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LOOP GRAVITY, TWISTED GEOMETRIES AND TORSION

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Introduction

Loop quantum gravity is a tentative theory of quantum gravity based on the canonical formalism, initially studied by Wheeler, deWitt in the ADM formulation [3, 4], that is the idea of first defining a kinematical Hilbert space for the theory, and then implementing the constraints imposing the invariance under diffeomorphisms. This program encoutered a number of technical difficulties, most notably in the construction of the kinematical scalar product [5]. The main advance occurred with the reformulation of general relativity in terms of SU(2) connections, by Ashtekar [6]. The result recasted general relativity in the guise of a gauge theory and made tools from quantum field theory available to construct the scalar product. In particular, thanks to the "Loops representations" [7], Wilson loops provide a complete description of the kinematical Hilbert space, and a convenient basis is given by Penrose's spin networks [9] and the associated holonomy-flux algebra. This basis diagonalizes geometric operators such as areas and volumes, and the resulting spectra are notably discrete, with minimal eigenvalues proportional to the Planck length, which in the theory is the only dimensionful constant, to be fixed by observations.

To be more precise, in real Ashtekar-Barbero variables [10] general relativity can be seen as an Hamiltonian system with three constraints enforcing the gauge symmetries inside the theory: an $SU(2)$ symmetry ($\mathcal{G}_i$ - *Gauss constraint*) and the diffeomorphism symmetry which split into the spatial diffeomorphisms (H_a) and the Hamiltonian constraint (H). The canonical quantisation of the space-time was achieved (in the late 90's), so we have a rigorous characterisation of the $SU(2)$-invariant Hilbert space $\mathcal{H}^0$ and of its "kinematical states" as the solutions of the quantum Gauss constraint equation. A basis for such Hilbert space was found, that inherits the $SU(2)$ structure introduced by the Asktekar variables: these are the *spin-networks*. These states $\psi_\Gamma(\{j_l\}\ ,\ \{i_n\})$ are described by a collection of *links* and *nodes* arranged in a *graph* $\Gamma(L, V)$, with two sets of quantum numbers $(\{j_l, i_n\})$, associated with each link and each node, respectively. The $j_l \quad l = 1, \ldots, L$ are the "spins", semi-integers quantum numbers associated to each link, standing for the $SU(2)$ representations that one uses to perform parallel transport on that specific link while the $i_n \quad n = 1, \ldots, V$ are the "intertwiners" quantum numbers associated to the nodes. They are related to the Clebsh-Gordan coefficients and define the way in which the $SU(2)$ representations are contracted over the spin-network state. Finally, the Hilbert space of the full theory is the direct sum of the Hilbert spaces $\mathcal{H}_\Gamma$ associated to each graph Γ.

$$\mathcal{H} = \bigoplus_{\Gamma} \mathcal{H}_\Gamma$$

The picture that emerges has a rich phenomenology even if the solution of the full dynamic is still lacking. The quantisation of space-time is achieved and we are able to compute the spectra of the *geometric operators* such as the length, area and volume operators. For example, the spectrum of the area operator has a tidy expression, since in the spin-network basis it is diagonal:

$$\hat{A}\,\psi_\Gamma = 8\pi\gamma L_P^2 \sum_k \sqrt{j_k(j_k+1)}\psi_\Gamma$$

Where k runs over the representations of $SU(2)$ labelling the links of the graph Γ, γ is the *Immirzi parameter* and $L_p = \sqrt{\hbar G/c^3}$ is the *Planck length*.

It should be appreciated that this picture of "quantum geometry" has three different aspects in which the word "quantum" plays a role. First, the spectra are discrete. Second, not all classical (kinematical) observables commute. Third, the basis is labelled by a specific graph. We can loosely speak of an analogy between the number of nodes and links in the graph and the number of particles in a Fock state, and thus refer with some abuse of language to the graph itself as a quantum number. Fixing a given graph then corresponds to a truncation of the theory to a finite number of degrees of freedom. This truncation is at the basis of the most developed description of the dynamics of the theory, the so-called spin foam formalism [11, 12, 13, 14]. This is a generalization of Feynman graphs to 2-dimensional complexes. The formalism provides a projector on the physical Hilbert space for the kinematical spin networks states, expressed as an infinite sum over all possible 2-complexes (or "foams") compatible with the spin network's graphs.

