weyl chamber of G is defined as

$$D = \{\sum x_i\lambda_i | x_i \geq 0\}$$

and the weights in D are precisely the dominant weights. The group G is simply connected if the character lattice coincides with the weight lattice.

The irreducible representations of G correspond to the dominant weights: for each dominant weight λ we denote by Γ^λ the corresponding irreducible representation.

Example 5.1.2 (The special linear group - II). We continue with our previous example. The Lie algebra of $G = SL(n+1)$ is given by the matrices with trace zero

$$Lie(G) = \mathfrak{sl}_{n+1} = \{ A \in M(n+1, \mathbf{k}) \mid \operatorname{Tr} A = 0) \}$$

and the torus T acts over it by conjugation: if $A = (a_{ij}) \in \mathfrak{sl}_{n+1}$ and $t \in T$ then

$$t \cdot A = tAt^{-1} = (t_i t_j^{-1} a_{ij})$$

so that the set of roots of T is

$$\Phi = \{ \chi_i - \chi_j \mid i \neq j \}$$

observe that these roots do span $\Lambda_{\mathbb{R}}(T)$, for example because they contain the basis $\alpha_1, \dots, \alpha_n$.

Now it is clear that the Lie algebra of B is given by the lower-triangular matrices with trace zero, so that

$$\Phi^+ = \Phi(B) = \{ \chi_i - \chi_j \mid i > j \}$$

and the subset of simple roots is given precisely by $\Delta = (\alpha_1, \dots, \alpha_n)$. The Cartan matrix is given by

$$C = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$$

so that, for every $i = 1, \dots, n$ we have

$$\alpha_i = -\lambda_{i-1} + 2\lambda_i - \lambda_{i+1}$$

with the convention that $\lambda_0 = \lambda_{n+1} = 0$.

If $(\lambda_1, \dots, \lambda_n)$ are the fundamental weights and $p_1\lambda_1 + \cdots + p_n\lambda_r$ with $p_i \in \mathbb{N}$ is a dominant weight, then we denote by $\Gamma^{(p_1,\dots,p_n)}$ the irreducible representation of G corresponding to $p_1\lambda_1 + \cdots + p_n\lambda_n$. In the language of Schur functors one can see (for example from [FH91]) that

$$\Gamma^{(p_1,\dots,p_n)} = S^\lambda(V) \qquad \text{where } \lambda = (p_1 + \cdots + p_n, p_1 + \cdots + p_{n-1}, \dots, p_1)$$

and $V = \mathbf{k}^{n+1}$ is the standard representation of G.

We can also study the action of the Weyl group: for every root $\alpha \in \Phi$, we denote by $r_\alpha \in W$ the reflection with respect to the hyperplane orthogonal to α. Then for every simple root $\alpha_i \in \Delta$ we have

$$r_{\alpha_i}(\lambda_j) = \lambda_j - \frac{2(\alpha_i, \lambda_j)}{(\alpha_i, \alpha_i)}\alpha_i = \lambda_j - \delta_{ij}\alpha_i = \lambda_j - \delta_{ij}(-\lambda_{i-1} + 2\lambda_i - \lambda_{i+1})$$

so that

$$r_{\alpha_i}(\lambda_j) = \begin{cases} \lambda_{j-1} - \lambda_j + \lambda_{j+1}, & \text{if } i = j \\ \lambda_j, & \text{if } i \neq j \end{cases}$$

As a particular example, we consider the case $n = 2$: the Cartan matrix and its inverse are

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad C^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

so that the fundamental weights are given by

$$\lambda_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2 \qquad \lambda_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$$

and we can represent the weight lattice by

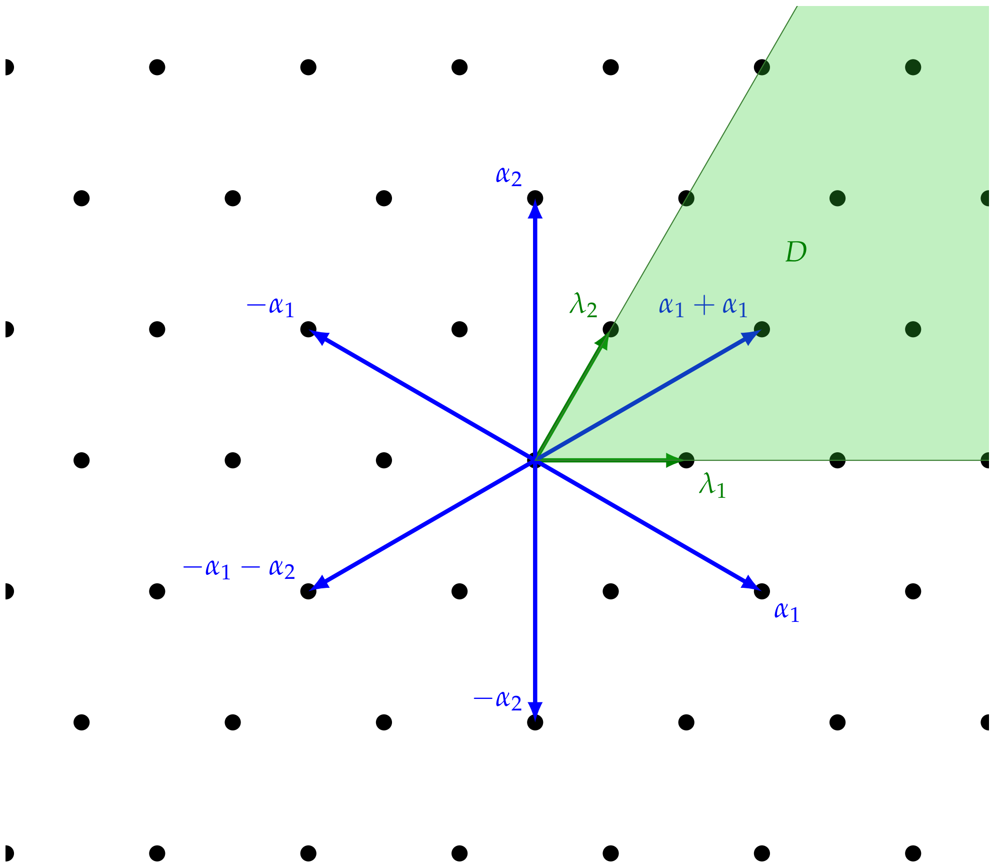


Figure 5.1: Weight lattice of $SL(3)$: the fundamental weights are depicted in green, the root system in blue and the shaded part corresponds to the Weyl chamber D

Recall that a parabolic subgroup $P < G$ is a connected closed subgroup such that the quotient G/P is projective. Equivalently, it is a connected closed subgroup that contains a Borel subgroup.

Fix now an arbitrary subset of simple roots $\Sigma \subseteq \Delta$: then we set

$$\Phi(\Sigma) = \left\{ \alpha \in \Phi^+ \middle| \alpha = \sum_{\alpha_i \in \Delta \setminus \Sigma} n_i \alpha_i \right\}$$

and we define $P(\Sigma)$ as the connected closed subgroup of G such that

$$Lie(P(\Sigma)) = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(\Sigma)} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$$

Then $P(\Sigma)$ is a parabolic subgroup: indeed, since $\Phi^+ \subseteq \Phi$ is a positive subsystem of roots, then $-\Phi^+$ is a positive subsystem as well, so that there is a Borel subgroup $B^- < G$ such that

$Lie(B^-) = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha}$, and then it is clear that $B^- < P(\Sigma)$ by construction. Conversely, every parabolic subgroup of G is conjugated to a parabolic subgroup of the form $P(\Sigma)$ for a certain $\Sigma \subseteq \Delta$ (cfr. [Ott]).

Now, let $P = P(\Sigma)$ and let $N \lhd P$ be the unipotent radical of P, that is the maximal connected unipotent normal subgroup of P. Then its Lie algebra is

$$Lie(N) = \mathfrak{n} = \bigoplus_{\alpha\in\Phi^+\setminus\Phi(\Sigma)} \mathfrak{g}_{-\alpha}$$

so that the quotient $Lie(P)/Lie(N)$ is a subalgebra and then we know that there is a closed reductive subgroup $R < P$ such that

$$Lie(R) = \mathfrak{r} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi(\Sigma)} \mathfrak{g}_{\alpha} \oplus \bigoplus_{\alpha\in\Phi(\Sigma)} \mathfrak{g}_{-\alpha} \qquad P = N \rtimes R$$

This is called a Levi decomposition of P (for a parabolic subgroup there is always one, even in positive characteristic [Hum81]).

Now, observe that the group P is not reductive in general, however thanks to the Levi decomposition we can say which representations of P are completely reducible, that is, those that can be split into irreducible ones.

Theorem 5.1.1 (Ise's theorem)**.** *A representation $\rho\colon P \longrightarrow GL(V)$ is completely reducible if and only if the induced representation $\rho_{|N}\colon N \longrightarrow GL(V)$ is trivial.*

Proof. It is clear that if a representation is trivial on N then it is completely reducible, as R is reductive. To prove the converse, we may suppose that the representation is irreducible: if N acts nontrivially on V then we see that V^N is a subrepresentation of P (as N is normal in G), which is neither the whole V nor zero (as N is unipotent), and this is absurd. □

Remark 5.1.1. In particular, from Theorem 5.1.1 it follows that the irreducible representations of P correspond exactly to the irreducible representations of R.

Observe that T is a maximal torus for R as well and that the root system of R corresponds precisely to $\Phi(\Sigma)$, and a set of simple roots is given by $\Delta \setminus \Sigma$. The irreducible representations of R correspond precisely to the weights of T of the form

$$\lambda = \sum n_i\lambda_i \qquad n_i \in \mathbb{Z}, \qquad n_i \geq 0 \text{ if } \alpha_i \in \Delta\setminus\Sigma$$

the fundamental Weyl chamber of R is

$$D' = \{\sum x_i\lambda_i |\; x_i \geq 0 \text{ if } \alpha_i \in \Delta\setminus\Sigma\}$$

and the dominant weights of R are precisely those in D'. We also define

$$W' = \{\, w \in W \mid wD \subseteq D' \,\}$$

Example 5.1.3 (The projective space - I)*.* Consider in the group $G = SL(n+1)$ the subgroup

$$P = \left\{ \left(\begin{array}{c|c} d^{-1} & x_1 \cdots x_n \\ \hline 0 & \\ \vdots & \mathbf{A} \\ 0 & \end{array} \right) \middle|\; A \in GL(n),\; d = \det A,\; x_i \in \mathbf{k} \right\}$$

then, in our previous notations we see that this is the parabolic subgroup $P = P(\{\,\alpha_1\,\})$ and indeed $G/P \cong \mathbb{P}^n$ is projective.

The unipotent radical of $N \lhd P$ and the Levi complement $R < P$ are given by

$$N = \left\{ \left(\begin{array}{c|c} 1 & x_1 \cdots x_n \\ \hline 0 & \\ \vdots & \mathbf{I} \\ 0 & \end{array} \right) \middle|\; x_i \in \mathbf{k} \right\} \qquad R = \left\{ \left(\begin{array}{c|c} d^{-1} & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \mathbf{A} \\ 0 & \end{array} \right) \middle|\; A \in GL(n),\; d = \det A \right\} \cong GL(n)$$

and the dominant weights for R are those of the form $p_1\lambda_1 + \cdots + p_n\lambda_n$ with $p_i \in \mathbb{Z}$ and $p_i \geq 0$ for all $i \geq 2$. In the language of Schur functors, the irreducible representation of R corresponding to $p_1\lambda_1 + \cdots + p_n\lambda_n$ is

$$S^\mu(U) \otimes \det{}^{p_1}(U)$$

where $U = V/\langle e_0 \rangle$, $\mu = (p_2 + \cdots + p_n, p_2 + \cdots + p_{n-1}, p_2)$ and $\det(U)$ is the determinantal representation.

For instance, if $n = 2$ we have

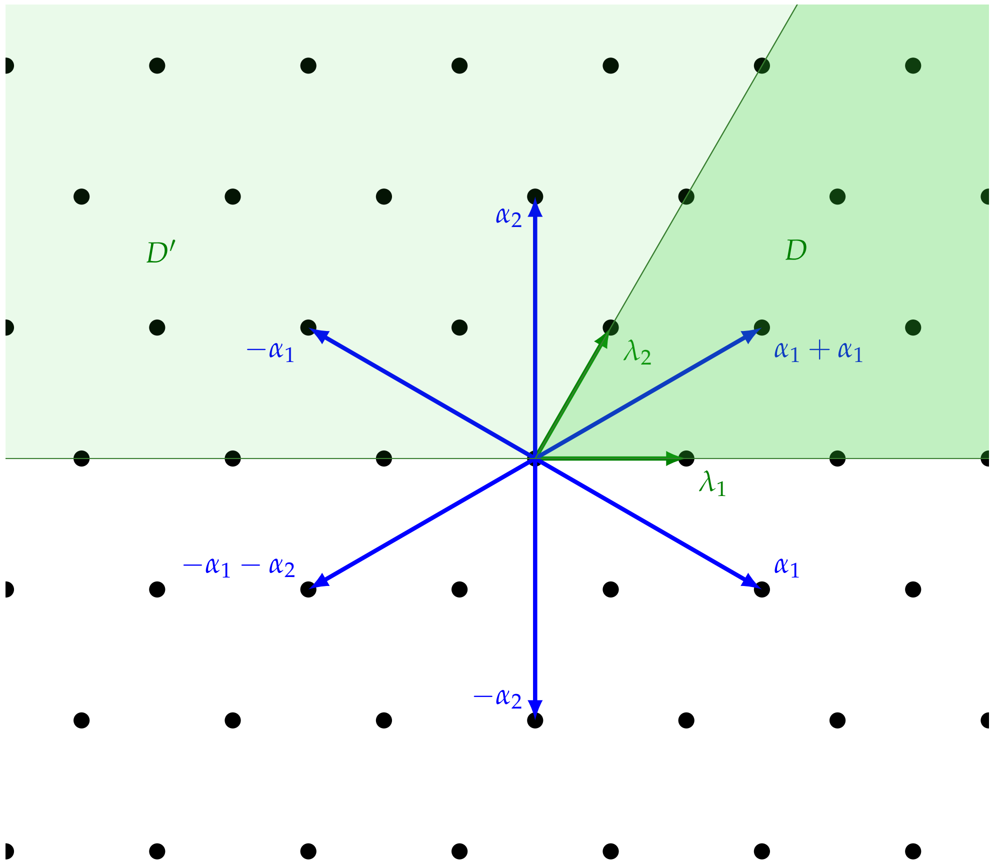


Figure 5.2: The lightly shaded part D' correspond to the fundamental Weyl chamber of $R = GL(2)$. The irreducible representation corresponding to the weight $m\lambda_1 + n\lambda_2 \in D'$ is $S^n U \otimes \det^m(U)$

5.2 Vector bundles and representations

We are interested in studying vector bundles on the smooth projective variety $X = G/P$. In all what follows, we will denote by $\pi\colon G \longrightarrow X$ the natural projection and by $z \in X$ the point fixed by P, corresponding to the lateral class $1_G P$.

5.2.1 A distinguished open cover of G/P

From what we have seen above, we have the decomposition

$$\mathfrak{g} = \mathfrak{p} \oplus \bigoplus_{\alpha \in \Phi^+ \setminus \Phi(\Sigma)} \mathfrak{g}_\alpha$$

and we define the unipotent connected closed subgroup $U_P < G$ by the property

$$Lie(U_P) = \bigoplus_{\alpha \in \Phi^+ \setminus \Phi(\Sigma)} \mathfrak{g}_\alpha$$

Lemma 5.2.1. *We have that*

1. $U_P \cap P = \{ 1_G \}$.
2. $U_P \cdot P$ *is open in* G.
3. $U_P \cdot z$ *is open in* X.

Proof.

1. First we observe that $Lie(U_P \cap P) = Lie(U_P) \cap Lie(P) = 0$ so that $U_P \cap P$ must have dimension 0. Since every subgroup of an unipotent subgroup is connected, this implies the thesis.
2. The group $U_P \times P$ acts on G by
$$(U_P \times P) \times G \longrightarrow G \qquad ((u,p),g) \mapsto ugp^{-1}$$
and $U_P \cdot P$ is precisely the orbit of 1_G under this action. In particular, we know that $U_P \cdot P$ is open in its closure, but as $\dim(U_P \cdot P) = \dim U_P + \dim P = \dim G$, we see that $\overline{U_P \cdot P} = G$ and we conclude.
3. This follows as in the previous point, considering the natural action of U_P on X and the orbit of z.

□

Now, for every $g \in G$ we can consider the open subset $A_g = gU_P \cdot z$. It is clear that the A_g form an open cover of X and, moreover, we have isomorphisms

$$U_P \xrightarrow{\sim} A_g \qquad u \mapsto guP$$

so that this is an affine open cover of X. Moreover, it is clear that we have isomorphisms

$$\pi^{-1}(A_g) = gU_P P \longrightarrow U_P \times P \qquad gup \mapsto (u,p)$$

5.2.2 Homogeneous vector bundles and representations

First we define homogeneous vector bundles on X.

Definition 5.2.1 (Homogeneous vector bundles)**.** Let $X = G/P$ as before. An **homogeneous vector bundle** on X is a vector bundle $E \longrightarrow X$ together with a regular action $\sigma\colon G \times E \longrightarrow E$ of G such that the following diagram commutes

$$\begin{array}{ccc} G \times E & \xrightarrow{\sigma} & E \\ \downarrow & & \downarrow \\ G \times X & \longrightarrow & X \end{array}$$

and the maps

$$\sigma(g, \cdot)\colon E_x \longrightarrow E_{g(x)}$$

are linear for every $g \in G$ and $x \in X$.

Remark 5.2.1. Obviously, a morphism between two homogeneous vector bundles $\sigma_1\colon G \times E_1 \longrightarrow E_1$ and $\sigma_2\colon G \times E_2 \longrightarrow E_2$ on X is a morphism of vector bundles $f\colon E_1 \longrightarrow E_2$ such that the following diagram commutes

$$\begin{array}{ccc} G \times E_1 & \xrightarrow{\sigma_1} & E_1 \\ {\scriptstyle \mathrm{id}_G \times f}\downarrow & & \downarrow{\scriptstyle f} \\ G \times E_2 & \xrightarrow{\sigma_2} & E_2 \end{array}$$

It is easy to show that homogeneous vector bundles on X form a category that we denote by $\mathbf{HomVec}_X$.

Remark 5.2.2. let E be an homogeneous vector bundle on X and let $z \in X$ denote the class of P. Then the fiber E_z is naturally a representation of P. It is easy to see that this gives a functor

$$\mathbf{HomVec}_X \longrightarrow \mathbf{Mod}_P \qquad E \mapsto E_z$$

we want to show now that this functor is an equivalence of categories.

Let $\rho\colon P \longrightarrow GL(V)$ be a regular representation of P, we want to define an homogeneous vector bundle E_ρ on X such that $E_{\rho,z} \cong V$.

To present the idea, let us work just with sets: we have a natural action of P over $G \times V$ given by

$$P \times (G \times V) \longrightarrow G \times V \qquad (p,(g,v)) \mapsto (gp^{-1}, p \cdot v)$$

and we denote $G \times_P V \overset{\text{def}}{=} (G \times V)/P$. Notice that we have a natural map

$$p\colon G \times_P V \longrightarrow X \qquad [g,v] \mapsto gP$$

and a natural action

$$G \times (G \times_P V) \longrightarrow G \times_P V \qquad (g,[h,v]) \mapsto [gh,v]$$

such that the following diagram commutes

$$\begin{array}{ccc} G \times (G \times_P V) & \longrightarrow & G \times_P V \\ {\scriptstyle \mathrm{id}_G \times p}\downarrow & & \downarrow{\scriptstyle p} \\ G \times X & \longrightarrow & X \end{array}$$

and moreover we see that the fiber $p^{-1}(z) = \{\, [1,v] \mid v \in V \,\}$ is naturally isomorphic to V as a P-module.

To actually build $G \times_P V$, consider the open affine cover of X given by the $(A_g)_{g\in G}$ and the isomorphisms

$$U_P \longrightarrow A_g \qquad u \mapsto guP$$

then over U_P consider the vector bundle given by $U_P \times V \longrightarrow U_P$. It is easy to show that these maps can be glued together over the open affine subsets A_g to form a vector bundle $E_\rho \longrightarrow X$ on X that corresponds to $G \times_P V$ at the level of closed points. We denote the whole of E_ρ again by $G \times_P V$, as there is no risk of confusion.

In this way, we have defined a functor

$$\mathbf{Mod}_P \longrightarrow \mathbf{HomVec}_X \qquad V \mapsto G \times_P V$$

Proposition 5.2.1. *The two functors*

$$\boldsymbol{HomVec}_X \longrightarrow \boldsymbol{Mod}_P \qquad E \mapsto E_z$$

and

$$\boldsymbol{Mod}_P \longrightarrow \boldsymbol{HomVec}_X \qquad V \mapsto G \times_P V$$

define an equivalence of categories between P-modules and homogeneous vector bundles over G/P.

Proof. We have already showed before that if V is a P-module, then $(G \times_P V)_z$ is naturally isomorphic to V as a P-module. Conversely, take an homogeneous vector bundle E over $X = G/P$: then it is easy to show that we have a morphism of vector bundles

$$G \times_P E_z \longrightarrow E \qquad [g,v] \mapsto g \cdot v$$

and this is bijective (at the level of closed points) between smooth varieties of the same dimension. Then, by Zariski's Main Theorem, it is an isomorphism. It is clear that this choice of an isomorphism is natural, and this concludes the proof. ☐

Example 5.2.1 (Tangent bundle and cotangent bundle). Let T_X be the tangent bundle of X. Then it is easy to see that we have an isomorphism of P-modules $T_{X,z} \cong \mathfrak{g}/\mathfrak{p}$, so that T_X corresponds to the representation

$$\mathfrak{g}/\mathfrak{p} = \bigoplus_{\alpha \in \Phi^+ \setminus \Phi(\Sigma)} \mathfrak{g}_\alpha$$

whereas Ω^1_X corresponds to the representation

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+ \setminus \Phi(\Sigma)} \mathfrak{g}_{-\alpha}$$

Example 5.2.2 (The projective space - II). Let us consider the case of $X = \mathbb{P}^n = SL(n+1)/P(\alpha_1)$. For every $a \in \mathbb{Z}$ we see that the line bundle $\mathcal{O}_{\mathbb{P}^r}(a)$ corresponds to the representation $\det^a$ of R.