To compute these transition amplitudes, it is necessary to understand the theory on a fixed graph. Because of the truncation, we are only capturing a finite number of degrees of freedom, thus the fixed-graph theory can be seen as a discretisation of general relativity. It is natural to associate such discretisation with a cellular decomposition of the space manifold, defined for instance by the Voronoi dual to the spin network's graph. Consider then the kinematical semi-classical limit of $\mathcal{H}_\Gamma$, that is the phase space out of which $\mathcal{H}_\Gamma$ and its holonomy-flux algebra are obtained through quantisation. This is simply the symplectic manifold $P_\Gamma = \oplus_l T^*SU(2)$, where l are the links of the graph, divided by the action of the Gauss law at each node of the graph. It is indeed the same phase space appearing for instance in lattice gauge theories. It was shown in [21] that the data on the phase space admit a clear geometric interpretation: they define a collection of flat polyhedra, associated with the dual to the graph, and endowed with a notion of embedding in a 4-dimensional Lorentzian manifold. Therefore, the phase space on a given graph provides a discrete version of the ADM phase space, with 3-dimensional intrinsic and extrinsic geometry as data. Such discrete geometries are called twisted geometries. In the special case of 4-valent graphs, whose dual is a triangulation, they can be directly compared with the geometries used in Regge calculus, a well-known discretisation of general relativity used since the 60's. It turns out that twisted geometries are more general than Regge geometries, in the following sense: a triangles shared by two adjacent tetrahedra has a unique area, but a priori different shapes when seen in the two frames. Therefore, a piecewise flat but possibly discontinuous metric is defined. Special shape-matching conditions can be imposed, and reduce the twisted geometries to Regge geometries.

Such generalisation of the discrete geometry can be seen as a consequence of the fact that one is truly working with a first order formalism for general relativity, in which tetrad and connection are initially independent degrees of freedom. Now, in the continuum it is well known that the first order formulation is equivalent to the second order one, because variation with respect to an independent connection gives a condition of vanishing torsion, for which the Levi-Civita connection is known to be the only solution, and one retrieves immediately the Einstein's equation for the remaining variation with respect to the metric. This immediately raises the question of whether the shape mismatch are just kinematical artefacts, and a proper imposition of the dynamics will reduce the states to the Regge subspace. A positive hint in this direction comes from the quantum dynamics and its description by the so-called ERPL model [38, 39]. In fact, it was shown in [43, 44] that the the fundamental amplitude (the equivalent of a single vertex amplitude in QFT) is dominated, in the limit of large spin numbers, by exponentials of the Regge action. Therefore, if large spins dominate the semi-classical limit, this is a strong indication that shape-matching conditions are dynamically imposed, and furthermore the theory leads to general relativity in the large scale limit by virtue of the good continuum properties of the Regge action.

While the result of [43, 44] is mathematically rigorous, its relation to twisted geometries and in particular shape-matching conditions is somewhat intricate. Furthermore, the complexity of the quantum amplitudes make it difficult to extend the result as needed. In fact, it has been argued in [45] that while a single vertex amplitude correctly reproduces the Regge action, gluing together many vertices in a large foam is only compatible with flat solutions, with vanishing curvature. Furthermore, the formalism per se is defined on arbitrary vertices, whereas the result of [43, 44] and the link to the Regge action is only defined for vertices whose dual is a 4-simplex. A possible path to make progress on these important open questions is to better understand the classical dynamics of twisted geometries, beyond the Regge sector. This brings us back to reconsider the question of the role of the shape matching conditions. As mentioned above, the first order action is equivalent to general relativity thanks to the vanishing torsion condition arising as one of the field equations. At the canonical level, the situation is a bit more complicated: the (spatial pull-back of) torsionless equation arises as a secondary constraint, obtained by requiring Poisson-commutation of the primary constraints with the Hamiltonian constraint.[1] Specifically, the constraints whose preservation leads to the vanishing of the spatial torsion are the so-called *simplicity* constraints which make sure that the variable conjugated to the connection is built out of tetrads. As it turns out, the EPRL model only imposes the primary simplicity constraints, under the logic that imposing them "at all times", or more precisely at every relevant edge of the 2-complex, makes it unnecessary to further impose their 'time-preservation', or the secondary ones, a logic that goes back to the first 4-dimensional spin foam model by Barrett and Crane [43, 44]. This approach is based on evidence provided by the case of a 4-simplex, and it is conceivable that the difficulties mentioned above are precisely related to an incorrect use of the argument. Indeed, the lack of proper secondary constraints is the main criticism raised to the spin foam formalism in [22].

We then come to the core question of the thesis. Is it possible that the secondary constraints impose the shape matching conditions? The question is not so easy to answer, because of an obvious problem: the truncation to a fixed graph breaks diffeomorphism invariance, therefore one does not have a Hamiltonian constraint at disposal, to study the preservation of the primary constraints and eventually derive the existence of secondary ones. The problem has been first addressed by Dittrich and Ryan in [47]. They proposed to bypass the difficulty by directly discretising the continuum secondary constraints, and were able to solve them only if the shape matching conditions hold, thus concluding that

$$T = 0 \qquad \Longleftrightarrow \qquad \mathrm{shape-matching} \tag{1}$$

Conversely, the Marseille collaboration Haggard-etc [46] proposed a meaning for the torsionless equation in a discontinuous setting as a distributional equation, and showed that it can be solved *without* imposing shape matching. They thus concluded that

$$T = 0 \qquad \neq \qquad \mathrm{shape-matching} \tag{2}$$

[1] Secondary constraints are a notion familiar from Dirac's extended formalism. For instance, also the Gauss law in electromagnetism can be seen as a secondary constraint associated with the vanishing of the momentum conjugated to A_0.

The purpose of the thesis is to clarify this conundrum and give a solution to it. First of all, the origin of the opposite results can be traced back to the different notions of discrete torsion used. Some type of arbitrariness such as this one is unfortunately unavoidable when one is discretising a continuum theory. Therefore the only way out is to derive the constraints from a dynamical system, as preservation equations for the primary constraints. There is indeed one simple case in which a Hamiltonian is well defined: the case of a flat evolution. This restricted setting serves the purpose of a case study, to understand in the only situation in which we completely control the system, to give a sharp answer to the question. And can then serve as guidance in the most interesting case. What we find solves the conundrum in a non-trivial way. First, the secondary constraints have the role of embedding in a non-trivial way the SU(2) spin networks in the Lorentzian phase space, in particular allowing the SU(2) variables to probe the boost degrees of freedom, precisely mimicking what happens in the continuum theory with the use of Ashtekar-Barbero variables. Therefore, the permit to give a covariant space-time interpretation to the holonomy appearing in spin network states. Second, they can be consistently solved only when the shapes match. Therefore, the shape matching conditions do not arise directly as constraint equations, but rather as consistency conditions when one tries to interpret the solution in space-time and not just space. The result explains the difference in the findings of Dittrich and Ryan[47] and Haggard *et al.* [46], solves the controversy, and furthermore extends their study to the Lorentzian signature. While the results are based on a toy model with flat dynamics, they give a clear prescription on what to expect in the general case, and in the conclusions we will discuss how our results can be extended to the curved case, by means of the notion of "pseudo-constraints" introduced by Dittrich in [23].

The dissertation is conceptually divided in three parts. The first one is made by the first three chapters and it is an overview of loop quantum gravity in its standard formulation: in the first chapter we give an introduction to the path-integral approach to loop quantum gravity, the spin-foam formulation, while in the second and third chapter we will constructively present the canonical formulation of the theory, starting from the Einstein-Hilbert formulation of General Relativity. On this ground we will develop the second part, made by the next three chapters. In the fourth chapter, after an introduction to Regge calculus we present the recent geometric interpretation of classical discrete phase space behind the loop quantum gravity and the fascinating relation with Twistors strictly connected to it. In the fifth and sixth chapters we will review the extension of such a formalism to the *covariant phase space* of loop gravity and its geometrical interpretations in terms of *twistor networks*. Finally in the last two chapters we will propose a model to investigate the mentioned problem of the presence of secondary constraints and summarise the conclusions that can be drawn from this work.

Chapter 1

From General Relativity to Loop Quantum Gravity

Loop quantum gravity is a tentative theory of quantum gravity that aims to describe what happens to the gravitational field in the Planck regime, where General Relativity and Quantum Mechanics are supposed to be equally important. The original formulation can be seen as originating from Dirac's efforts in quantising General Relativity exploiting the theory of the Hamiltonian extended systems and the first attempt in this direction was performed by deWitt in 1967 [3] who implemented the Dirac's quantisation program on the Hamiltonian formulation of General Relativity developed by Arnowitt, Deser and Misner [4] few years before.

$$
\begin{aligned}
&\hat{H}\left(h_{ab}, \frac{\delta}{\delta h_{ab}}\right) = -\frac{\hbar^2}{2} G^{abcd} \ : \ \frac{1}{\det(\hat{g})} \frac{\delta^2}{\delta h_{ab} \delta h_{cd}} \ : \ -\sqrt{\det(\hat{g})} R(\hat{g}) \\
&\hat{H}\left(h_{ab}, \frac{\delta}{\delta h_{ab}}\right) \Psi[h] = 0
\end{aligned}
$$

This eventually led to the famous Wheeler-deWitt equation which however was soon realised to be very ill-defined and indeed it was not possible to find any realistic solution. The turning point was in the late '80, when two fundamental works appeared: from one hand Ashtekar proposed to treat General Relativity exploiting $SU(2)$ variables [6], on the other hand a seminal paper by Rovelli and Smolin proposed the "Loop representation" for General Relativity[7]. Together, these two works started the whole program of Loop Quantum Gravity and finally the non-perturbative and background independent, canonical quantisation of space-time was achieved.