Instead, the standard representation U of R corresponds to quotient bundle Q defined by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^r}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^r} \longrightarrow Q \longrightarrow 0$$

(observe that we are now seeing $\mathbb{P}^r$ as the set of lines in $\mathbf{k}^{r+1}$, this is for simplicity as otherwise we should put a dual in all the representations).

To determine the cotangent bundle, we observe that

$$\Phi^+ \setminus \Phi(\alpha_1) = \{ \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots \alpha_1 + \cdots + \alpha_n \}$$

so that the weights $\xi_1, \ldots, \xi_n$ of the cotangent bundle $\Omega^1_{\mathbb{P}^n}$ are

$$\xi_i = -(\alpha_1 + \cdots + \alpha_i) = -\lambda_1 - \lambda_i + \lambda_{i+1}$$

In particular, we see that the dominant weight is $\xi_1 = -2\lambda_1 + \lambda_2$, so that the cotangent bundle correspond to the irreducible representation $\wedge^{n-1}U \otimes \det(U)^{-2} \cong (U)^\vee \otimes \det(U)^{-1}$. From what we have said before, it follows that $\Omega^1_{\mathbb{P}^r} \cong \wedge^{n-1}Q(-2)$.

We also see that all the bundles $\Omega^j_{\mathbb{P}^r}$ are irreducible, as they correspond to irreducible representations of R.

5.3 Hermitian symmetric varieties and Higgs bundles

We want to study homogeneous bundles beyond the irreducible ones, that is the same as studying arbitrary representations of P.

We start by introducing an useful notation:

5.3.1 The gr functor

Definition 5.3.1 (The *gr* functor)**.** We define the functor

$$\mathrm{gr}\colon \mathbf{Mod}_P \longrightarrow \mathbf{Mod}_R \qquad V \mapsto \mathrm{gr}\, V$$

that simply sends a representation of P to its restriction as a representation of R.

Remark 5.3.1. Thanks to Proposition 5.2.1, we can look at this functor also as a functor from thecategory of homogeneous vector bundles on X to the category of completely reducible homogeneous vector bundles on X. Moreover, the functor restricts to the identity on the irreducible homogeneous bundles.

Remark 5.3.2. From Theorem 5.1.1 it is clear that *gr* gives a correspondence between completely reducible representations of P and representations of R. In particular, this restrict to a correspondence between irreducible representations of P and irreducible representations of R.

Remark 5.3.3. It is clear that *gr* is an exact functor. Moreover it is easy to check that we have the following properties

- $\mathrm{gr}(E^\vee) = \mathrm{gr}(E)^\vee$
- $\mathrm{gr}(E_1 \oplus E_2) = \mathrm{gr}(E_1) \oplus \mathrm{gr}(E_2)$
- $\mathrm{gr}(E_1 \otimes E_2) = \mathrm{gr}(E_1) \otimes \mathrm{gr}(E_2)$
- $\mathrm{gr}(\wedge^p E) = \wedge^p \mathrm{gr}(E)$
- $\mathrm{gr}(S^p E) = S^p(\mathrm{gr}(E))$

Remark 5.3.4. We can compute *gr* also by means of filtrations: for every P-module E there is a filtration

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{p-1} \subseteq E_p = E$$

of subrepresentations such that E_i/E_{i-1} are irreducible P-modules for every $i = 1, \dots, p$. Then it is clear that

$$\mathrm{gr}(E) = \bigoplus_{i=1}^{p} E_i/E_{i-1}$$

Moreover, this shows that the right hand side does not depend on the chosen filtration.

5.3.2 Extending representations

Recall that we have a fixed Levi decomposition $P = N \rtimes R$, so that R acts on N by conjugation. Now observe that for every representation

$$\rho \colon P \longrightarrow GL(V)$$

of P, we have the two induced representations

$$\rho_{|N} \colon N \longrightarrow GL(V) \qquad \mathrm{gr}(\rho) \colon R \longrightarrow GL(V)$$

and it is clear that the map $\rho_{|N} \colon N \longrightarrow GL(V)$ is R-equivariant, where R acts by conjugation on both sides. We can recover the representation ρ from the two representations $\rho_{|N}$ and $\mathrm{gr}(\rho)$, and in fact this gives a complete characterization of all the representations of P.

To state this precisely, we define the category $\mathbf{Ext}_N\mathbf{Mod}_R$ as the one whose objects are couples (V, α), where V is an R-module and $\alpha \colon N \longrightarrow GL(V)$ is a representation of N that is R-equivariant (with respect to the conjugation on both sides). The definition of a morphism in this category is clear, and it is also clear that the restriction of a representation of P to the representations of R and N defines a functor. It is easy to see that this is actually an equivalence of categories:

Lemma 5.3.1. *The functor*

$$\mathbf{Mod}_P \longrightarrow \mathbf{Ext}_N\mathbf{Mod}_R \qquad \rho \mapsto (gr\ (\rho), \rho_{|N})$$

is an equivalence of categories.

Proof. Take an element (V, α) in $\mathbf{Ext}_N\mathbf{Mod}_R$, for simplicity we denote by $\beta \colon R \longrightarrow GL(V)$ the representation of R. We can define a representation of P on V as

$$\rho \colon P = (N \rtimes R) \longrightarrow GL(V) \qquad nr \mapsto \alpha(n)\beta(r)$$

To show that this is actually an action observe that for every $r_1, r_2 \in R, n_1, n_2 \in N$ we have that

$$\begin{aligned}\rho(n_1 r_1 n_2 r_2) &= \rho(n_1 r_1 n_2 r_1^{-1} r_1 r_2) = \alpha(n_1 r_1 n_2 r_1^{-1})\beta(r_1 r_2) = \alpha(n_1)\alpha(r_1 n_2 r_1^{-1})\beta(r_1)\beta(r_2) \\ &= \alpha(n_1)\beta(r_1)\alpha(n_2)\beta(r_1)^{-1}\beta(r_1)\beta(r_2) = \alpha(n_1)\beta(r_1)\alpha(n_2)\beta(r_2) = \rho(n_1 r_1)\rho(r_2 n_2)\end{aligned}$$

It is now very easy to show that in this way we have defined a functor

$$\mathbf{Ext}_N\mathbf{Mod}_R \longrightarrow \mathbf{Mod}_P$$

that is the inverse of the one above. □

Since N is simply connected, to give an R-equivariant morphism of groups $\beta\colon N \longrightarrow GL(V)$ is the same as giving an R-equivariant morphism of Lie algebras $\theta\colon \mathfrak{n} \longrightarrow \mathfrak{gl}(V)$. More precisely, we define another category $\mathbf{Ext}_{\mathfrak{n}}\mathbf{Mod}_R$ whose objects are couples (V,θ), where V is an R-module and $\theta\colon \mathfrak{n} \longrightarrow \mathfrak{gl}(V)$ is an R-equivariant morphism of Lie algebras. Then we have

Lemma 5.3.2. *The categories* $\mathbf{HomVec}_X, \mathbf{Mod}_P, \mathbf{Ext}_r\mathbf{Mod}_R, \mathbf{Ext}_{\mathfrak{n}}\mathbf{Mod}_R$ *are all equivalent.*

5.3.3 Higgs bundles on Hermitian symmetric varieties

From now on, we will restrict our attention to a particular class of homogeneous varieties.

Definition 5.3.2 (Hermitian symmetric variety)**.** The homogeneous variety $X = G/P$ is said to be an **Hermitian symmetric variety** if $[\mathfrak{n},\mathfrak{n}] = 0$.

Remark 5.3.5*.* By [Hum81] the condition $[\mathfrak{n},\mathfrak{n}] = 0$ is equivalent to N being abelian, that is, the adjoint action of G on N is trivial when restricted to N.

In view of Theorem 5.1.1 and Example 5.2.1 , another equivalent reformulation is that $X = G/P$ is Hermitian symmetric if and only if Ω^1_X is completely reducible as an homogeneous bundle.

Definition 5.3.3 (Irreducible hermitian symmetric variety)**.** An Hermitian symmetric variety is said to be **irreducible** if Ω^1_X is an irreducible homogeneous bundle, that is, if $\mathfrak{n}$ is irreducible as a P-module.

Remark 5.3.6*.* It has been shown by Cartan that any Hermitian symmetric variety is the product of irreducible ones and moreover the irreducible ones are Grassmannians, quadrics, spinor varieties, maximal Lagrangian Grassmannians and two exceptional varieties of dimension 16 and 27. For references consult [Kos61].

The advantage of Hermitian symmetric varieties is that the equivalence of categories of Lemma 5.3.2 can be greatly simplified. Indeed suppose that we are given an R-module V and an R-equivariant linear map

$$\theta\colon \mathfrak{n} \longrightarrow \mathfrak{gl}(V)$$

Since $[\mathfrak{n},\mathfrak{n}] = 0$, for this to be a morphism of Lie algebras it is sufficient that $[\theta(n_1),\theta(n_2)] = 0$ for all $n_1, n_2 \in \mathfrak{n}$. Equivalently, this can be rephrased by saying that

$$\theta \wedge \theta = 0 \qquad \text{in } \mathrm{Hom}(\mathfrak{n} \wedge \mathfrak{n}, \mathfrak{gl}(V))^P$$

Now, we can write θ also as a P-equivariant linear map

$$\theta\colon \mathfrak{n} \otimes V \longrightarrow V$$

or, working with homogeneous bundles, as an equivariant morphism of vector bundles

$$\theta\colon \Omega^1_X \otimes V \longrightarrow V \qquad \text{or} \qquad \theta\colon V \longrightarrow T_X \otimes V$$

hence, we give the following definition

Definition 5.3.4 (Higgs bundles)**.** The category of Higgs bundles on the Hermitian symmetric variety X is the category $\mathbf{Higgs}_X$ whose objects are couples (V,θ), where V is an R-module and $\theta\colon \Omega^1_X \otimes V \longrightarrow V$ is a morphism of homogeneous vector bundles such that $\theta \wedge \theta = 0$.

From the above discussion, we have the following result:

Proposition 5.3.1. *Let X be an Hermitian symmetric variety. Then the categories* $\mathbf{HomVect}_X, \mathbf{Mod}_P$ *and* $\mathbf{Higgs}_X$ *are equivalent.*

5.4 Bott-Borel-Weil Theorem

We have seen above that homogeneous vector bundles on X are equivalent to P-modules. In particular, by Theorem 5.1.1, the irreducible homogeneous bundles on X correspond to irreducible R-modules.

The theorem of Bott-Borel-Weil gives a way to compute the cohomology $H^j(X, E)$ of an irreducible vector bundle by means of this identification.

We denote by $g = \sum_{i=1}^n \lambda_i$ the sum of all the fundamental weights.

Theorem 5.4.1 (Bott). *Let $\lambda \in D'$ be a dominant weight for R. Then there is an unique element $w \in W$ such that $w(\lambda + g) \in D$ (in particular $w^{-1} \in W'$). Set $\nu = w(\lambda + g) - g$, then*

1. *If $\nu \in D$ then*

$$H^j(X, E_\lambda) = \begin{cases} \Gamma^\nu & \text{if } j = \ell(w) \\ 0 & \text{if } j \neq \ell(w) \end{cases}$$

2. *If $\nu \notin D$, then*

$$H^j(X, E_\lambda) = 0 \qquad \text{for all } j$$

Proof. See [Ott]. □

As a corollary of this theorem we recover the previous theorem due to Borel and Weil

Theorem 5.4.2 (Borel-Weil). *Let $\lambda \in D$ be a dominant weight for P. Then*

$$H^j(X, E_\lambda) = \begin{cases} \Gamma^\lambda & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}$$

To understand better what is happening we set some notation: for every root $\alpha \in \Phi$ we denote by H_α the hyperplane in $\Lambda_{\mathbb{R}}(T)$ orthogonal to α and by r_α the reflection with respect to H_α. We also denote by $Y_\alpha = H_\alpha - g$ the affine hyperplane orthogonal to α passing thorugh $-g$ and by s_α the reflection with respecto to Y_α. It is clear that, if $\alpha_1, \ldots, \alpha_h \in \Phi$ are roots then

$$(s_{\alpha_1} \circ \cdots \circ s_{\alpha_h})(x) = (r_{\alpha_1} \circ \cdots \circ r_{\alpha_h})(x + g) - g$$

and, as every element $w \in W$ is given by composition of reflections r_α, we see that every transformation $x \mapsto w(x + g) - g$ is given by composition of reflections s_α.

Remark 5.4.1. Let us keep the same notation as in Bott Theorem: then we notice that, as $w(\lambda + g) \in D$ then $\nu = w(\lambda + g) - g \notin D$ if and only if $w(\lambda + g) - g \in \partial D - g$. Hence, the weights $\lambda \in D'$ such that $H^j(X, E_\lambda) = 0$ for all j correspond exactly to

$$D' \cap \bigcup_{w \in W'} w(\partial D + g) - g = D' \cap \bigcup_{\alpha \in \Phi} Y_\alpha$$

Hovever, we notice that for every $\alpha \in \Phi(\Sigma)$, we have that $D' \cap Y_\alpha = \emptyset$: indeed, any such α can be written in the form $\alpha = \sum_{\alpha_j \in \Sigma} n_j \alpha_j$ for certain $n_j \geq 0$ and then, if $x = \sum x_i \lambda_i \in D'$ we know by definition that $x_i \geq 0$ for all $\alpha_i \in \Sigma$, so that

$$(x + g, \alpha) = \left(\sum (x_i + 1) \lambda_i, \sum_{\alpha_j \in \Sigma} n_j \alpha_j \right) = \sum_{\alpha_j \in \Sigma} (x_j + 1) n_j > 0$$

as at least one of the n_j is strictly positive.

This shows that

$$\{ \lambda \in D' \mid H^\bullet(X, E_\lambda) \neq 0 \} = D' \setminus \bigcup_{\alpha \in \Phi^+ \setminus \Phi(\Sigma)} Y_\alpha = D' \setminus \bigcup_{i=1}^n Y_{\xi_i}$$

and motivates the following definition

Definition 5.4.1 (Bott Chambers)**.** The connected components of $D' \setminus \bigcup_{i=1}^{n} Y_{\xi_i}$ are called the **extended Bott chambers** of X. Every extended Bott chamber contains an unique subset of the form $w(D+g)-g$ for a certain $w \in W'$: these subsets are called the **Bott chambers** of X. We say that two Bott chambers are **adjacent** if their extended Bott chambers have a common hyperplane in their boundary.

Remark 5.4.2. For every Bott chamber B we define its length $\ell(B)$ as the length $\ell(w)$ of the element $w \in W'$ such that $B = w(D+g)-g$. Then Bott's theorem tells us that the irreducible bundles E_λ such that $H^j(X, E_\lambda) \neq 0$ are precisely those contained in the Bott chambers B such that $\ell(B) = j$.

5.4.1 Bott Theorem on projective space

As an example, we want to apply Bott's theorem to the case of projective space $\mathbb{P}^n = SL(n+1)/P(\alpha_1)$. First we recall a result of Kostant

Theorem 5.4.3. *Suppose that* $\mathbf{k} = \mathbb{C}$*. Then*

$$\#\{\, w \in W' \mid \ell(w) = j \,\} = \dim H^{2j}(X, \mathbb{C})$$

and in particular

$$\#W' = \chi(X, \mathbb{C})$$

Proof. See [Kos61]. □

In particular, for projective spaces we obtain that

Corollary 5.4.1. *Let* $\mathbf{k}$ *be any algebraically closed field of characteristic* 0 *and consider* $\mathbb{P}^r = SL(n+1)/P(\alpha_1)$*. Then for every* $j = 0, 1, \ldots, n$ *we have*

$$\#\{\, w \in W' \mid \ell(w) = j \,\} = 1$$

and in particular

$$\#W' = n+1$$

Proof. Since the data $\#\{\, w \in W' \mid \ell(w) = j \,\}$ and $\#W'$ are purely combinatorial, they do not depend on the field $\mathbf{k}$, so that we can suppose $\mathbf{k} = \mathbb{C}$. Then the result follows immediately from Theorem 5.4.3. □

Thanks to this result we can determine the set W' explicitly

Lemma 5.4.1. *For the projective space* $\mathbb{P}^n = SL(n+1)/P(\alpha_1)$ *we have*

$$W' = \{\, id, r_{\alpha_1}, r_{\alpha_1} r_{\alpha_2}, \ldots, r_{\alpha_1} \ldots r_{\alpha_n} \,\}$$

and $\ell(r_{\alpha_1} \ldots r_{\alpha_i}) = i$*.*

Proof. First we prove that for every $i = 1, \ldots, n$ we have

$$(r_{\alpha_1} \ldots r_{\alpha_i}) \left(\sum_{j=1}^{n} p_j \lambda_j \right) = \left(-\sum_{j=1}^{i} p_j \right) \lambda_1 + \sum_{j=1}^{i} p_j \lambda_{j+1} + \sum_{j=i+1}^{n} p_j \lambda_j$$

We proceed by induction on i: if $i = 1$ this is clear from Example 5.1.2, so that we suppose $i > 1$ and then using again Example 5.1.2 and the inductive hypothesis we get

$$\begin{aligned}
(r_{\alpha_1} \ldots r_{\alpha_i}) \left(\sum_{j=1}^{n} p_j \lambda_j \right) &= (r_{\alpha_1} \ldots r_{\alpha_{i-1}}) \left(\sum_{j=1}^{i-2} p_j \lambda_j + (p_{i-1} + p_i)\lambda_{i-1} - p_i \lambda_i + (p_i + p_{i+1})\lambda_{i+1} + \sum_{j=i+2}^{n} p_j \lambda_j \right) \\
&= \left(-\sum_{j=1}^{i-2} p_j - (p_{i-1} + p_i) \right) \lambda_1 + \sum_{j=1}^{i-2} p_j \lambda_{j+1} + (p_{i-1} + p_i)\lambda_i - p_i \lambda_i + (p_i + p_{i+1})\lambda_{i+1} + \sum_{j=i+2}^{n} p_j \lambda_j \\
&= \left(-\sum_{j=1}^{i} p_j \right) \lambda_1 + \sum_{j=1}^{i} p_j \lambda_{j+1} + \sum_{j=i+1}^{n} p_j \lambda_j
\end{aligned}$$

This shows that the elements $r_{\alpha_1}, \ldots, r_{\alpha_1} \ldots r_{\alpha_n}$ are all distinct and contained in W'. As we know from Corollary 5.4.1 that $\#W' = n+1$, we see that $W' = \{ \mathrm{id}, r_{\alpha_1}, \ldots, r_{\alpha_1} \ldots r_{\alpha_n} \}$. For the statement about length, we proceed by induction on i: it is clear that $\ell(r_{\alpha_1}) = 1$, and if $i > 1$ then by definition we know that $\ell(r_{\alpha_1} \ldots r_{\alpha_i}) \leq i$. Now, from Corollary 5.4.1, we know that $\#\{ w \in W' \mid \ell(w) = j \} = 1$ for every $1 \leq j \leq n$ and by inductive hypothesis $\ell(r_{\alpha_1} \ldots r_{\alpha_j}) = j$ for all $1 \leq j \leq i$. Then by pigeonhole it must be that $\ell(r_{\alpha_1} \ldots r_{\alpha_i}) = i$.

□

Lemma 5.4.2. *Let $\xi_1, \ldots, \xi_n$ be the weights of the cotangent bundle $\Omega^1_{\mathbb{P}^n}$. Then for every $i = 1, \ldots, n$ it holds that*

1. $\xi_i = -(\alpha_1 + \cdots + \alpha_i) = -\lambda_1 - \lambda_i + \lambda_{i-1}$.
2. $\alpha_1 + \cdots + \alpha_{i+1} = (r_{\alpha_1} \ldots r_{\alpha_i})(\alpha_{i+1})$
3. $r_{\xi_{i+1}} = (r_{\alpha_1} \ldots r_{\alpha_i}) r_{\alpha_{i+1}} (r_{\alpha_1} \ldots r_{\alpha_i})^{-1}$.
4. $r_{\alpha_i} \ldots r_{\alpha_1} = r_{\xi_1} \ldots r_{\xi_i}$.

Proof.
1. We have already done this computation in Example 5.2.2.
2. We know from Example 5.1.2 that $\alpha_{i+1} = -\lambda_i + 2\lambda_{i+1} - \lambda_{i+2}$ and then from the proof of Lemma 5.4.1 we see that
$$(r_{\alpha_1} \ldots r_{\alpha_i})(\alpha_{i+1}) = (r_{\alpha_1} \ldots r_{\alpha_i})(-\lambda_i + 2\lambda_{i+1} - \lambda_{i+2}) = \lambda_1 + \lambda_{i+1} - \lambda_{i+2} = -\xi_{i+1}$$
3. This follows from the fact that $r_{\xi_{i+1}} = r_{(r_{\alpha_1} \ldots r_{\alpha_i})(\alpha_{i+1})}$, thanks to point (2).
4. This follows from point (3).

□

Then we can express Bott's theorem as follows:

Proposition 5.4.1. *Consider $\mathbb{P}^n = SL(n+1)/P(\alpha_1)$. Then for every $i = 0, \ldots, n$ there is an unique Bott chamber B_i of length i, and it is given by*

$$B_i = (r_{\alpha_1} \ldots r_{\alpha_i})(D+g) - g = (r_{\xi_i} \ldots r_{\xi_1})(D+g) - g = (s_{\xi_i} \ldots s_{\xi_1})(D)$$

Proof. It follows immediately from Bott's Theorem 5.4.1, Lemma 5.4.1 and Lemma 5.4.2. □

Remark 5.4.3. In particular, the above result tells us that the vertex of the Bott chamber B_i is given by
$$\mu_i = (r_{\alpha_1} \ldots r_{\alpha_i})(g) - g = (r_{\xi_i} \ldots r_{\xi_1})(g) - g = \xi_1 + \cdots + \xi_i$$
and, from Example 5.2.2 we see that this is precisely the dominant weight of the irreducible homogeneous bundle $\Omega^i_{\mathbb{P}^n}$.

Proposition 5.4.2. *Let $0 \leq i \leq n-1$ let $\lambda \in D'$ and let $\mu = s_{\xi_{i+1}}(\lambda)$. Then $H^i(\mathbb{P}^n, E_\lambda) \cong H^{i+1}(\mathbb{P}^n, E_\mu)$. If moreover $H^i(\mathbb{P}^n, E_\lambda) \neq 0$ then μ is the unique element in $\lambda' \in D'$ such that $H^i(\mathbb{P}^n, E_\lambda) \cong H^{i+1}(\mathbb{P}^n, E_{\lambda'})$.*

Observe that λ and μ differ by a multiple of ξ_{i+1}.

Proof. Suppose first that $H^i(\mathbb{P}^n, E_\lambda) \neq 0$. Then we know from Bott's theorem that $\lambda = s_{\xi_i} \ldots s_{\xi_1}(\nu)$ for a certain $\nu \in D$ and that $H^i(\mathbb{P}^n, E_\lambda) \cong \Gamma^\nu$, and again from Bott's theorem we know that the unique element $\lambda' \in D'$ such that $H^{i+1}(\mathbb{P}^n, E_{\lambda'}) \cong \Gamma^{vu}$ is $s_{\xi_{i+1}} s_{\xi_i} \ldots s_{\xi_1}(\nu) = s_{\xi_{i+1}}(\lambda) = \mu$. If $H^i(\mathbb{P}^n, \mathbb{E}_\lambda) = 0$, the statement follows from Bott's theorem. □

Example 5.4.1 (Bott's Theorem on the projective plane). Now we focus on the projective plane $\mathbb{P}^2$. In this case we see that

$$D = \{ a_1\lambda_1 + a_2\lambda_2 \mid a_i \geq 0 \}$$

and, for every $\alpha = a_1\lambda_1 + a_2\lambda_2 \in D$, the corresponding irreducible representation of $G = SL(3)$ is

$$\Gamma^\alpha = \Gamma^{(a_1,a_2)} = S^{(a_1+a_2,a_1)}(V)$$

Instead , we have

$$D' = \{ b_1\lambda_1 + b_2\lambda_2 \mid b_2 \geq 0 \}$$

and for every $\beta = b_1\lambda_1 + b_2\lambda_2 \in D'$ the corresponding irreducible homogeneous vector bundle on $\mathbb{P}^2$ is

$$E_\beta = S^{b_2}Q(b_1)$$

where Q is the quotient bundle on $\mathbb{P}^2$ defined by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow Q \longrightarrow 0$$

Then Bott's Theorem can be stated as follows: let $n \geq 0$ and $m \in \mathbb{Z}$, then

$$H^0(\mathbb{P}^2, S^nQ(m)) = \begin{cases} S^{(n+m,m)}(V), & \text{if } n \geq 0, m \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$H^1(\mathbb{P}^2, S^nQ(m)) = \begin{cases} S^{(n-1,-m-2)}(V), & \text{if } n \geq 1, -n-1 \leq m \leq -2 \\ 0, & \text{otherwise} \end{cases}$$

$$H^2(\mathbb{P}^2, S^nQ(m)) = \begin{cases} S^{(-m-3,n)}(V), & \text{if } n \geq 0, m \leq -n-3 \\ 0, & \text{otherwise} \end{cases}$$

However the content of the theorem can be more effectively expressed by Figure 5.3.

5.4.2 Bott Theorem on Hermitian symmetric varieties

Now we want to state some consequences of Bott's Theorem on irreducible Hermitian symmetric varieties, but we restrict further to the case of irreducible Hermitian symmetric variety of type ADE: these are the irreducible Hermitian symmetric varieties G/P such that the Dynkin diagram of G is of type ADE . It can be seen that these varieties are Grassmannians, quadrics of even dimension, spinor varieties and the two exceptional cases.

If X is irreducible of type ADE, then $\text{Pic}(X) = \mathbb{Z}$ (see [OR06]), so that we can define the slope of a vector bundle E on X as the rational number

$$\mu(E) \stackrel{\text{def}}{=} \frac{c_1(E)}{\text{rk}(E)} \in \mathbb{Q}$$

Proposition 5.4.3. *Let $X = G/P$ be an Hermitian symmetric variety. Then for every $j \geq 0$ the irreducible bundles E_λ such that $H^j(X, E_\lambda)^G \neq 0$ are precisely the direct summands of Ω^j_X. Moreover, for every such E_λ we have that $H^j(X, E_\lambda)^G \cong \mathbf{k}$.*

Proof. We are going to prove this theorem in the case of projective space $X = \mathbb{P}^n$. For the general case see [OR06] or [Kos61]. We know from before that for every $j \geq 0$ the bundle $\Omega^j_{\mathbb{P}^r}$ is irreducible and that it is the vertex of the Bott chamber B_j. Then from Bott's theorem it follows that

$$H^j(\mathbb{P}^r, \Omega^j_{\mathbb{P}^r}) \cong H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}) = \mathbf{k}$$

Conversely, let E_λ be an irreducible homogeneous bundle: then by Bott's theorem $H^j(X, E_\lambda)$ is an irreducible G-module, so that $H^j(\mathbb{P}^n, E_\lambda)^G \neq 0$ if and only if $H^j(\mathbb{P}^n, \mathbb{E}_\lambda) = \mathbf{k}$. This means that E_λ is the vertex of the Bott chamber B_j, that is $E_\lambda = \Omega^j_{\mathbb{P}^r}$. $\square$

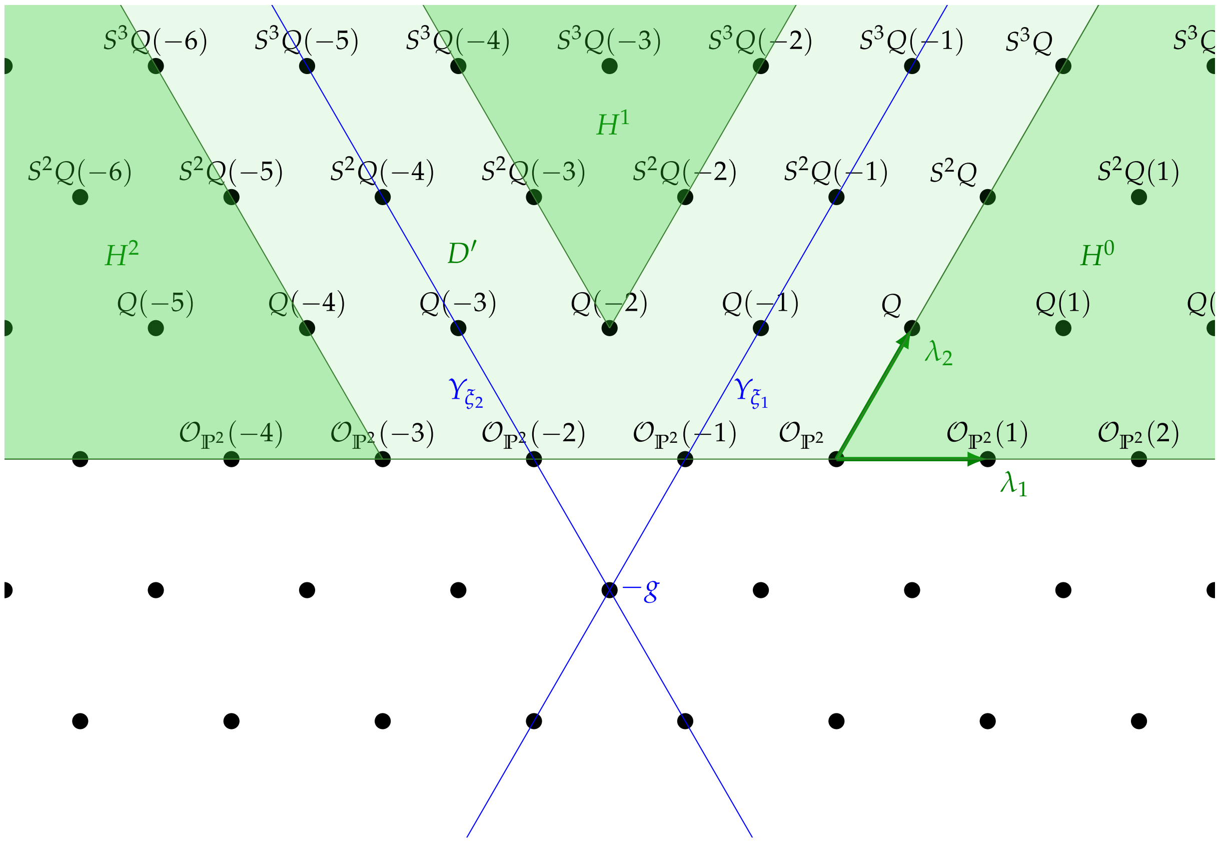


Figure 5.3: The darker shaded parts are the Bott chambers corresponding to irreducible homogeneous bundles with nonzero H^0, H^1 and H^2. The Bott chamber with nonzero H^0 corresponds to the Weyl chamber D. The hyperplanes corresponding to bundles with zero cohomology are represented in blue.

Corollary 5.4.2. *Let E be a completely reducible homogeneous vector bundle on an irreducible Hermitian symmetric variety X. Then $H^j(X,E)^G \cong Hom(\Omega^j_X, E)^G$.*

Proof. We can suppose that $E = E_\lambda$ is irreducible and now the thesis follows immediately from Proposition 5.4.3. □

Proposition 5.4.4. *Let X be an Hermitian symmetric variety and let $\lambda, \nu \in D'$.*

1. *$Hom(\Omega^i_X \otimes E_\lambda, E_\nu)^G \cong Ext^i(E_\lambda, E_\nu)^G$. If $i = 1$ then both spaces have dimension 0 or 1.*
2. *If X is irreducible, and $Ext^i(E_\lambda, E_\nu)^G \neq 0$, then $\mu(E_\nu) = \mu(E_\lambda) + i\mu(\Omega^1_X)$.*

Proof. Let $E = E_\nu \otimes E_\lambda^\vee$. Then E is completely reducible and by the previous corollary we have that $\mathrm{Hom}(\Omega^i_X \otimes E_\lambda, E_\nu)^G \cong \mathrm{Hom}(\Omega^i_X, E)^G \cong H^j(X,E)^G \cong \mathrm{Ext}^j(\Omega_X, E)^G \cong \mathrm{Ext}^j(E_\lambda, E_\nu)^G$.

Now, if $i = 1$ then we know that Ω^1_X is irreducible. Let $\xi_1, \dots, \xi_m$ be the weights, counted with multiplicity, corresponding to the bundle Ω^1_X, that is, to the representation $\mathfrak{n}$. We notice that the ξ_i are all distinct, since $\mathfrak{n} \subseteq \mathfrak{g}$ and all root spaces of a semisimple Lie group have dimension 1 (cfr. [Hum81]). Then we know from the generalized Littlewood-Richardson rule (Theorem 1.1.4) that

$$E_\lambda \otimes \Omega^1_X = \bigoplus_{j \in J} E_{\lambda + \xi_j}$$

for a certain subset $J \subseteq \{1, \dots, m\}$. Since the weights $\lambda + \xi_i$ are all distinct, this concludes the proof of the first point.

For the second point, observe that $\mathrm{Ext}^i(E_\lambda, E_\nu)^G \neq 0$ if and only if $E_\nu = E_{\lambda+\xi_r}$ for a certain $r \in J$. Now, by a theorem of Ramanan (see [Ram66] or [Ott]) every irreducible homogenous bundle is stable, and since the tensor product of stable bundles is polystable (cfr. [HL96]) we see that $\mu(E_v) = \mu(\Omega^j_X \otimes E_\lambda) = \mu(E_\lambda) + j\mu(\Omega^1_X)$. □

Corollary 5.4.3. *Let $0 \leq i < n$ and let $\lambda \in D'$. Then there are μ and s_{ξ_j} such that $\lambda' = s_{\xi_j}(\lambda)$ and $H^i(X, E_\lambda) = H^{i+1}(X, E_\mu)$ or $H^i(X, E_\lambda) = H^{i-1}(X, E_\mu)$. In particular lambda and μ differ by a multiple of ξ_j. There is exactly one such μ in every Bott chamber adjacent to the Bott chamber of E_λ.*

Proof. Let λ_0 be the vertex of the Bott chamber that contains λ. Then take the ξ_j such that $s_{\xi_j}(\lambda_0)$ is the maximal weight of a direct summand of Ω^{i+1}_X. □

5.5 Quivers and relations

Now we are going to describe an useful tool for the study of an homogeneous vector bundle.

5.5.1 Basic definitions

We give some basic definitions for quivers.

Definition 5.5.1 (Quiver)**.** A **quiver** is an oriented graph $\mathcal{Q}$. We denote by $\mathcal{Q}_0$ the set of vertices, by $\mathcal{Q}_1$ the set of arrows and by $t\colon \mathcal{Q}_1 \longrightarrow \mathcal{Q}_0$ and $h\colon \mathcal{Q}_1 \longrightarrow \mathcal{Q}_0$ the maps that to each arrow associate the head and the tail.

Suppose that there is a function $s\colon \mathcal{Q}_0 \longrightarrow \mathbb{Q}$ such that, if there is an arrow from v to w then $s(w) = s(v) + 1$. Then we say that $\mathcal{Q}$ is **leveled** by s.

Definition 5.5.2 (Paths)**.** A **path** in a quiver $\mathcal{Q}$ is a sequence $a_1 \cdot \dots \cdot a_r$ of arrows where the head of each arrow is the tail of the next one. If $p = a_1 \cdot \dots \cdot a_r$ is a path we denote by $h(p) = h(a_r)$ its head and by $t(p) = t(a_1)$ its tail. The **length** of a path is the number of arrows that compose it.

Definition 5.5.3 (Path Algebra)**.** For any pair $(a, b) \in \mathcal{Q}_0^2$ we denote by

$$\mathcal{Q}_{(a,b)} = \{\, p \text{ path } \mid t(p) = a,\ h(p) = b \,\}$$

and we define the graded vector space

$$\mathbf{k}[\mathcal{Q}] = \bigoplus_{(a,b)\in\mathcal{Q}_0^2} \mathcal{Q}_{(a,b)}$$

If p, q are two paths then we define their product $p \cdot q$ as their concatenation, if $h(p) = t(q)$ and 0 otherwise. This operation can be extended to $\mathbf{k}[\mathcal{Q}]$ that becomes an associative graded algebra, called the path algebra.

Definition 5.5.4 (Relations)**.** A **relation** in $\mathbf{k}[\mathcal{Q}]$ is an homogenous element of $\mathbf{k}[\mathcal{Q}]$, i.e. a linear combination

$$\lambda_1 p_1 + \dots + \lambda_r p_r$$

where $\lambda_i \in \mathbf{k}$ and p_i are paths with same head and same tail.

Definition 5.5.5 (Quiver representation)**.** A **representation of a quiver** $\mathcal{Q}$ is the datum of a family of finite-dimensional vector space $(V_v)_{v\in\mathcal{Q}_0}$ and linear maps $\phi_a\colon V_{t(a)} \longrightarrow V_{h(a)}$ for every $a \in \mathcal{Q}_1$. A representation of a quiver $\mathcal{Q}$ is said to be of finite type if $V_v = 0$ for all $v \in \mathcal{Q}_0$ apart for a finite number at most.

Definition 5.5.6 (Representations of quivers with relations)**.** Let $\mathcal{R} \subseteq \mathbf{k}[\mathcal{Q}]$ be an homogeneous ideal. A **representation of $\mathcal{Q}$ with relations** $\mathcal{R}$ is a representation $(V_v, \phi_a)_{v\in\mathcal{Q}_0, a\in\mathcal{Q}_1}$ of $\mathcal{Q}$ such that

$$\lambda_1 \phi_{p_1} + \dots + \lambda_s \phi_{p_s} = 0$$

for every relation $\lambda_1 p_1 + \dots + \lambda_s p_a \in \mathcal{R}$.

***Remark* 5.5.1.** If we have a representation (V_x, ϕ_a) of a quiver $\mathcal{Q}$, and $p = a_1 \cdot \dots \cdot a_r$ is a path in $\mathcal{Q}$ we use the notation $\phi_p \stackrel{\text{def}}{=} \phi_{a_r} \circ \dots \circ \phi_{a_1}$.

Definition 5.5.7 (Morphism of representations)**.** Let $(V_v, \phi_a)_{v \in \mathcal{Q}_0, a \in \mathcal{Q}_1}$ and $(W_v, \psi_a)_{v \in \mathcal{Q}_0, a \in \mathcal{Q}_1}$ be two representations (with relations $\mathcal{R}$) of a quiver $\mathcal{Q}$. Then a **morphism of representations** $f\colon (V, \phi) \longrightarrow (W, \psi)$ is the datum of linear maps $f_v\colon V_v \longrightarrow W_v$ for every $v \in \mathcal{Q}_0$ such that the following diagram

$$\begin{array}{ccc} V_{t(a)} & \xrightarrow{\phi_a} & V_{h(a)} \\ {\scriptstyle f_{t(a)}}\downarrow & & \downarrow{\scriptstyle f_{h(a)}} \\ W_{t(a)} & \xrightarrow{\psi_a} & W_{h(a)} \end{array}$$

is commutative for every $a \in \mathcal{Q}_1$.

We denote the category of representations of a quiver $\mathcal{Q}$ with relations $\mathcal{R}$ as $\mathbf{Rep}_{(\mathcal{Q},\mathcal{R})}$.

***Remark* 5.5.2.** There is an equivalence of categories between $\mathbf{Rep}_{(\mathcal{Q},\mathcal{R})}$ and $\mathbf{Mod}_{\mathbf{k}[\mathcal{Q}]/\mathcal{R}}$.

5.5.2 The quiver associated to an homogeneous variety

Consider the Hermitian symmetric variety $X = G/P$. In order to describe homogeneous bundles on X, we are going to associate to X a quiver.

Definition 5.5.8 (Quiver associated to an homogeneous variety)**.** To every homogeneous variety $X = G/P$ we associate the quiver $\mathcal{Q}_X$ that has as vertices the irreducible homogeneous vector bundles E_λ and such that there is an arrow $E_\lambda \longrightarrow E_\mu$ between two irreducible bundles if and only if $\operatorname{Ext}^1(E_\lambda, E_\mu)^G \neq 0$.

***Remark* 5.5.3.** Let $\xi_1, \dots, \xi_m$ be the weights of the homogeneous bundle Ω^1_X. Then Proposition 5.4.4 shows that there is an arrow in $\mathcal{Q}_X$ from E_λ to E_ν if and only if $\nu = \lambda + \xi_i$ for a certain $i \in \{1, \dots, m\}$. In particular we see that the quiver is leveled.

Corollary 5.5.1. *The quiver $\mathcal{Q}_X$ is leveled by the function*

$$\widetilde{\mu}(E_\lambda) = \frac{\mu(E_\lambda)}{\mu(\Omega^1_X)}$$

Proof. Follows immediately from Proposition 5.4.4. □

Now, let E be an homogeneous bundle on X : then we can write

$$\operatorname{gr} E = \bigoplus_\lambda E_\lambda \otimes V_\lambda$$

where the sum is over all dominant weights $\lambda \in D'$ and the vector spaces $V_\lambda = \operatorname{Hom}(\operatorname{gr} E, E_\lambda)^G$ count the multiplicity of E_λ in $\operatorname{gr} E$.

Then we associate to E a representation of $\mathcal{Q}$ in this way:

- To every point $\lambda \in \mathcal{Q}_{X,0}$ we associate the vector space V_λ.
- For every $\lambda \in \mathcal{Q}_{X,0}$ choose a maximal vector $v_\lambda \in \Gamma^\lambda$ and for any ξ_i an element $n_i \in \mathfrak{g}_{\xi_i}$. Now, we know that if $\operatorname{Ext}(E_\lambda, E_\nu)^G \cong \operatorname{Hom}(\Omega^1_X \otimes E_\lambda, E_\nu)^G \neq 0$, then $\nu - \lambda = \xi_i$ for a certain i. In this case we fix an element $m_{\lambda,\nu} \in Hom(\Omega^1_X \otimes E_\lambda, E_\nu)^G$ such that

$$\xi_i \otimes m_{\lambda,\nu}\colon v_\lambda \mapsto v_\nu$$

  Notice that $m_{\lambda,\nu}$ is actually a generator, since we know that $Hom(\Omega^1_X \otimes E_\lambda, E_\nu)^G \cong \mathbf{k}$.

Now, observe that

$$\mathrm{Ext}^1(\mathrm{gr}\ E, \mathrm{gr}\ E)^G = \bigoplus_{(\lambda,\nu)} \mathrm{Hom}(V_\lambda, V_\nu) \otimes \mathrm{Ext}^1(E_\lambda, E_\nu)^G$$

so that, to give an element in every $\mathrm{Hom}(V_\lambda, V_\nu)$ it is sufficient to give an element $[E] \in \mathrm{Ext}^1(\mathrm{gr}\ E, \mathrm{gr}\ E)^G$.

To this end, observe that from Proposition 5.4.4, we have an isomorphism

$$\mathrm{Ext}^1(\mathrm{gr}\ E, \mathrm{gr}\ E)^G \cong \mathrm{Hom}(\Omega^1_X \otimes \mathrm{gr}\ E, \mathrm{gr}\ E)^G$$

and, thanks to Theorem 5.3.1, we see that to the homogeneous vector bundle E corresponds naturally the equivariant morphism

$$\theta\colon \Omega^1_X \otimes \mathrm{gr}\ E \longrightarrow \mathrm{gr}\ E$$

It is easy to verify that in this way we have defined a functor

$$\mathbf{HomVec}_X \longrightarrow \mathbf{Rep}_{\mathcal{Q}_X}$$

Now, we need to recall briefly the Yoneda product, according to [Eis95]: for any E, F, K homogeneous vector bundles on X we have an equivariant morphism

$$\mathrm{Ext}^i(E, F) \otimes \mathrm{Ext}^j(F, K) \longrightarrow \mathrm{Ext}^{i+j}(E, K)$$

and moreover this product is associative. In particular, for any homogeneous vector bundle E we get a bilinear map

$$\mathrm{Ext}^1(E, E) \otimes \mathrm{Ext}^1(E, E) \longrightarrow \mathrm{Ext}^2(E, E)$$

whose symmetric part induces a quadratic morphism

$$\mathrm{Ext}^1(E, E) \longrightarrow \mathrm{Ext}^2(E, E)$$

In particular, singe the Yoneda product is equivariant, we get an induced morphism

$$m\colon \mathrm{Ext}^1(E, E)^G \longrightarrow \mathrm{Ext}^2(E, E)$$

that we call the equivariant Yoneda morphism.

Proposition 5.5.1. *Let $X = G/P$ be an Hermitian symmetric variety.*

1. *For any homogeneous bundle E, we have $m([E]) = 0$, where m is the equivariant Yoneda morphism*

$$m\colon Ext^1(gr\ E, gr\ E)^G \longrightarrow Ext^2(gr\ E, gr\ E)^G$$

2. *Conversely, for any R-module F and for every $e \in Ext^1(E, E)^G$ such that $m(e) = 0$ there exists an homogeneous vector bundle E on X such that $gr\ E = F$ and $e = [E]$.*

Idea of proof. It can be shown that under the isomorphism $\mathrm{Ext}^i(\mathrm{gr}\ E, \mathrm{gr}\ E)^G \cong \mathrm{Hom}(\Omega^i_X \otimes \mathrm{gr}\ E, \mathrm{gr}\ E)$ of Proposition 5.4.4, the Yoneda product corresponds to the map

$$m\colon \mathrm{Hom}(\Omega^1_X \otimes \mathrm{gr}\ E, \mathrm{gr}\ E) \longrightarrow \mathrm{Hom}(\Omega^2_X \otimes \mathrm{gr}\ E, \mathrm{gr}\ E) \qquad \theta \mapsto \theta \wedge \theta$$

so that the conclusion follows immediately from Theorem 5.3.1. □

Remark 5.5.4. The above result tells us which relations we should put on the quiver in order to get an equivalence of categories: we can write any $e \in \mathrm{Ext}^1(\mathrm{gr}\, E, \mathrm{gr}\, E)^G$ as

$$e = \sum_{(\lambda,\nu)} g_{\lambda,\nu} \otimes m_{\lambda,\nu}$$

for certain $g_{\lambda,\nu} \in \mathrm{Hom}(V_\lambda, V_\nu)$. Then the condition $m(e) = 0$ becomes

$$\sum_{\lambda,\nu} (\sum_{\gamma} (g_{\gamma\nu} g_{\lambda,\gamma})(m_{\gamma\nu} \wedge m_{\lambda\gamma})) = 0$$

Hence,we define the ideal of the relations $\mathcal{R}_X$ as the ideal in $\mathbf{k}[\mathcal{Q}_X]$ generated by the quadratic equations

$$\sum_{\gamma} (g_{\gamma\nu} g_{\lambda,\gamma})(m_{\gamma\nu} \wedge m_{\lambda\gamma})$$

for any λ, ν.

Theorem 5.5.1. *There is an equivalence of categories between* $\textbf{\textit{HomVec}}_X, \textbf{\textit{Higgs}}_X, \textbf{\textit{Rep}}_{(\mathcal{Q}_X, \mathcal{R}_X)}$.

Proof. Follows from Theorem 5.3.1 and from the above discussion. □

Example 5.5.1 (The projective plane). On the projective plane $\mathbb{P}^2 = SL(3)/P(\alpha_1)$ the corresponding quiver is given by Figure 5.4.

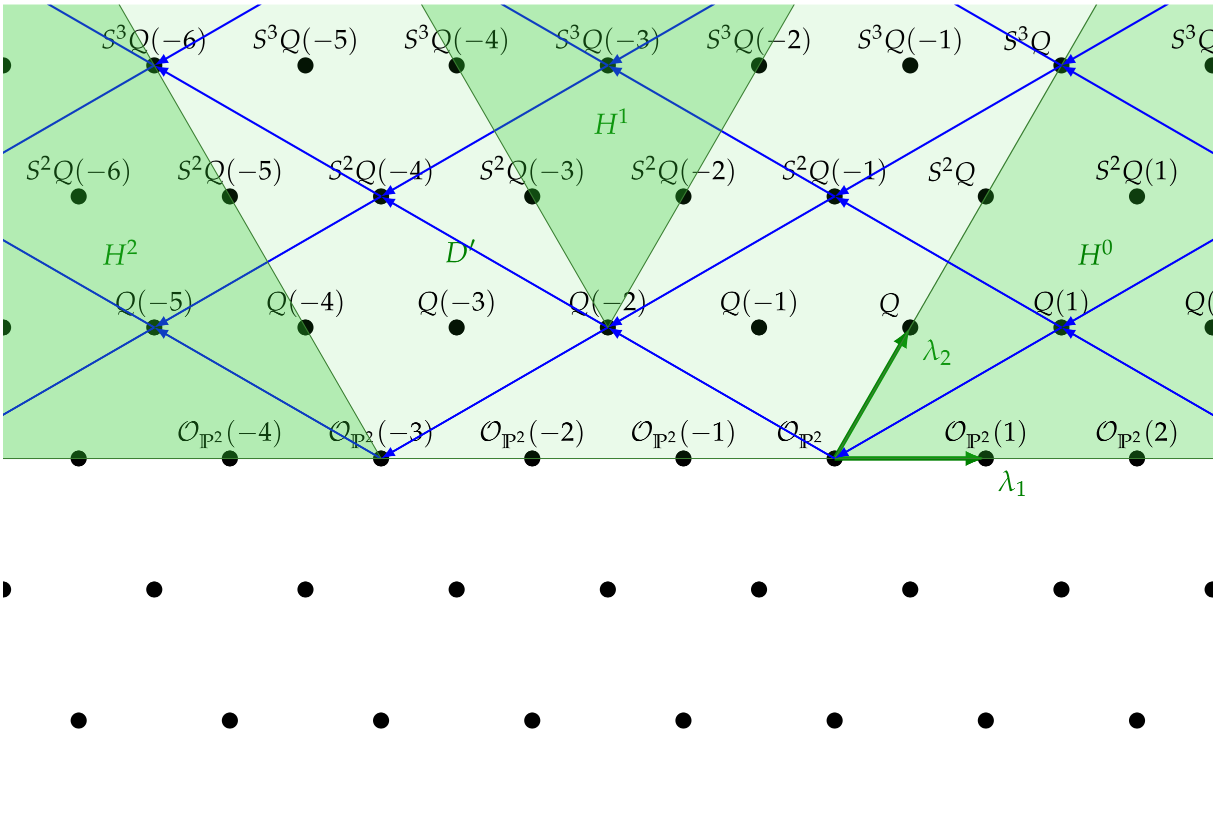


Figure 5.4: For simplicity, there is only one connected component of the quiver represented

Moreover, the relations on this quiver are those given by the commutativity of all squares

$$\begin{array}{ccc} E_\lambda & \longrightarrow & E_{\lambda+\xi_1} \\ \downarrow & & \downarrow \\ E_{\lambda+\xi_2} & \longrightarrow & E_{\lambda+\xi_1+\xi_2} \end{array}$$

since we will not need this we will not give the proof, but the interested reader can find it in [OR06].

5.6 Cohomology of homogeneous vector bundles

In all what follows $X = G/P$ will be an irreducible Hermitian symmetric variety of type ADE and of dimension n. We have seen that Bott's theorem gives a complete description of the cohomology of a completely reducible homogeneous bundle on X, now we are going to present a method, due to Ottaviani and Rubei to compute the cohomology of any vector bundle from its quiver representation. The exposition follows faithfully the article [OR06].

First we need a result in homological algebra: let E be a vector bundle on X, not necessarily homogeneous, and consider a filtration

$$0 = E_0 \subseteq E_1 \cdots \subseteq E_r = E$$

Then we have the exact sequences

$$\begin{array}{c} 0 \longrightarrow E_p \longrightarrow E_{p+1} \longrightarrow E_{p+1}/E_p \longrightarrow 0 \\ 0 \longrightarrow E_{p+1}/E_p \longrightarrow E/E_p \longrightarrow E_p/E_{p+1} \longrightarrow 0 \end{array}$$

and from the corresponding exact sequences in cohomology we define

$$Z_j^p \overset{\text{def}}{=} \text{Ker}\ (H^j(X, E_{p+1}/E_p) \longrightarrow H^{j+1}(X, E_p)),$$
$$B_j^p \overset{\text{def}}{=} \text{Im}\ (H^{j-1}(X, E/E_{p+1}) \longrightarrow H^j(X, E_{p+1}/E_p))$$

Theorem 5.6.1. *With notations as before, we have $B_j^p \subseteq Z_j^p$ and*

$$H^j(X, E) \cong \bigoplus_{p=0}^{r-1} Z_j^p / B_j^p$$

Proof. See [OR06]. □

Now, we go back to homogeneous vector bundles on X. We denote by $\xi_1, \ldots, \xi_n$ the weights of the cotangent bundle Ω_X^1.

Proposition 5.6.1. *Let E_λ and E_μ be in two adjacent Bott chambers such that $H^i(X, E_\lambda) \cong H^{i+1}(X, E_\mu)$. Then $\mu - \lambda = k\xi_j$ for some $k \in \mathbb{Z}$ and a certain root ξ_j of Ω_X^1 and we have*

$$\dim_{\mathbf{k}} Hom\ (E_\lambda \otimes S^k(\Omega_X^1), E_\mu)^G = 1$$

Proof. The fact that $\mu - \lambda = k\xi_j$ follows from Corollary 5.4.3. From the generalized Littlewood-Richardson rule (Theorem 1.1.4) we see that

$$E_\lambda \otimes S^k(\Omega_X^1) = \bigoplus_{\nu \in J} E_{\lambda+\nu}$$

where

$$J \subseteq \{ a_{i_1}\xi_{i_1} + \cdots + a_{i_h}\xi_{i_h} \mid a_{i_j} \geq 0, a_{i_1} + \cdots + a_{i_h} = k \}$$

and to conclude we observe that if $a_{i_q} \neq 0$ for a certain $i_q \neq j$ then

$$|a_{i_1}\xi_{i_1} + \cdots + a_{i_h}\xi_{i_h}| < |k\xi_j|$$

as all the roots ξ_i have the same length, since X is of type ADE. □

Proposition 5.6.2. *For every weight ξ_j of Ω^1_X we have*

$$Ext^2(E_\lambda, E_{\lambda+2\xi_j})^G = Hom(E_\lambda \otimes \Omega^2_X, E_{\lambda+2\xi_j})^G = 0$$

Proof. The first equality follows from Proposition 5.4.4. For the second one, using the generalized Littlewood-Richardson rule (Theorem. 1.1.4) we see that

$$E_\lambda \otimes \Omega^2_X = \bigoplus_{(i,j)\in J} E_{\lambda+\xi_i+\xi_j}$$

where $J \subseteq \{ (h,k) \mid 1 \leq h \leq k \leq n \}$. Then it is enough to prove that $\xi_h + \xi_k \neq 2\xi_j$ for every $h \neq k$, but since X is of type ADE all the roots have the same length so that $|\xi_h + \xi_k| < |2\xi_j|$ if $h \neq k$. □

Keeping the same notations as in Propositions 5.6.1 and 5.6.2 we see that to build the quiver representation associated to E we have chosen distinguished elements in $\mathrm{Hom}(E_{\lambda+p\xi_j} \otimes \Omega^1_X, E_{\lambda+(p+1)\xi_j})^G$ and these give a distinguished element in $\mathrm{Hom}(E_\lambda \otimes S^k(\Omega^1_X), E_\mu)^G$, that is one dimensional by Proposition 5.6.1. Thanks to these elements, we can define extensions

$$0 \longrightarrow E_{\lambda+(p+1)\xi_j} \longrightarrow Z_p \longrightarrow E_{\lambda+p\xi_j} \longrightarrow 0$$

and using Proposition 5.5.1, we see that these extension fit together, as the Ext^2 vanish by Proposition 5.6.2. This way, we have obtained a bundle P' such that $\mathrm{gr}\, P = \sum_{p=0}^{k-1} E_{\lambda+p\xi_j}$ and two exact sequences

$$0 \longrightarrow Z' \longrightarrow P' \longrightarrow E_\lambda \longrightarrow 0$$
$$0 \longrightarrow E_\mu \longrightarrow Z' \longrightarrow Z/E_\mu \longrightarrow 0$$

Now, let λ' and μ' be the vertices of the Bott chambers containing λ and μ respectively and let A be the unique indecomposable bundle in the extension

$$0 \longrightarrow E_{\mu'} \longrightarrow A \longrightarrow E_{\lambda'} \longrightarrow 0$$

Lemma 5.6.1. *With the above notations we have*

$$H^i(X, A) = 0 \qquad \text{for every } i$$

Proof. See [OR06] and [Hil82]. □

Now, let $U = H^i(X, E_\lambda) = H^{i+1}(X, E_\mu)$. Then

Lemma 5.6.2. *The only direct summand of $gr(E_{\mu'} \otimes U)$ such that $H^\bullet(X, \cdot) \cong U$ is E_μ.*

Proof. See [OR06]. □

For what follows, it will be useful to set up a piece of notation: if V is a representation of G, we denote by V^U the direct sum of all the subrepresentations of V isomorphic to U.

We also need a preliminary result.

Definition 5.6.1 (A_m-quiver)**.** Let m be a positive integer. The A_m quiver is the quiver

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet$$

with m vertices.

Then we have the following result

Theorem 5.6.2. *Every representation of the A_m-quiver can be decomposed into indecomposable representations. Every indecomposable representation is of type*

$$0 \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow V_1 \xrightarrow{f_1} V_2 \longrightarrow \ldots V_{r-1} \xrightarrow{f_{r-1}} V_r \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0$$

where the V_i have dimension 1 and the maps f_i are isomorphisms for every $i = 1, \ldots, r-1$.

Proof. See [GR92]. □

Theorem 5.6.3. *With the same notations as before*

$$H^j(X, P') = 0 \qquad \textit{for all } j$$

Proof. Let K be the subbundle in $A \otimes U$ generated by all the direct summands isomorphic to E_λ. Then there is an exact sequence

$$0 \longrightarrow K \longrightarrow A \otimes U \longrightarrow Q \longrightarrow 0$$

and by Lemma 5.6.2 it follows that $H^j(X, K)^U$ and $H^j(X, Q)^U$ are nonzero at most for $j = i$ or $j = i + 1$.

We want to show that gr K contains all the direct summands of gr $(A \otimes U)$ isomorphic to E_μ. Otherwise, we would have $H^{i+1}(X, Q)^U \neq 0$ and then Lemma 5.6.1 would give that $H^{i+2}(X, K)^U \neq 0$, that is a contradiction. Thus, we see that $H^j(X, Q)^U = 0$ for all j and consequently $H^j(X, K)^U \cong 0$ for every j.

Then, let S' be the quotient of K obtained restricting the quiver representation of E to the path that connect the vertices E_λ and E_μ. Decomposing S' into irreducible representations (cfr. Theorem 5.6.2) we see that S' is isomorphic to the direct sum of various copies of P'.

Now, we have the exact sequence

$$0 \longrightarrow K' \longrightarrow K \longrightarrow S' \longrightarrow 0$$

and since $H^j(X, \mathrm{gr}\ K')^U = 0$ for every j, it follows that $H^j(X, K')^U = 0$ for every j as well, so that $H^j(X, S') = 0$ for all j. □

Now, applying Theorem 5.6.3 to the exact sequence in cohomology associated to the short exact sequence

$$0 \longrightarrow Z' \longrightarrow P' \longrightarrow E_\lambda \longrightarrow 0$$

we get an isomorphism

$$H^i(X, E_\lambda) \xrightarrow{\sim} H^{i+1}(X, Z')^U$$

and from the exact sequence

$$0 \longrightarrow E_\mu \longrightarrow Z' \longrightarrow Z'/E_\mu \longrightarrow 0$$

we get another isomorphism

$$H^{i+1}(X, E_\mu) \xrightarrow{\sim} H^{i+1}(X, Z')$$

so that we get a distinguished isomorphism

$$j_{\lambda\mu} \colon H^i(X, E_\lambda) \longrightarrow H^{i+1}(X, E_\mu)$$

Now, denote by P the homogeneous bundle corresponding to the A_m-quiver from E_λ to E_μ, with the same representation maps as for E.

Lemma 5.6.3. *With the above notations, the boundary map*

$$U \otimes V_\lambda = H^i(X, P/V_\mu \otimes E_\mu)^U \longrightarrow H^i(X, V_\mu \otimes E_\mu) = U \otimes V_\mu$$

is the tensor product of the distinguished isomorphism $j_{\lambda\mu}$ and the composition of the maps in the quiver representation.

Proof. Suppose first that P is irreducible. We can assume that $\dim V_{\lambda+p\xi_j} = 1$ for $0 \le p \le k$ and that $\lambda + k\xi_j = \mu$. We observe that P defines nonzero elements in the one-dimensional spaces

$$\mathrm{Hom}(V_{\lambda+p\xi_j} \otimes E_{\lambda+p\xi_j} \otimes \Omega^1_X, V_{\lambda+(p+1)\xi_j} \otimes E_{\lambda+(p+1)\xi_j})^G$$
$$= \mathrm{Hom}(V_{\lambda+p\xi_j}, V_{\lambda+(p+1)\xi_j}) \otimes \mathrm{Hom}(E_{\lambda+p\xi_j} \otimes \Omega^1_X, E_{\lambda+(p+1)\xi_j})^G$$

We have a natural isomorphism

$$\bigotimes_{i=0}^{k-1} \mathrm{Hom}(V_{\lambda+p\xi_j} \otimes E_{\lambda+\xi_j} \otimes \Omega^1_X, V_{\lambda+(p+1)\xi_j} \otimes E_{\lambda+(p+1)\xi_j})^G \cong \mathrm{Hom}(V_\lambda \otimes E_\lambda \otimes S^k(\Omega^1_X), V_\mu \otimes E_\mu)^G$$
$$= \mathrm{Hom}(V_\lambda, V_\mu) \otimes \mathrm{Hom}(E_\lambda \otimes S^k(\Omega^1_X), E_\mu)^G$$

where in $\mathrm{Hom}(V_\lambda, V_\mu)$ we perform the composition of the quiver maps. From the above remarks and this isomorphism we get an element in $\mathrm{Hom}(V_\lambda, V_\mu) \otimes \mathrm{Hom}(E_\lambda \otimes S^k(\Omega^1_X), E_\mu)^G$ from which we can reconstruct P.

Now from the exact sequence

$$0 \longrightarrow Z \longrightarrow P \longrightarrow V_\lambda \otimes E_\lambda \longrightarrow 0$$

we get the boundary map

$$H^i(X, E_\lambda \otimes V_\lambda) \longrightarrow H^{i+1}(X, Z)^U$$

and from the exact sequence

$$0 \longrightarrow V_\mu \otimes E_\mu \longrightarrow Z \longrightarrow P' \longrightarrow 0$$

and Theorem 5.6.3 we get an isomorphism

$$H^{i+1}(X, E_\mu \otimes V_\mu) \xrightarrow{\sim} H^{i+1}(X, Z)$$

and we obtain a map

$$c_{\lambda\mu} \colon H^i(X, E_\lambda \otimes V_\lambda) \longrightarrow H^{i+1}(X, E_\mu \otimes V_\mu)$$

that by construction is the tensor product of the isomorphism $j_{\lambda\mu}$ and the maps in the quiver representation.

Now, for the general case, it is enough to split P into the direct sum of irreducible representations, using Theorem 5.6.2. □

Definition 5.6.2. We define the maps $c_i \colon H^i(X, \mathrm{gr}\ E) \longrightarrow H^{i+1}(X, \mathrm{gr}\ E)$ as

$$c_i = \sum c_{\lambda\mu}$$

where the sum is over all couples (λ, μ) in adjacent Bott chambers such that $H^i(X, E_\lambda) \cong H^{i+1}(X, E_\mu)$ and the maps $c_{\lambda\mu}$ are defined as in the proof of Lemma 5.6.3.

Now we are going to show that in this way we have defined a complex

$$0 \longrightarrow H^0(X, \mathrm{gr}\ E) \xrightarrow{c_0} H^1(X, \mathrm{gr}\ E) \longrightarrow \ldots \longrightarrow H^{n-1}(X, \mathrm{gr}\ E) \xrightarrow{c_{n-1}} H^n(X, \mathrm{gr}\ E) \longrightarrow 0$$

whose cohomology gives the cohomology of E. To begin with, we observe that the construction of the complex $H^\bullet(X, \mathrm{gr}\ E)$ is functorial in E.

We need an auxiliary construction

Definition 5.6.3 (Bundle generated by arrows)**.** Let E be an homogeneous bundle on X with $\mathrm{gr}\ E = \bigoplus V_\lambda \otimes E_\lambda$. Then $V = \bigoplus V_\lambda$ is a $\mathbf{k}[\mathcal{Q}_\mathcal{X}]$-module. Consider a weight $\lambda' \in D'$, then we define the subbundle of E generated by all the arrows starting from λ as the bundle corresponding to the sub $\mathbf{k}[\mathcal{Q}_X]$-module generated by $V_\lambda \subseteq V$.

Now we can prove the theorem:

Theorem 5.6.4. *The sequence* $(H^\bullet(X, gr\ E), c_\bullet)$ *is a complex with cohomology*

$$\frac{Ker\ c_i}{Im\ c_{i-1}} \cong H^i(X, E)$$

Proof. Let U be an irreducible G-module. It is enough to prove that

$$H^i(X,E)^U = \frac{\text{Ker } (H^i(X, \text{gr } E)^U \xrightarrow{c_i} H^{i+1}(X, \text{gr } E)^U)}{\text{Im } (H^{i-1}(X, \text{gr } E)^U \xrightarrow{c_i} H^i(X, \text{gr } E)^U)}$$

In order to do this we consider the following filtration of E:

- E_1 is defined taking all the arrows starting from any direct summand $F \subseteq \text{gr } E$ such that $H^n(X, F)^U \neq 0$.
- E_2 is defined taking all the arrows starting from any direct summand $F \subseteq \text{gr } E$ such that $H^{n-1}(X, F)^U \oplus H^n(X, F)^U \neq 0$.
- In general, E_i is defined by taking all the arrows starting from any direct summand $F \subseteq \text{gr } E$ such that $\oplus_{k=0}^{i-1} H^{n-i}(X, F)^U \neq 0$.

Then we have that

$$H^j(X, \text{gr } E_{i+1}/E_i)^U = \begin{cases} H^{n-1}(X, \text{gr } E)^U, & \text{if } j = n-i \\ 0 & \text{if } i \neq n-i \end{cases}$$

so that, from the spectral sequence associated to the filtration, we see have

$$H^j(X, E_{i+1}/E_i)^U = \begin{cases} H^{n-1}(X, \text{gr } E)^U, & \text{if } j = n-i \\ 0 & \text{if } i \neq n-i \end{cases}$$

We have the commutative diagram

$$\begin{array}{ccccc}
H^{i-1}(X, E_{n-i+2}/E_{n-i+1})^U & & & & \\
\| & & & & \\
H^{i-1}(X, \text{gr } E/E_{n-i+1})^U & \searrow & & & \\
\downarrow f & & & & \\
H^i(X, E/E_{n-i+1})^U & \longrightarrow & H^i(X, E_{n-i+1}/E_{n-i})^U & \longrightarrow & H^{i+1}(X, E_{n-i})^U \\
 & & & \searrow & \downarrow g \\
 & & & & H^{i+1}(X, \text{gr } E_{n-i})^U \\
 & & & & \| \\
 & & & & H^{i+1}(X, E_{n-1}/E_{n-i-1})^U
\end{array}$$

where f is given by the spectral sequence and it is surjective, as $H^i(X, \text{gr } E/E_{n-i+1})^U = 0$, and the map g is injective as $H^i(X, \text{gr } E_{n-i})^U = 0$. Furthermore, we observe that the central term is

$$H^i(X, E_{n-i+1}/E_{n-i})^U = H^i(X, \text{gr } E)^U$$

Now, from Theorem 5.6.1 and the above diagram it follows that

$$H^i(X,E)^U \cong Z_i^{n-i}/B_i^{n-i} = \frac{\text{Ker } (H^i(X, E_{n-i+1}/E_{n-i})^U \longrightarrow H^{i+1}(X, E_{n-i}/E_{n-i-1})^U)}{\text{Im } (H^i(X, E_{n-i+2}/E_{n-i+1})^U \longrightarrow H^i(X, E_{n-i+1}/E_{n-i})^U)}$$

We just need to show that the boundary map

$$H^i(X, E_{n-i+2}/E_{n-i+1})^U \longrightarrow H^i(X, E_{n-i+1}/E_{n-i})^U$$

induced by the exact sequence

$$0 \longrightarrow E_{n-i+1}/E_{n-i} \longrightarrow E_{n-i+2}/E_{n-i} \longrightarrow E_{n-i+2}/E_{n-i+1} \longrightarrow 0$$

is indeed the composition of the quiver maps tensored with the distringuished isomorphism $j_{\lambda\mu}$. By Lemma 5.6.3, we know that this is true if the quiver representation has support on a A_m-quiver and we want to reduce ourselves to this case. take two subbundles $V_\lambda \otimes E_\lambda \subseteq \operatorname{gr} E_{n-i+2}/E_{n-i+1}$ and $V_\mu \otimes \operatorname{gr} E_\mu \subseteq E_{n-i+1}/E_{n-i}$ such that $U \cong H^{i-1}(X, E_\lambda) \cong H^i(X, E_\mu)$.

We need to prove that the composition of the maps

$$H^{i-1}(X, V_\lambda \otimes E_\lambda) \longrightarrow H^{i-1}(X, E_{n-i+2}/E_{n-i+1})^U \longrightarrow H^i(X, E_{n-i+1}/E_{n-i})^U \longrightarrow H^i(X, V_\mu \otimes E_\mu)$$

is obtained by composing the maps that go from V_λ to V_μ in the quiver representation.

Consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K \cap (E_{n-i+1}/E_{n-i}) & \longrightarrow & E_{n-i+1}/E_{n-i} & \longrightarrow & Q' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & E_{n-i+2}/E_{n-i} & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & E_{n-i+2}/E_{n-i+1} & \longrightarrow & Q'' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where W is the quotient of E_{n+i-2}/E_{n-i} obained by taking all the arrows arriving in E_μ. This diagram induces another commutative diagram

$$\begin{array}{ccccc}
 & & H^i(X, V_\lambda \otimes E_\lambda) & \searrow & \\
 & & \downarrow & & \\
 & & H^i(X, E_{n-i+2}/E_{n-i+1})^U & \longrightarrow & H^i(X, Q'')^U \\
 & & \downarrow & & \downarrow \\
H^{i+1}(X, K \cap E_{n-i+1}/E_{n-i})^U & \xrightarrow{f} & H^{i+1}(X, E_{n-i+1}/E_{n-i})^U & \longrightarrow & H^{i+1}(X, Q')^U \\
 & & \downarrow h & & \\
 & & H^{i+1}(X, V_\mu \otimes E_\mu) & &
\end{array}$$

Then we see that $h \circ f = 0$ because E_μ is not a direct summand of K. Then the map h lifts to

$$\begin{array}{ccc}
H^i(X, V_\lambda \otimes E_\lambda) & \overset{r}{\searrow} & \\
\downarrow & & \\
H^i(X, E_{n-i+2}/E_{n-i+1})^U & \longrightarrow & H^i(X, Q'')^U = H^i(X, V_\lambda \otimes E_\lambda) \\
\downarrow & & \downarrow \\
H^{i+1}(X, E_{n-i+1}/E_{n-i})^U & \longrightarrow & H^{i+1}(X, Q')^U \\
\downarrow & \overset{g}{\swarrow} & \\
H^{i+1}(X, V_\mu \otimes E_\mu) & &
\end{array}$$

For the final step, we consider the subbundle $P \subseteq Q$ obtained taking all the arrows starting from λ. Then P satisfies the assumptions of Lemma 5.6.3. We have a commutative diagram

$$\begin{array}{ccc} H^i(X, P/P\cap Q')^U \cong V_\lambda \otimes W & \xrightarrow{\ r\ } & H^i(X, Q'')^U \\ \downarrow & & \downarrow \\ H^{i+1}(X, V_\mu \otimes E_\mu)^U & \xrightarrow{\ k\ } & H^{i+1}(X, Q')^U \\ & & \downarrow g \\ & & H^{i+1}(X, V_\mu \otimes E_\mu) \end{array}$$

where k and r are induced in cohomology by the inclusions. By construction of Q, it follows that $H^{i+1}(X, \mathrm{gr}\, Q')^U = H^{i+1}(X, V_\mu \otimes E_\mu)$; hence it follows that $H^{i+1}(X, Q')^U = H^{i+1}(X, V_\mu \otimes E_\mu)$ where the equality is given by g and $g \circ k = \mathrm{id}$.

By Lemma 5.6.3 the map in the leftmost column in the last diagram is the composition of the quiver maps tensored with the distinguished isomorphism $j_{\lambda\mu}$. Then the proof is completed by diagram chasing in the other two diagrams. □

5.7 Example: the projective line

To illustrate the theory that we have developed so far, we consider the case of the projective line $\mathbb{P}^1$.

Let $V = \mathbf{k}^2$ and consider the group $G = SL(2)$. As we did for the n-dimensional case, we set in G the torus T and the Borel subgroup

$$T = \left\{ \begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix} \middle| \ t_i \in \mathbf{k}^*, t_0 t_1 = 1 \right\} \qquad B = \left\{ \begin{pmatrix} t_0 & 0 \\ x & t_1 \end{pmatrix} \middle| \ x \in \mathbf{k}, t_i \in \mathbf{k}^*, t_0 t_1 = 1 \right\}$$

We consider the characters $\chi_i \in X^*(T)$ defined by

$$\chi_i \colon T \longrightarrow \mathbf{k}^* \qquad t \mapsto t_i$$

for $i = 0, 1$ and we set $\alpha_1 = \chi_1 - \chi_0$. Then $\Phi = \{\alpha_1, -\alpha_1\}$ and $\Phi(B) = \Phi^+ = \Delta = \{\alpha_1\}$. The Weyl group is generated by the reflection

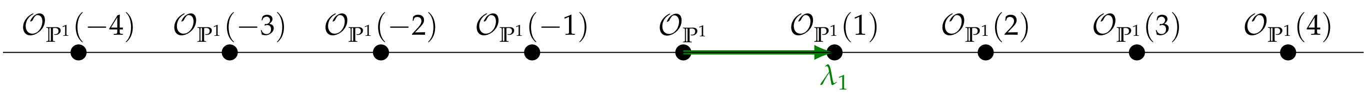
$$r \colon \alpha_1 \mapsto -\alpha_1$$

The fundamental weight corresponding to α_1 is $\lambda_1 = \frac{1}{2}\alpha_1$ and the dominant weights are those of the form $p\lambda_1$ for $p \geq 0$. The irreducible representation of G corresponding to the dominant weigth $p\lambda_1$ is the symmetric power $S^p(V)$.

Now, in this case the parabolic group P coincides with the Borel subgroup: in particular the Levi decomposition is given by $B = N \rtimes R$ with

$$N = \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \middle| \ x \in \mathbf{k} \right\} \qquad R = \left\{ \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \middle| \ t \in \mathbf{k}^* \right\}$$

and we see that the Weyl chamber of R is the whole of $\Lambda_{\mathbb{R}}(T)$: the irreducible representation of R corresponding to the weight $a\lambda_1$ is $\det^a$, that corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^1}(a)$.



In this setting, Bott's Theorem is very simple: observe that $g = \lambda_1$ and that $W' = W$, so that we can express it in the following form.

Theorem 5.7.1. *It holds that*

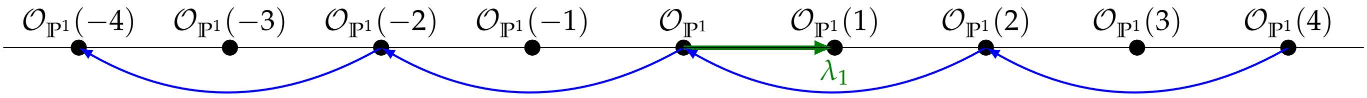
$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) = \begin{cases} S^a(V) & \text{if } a \geq 0 \\ 0 & \text{if } a < 0 \end{cases} \qquad H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a)) = \begin{cases} S^{-a-2}(V) & \text{if } a \leq -2 \\ 0 & \text{if } a > -2 \end{cases}$$

In particular, we see that the Bott chambers of $\mathbb{P}^1$ of length 0 and 1 respectively are

$$B_0 = \{ p\lambda_1 \mid p \geq 0 \} \qquad B_1 = \{ p\lambda_1 \mid p \leq -2 \}$$

and their vertices are indeed $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-2) = \Omega^1_{\mathbb{P}^1}$.

Now, since $\Omega^1_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ we see that the quiver of $\mathbb{P}^1$ is given by the following diagram (for clarity, only one connected component of the quiver is showed):



5.7.1 Plethysm of $\wedge^n(S^m(V))$

As an application of Theorem 5.6.4 we are going to obtain a plethysm formula for $\wedge^n S^m(V)$, i.e. a formula for its decomposition into irreducible representation. We are going to use the same notation of Cayley-Sylvester formula 1.1.5, that is we set

$$p(n, m, e) \overset{\text{def}}{=} \# \{ (\mu_1, \ldots, \mu_n) \mid m \geq \mu_1 \geq \cdots \geq \mu_n \geq 0, \ \mu_1 + \cdots + \mu_n = e \}$$

for every nonnegative integers n, m, e.

The result that we want to prove is the following:

Proposition 5.7.1. *Let n, m be two nonnegative integers. Then*

$$\wedge^n(S^m(V)) = \bigoplus_{e \geq 0} S^e(V)^{\oplus M(n,m,e)}$$

where

$$M(n, m, e) \overset{def}{=} p\left(n, m-n+1, \frac{n(m-n+1)-e}{2}\right) - p\left(n, m-n+1, \frac{n(m-n+1)-e}{2} - 1\right)$$

in particular, we have the isomorphism of $SL(V)$-representations

$$\wedge^n(S^m(V)) \cong S^n(S^{m-n+1}(V))$$

Proof. Consider the vector bundle $\wedge^n(S^m(V)) \otimes \mathcal{O}_{\mathbb{P}^1}$ on $\mathbb{P}^1$. We know from Bott's Theorem that $H^0(\mathbb{P}^1, \wedge^n(S^m(V)) \otimes \mathcal{O}_{\mathbb{P}^1}) = \wedge^n(S^m(V)$ and from Theorem 5.6.4, we know that we can compute $H^0(\mathbb{P}^1, \wedge^n(S^m(V)))$ as the 0-th cohomology of the complex

$$0 \longrightarrow H^0(\mathbb{P}^1, \mathrm{gr}\ \wedge^n (S^m(V)) \otimes \mathcal{O}_{\mathbb{P}^1}) \xrightarrow{c_0} H^1(\mathbb{P}^1, \ \mathrm{gr}\ \wedge^n (S^m(V)) \otimes \mathcal{O}_{\mathbb{P}^1}) \longrightarrow 0$$

From Theorem 5.6.2 it follows that the map c_0 has maximal rank so that the computation of $H^0(\mathbb{P}^1, \wedge^n(S^m(V)) \otimes \mathcal{O}_{\mathbb{P}^1})$ reduces to a combinatorial problem: for every $e \geq 0$ and $i = 0, 1$ we define $m_i(e)$ as the number of times that the irreducible representation $S^e(V)$ appears on $H^i(\mathbb{P}^1, \mathrm{gr}\ \wedge^n (S^m(V)) \otimes \mathcal{O}_{\mathbb{P}^1})$, then

$$\wedge^n(S^m(V)) = H^0(\mathbb{P}^1, \wedge^n(S^m(V)) \otimes \mathcal{O}_{\mathbb{P}^1}) = \bigoplus_{e \geq 0} S^p(V)^{\oplus \max\{ m_0(e) - m_1(e), 0 \}}$$

But now we can observe that $H^1(\mathbb{P}^1, \wedge^n(S^m(V)) \otimes \mathcal{O}_{\mathbb{P}^1}) = 0$ and this means that $m_0(e) \geq m_1(e)$ for all $e \geq 0$. Hence

$$\wedge^n(S^m(V)) = H^0(\mathbb{P}^1, \wedge^n(S^m V))) = \bigoplus_{e\geq 0} S^e(V)^{\oplus(m_0(e)-m_1(e))}$$

Now, taking the Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^1}(1) \longrightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0$$

and then dualizing we get the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow Q \longrightarrow 0$$

where $Q = \Omega^1_{\mathbb{P}^1}{}^\vee(-1) = \mathcal{O}_{\mathbb{P}^1}(1)$ so that

$$\text{gr } V \otimes \mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

Then

$$\text{gr}(S^m(V)) = S^m(\text{gr } V) = S^m(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = \bigoplus_{h=0}^{m} S^{m-h}(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes S^h(\mathcal{O}_{\mathbb{P}^1}(-1)) = \bigoplus_{h=0}^{m} \mathcal{O}_{\mathbb{P}^1}(m-2h)$$

Suppose for simplicity that $m = 2d$ is even: then we can write

$$\text{gr}(S^{2d}(V)) = \bigoplus_{h=-d}^{d} \mathcal{O}_{\mathbb{P}^1}(2h)$$

so that

$$\text{gr } (\wedge^n(S^{2d}(V))) = \wedge^n \text{gr } (\bigoplus_{h=-d}^{d} \mathcal{O}_{\mathbb{P}^1}(2h)) = \bigoplus_{(a_0,\ldots,a_{2d})} (\wedge^{a_0}\mathcal{O}_{\mathbb{P}^1} \otimes \ldots \otimes \wedge^{a_{2d}}\mathcal{O}_{\mathbb{P}^1}) \left(2\sum_{h=0}^{2d} h a_h\right)$$

where the sum is over all $(a_0, \ldots, a_{2d})$ such that $a_i \geq 0$ and $a_0 + \cdots + a_{2d} = n$. Since $\wedge^a \mathcal{O}_{\mathbb{P}^1} = 0$ as soon as $a \geq 2$, we can rewrite the above as

$$\text{gr } (\wedge^n(S^{2d}(V))) = \bigoplus_{(i_1,\ldots,i_n)} \mathcal{O}_{\mathbb{P}^1}\left(2\sum_{h=1}^{n} i_h\right)$$

where the sum is over all elements

$$(i_1, \ldots, i_n) \qquad \text{such that } d \geq i_1 > i_2 > \cdots > i_n \geq -d$$

that represent the subsets of $\{-d, \ldots, d\}$ of cardinality n.

Now, if we define the numbers

$$q(n,d,w) \stackrel{\text{def}}{=} \#\left\{(i_1,\ldots,i_n) \middle| d \geq i_1 > i_2 > \cdots > i_n \geq -d,\ \sum_{h=1}^{n} i_h = w\right\}$$

then we can write

$$\text{gr } (\wedge^n(S^m(V))) = \bigoplus_{w\in\mathbb{Z}} \mathcal{O}_{\mathbb{P}^1}(2w)^{\oplus q(n,d,w)}$$

so that

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\text{gr } (\wedge^n(S^m(V)))) = \bigoplus_{w\geq 0} S^{2w}(V)^{\oplus q(n,d,w)}$$
$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\text{gr } (\wedge^n(S^m(V)))) = \bigoplus_{w\geq 0} S^{2w}(V)^{\oplus q(n,d,-w-1)}$$

Now we observe that $q(n,d,w)$ is clearly an even function on w, so that, in conclusion

$$\wedge^n(S^{2d}V) = \bigoplus_{w\geq 0} S^{2w}(V)^{\oplus q(n,d,w)-q(n,d,w+1)}$$

Now, to complete the proof it is enough to show that

$$q(n,d,w) = p\left(n, 2d-n+1, \frac{n(2d-n+1)-2w}{2}\right)$$

for every n,d,w. To this end, define the sets

$$Q(n,d,w) = \left\{ (j_1,\dots,j_n) \,\middle|\, 2d \geq j_1 > \dots > j_n \geq 0,\ \sum_{h=1}^n j_h = -w + nd \right\}$$
$$P(n,d,w) = \left\{ (\lambda_1,\dots,\lambda_n) \,\middle|\, 2d-n+1 \geq \lambda_1 \geq \dots \geq \lambda_n \geq 0,\ \sum_{h=1}^n \lambda_h = \frac{n(2d-n+1)-2w}{2} \right\}$$

and observe that $p\left(n, 2d-n+1, \frac{n(2d-n+1)-2w}{2}\right) = \#P(n,d,w)$ by definition, whereas the bijection

$$\left\{ (i_1,\dots,i_n) \,\middle|\, d \geq i_1 > \dots > i_n \geq -d,\ \sum_{h=1}^n i_h = -w \right\} \longrightarrow Q(n,d,w) \qquad (i_1,\dots,i_n) \mapsto (i_1+d,\dots,i_n+d)$$

shows that $q(n,d,-w) = \#Q(n,d,w)$ and then $q(n,d,w) = \#Q(n,d,w)$ by parity. Consider the two maps

$$F\colon Q(n,d,w) \longrightarrow P(n,d,w) \qquad (j_1,\dots,j_n) \mapsto (j_1-n+1,\dots,j_{n-1}-1,j_n)$$
$$G\colon P(n,d,w) \longrightarrow Q(n,d,w) \qquad (\lambda_1,\dots,\lambda_m) \mapsto (\lambda_1+n-1,\dots,\lambda_{n-1}+1,\lambda_n)$$

We show that these are well-defined: suppose that $(j_1,\dots,j_n) \in Q(n,d,w)$, then we know that $j_h > j_{h+1}$ for all $h = 2,\dots,n$, so that $j_h - h - 1 \geq j_{h+1} - h$ and it is also obvious that $j_n - n + 1 \leq 2d - n + 1$, while $j_1 \geq 0$ by definition. Moreover, we see that

$$\sum_{h=1}^n j_h - (h-1) = \sum_{h=1}^n j_h - \sum_{h=1}^n (h-1) = -w + nd - \frac{n(n-1)}{2} = \frac{n(2d-n+1)-2w}{2}$$

and this proves that F is well-defined. In a similar way, one can show that G is well-defined, and then it is obvious that they are one the inverse of the other. This concludes the proof of decomposition in the case of $m = 2d$ even, for m odd we can get to the conclusion by a similar reasoning.

For the isomorphism $\wedge^n S^m(V) \cong S^n(S^{m-n+1}(V))$, it is enough to confront the above formula with the Cayley-Sylvester formula 1.1.5, and see that they coincide. □

Chapter 6

The Veronese surface

In this chapter we want to present a method to compute the Betti table of the Veronese surface $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$, under an additional hypothesis on the $SL(3)$-morphisms involved.

6.1 Working on the projective plane

Let $V = \mathbf{k}^3$ and let $\mathbb{P}^2 = \mathbb{P}(V)$ be the projective space of lines of V. Then for every $d \geq 1$ consider the line bundle $\mathcal{O}_{\mathbb{P}^2}(d)$. We want to study the Betti table of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$: from Remark 3.1.1 we know that this Betti table has shape

	0	1	2	...	$3d-3$	$3d-2$	$3d-1$	...	r_d-d	r_d-d+1	...	r_d-3	r_d-2
0	1	–	–	–	–	–	–	–	–	–	–	–	–
1	–	*	*	...	*	*	*	...	*	–	...	–	–
2	–	–	–	...	–	*	*	...	*	*	...	*	*

where $r_d = \dim S^d(\mathbf{k}^3) - 1 = \binom{d+2}{2} - 1 = \frac{(d+2)(d+1)}{2} - 1$. In particular, we see that as soon as $d \geq 5$ there are diagonals $p+q = s$ with more than one nonzero Betti number, so that the techniques that we have used in Subsection 2.7.4 to compute the Betti numbers do not work anymore.

Then we need another method: let M_d be the syzygy bundle defined by the exact sequence

$$0 \longrightarrow M_d \longrightarrow S^d V^\vee \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(d) \longrightarrow 0$$

then by Example 2.1.1, we know that

$$K_{p,q}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = H^q(\mathbb{P}^2, \wedge^{p+q} M_d)$$

The key point is that M_d is an homogeneous bundle, so that we can compute its cohomology using Theorem 5.6.4. More specifically, for any $s = p+q$ we consider the complex

$$0 \longrightarrow H^0(\mathbb{P}^2, \mathrm{gr}\ \wedge^s M_d) \xrightarrow{c_0} H^1(\mathbb{P}^2, \mathrm{gr}\ \wedge^s M_d) \xrightarrow{c_1} H^2(\mathbb{P}^1, \mathrm{gr}\ \wedge^s M_d) \longrightarrow 0 \qquad (6.1)$$

and then Theorem 5.6.4 tells us that

$$H^0(\mathbb{P}^2, \wedge^s M_d) = \mathrm{Ker}\ c_0 \qquad H^1(\mathbb{P}^2, \wedge^s M_d) = \frac{\mathrm{Ker}\ c_1}{\mathrm{Im}\ c_0} \qquad H^2(\mathbb{P}^2, \wedge^s M_d) = \frac{H^2(\mathbb{P}^2, \mathrm{gr}\ \wedge^s M_d)}{\mathrm{Im}\ c_1}$$

The additional hypothesis that we make is that the maps c_0, c_1 have maximal rank, so that, once we know the decomposition into irreducible representation of $H^i(\mathbb{P}^2, \mathrm{gr}\ \wedge^s M_d)$ we can compute the cohomology of the complex in a purely combinatorial way as in the proof of Proposition 5.7.1. To understand better what we mean, we provide an example for the case $d = 2$.

6.1.1 The case $d = 2$

We want to work out again the Betti table of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. First, we need to compute gr M_2: since gr is an exact functor, we have an exact sequence

$$0 \longrightarrow \text{gr } M_2 \longrightarrow \text{gr } (S^2V^\vee \otimes \mathcal{O}_{\mathbb{P}^2}) \longrightarrow \text{gr } \mathcal{O}_{\mathbb{P}^2}(2) = \mathcal{O}_{\mathbb{P}^2}(2) \longrightarrow 0 \tag{6.2}$$

and then, from the Euler sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^2}(1) \longrightarrow V^\vee \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \longrightarrow 0$$

we get that

$$\text{gr } (V^\vee \otimes \mathcal{O}_{\mathbb{P}^2}) = \Omega^1_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$$

It will be convenient for later to work with the quotient bundle Q instead than the cotangent bundle. Taking the dual of the Euler sequence, we see that $Q = \Omega^1_{\mathbb{P}^2}(1)^\vee$, hovever, on $\mathbb{P}^2$ we know that $\Omega^2_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$ so that $\Omega^1_{\mathbb{P}^2}{}^\vee = \Omega^1_{\mathbb{P}^2}(3)$ and then $\Omega^1_{\mathbb{P}^2} = Q(-2)$. Then, we see that

$$\text{gr } (S^2V^\vee \otimes \mathcal{O}_{\mathbb{P}^2}) = S^2(\text{gr } V^\vee \otimes \mathcal{O}_{\mathbb{P}^2}) = S^2(Q(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)) = S^2Q(-2) \oplus Q \oplus \mathcal{O}_{\mathbb{P}^2}(2)$$

and from the exact sequence 6.2 we see that

$$\text{gr } M_2 = S^2Q(-2) \oplus Q$$

for later use, we remark that rk $S^2Q(-2) = 3$ and that rk $Q = 2$.

Now we need to compute the cohomology of $\wedge^s$gr M_2, note that, as rk $M_2 = 5$ it will be enough to consider $s = 0, 1, 2, 3, 4, 5$.

- $s = 0$: in this case we have to consider $\wedge^0$gr $M_2 = \mathcal{O}_{\mathbb{P}^2}$. The complex 6.1 in this case is

  $$0 \longrightarrow \mathbf{k} \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

  and then taking cohomology we see that

  $$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = \mathbf{k} \qquad H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0 \qquad H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$$

- $s = 1$: in this case we have to consider gr $M_2 = S^2Q(-2) \oplus Q$. Then from Example 5.4.1, we can compute the complex in cohomology and see that it is given by

  $$0 \longrightarrow V \longrightarrow V \longrightarrow 0 \longrightarrow 0$$

  so that

  $$H^0(\mathbb{P}^2, M_2) = 0 \qquad H^1(\mathbb{P}^2, M_2) = 0 \qquad H^2(\mathbb{P}^2, M_2) = 0$$

- $s = 2$: in this case we have to consider $\wedge^2(S^2Q(-2) \oplus Q)$. We have the decomposition

  $$\wedge^2(S^2Q(-2) \oplus Q) = \wedge^2(S^2Q)(-4) \oplus [S^2Q \otimes Q](-2) \oplus \wedge^2 Q$$

  Now, we neet to decompose $\wedge^2(S^2Q)$ and $S^2Q \otimes Q$ as direct sum of irreducible homogeneous bundles. This is a problem of representations of $GL(2)$, viewing Q as the standard representation: in particular, using Proposition 5.7.1 for the plethysm $SL(2)$ we see that

  $$\wedge^2(S^2Q) = S^2Q(a)$$

  for a certain $a \in \mathbb{Z}$ that corresponds to a power of the determinant representations of $GL(2)$. Now, to compute the right a, we observe that $\wedge^2(S^2Q)$ is polystable, since irreducible homogeneous bundles are stable (see [Ram66] and [Ott]) and tensor powers, symmetric powers and wedge powers of stable bundles are stable (see [HL96]). Hence

  $$\mu(\wedge^2(S^2Q)) = \mu(S^2Q(a)) \tag{6.3}$$

Now, we see that $\mu(\mathcal{O}_{\mathbb{P}^2}(a)) = \frac{c_1(\mathcal{O}_{\mathbb{P}^2}(a))}{\operatorname{rk}\mathcal{O}_{\mathbb{P}^2}(a)} = a$, whereas the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \longrightarrow V \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow Q \longrightarrow 0$$

tells us that $c_1(Q) = c_1(V \otimes \mathcal{O}_{\mathbb{P}^2}) - c_1(\mathcal{O}_{\mathbb{P}^1}(-1)) = 0 - (-1) = 1$ so that $\mu(Q) = \frac{1}{2}$. Then we see that $\mu(\wedge^2(S^2Q)) = 2\mu(S^2(Q)) = 2 \cdot 2 \cdot \mu(Q) = 2$ and $\mu(S^2Q(a)) = \mu(S^2(Q)) + \mu(\mathcal{O}_{\mathbb{P}^2}(a)) = 1 + a$ so that the equation 6.3 tells us that $a = 1$ and then

$$\wedge^2(S^2(Q)) = S^2Q(1)$$

With a similar reasoning we can compute the decomposition of $S^2Q \otimes Q$: from the Pieri rule 1.1.3 we have that $S^2Q \otimes Q = S^3(Q)(a) \oplus S^{(2,1)}Q(b)$ for certain $a, b \in \mathbb{Z}$. Now we observe that, as representations of $GL(2)$ we have $S^{(2,1)}Q = Q(c)$ for a certain $c \in \mathbb{Z}$, and to compute $a, c \in \mathbb{Z}$ we can use the same strategy with the slope as before: we see that $\mu(S^2Q \otimes Q) = \mu(S^2Q) + \mu(Q) = 1 + \frac{1}{2} = \frac{3}{2}$ whereas $\mu(S^3Q(a)) = \frac{3}{2} + a$ and $\mu(Q(c)) = \frac{1}{2} + c$. Then from the identities

$$\mu(S^2Q \otimes Q) = \mu(S^3Q(a)) \qquad \mu(S^2Q \otimes Q) = \mu(Q(c))$$

we see that $a = 0$ and $c = 1$. Then

$$S^2Q \otimes Q = S^3Q \oplus Q(1)$$

To conclude, we see that $\wedge^2 Q = \mathcal{O}_{\mathbb{P}^2}(a)$ for a certain $a \in \mathbb{Z}$, and using the slope as before we see that $a = 1$. Then, we have shown that

$$\mathrm{gr}\ \wedge^2 M_2 = S^3Q(-2) \oplus S^2Q(-3) \oplus Q(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$$

And taking cohomology with Bott's theorem (see Example 5.4.1) we see that the cohomology complex of $\mathrm{gr}\ \wedge^2 M_2$ is

$$0 \longrightarrow \wedge^2 V \longrightarrow \wedge^2 V \oplus S^2V \longrightarrow 0 \longrightarrow 0$$

and with our hypothesis of maximal rank maps we see that

$$H^0(\mathbb{P}^2, \wedge^2 M_2) = 0 \qquad H^1(\mathbb{P}^2, \wedge^2 M_2) = S^2V \qquad H^2(\mathbb{P}^2, \wedge^2 M_2) = 0$$

- $s = 3$: in this case we have to consider $\wedge^3(S^2Q(-2) \oplus Q)$ we see that

$$\wedge^3(S^2Q(-2) \oplus Q) = \wedge^3(S^2Q)(-6) \oplus [\wedge^2(S^2Q) \otimes Q](-4) \oplus [S^2Q \otimes \wedge^2 Q](-2)$$

Now we see that $\wedge^3(S^2Q) = \mathcal{O}_{\mathbb{P}^2}(a)$ for a certain $a \in \mathbb{Z}$ and since $\mu(\wedge^3(S^2Q)) = 6 \cdot \frac{1}{2} = 3$ and $\mu(\mathcal{O}_{\mathbb{P}^2}(a)) = a$, it follows that $a = 3$.

We proceed to decompose $\wedge^2(S^2Q) \otimes Q$: we have already seen (in the case $d = 2$) that $\wedge^2(S^2Q) = S^2Q(1)$ and then using Pieri's rule we see that $S^2Q(1) \otimes Q = (S^3Q)(a) \oplus Q(b)$ for certain $a, b \in \mathbb{Z}$. But $\mu(S^2Q(1) \otimes Q) = \frac{5}{2}$ and $\mu(S^3Q(a)) = \frac{3}{2} + a$ and $\mu(Q(b)) = \frac{1}{2} + b$ so that $S^2Q(1) \otimes Q = S^3Q(1) \oplus Q(2)$.

The last remaining is $S^2Q \otimes \wedge^2 Q$: we observe that $\wedge^2 Q = \mathcal{O}_{\mathbb{P}^2}(1)$ since $\mu(\wedge^2 Q) = 1$ and then $S^2Q \otimes \wedge^2 Q = S^2Q(1)$.

To conclude, we have shown that

$$\mathrm{gr}\ \wedge^3 M_2 = S^3Q(-3) \oplus S^2Q(-1) \oplus Q(-2) \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$$

and taking cohomology with Bott's theorem we see that the cohomology complex is given by

$$0 \longrightarrow 0 \longrightarrow S^{(2,1)}V \oplus \mathbf{k} \longrightarrow \mathbf{k} \longrightarrow 0$$

so that

$$H^0(\mathbb{P}^2, \wedge^3 M_2) = 0 \qquad H^1(\mathbb{P}^2, \wedge^3 M_2) = S^{(2,1)}V \qquad H^2(\mathbb{P}^2, \wedge^3 M_2) = 0$$

- $s = 4$: in this case we have to study $\wedge^4(S^2Q(-2) \oplus Q)$: we see that

$$\wedge^4(S^2Q(-2) \oplus Q) = [\wedge^3(S^2Q) \otimes Q](-6) \oplus [\wedge^2(S^2Q) \otimes \wedge^2Q](-4)$$

  and we know from computations that we have already done that $\wedge^3(S^2Q) = \mathcal{O}_{\mathbb{P}^2}(3)$, $\wedge^2(S^2Q) = S^2Q(1)$ and $\wedge^2Q = \mathcal{O}_{\mathbb{P}^2}(1)$ so that

$$\text{gr } \wedge^4 M_2 = S^2Q(-2) \oplus Q(-3)$$

  and the cohomology complex is

$$0 \longrightarrow 0 \longrightarrow V \longrightarrow 0 \longrightarrow 0$$

  so that

$$H^0(\mathbb{P}^2, \wedge^4 M_2) = 0 \qquad H^1(\mathbb{P}^2, \wedge^4 M_2) = V \qquad H^2(\mathbb{P}^2, \wedge^4 M_2) = 0$$

- $s = 5$: in this last case we have to study $\wedge^5(S^2Q(-2) \oplus Q)$ and we see that

$$\wedge^5(S^2Q(-2) \oplus Q) = [\wedge^3(S^2(Q)) \otimes \wedge^2Q](-6) = \mathcal{O}_{\mathbb{P}^2}(-2)$$

  and then the cohomology complex is

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0$$

  so that

$$H^0(\mathbb{P}^2, \wedge^5 M_2) = 0 \qquad H^1(\mathbb{P}^2, \wedge^5 M_2) = 0 \qquad H^2(\mathbb{P}^2, \wedge^5 M_2) = 0$$

Then, we see that the Betti table of $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is given representation-theoretically by

	0	1	2	3
0	$\mathbf{k}$	–	–	–
1	–	S^2V	$S^{(2,1)}V$	V

and computing dimensions from Proposition 1.1.1, we see that the Betti numbers are given by

	0	1	2	3
0	1	–	–	–
1	–	6	8	3

in accordance to what we have proved before. In particular this shows that our assumption that the maps in the cohomology complex have maximal rank is verified in this case.

6.1.2 The general case

Now we want to apply this strategy to the case of an arbitrary $d \geq 1$. The main points are:

1. Compute the decomposition into irreducible bundles of gr $\wedge^s M_d$. To do this we can exploit two auxiliary tools:
   - The formula for the plethysm of $\wedge^n(S^m(\mathbf{k}^2))$ that we have obtained in Proposition 5.7.1.
   - The fact that irreducible homogeneous bundle on $\mathbb{P}^2$ are stable (see [Ram66], [Ott]). So that their tensor powers, alternating powers and symmetric powers splits as the direct sums of homogeneous vector bundles with the same slope.
2. Compute the cohomology $H^i(\mathbb{P}^2, \text{gr } M_d)$ thanks to Bott's Theorem in Example 5.4.1.

3. Compute the cohomology of the complex

$$0 \longrightarrow H^0(\mathbb{P}^2, \mathrm{gr}\ \wedge^2 M_d) \xrightarrow{c_0} H^1(\mathbb{P}^2, \mathrm{gr}\ \wedge^2 M_d) \xrightarrow{c_1} H^2(\mathbb{P}^2, \mathrm{gr}\ \wedge^s E_d) \xrightarrow{c_2} 0$$

   under the assumption that the maps c_0 an c_1 have maximal rank. Then use Theorem 5.6.4, which tells us that

$$H^i(\mathbb{P}^2, \wedge^s M_d) \cong \frac{\mathrm{Ker}\ c_i}{\mathrm{Im}\ c_{i-1}}$$

4. Compute the Koszul cohomology through Example 2.1.1, that tells us that

$$K_{p,q}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \cong H^q(\mathbb{P}^2, \wedge^{p+q} M_d)$$

Now, the first point is to compute the decomposition of gr $\wedge^s M_d$ into irreducible representations. We have already seen in the case $d = 2$ that

$$\mathrm{gr}\ (V^\vee \otimes \mathcal{O}_{\mathbb{P}^2}) = Q(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$$

and then

$$\mathrm{gr}\ (S^d V^\vee \otimes \mathcal{O}_{\mathbb{P}^2}) = S^d(\mathrm{gr}\ V^\vee \otimes \mathcal{O}_{\mathbb{P}^2}) = S^d(Q(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)) = \bigoplus_{h=0}^{d} S^h Q(d-2h)$$

Now, the exact sequence

$$0 \longrightarrow M_d \longrightarrow S^d V^\vee \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(d) \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathrm{gr}\ M_d \longrightarrow \mathrm{gr}\ S^d V^\vee \otimes \mathcal{O}_{\mathbb{P}^2} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(d) \longrightarrow 0$$

so that

$$\mathrm{gr}\ M_d = \bigoplus_{h=1}^{d} S^h Q(d-2h)$$

We need to study the wedge power gr $\wedge^s M_d$ for every $s \geq 0$. We see that

$$\mathrm{gr}\ \wedge^s M_d = \wedge^s(\mathrm{gr}\ M_d) = \bigwedge^s \left(\bigoplus_{h=1}^{d} S^h Q(d-2h) \right) = \bigoplus_{(a_1,\dots,a_d)} \bigotimes_{h=1}^{d} \wedge^{a_h} S^h Q(d-2h)$$

$$= \bigoplus_{(a_1,\dots,a_d)} \bigotimes_{h=1}^{d} (\wedge^{a_h} S^h Q)(a_h(d-2h)) = \bigoplus_{(a_1,\dots,a_d)} \left[\bigotimes_{h=1}^{d} \wedge^{a_h} S^h Q \right] \left(sd - 2\sum_h a_h h \right)$$

where the sum is over the set

$$\{ a = (a_1, \dots, a_d) \mid a_h \geq 0,\ a_1 + \cdots + a_d = s \}$$

But since rk $S^h Q = h + 1$ we can limit ourselves to sum over the set

$$A_{d,s} = \{ a = (a_1, \dots, a_d) \mid 0 \leq a_h \leq h+1,\ a_1 + \cdots + a_d = s \}$$

hence

$$\mathrm{gr}\ \wedge^s M_d = \bigoplus_{a \in A_{d,s}} \left[\bigotimes_{h=1}^{d} \wedge^{a_h} S^h Q \right] \left(sd - 2\sum_h a_h h \right)$$

Now, we need to find the decomposition of $\otimes_{h=1}^{d} \wedge^{a_h} S^h Q$: we can write formally

$$\bigotimes_{h=1}^{d} \wedge^{a_h} S^h Q = \bigoplus_{n\geq 0} S^n Q(b_n) \otimes_{\mathbf{k}} M(a,n)$$

where $M(a,n)$ is a vector space that counts the multiplicity of the $SL(2)$-representation $S^n(\mathbf{k}^2)$ in $\otimes_{h=1}^{d} \wedge^{a_h} S^h(\mathbf{k}^2)$ and the b_h are certain integers. To determine the b_h we can use the slope: we see that

$$\mu(\otimes_{h=1}^{d} \wedge^{a_h} S^h Q) = \sum_{h=1}^{d} \mu(\wedge^{a_h} S^h Q) = \sum_{h=1}^{d} \frac{1}{2} a_h h = \frac{1}{2} \sum_{h=1}^{d} a_h h$$

and then the equation $\mu(S^n Q(b_n)) = \mu(\otimes_{h=1}^{d} \wedge^{a_h} S^h Q)$ tells us that

$$b_n = \frac{1}{2}\left(\sum_{h=1}^{d} a_h h - n\right)$$

Summing up, we can write

$$\operatorname{gr} \wedge^s M_d = \bigoplus_{a\in A_{d,s}} \bigoplus_{n\geq 0} S^n Q\left(ds - \frac{3}{2}\sum_{h=1}^{d} a_h h - \frac{n}{2}\right) \otimes_{\mathbf{k}} M(a,n) \tag{6.4}$$

Now we have to compute the vector space $M(a,n)$, which is a problem in representation theory of $SL(2)$. However, the above expression tells us already that

$$M(a,n) = 0 \qquad \text{if} \quad \sum_{h=1}^{d} h a_h - n \equiv 1 \bmod 2$$

Now, let $W = \mathbf{k}^2$ be the standard representation of $SL(2)$. Then we need to compute the decomposition into irreducible representations of $\bigotimes_{h=1}^{d} \wedge^{a_h} S^h W$. For every $a = (a_1, \dots, a_d) \in A_{d,s}$. We can write formally

$$\wedge^{a_h} S^h W = \bigoplus_{e\geq 0} S^e W \otimes J(a_h, h, e)$$

where $J(a_h, h, e)$ is the vector space that counts the multiplicity of $S^e W$ in $\wedge^{a_h} S^h W$. And now we see that

$$\bigotimes_{h=1}^{d} \wedge^{a_h} S^h W = \bigotimes_{h=1}^{d} \left(\bigoplus_{e\geq 0} S^e W \otimes J(a_h,h,e)\right) = \bigoplus_{e} S^n W \otimes \left[K^{(e,n)} \otimes \bigotimes_{h=1}^{d} J(a_h,h,e_h)\right]$$

where $K^{(e,n)}$ is the vector space that counts the multiplicity of $S^n W$ in $S^{e_1} W \otimes \ldots \otimes S^{e_d} W$, and the sum is over all $e = (e_1, \dots, e_d), e_h \geq 0$. Thus, in our above notation we can write

$$M(a,n) = \bigoplus_{e} \left[K^{(e,n)} \otimes \bigotimes_{h=1}^{d} J(a_h,h,e_h)\right] \tag{6.5}$$

The vector spaces $K^{(e,n)}$ can be computed inductively thanks to the Clebsch-Gordan formula:

Proposition 6.1.1 (Clebsch-Gordan formula)**.** *Let $a \geq b \geq 0$ be two integers. Then*

$$S^a W \otimes S^b W = \bigoplus_{h=0}^{b} S^{a+b-2h} W$$

Proof. This is a simple application of the Pieri's formula 1.1.3: we see that

$$S^aW \otimes S^bW = S^{a+b}W \oplus S^{(a+b-1,1)}W \oplus \ldots \oplus S^{(a,b)}W$$

but as $SL(2)$-representations, we see that

$$S^{(h,k)}W = S^{h-k}W$$

for all $h \geq k \geq 0$ and we get the formula. □

In particular, we can say when $K^{(e,n)} \neq 0$: this was explained to us by A. Maffei.

Corollary 6.1.1. *With the notations as above, let $e = (e_1, \ldots, e_d)$ and $n \geq 0$. Set $r = e_1 + \cdots + e_d$ and $m = \max_{h=1}^{d} \{ 2e_h - r \}$. Then*

- *If $e_1 + \cdots + e_d \equiv 0 \pmod 2$, then $K^{(e,n)} \neq 0$ if and only if*

$$n \equiv 0 \pmod 2, \ \max\{ m, 0 \} \leq n \leq r$$

- *If $e_1 + \cdots + e_d \equiv 1 \pmod 2$, then $K^{(e,n)} \neq 0$ if and only if*

$$n \equiv 1 \pmod 2, \ \max\{ m, 1 \} \leq n \leq r$$

Proof. We proceed to prove the thesis by induction on d. The case $d = 1$ is clear, and the case $d = 2$ follows from the Clebsch-Gordan formula 6.1.1. Then we suppose that $d > 1$ and that the thesis is true for $d - 1$. Now, we observe that $K^{(e,n)}$ is symmetric with respect to the e_h so that we can suppose $e_1 \geq e_2 \geq \cdots \geq e_d$. In particular $m = e_1 - e_2 - e_3 - \cdots - e_d$. Now, let $e' = (e_1, \ldots, e_{d-1})$, then

$$S^{e_1}W \otimes \ldots \otimes S^{e_d}W = \bigoplus_{p \geq 0} (S^pW \otimes K^{(e',p)}) \otimes S^{e_d}W = \bigoplus_{p \geq 0} \bigoplus_{n \geq 0} S^nW \otimes K^{((e_d,p),n)} \otimes K^{(e',p)}$$

hence

$$K^{(e,n)} = \bigoplus_{p \geq 0} K^{(e',p)} \otimes K^{((e_d,p),n)}$$

and we study the various cases.

- Suppose that $e_1 + \cdots + e_{d-1} \equiv 0 \bmod 2$: then we have the two subcases
  - $e_d \equiv 0 \bmod 2$: suppose first that $K^{(e,n)} \neq 0$. Then there exists a $p \geq 0$ such that $K^{(e',p)} \neq 0$ and $K^{((e_d,p),n)} \neq 0$. By inductive hypothesis we know from $K^{(e',p)} \neq 0$ that $p \equiv 0 \pmod 2$ and

    $$\max\{ e_1 - e_2 - \cdots - e_{d-1}, 0 \} \leq p \leq e_1 + \cdots + e_{d-1}$$

    and from $K^{((e_d,p),n)} \neq 0$ we know that $n \equiv p + e_d \equiv 0 \pmod 2$ and

    $$\max\{ p - e_d, e_d - p, 0 \} \leq n \leq p + e_d$$

    but then we see that $n \leq p + e_d \leq e_1 + \cdots + e_d$ and that $n \geq p - e_d \geq e_1 - e_2 - \cdots - e_d$, so that

    $$\max\{ m, 0 \} \leq n \leq r$$

    Conversely, suppose that $n \equiv 0 \pmod 2$ and $\max\{ m, 0 \} \leq n \leq r$: then we want to show that there is a $p \geq 0$ such that $K^{(e',p)} \neq 0$ and $K^{((e_d,p),n)} \neq 0$. By inductive hypothesis, these conditions are equivalent to $p \equiv 0 \pmod 2$ and

    $$\max\{ p - e_d, e_d - p, 0 \} \leq n \leq e_d + p, \qquad \max\{ e_1 - e_2 - \cdots - e_{d-1}, 0 \} \leq p \leq e_1 + \cdots + e_{d-1}$$

that is $p \equiv 0 \pmod 2$ and

$$\max\{n-e_d, e_d-n, m+e_d, 0\} \leq p \leq \min\{n+e_d, r\}$$

but now we observe that

$$\begin{aligned} (n+e_d)-(n-e_d) &= 2e_d & r-(n-e_d) &= r-n+e_d \\ (n+e_d)-(e_d-n) &= 2n & r-(e_d-n) &= r+n-e_d = e_1+\cdots+e_{d-1}+n \\ (n+e_d)-(m+e_d) &= n-m & r-(m+e_d) &= r-m-e_d = 2e_1+\cdots+2e_{d-1}+e_d \\ (n+e_d)-0 &= n+e_d & r-0 &= r \end{aligned}$$

so that in any case

$$\min\{n+e_d, r\} - \max\{n-e_d, e_d-n, m+e_d, 0\} \geq 0$$

and if they are equal, then we see that $n+e_d \equiv 0 \pmod 2$ and $r \equiv 0 \pmod 2$ by hypothesis, so that we always manage to find a p that works out.

  – $e_d \equiv 1 \pmod 2$: this case can be solved as the previous one.

- Suppose that $e_1+\cdots+e_{d-1} \equiv 1 \pmod 2$: then this case can be solved in a way similar to that for the one before.

□

Unfortunately, we have not managed to find an explicit formula for the dimension of $K^{(e,n)}$ and we do not have an explicit criterion for deciding when $J(a,b,c) \neq 0$ either. However, this strategy can be effectively implemented with the aid of a computer:

- We have written a program in Python that computes the set

$$A_{d,s} = \{ a = (a_1,\dots,a_d) \mid 0 \leq a_h \leq h+1,\ a_1+\cdots+a_d = s \}$$

  through the formula

$$A_{d+1,s} = \bigsqcup_{h=0}^{\min\{s,d+2\}} A_{d,s-h} \times \{h\}$$

  that allows to compute the $A_{d,s}$ starting from the $A_{1,\bullet}$, that are very easy to determine.

- We have written a program in Python that computes $\dim K^{(e,n)}$ using the Clebsch-Gordan formula 6.1.1. Again we note that the $K^{((e_1,\dots,e_d),n)}$ can be computed inductively from the $K^{((e_1,\dots,e_{d-1}),\bullet)}$.

- We have written a program in Python that computes $\dim J(a,b,c)$ through the formula of Proposition 5.7.1.

- We have written a program in Python that computes the decomposition $\operatorname{gr} \wedge^s M_d$ via the formulas 6.4,6.5 and the auxiliary programs listed before. Then we have written another program that computes the cohomology $H^i(\mathbb{P}^2, \operatorname{gr} E_d)$ via Bott's theorem as in Example 5.4.1 and computes the cohomology $H^i(\mathbb{P}^2, E_d)$ using Theorem 5.6.4, under the additional hypothesis that the maps have maximal rank.

For the cases $d = 2,3,4$ we have obtained the same results of Subsection 2.7.3, confirming the hypothesis of the maximal rank.

However with this method we have managed to compute the cases $d = 5,6$ as well, that our approach of before, or the standard approach through Macaulay2, could not break. We report here the graphs of $k_{p,1}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ and $k_{p,2}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d))$ in these cases.

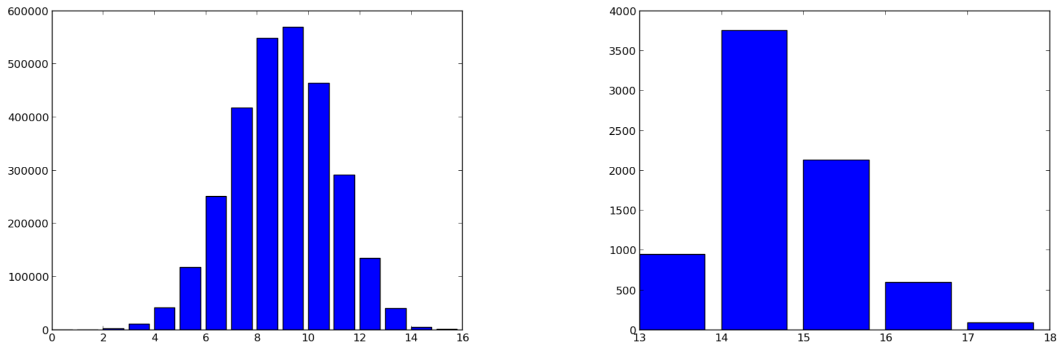


Figure 6.1: The plot on the left shows the Betti numbers $k_{p,1}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(5))$, whereas the plot on the right shows the Betti numbers $k_{p,2}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(5))$. These diagrams are obtained under the additional hypothesis that the maps c_i have maximal rank.

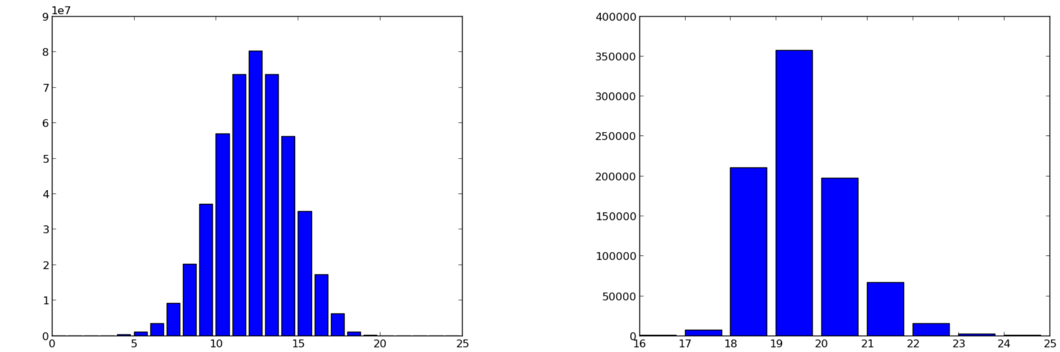


Figure 6.2: The plot on the left shows the Betti numbers $k_{p,1}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$, whereas the plot on the right shows the Betti numbers $k_{p,2}(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$. These diagrams are obtained under the additional hypothesis that the maps c_i have maximal rank.

6.2 Working on the flag variety

We consider now another approach to computing the cohomology of the vector bundles $\wedge^s M_d$ through homogeneous bundles on the flag variety over $\mathbb{P}^2$.

6.2.1 Homogeneous bundles on the flag variety

Take the Borel subgroup $B^- < G = SL(3)$ given by upper-triangular matrices

$$B^- = \left\{ \begin{pmatrix} t_0 & * & * \\ & t_1 & * \\ & & t_2 \end{pmatrix} \middle| t_0 t_1 t_2 = 1 \right\}$$

and we consider the flag variety $X = G/B^-$. Notice that there is a natural projection map

$$\pi\colon X = G/B^- \longrightarrow G/P = \mathbb{P}^2$$

X is naturally an homogeneous variety, so that we can study homogeneous bundles over it with the methods developed in Chapter 5.

Keeping the same notation of Examples 5.1.1, 5.1.2 and 5.1.3 we see that $B^- = P(\Delta)$ and in this case $D' = \Lambda_{\mathbb{R}}(T)$ and $W' = W$.

Moreover, the Levi decomposition of B^- is given by $B^- = U \rtimes T$, where T is the torus of diagonal matrices and U is the set of upper-triangular unipotent matrices

$$U = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$$

Then, we know from the discussion in Chapter 5 that the irreducible homogeneous bundles on X correspond to irreducible representations of T, that is, to characters $\lambda \in X^*(T)$; in particular, since every irreducible representations of T is one-dimensional, all these bundles are line bundles. For every character $\lambda = p_1\lambda_1 + p_2\lambda_2$ of T, we denote by $L_\lambda = L_{(p_1,p_2)}$ the corresponding homogeneous line bundle.

Bott's Theorem on X can be expressed as follows:

Proposition 6.2.1 (Bott's theorem on X). *Let $\lambda = p_1\lambda_1 + p_2\lambda_2$ be a character of T. Then*

$$H^0(X, L_\lambda) = \begin{cases} S^{(p_1+p_2,p_1)}(V), & \textit{if } p_1 + p_2 \geq p_1 \geq 0 \\ 0, & \textit{otherwise} \end{cases}$$

$$H^1(X, L_\lambda) = \begin{cases} S^{(p_2-1,-p_1-2)}(V), & \textit{if } p_2 - 1 \geq -p_1 - 2 \geq 0 \\ S^{(p_1-1,p_1+p_2+1)}(V) & \textit{if } p_1 - 1 \geq p_1 + p_2 + 1 \geq 0 \\ 0, & \textit{otherwise} \end{cases}$$

$$H^2(X, L_\lambda) = \begin{cases} S^{(-p_1-3,p_2)}(V), & \textit{if } -p_1 - 3 \geq p_2 \geq 0 \\ S^{(-p_2-3,-p_1-p_2-1)}(V) & \textit{if } -p_2 - 3 \geq -p_1 - p_2 - 3 \geq 0 \\ 0, & \textit{otherwise} \end{cases}$$

$$H^3(X, L_\lambda) = \begin{cases} S^{(-p_1-p_2-4,-p_2-2)}(V), & \textit{if } -p_1 - p_2 - 4 \geq -p_2 - 2 \geq 0 \\ 0, & \textit{otherwise} \end{cases}$$

Proof. It follows from the above remarks and Bott's Theorem 5.4.1. □

Remark 6.2.1. As for the projective plane, Bott's theorem is best understood graphically: see picture 6.3.

A particular class of homogeneous bundles on X are those coming from $\mathbb{P}^2$ by pullback. Indeed, it is easy to see that, if E is an homogeneous bundle on $\mathbb{P}^2$, then π^*E is an homogeneous bundle on X and, moreover the functor

$$\pi^* \colon \mathbf{HomVec}_{\mathbb{P}^2} \longrightarrow \mathbf{HomVec}_X$$

corresponds simply to the restriction

$$\mathbf{Mod}_P \longrightarrow \mathbf{Mod}_{B^-} \qquad (\rho \colon P \longrightarrow GL(V)) \mapsto (\rho_{|B^-} \colon B^- \longrightarrow GL(V))$$

In particular we see that for any homogeneous bundle E on $\mathbb{P}^2$ we have

$$\text{gr } \pi^*(E) = \pi^*(\text{gr } E)$$

indeed, both expressions amounts to taking a representation of P and restrict it to a representation of T.

We can describe explicitly the effect of the pullback on irreducible bundles. Consider a character $\lambda = p_1\lambda_1 + p_2\lambda_2$ such that $p_2 \geq 0$ and take the corresponding irreducible homogeneous bundle E_λ on $\mathbb{P}^2$. This is the bundle that corresponds to the irreducible representation V_λ of $R = GL(2)$ with highest weight λ, and then computing π^*E_λ is the same as computing the decomposition of V_λ into weight spaces, that is

$$\pi^*(E_\lambda) = \bigoplus_{n=0}^{\text{rk}E_\lambda - 1} L_{\lambda - n\alpha_2} = \bigoplus_{n=0}^{p_2} L_{(p_1+n,p_2-2n)} \tag{6.6}$$

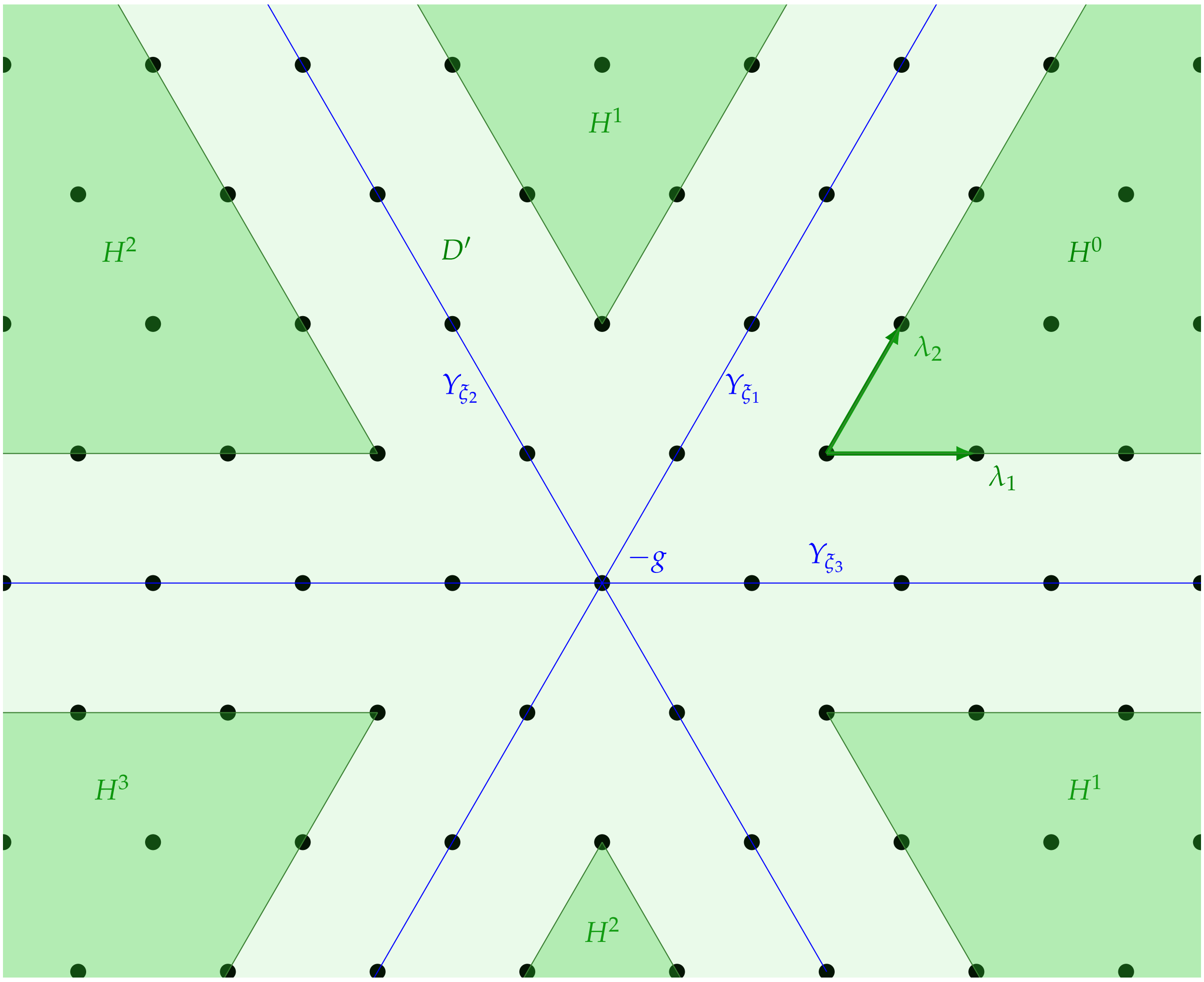


Figure 6.3: Bott chamber of X: the darker shaded parts are the Bott chambers corresponding to irreducible homogeneous bundles with nonzero H^0, H^1, H^2 and H^3. The Bott chamber with nonzero H^0 correspond to the Weyl chamber D. The hyperplanes corresponding to bundles with zero cohomology are represented in blue.

The reason why we are interested in working with the flag variety is that we can study the cohomology of homogeneous bundles on $\mathbb{P}^2$ by pulling them back to X:

Proposition 6.2.2. *Let E be an homogeneous vector bundle on $\mathbb{P}^2$. Then*

$$H^i(\mathbb{P}^2, E) \cong H^i(X, \pi^* E)$$

for every i.

Proof. This follows from the Leray spectral sequence. See [Ott]. □

6.2.2 Cohomology and Random triangles

From Proposition 6.2.2 that we can compute the cohomology $H^i(\mathbb{P}^2, \wedge^s M_d)$ as the i-th cohomology of the complex

$$0 \longrightarrow H^0(X, \wedge^s \pi^* \text{gr } M_d) \longrightarrow H^0(X, \wedge^s \pi^* \text{gr } M_d) \longrightarrow H^0(X, \wedge^s \pi^* \text{gr } M_d) \longrightarrow 0$$

that is obtained pulling back the corresponding complex on $\mathbb{P}^2$. From the above remarks, we see that $\pi^*(\text{gr } M_d)$ is a direct sum of line bundles, so that it is much easier to compute its alternating power $\wedge^s \pi^*(\text{gr } M_d)$, as there is no problem of plethysm or tensor decomposition.

Remark 6.2.2. We point out that for a general homogeneous vector bundle F on X it is not known whether the cohomology $H^i(X, F)$ coincides with the i-th cohomology of the complex

$$0 \longrightarrow H^0(X, \text{gr } F) \xrightarrow{c_0} H^1(X, \text{gr } F) \xrightarrow{c_1} H^2(X, \text{gr } F) \xrightarrow{c_2} 0$$

indeed, X is not an Hermitian symmetric variety, so that Theorem 5.6.4 does not apply.

However it was proved by A. Boralevi in [Bor] that there is still an isomorphism

$$H^0(X, F) \cong \text{Ker } c_0$$

Now we want to apply this strategy: we know from the previous section that we have a decomposition of homogeneous bundles on $\mathbb{P}^2$ given by

$$\text{gr } M_d = \bigoplus_{h=1}^{d} S^h Q(d-2h) = \bigoplus_{h=1}^{d} E_{(d-2h,h)}$$

so that, thanks to the formula 6.6 we see that

$$\text{gr } \pi^* M_d = \pi^*(\text{gr } M_d) = \bigoplus_{h=1}^{d} \bigoplus_{k=0}^{h} L_{(d-2h+k, h-2k)} = \bigoplus_{(a,b) \in T_d} L_{(a,b)}$$

where

$$T_d = \{ (d-2h+k)\lambda_1 + (h-2k)\lambda_2 \mid h = 1, \dots, d \; k = 0, \dots, h \}$$

is a triangle minus one vertex:

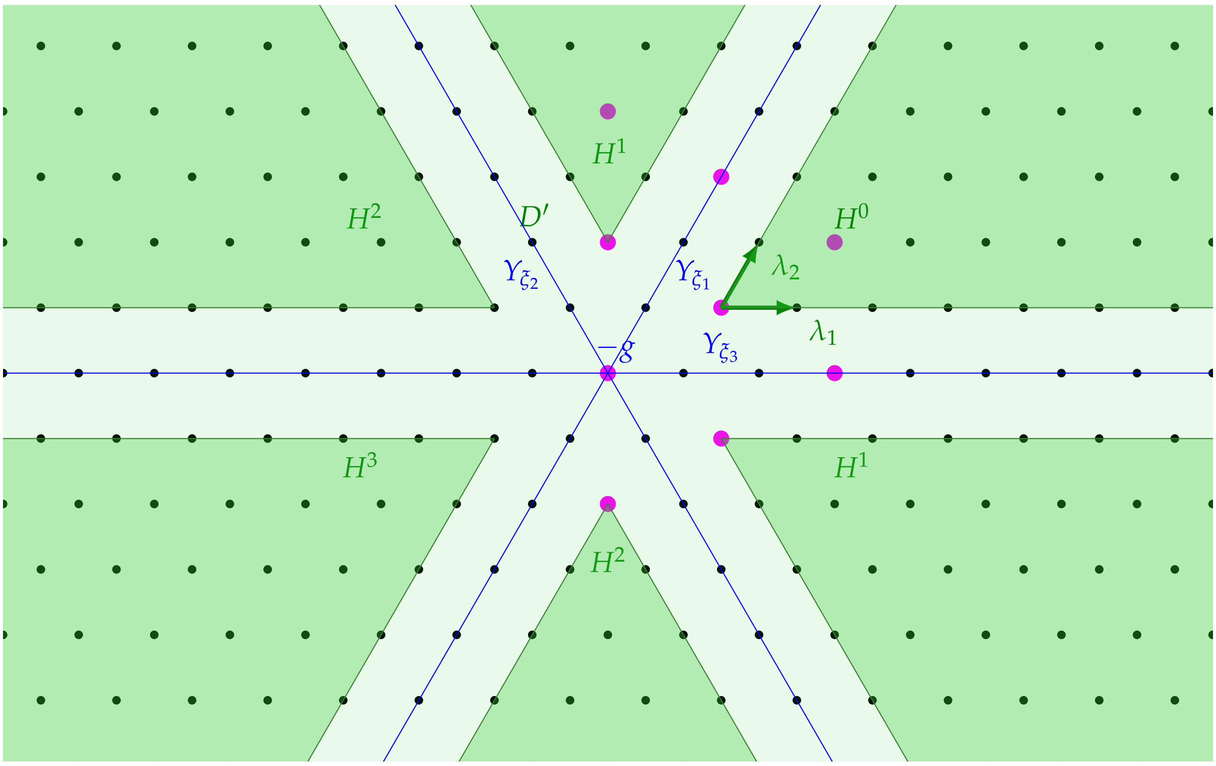


Figure 6.4: The bundle gr $\pi^* M_3$: the elements of T_3 are denoted by the fuchsia dots.

Now, since the $L_{(a,b)}$ are all vector bundles we see that

$$\wedge^s (\text{gr } \pi^* M_d) = \bigoplus_{\{\mu_1, \dots, \mu_s\}} L_{\mu_1 + \dots + \mu_s}$$

where the sum is over all distinct subsets of T_d of cardinality s. For example, the decomposition of $\wedge^2 \text{gr } \pi^* M_3$ is illustrated in the following figure.

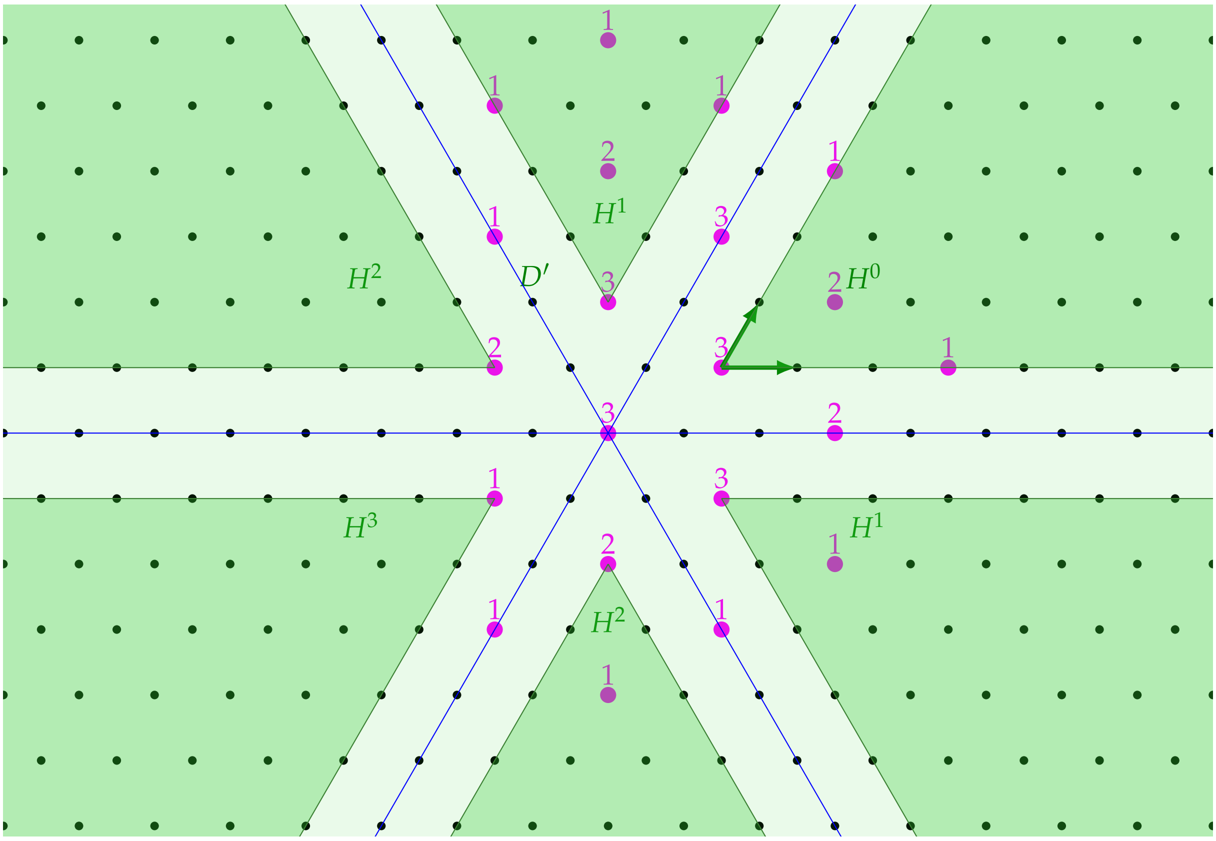


Figure 6.5: The bundle gr $\wedge^2 \pi^* M_3$: the numbers over each dot denote the multiplicity of the corresponding line bundle in the decomposition.

We see that the decomposition of the bundle gr $\wedge^s \pi^* M_d$ recalls somehow the Central Limit Theorem: indeed, summing over all subsets in T_d of cardinality s corresponds to considering the random sums $X_1 + \cdots + X_s$, where the X_i are random variables uniformly distributed over T_d, with the condition that all the X_i are distinct. This observation raises hope towards Conjecture 1, but a complete proof appears quite difficult anyway. Here we present two partial results in this direction.

Discrete random triangles

We have seen before that

$$\wedge^s(\mathrm{gr}\ \pi^* M_d) = \bigoplus_{\{\mu_1,\dots,\mu_s\}} L_{\mu_1+\cdots+\mu_s}$$

where the sum is over all distinct subsets of T_d of cardinality s.

The problem here comes from the fact that we are considering subsets of T_d of a fixed cardinality. However, if we consider all possible alternating powers at the same time we see that

$$\bigoplus_{s=0}^{+\infty} \wedge^s(\mathrm{gr}\ \pi^* M_d) = \bigoplus_{\{\mu_1,\dots,\mu_r\}} L_{\mu_1+\cdots+\mu_r}$$

where the second sum is over *all* subsets $\{\mu_1, \dots, \mu_r\} \subseteq T_d$. This means that

$$\bigoplus_{s=0}^{+\infty} \wedge^s(\mathrm{gr}\ \pi^* M_d) = \bigoplus_{\lambda} L_\lambda^{\oplus m(\lambda)}$$

where $m(d, \lambda)$ is the number of subsets of T_d the sum of whose element gives λ. More precisely

$$m(d, \lambda) = \# \left\{ I \subseteq T_d \,\middle|\, \sum_{\mu \in I} \mu = \lambda \right\}$$

This can be modelized probabilistically as follows: take for every $\mu \in T_d$ a random variable X_μ with distribution

$$\mathbb{P}(X_\mu = \mu) = \frac{1}{2} \qquad \mathbb{P}(X_\mu = 0) = \frac{1}{2}$$

and let

$$S_d = \sum_{\mu \in T_d} X_\mu$$

then it is easy to see that

$$\mathbb{P}(S_d = \lambda) = \frac{m(d,\lambda)}{\sum_\nu m(d,\nu)} \qquad \text{for all } \lambda$$

Proposition 6.2.3. *With the above notations, we have that, as $d \to +\infty$*

$$\frac{S_d - \mathbb{E}[S_d]}{d^2} \longrightarrow \mathcal{N}(0, Q)$$

in distribution, where

$$Q = \frac{1}{72}\begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$$

To prove this result we need some preparation first. To begin with, we recall a variant of the Central Limit Theorem due to Lyapounov.

Theorem 6.2.1 (Multidimensional Lyapounov's CLT)**.** *Let $\{ Y_{n,k} \mid n \geq 1, 1 \leq k \leq n \}$ be a triangular collection of random variables vith values in $\mathbb{R}^d$ such that the variables in the rows $Y_{n,1}, \dots, Y_{n,n}$ are independent and $\mathbb{E}[|Y_{n,k}|^3] < +\infty$. Set $T_n = Y_{n,1} + \cdots + Y_{n,n}$. If*

$$Var[T_n] \longrightarrow \Sigma \qquad \text{and} \qquad \lim_{n \to +\infty} \sum_{k=1}^{n} \mathbb{E}[|Y_{n,k} - \mathbb{E}[Y_{n,k}]|^{2+\delta}] = 0$$

for a certain $\delta > 0$, then

$$T_n - \mathbb{E}[T_n] \longrightarrow \mathcal{N}(0, \Sigma)$$

in distribution.

Proof. See [Bil95]. □

Then we consider the following situation: take a family of i.i.d. standard Bernoulli random variables $(B_{(a,b)})_{a \geq 0, b \geq 0}$ and consider for each $a \geq 0, b \geq 0$ the variable

$$Z_{(a,b)} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} B_{(a,b)} \\ B_{(a,b)} \end{pmatrix}$$

The meaning of this is that we have taken independent random variables $Z_{(a,b)}$ such that

$$\mathbb{P}\left(Z_{(a,b)} = \begin{pmatrix} a \\ b \end{pmatrix} \right) = \frac{1}{2} \qquad \mathbb{P}\left(Z_{(a,b)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right) = \frac{1}{2}$$

Now, for every $d \in \mathbb{N}$ consider the random variable

$$R_d = \sum_{\substack{a \geq 0, b \geq 0 \\ a+b \leq d}} Z_{(a,b)}$$

We want to investigate the asymptotic behaviour of R_d. To do this, it will be useful to define

$$Z_k = \sum_{\substack{a \geq 0, b \geq 0 \\ a+b = k}} Z_{(a,b)}$$

in particular we note that the Z_k are independent and that $R_d = Z_1 + \cdots + Z_d$.
First we compute the mean and the variance matrix.

Lemma 6.2.1. *Let Z_k and R_d be as before. Then the mean and the variance of the Z_k are given by*

$$\mathbb{E}[Z_k] = \frac{k(k+1)}{4}\begin{pmatrix}1\\1\end{pmatrix} \qquad Var[Z_k] = \frac{k(k+1)}{24}\begin{pmatrix}2k+1 & k-1\\ k-1 & 2k+1\end{pmatrix}$$

whereas the mean and the variance of the R_d are given by

$$\mathbb{E}[R_d] = \frac{d(d+1)(d+2)}{12}\begin{pmatrix}1\\1\end{pmatrix} \qquad Var[R_d] = \frac{d(d+1)(d+2)}{96}\begin{pmatrix}2d+2 & d-1\\ d-1 & 2d+2\end{pmatrix}$$

Proof. Let $B \sim B_{(a,b)}$ be a random variable with the same distribution as the $B_{(a,b)}$. Then we can write $B = (C\ D)^T$ and it is easy to see that C, D, CD are standard Bernoulli, so that

$$\mathbb{E}[C] = \mathbb{E}[D] = \frac{1}{2} \qquad \mathrm{Cov}(C,C) = \mathrm{Cov}(D,D) = \frac{1}{4} \qquad \mathrm{Cov}(C,D) = \mathbb{E}[CD] - \mathbb{E}[C]\mathbb{E}[D] = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Now, it is easy to see that

$$\mathbb{E}[Z_k] = \sum_{a+b=k}\mathbb{E}[Z_{(a,b)}] = \frac{1}{2}\left(\sum_{a+b=k}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\right)\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\frac{k(k+1)}{2} & 0\\ 0 & \frac{k(k+1)}{2}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{k(k+1)}{4}\begin{pmatrix}1\\1\end{pmatrix}$$

and

$$\begin{aligned}\mathrm{Var}[Z_k] = \sum_{a+b=k}\mathrm{Var}[Z_{(a,b)}] &= \frac{1}{4}\sum_{a+b=k}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} = \frac{1}{4}\sum_{a+b=k}\begin{pmatrix}a^2 & ab\\ ab & b^2\end{pmatrix} = \\ &= \frac{1}{4}\begin{pmatrix}\frac{k(k+1)(2k+1)}{6} & \frac{k(k+1)(k-1)}{6}\\ \frac{k(k+1)(k-1)}{6} & \frac{k(k+1)(2k+1)}{6}\end{pmatrix} = \frac{k(k+1)}{24}\begin{pmatrix}2k+1 & k-1\\ k-1 & 2k+1\end{pmatrix}\end{aligned}$$

From these, we can compute the mean and the variance of the R_d: the mean is given by

$$\mathbb{E}[R_d] = \sum_{k=1}^{d}\mathbb{E}[Z_k] = \left(\sum_{k=1}^{d}\frac{k(k+1)}{4}\right)\begin{pmatrix}1\\1\end{pmatrix} = \frac{d(d+1)(d+2)}{12}\begin{pmatrix}1\\1\end{pmatrix}$$

and the variance by

$$\begin{aligned}\mathrm{Var}[R_d] = \sum_{k=1}^{d}\mathrm{Var}[Z_k] &= \frac{1}{24}\sum_{k=1}^{d}\begin{pmatrix}k(k+1)(2k+1) & k(k+1)(k-1)\\ k(k+1)(k-1) & k(k+1)(2k+1)\end{pmatrix} = \\ &= \frac{d(d+1)(d+2)}{96}\begin{pmatrix}2d+2 & d-1\\ d-1 & 2d+2\end{pmatrix}\end{aligned}$$

□

Now we can prove that the R_d are asymptotically normal:

Lemma 6.2.2. *Let $R_d = Z_1 + \cdots + Z_n = \sum_{a+b\le n} Z_{(a,b)}$ be as before. Then*

$$\frac{R_d - \mathbb{E}[R_d]}{d^2} \longrightarrow \mathcal{N}(0,\Sigma)$$

in distribution, as $d \to +\infty$, where

$$\Sigma = \frac{1}{96}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

Proof. We want to apply the previous theorem: we set $Y_{d,k} = \frac{Z_k}{d^2}$ and $T_d = \sum_{k=1}^{d} Y_{d,k} = \frac{R_d}{d^2}$. Then it is clear that

$$\mathrm{Var}[T_d] = \frac{1}{d^4}\mathrm{Var}[R_d] \longrightarrow \Sigma = \frac{1}{96}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Now fix $k \geq 1$ and observe that

$$|Y_{d,k} - \mathbb{E}[Y_{d,k}]| = \frac{1}{d^2}|Z_k - \mathbb{E}[Z_k]| = \frac{1}{d^2}\left|\sum_{h=0}^{k}(Z_{(a,b)} - \mathbb{E}[Z_{(a,b)}])\right| = \frac{1}{d^2}\left|\sum_{h=0}^{k}\begin{pmatrix} h(B_{(h,k-h)} - \mathbb{E}[B_{(h,k-h)}]) \\ (k-h)(B_{(h,k-h)} - \mathbb{E}[B_{(h,k-h)}]) \end{pmatrix}\right| =$$

$$= \frac{1}{d^2}\left|\begin{pmatrix} \sum_{h=0}^{k} hC_h \\ \sum_{h=0}^{k}(k-h)C_h \end{pmatrix}\right| = \frac{1}{d^2}\left|\begin{pmatrix} A_k \\ B_k \end{pmatrix}\right| \leq \frac{1}{n^2}(|A_k| + |B_k|)$$

where $C_h = B_{(h,k-h)}$ and $A_k = \sum_{h=0}^{k} hC_h$ and $B_k = \sum_{h=0}^{k}(k-h)C_{k-h}$. In particular, A_k and B_k have the same distribution (as it is clear by their construction).

Now, we want to use the Lyapounov's CLT with $\delta = 2$: observe that we have

$$|Y_{d,k} - \mathbb{E}[Y_{d,k}]|^4 \leq \frac{1}{d^8}(|A_k| + |B_k|)^4 \leq \frac{2^3}{d^8}(|A_k|^4 + |B_k|^4)$$

where for the last inequality we have used the convexity of the function $x \mapsto x^4$ for $x \geq 0$. Now we can compute

$$\mathbb{E}[|Y_{d,k} - \mathbb{E}[Y_{d,k}]|^4] \leq \frac{2^3}{n^8}(\mathbb{E}[|A_k|^4] + \mathbb{E}[|B_k|^4]) = \frac{2^4}{n^8}\mathbb{E}[|A_k|^4]$$

Now, one can show (using cumulants, for example) that $\mathbb{E}[|A_k|^4] \sim Ck^6$ for a certain constant C so that

$$\sum_{k=0}^{d}\mathbb{E}[|Y_{d,k} - \mathbb{E}[Y_{d,k}]|^4] \sim B\frac{d^7}{d^8} \qquad \text{as } d \to +\infty$$

for another constant B, and then we conclude. □

Proposition 6.2.3 follows easily from the above result.

Proof of Proposition 6.2.3. Consider the affine maps

$$f_d \colon \Lambda_{\mathbb{R}}(T) \longrightarrow \Lambda_{\mathbb{R}}(T) \qquad x \mapsto Ax + b_d$$

where

$$A = \frac{1}{3\sqrt{2}}\begin{pmatrix} -1 & \sqrt{3} \\ -1 & -\sqrt{3} \end{pmatrix}, \qquad b_d = \frac{d}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Then it it easy to show that the set

$$T_d = \{\, (d - 2h + k)\lambda_1 + (h - 2k)\lambda_2 \mid h = 1, \dots, d \ k = 0, \dots, h \,\}$$

goes into the set

$$f_d(T_d) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| a \geq 0, b \geq 0, 1 \leq a + b \leq d \right\}$$

so that, in the notations of Proposition 6.2.3 and Lemma 6.2.2 we have that

$$S_d = f_d(R_d) = A \cdot R_d + b_d$$

Then

$$\frac{S_d - \mathbb{E}[S_d]}{d^2} = A \cdot \frac{R_d - \mathbb{E}[R_d]}{d^2}$$

and since we know from Lemma 6.2.2 that

$$\frac{R_d - \mathbb{E}[R_d]}{d^2} \longrightarrow \mathcal{N}(0, \Sigma)$$

in distribution, as $d \to +\infty$, it follows that

$$\frac{S_d - \mathbb{E}[S_d]}{d^2} \longrightarrow A^{-1} \cdot \mathcal{N}(0, \Sigma)$$

To conclude, it is enough to observe that

$$A^{-1} \cdot \mathcal{N}(0, \Sigma) = \mathcal{N}(0, A^{-1}\Sigma(A^{-1})^T) = \mathcal{N}(0, Q)$$

$\square$

Continuous random triangles

Consider again the decomposition

$$\wedge^s(\mathrm{gr}\ \pi^* M_d) = \bigoplus_{\{\mu_1, \dots, \mu_s\}} L_{\mu_1 + \dots + \mu_s}$$

where the sum is over all distinct subsets of T_d of cardinality s.

We have already said that this decomposition corresponds to taking the sum $S_{d,s} = X_{d,1} + \dots + X_{d,s}$ where the $X_{d,i}$ are i.i.d. random variables uniformly distributed on T_d, with the condition that $X_{d,i} \neq X_{d,j}$ if $i \neq j$. We would like to see an asymptotic normal behavior, but the condition $X_{d,i} \neq X_{d,j}$ is quite difficult to work with. We consider then a simplification of this situation: namely, we imagine to work not on a discrete triangle minus one point but on a continuous triangle. Then the condition $X_{d,i} \neq X_{d,j}$ becomes meaningless, and we just have to consider a sum of i.i.d. random variables without any restriction.

More precisely,, we consider for every d the full triangle $\widetilde{T}_d$ defined as the convex envelope of $T_d \cup \{ d\lambda_1 \}$ and we take a sequence of i.i.d. random variables $(X_{d,i})_{i \geq 1}$ with uniform distribution on $\widetilde{T}_d$. Then for every $s \geq 0$ we define the sum

$$S_{d,s} = X_{d,1} + \dots + X_{d,s}$$

and we want to investigate the asymptotic distribution of $S_{s,d}$ as $s \to +\infty$. Thanks to Lyaponouv's Theorem 6.2.1 we can prove that there is asymptotic normality in the case $s = d$.

Proposition 6.2.4. *With notations as before, we have that, as $d \to +\infty$*

$$\frac{S_{d,d} - \mathbb{E}[S_{d,d}]}{d^{\frac{3}{2}}} \longrightarrow \mathcal{N}(0, Q)$$

in distribution, where

$$Q = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Proof. First we consider the following situation: for every $d \geq 1$ let C_d be the triangle

$$C_d = \{ (x, y) \mid x \geq 0, y \geq 0, x + y \leq d \}$$

and consider a sequence $(Z_{d,i})_{i \geq 1}$ of i.i.d. random variables uniformly distributed over C_d. Then define

$$T_d = Z_{d,1} + \dots + Z_{d,d}$$

We want to compute the mean and the variance of T_d: let $Z = (A, B)$ be a random variable with uniform distribution over C_d. Then we see that

$$\mathbb{E}[A] = \frac{2}{d^2} \int_{C_d} x dx dy = \frac{2}{d^2} \int_0^d x(d-x) dx = \frac{2}{d^2} \cdot \frac{d^3}{6} = \frac{d}{3}$$
$$\mathbb{E}[B] = \frac{2}{d^2} \int_{C_d} y dx dy = \frac{2}{d^2} \int_0^d y(d-y) dy = \frac{2}{d^2} \cdot \frac{d^3}{6} = \frac{d}{3}$$

so that
$$\mathbb{E}[Z] = \frac{d}{3}\begin{pmatrix}1\\1\end{pmatrix}, \qquad \mathbb{E}[T_d] = d\mathbb{E}[Z] = \frac{d^2}{3}\begin{pmatrix}1\\1\end{pmatrix}$$
To compute the variance we see that
$$\begin{aligned}
\mathbb{E}[A^2] &= \frac{2}{d^2}\int_{C_d} x^2 dxdy = \frac{2}{d^2}\int_0^d x^2(d-x)dx = \frac{d^2}{6}\\
\mathbb{E}[B^2] &= \frac{2}{d^2}\int_{C_d} y^2 dxdy = \frac{2}{d^2}\int_0^d y^2(d-y)dy = \frac{d^2}{6}\\
\mathbb{E}[AB] &= \frac{2}{d^2}\int_{C_d} xydxdy = \frac{2}{d^2}\int_0^d x\frac{(d-x)^2}{2}dx = \frac{d^2}{12}
\end{aligned}$$

so that
$$\mathrm{Var}[Z] = \begin{pmatrix}\frac{d^2}{18} & -\frac{d^2}{36}\\ -\frac{d^2}{36} & \frac{d^2}{18}\end{pmatrix}, \qquad \mathrm{Var}[T_d] = d\mathrm{Var}[Z] = \begin{pmatrix}\frac{d^3}{18} & -\frac{d^3}{36}\\ -\frac{d^3}{36} & \frac{d^3}{18}\end{pmatrix}$$
Now, we define the variables $W_{d,i} = \frac{Z_{d,i}}{d^{\frac{3}{2}}}$ and we want to apply Theorem 6.2.1 to them. Observe that $W_{d,1} + \cdots + W_{d,d} = d^{-\frac{3}{2}}T_d$ and that
$$\mathrm{Var}[d^{-\frac{3}{2}}T_d] = \Sigma = \frac{1}{36}\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$$
Now we need to show that
$$\lim_{d\to+\infty}\frac{1}{d^{\frac{3}{2}(2+\delta)}}\sum_{i=1}^{d}\mathbb{E}[|Z_{d,i} - \mathbb{E}[Z_{d,i}]|^{2+\delta}] = 0$$
for a certain $\delta > 0$. For any $\delta > 0$ we have that
$$|Z - \mathbb{E}[Z]|^{2+\delta} \le (|A - \mathbb{E}[A]| + |B - \mathbb{E}[B]|)^{2+\delta} \le 2^{1+\delta}(|A - \mathbb{E}[A]|^{2+\delta} + |B - \mathbb{E}[B]|^{2+\delta})$$
by convexity of the function $x \mapsto x^{2+\delta}$. Then, since A and B have the same distribution, we see taking expectations that
$$\mathbb{E}[|Z - \mathbb{E}[Z]|^{2+\delta}] \le 2^{2+\delta}\mathbb{E}[|A - \mathbb{E}[A]|^{2+\delta}]$$
but since $|A| \le d$ it follows that
$$\mathbb{E}[|Z - \mathbb{E}[Z]|^{2+\delta}] \le 4^{2+\delta}d^{2+\delta}$$
Then
$$\lim_{d\to+\infty}\frac{1}{d^{\frac{3}{2}(2+\delta)}}\sum_{i=1}^{d}\mathbb{E}[|Z_{d,i} - \mathbb{E}[Z_{d,i}]|^{2+\delta}] \le \lim_{d\to+\infty} 4^{\delta+2}\frac{d^{3+\delta}}{d^{3+\frac{3}{2}\delta}} = 0$$
This shows that
$$\frac{T_d - \mathbb{E}[T_d]}{d^{\frac{3}{2}}} \longrightarrow \mathcal{N}(0,\Sigma)$$
in distribution, for $d \to +\infty$. To conclude, it is enough to observe that the affine maps
$$f_d\colon \Lambda_{\mathbb{R}}(T) \longrightarrow \Lambda_{\mathbb{R}}(T) \qquad x \mapsto Ax + b_d$$
that we have seen in the proof of Proposition 6.2.3 send $\widetilde{T}_d$ to C_d, so that $T_d = f(S_{d,d})$. Then, we get the desired result as in the proof of Proposition 6.2.3. □

Remark 6.2.3. With the same technique, one can prove a similar statement for the $S_{d,s}$ with $s = md$ for a certain constant m.

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