Today Loop Quantum Gravity comes basically in two version: the path-integral one, which is commonly called the *Covariant approach* and the Hamiltonian one, which we will refer to as the *Canonical approach*. The first one originates directly from the Feynman formulation of quantum mechanics and its main purpose is to correctly define a functional integral for General Relativity and compute the transition amplitudes between configurations of the gravitational field. The landscape of this approach is the *Spinfoam* formulation of Loop Quantum Gravity. From the other side, the Hamiltonian approach is the natural evolution of the work by deWitt and accomplished the Dirac's canonical quantisation program for constrained hamiltonian theories. Since the canonical formulation will be presented in the body of the thesis in this chapter we give a general overview of the covariant approach, together with the main results of the theory.

Before we introduce the covariant formulation of loop quantum gravity, we would like to present an heuristic argument that captures the core of the theory and which is useful to understand the Loop Quantum Gravity point of view: the Bronstein argument.

1.1 Invitation - the Bronstein argument

The argument is quite famous among Loop Quantum Gravity people and the story usually starts with a work of Landau in 1931 when, we remember, Quantum Field Theory was not developed. Together with Peierls [24] Landau suggested that once one apply the canonical commutation relation proposed by Heisenberg to the electromagnetic field, these relations would spoil the measurability of the field with arbitrary accuracy. Landau's idea was that a sharp localisation in space-time to reveal a particle could have been in contradiction with the Heisenberg uncertainty relation and thus with the commutation relations themselves. As we know, the idea is not correct and this was explicitly showed by Bohr and Rosenfeld in the case of the electromagnetic field [25]. Matvei Bronstein[26] was close to Landau, he studied the analysis proposed by Bohr-Rosenfeld, applied it to the gravitational field and understood that in this case their argument fails while Landau's intuition was correct. We give here a modern version of the Bronstein treatment [27], which is at the core of Loop Quantum Gravity and it is implemented at the most fundamental level.

Suppose one wants to measure the value of a field at some point x, with a definite accuracy $\Delta x < L$, this will imply a spreading in the momentum $\Delta p > \frac{\hbar}{L}$. We can see this from the point of view of accelerators physics where in order to probe "small regions" a sharpen localisation and thus a high momentum and energy are required $p^2 > \frac{\hbar^2}{L^2}$. All of this is the core of standard quantum theory to which some "General Relativistic element" is added saying that every form of energy act as a gravitational mass $M \sim \frac{E}{c^2}$, deforming the structure of the space-time around itself: the more the energy is concentrated the more the space around it is deformed, until it reaches the Schwarzschild radius and creates a black-hole $R \sim GM/c^2$. Putting the two considerations together one conclude that, in order to obtain sharp localisation one must concentrate high energy within the accuracy limit required L but this will make R to grow until it reaches the very value of L and the particle we want to measure will be hidden inside the horizon of its own minimal black-hole.

Combining the relations above one obtain a minimum limit in which any particle can be localised without being hidden by its horizon, which is the Planck length:

$$L \sim \frac{MG}{c^2} \sim \frac{EG}{c^4} \sim \frac{pG}{c^3} \sim \frac{\hbar G}{Lc^3} \qquad\qquad L_p = \sqrt{\frac{\hbar G}{c^2}} \sim 1.6 \times 10^{-35}\text{m} \tag{1.1}$$

The argument is clearly an extrapolation from semiclassical physics but it catches the core around which Loop Quantum Gravity is built: a deep revision of the concept of gravitational field is required when one wants to deal with quantum gravity, it is not possible to treat gravitational field as a quantum field on a smooth space-time but rather it is *a quantum state of space-time* and in order to understand its properties we are required to modify our "classical geometrodynamics" point of view, moving towards a genuine quantum notion of geometry, in which the continuous space-time will emerge as a semiclassical approximation.

1.2 Covariant Approach

Without any claim of rigour, in this brief section we will give a review of the basic elements of *Covariant Loop Quantum Gravity* or *Spinfoam* gravity theory, in a constructive way. It is important to know that a more rigorous treatment, starting from General Relativity, is available but it requires much more details that are not necessary for our purposes. We start mentioning that canonical and covariant Loop Quantum Gravity agree for what concerns the kinematical states of the theory so everything we are saying here that concerns the kinematics will be developed in detail in the body of the dissertation when we will present the canonical approach. The main difference among the two formulations lies in the very definition of the dynamics: in one case it amounts to find solutions of the Hamiltonian constraints whether the second approach aims to rigorously define transition amplitudes among kinematical states, arising from a path integral definition of the partition function of the theory.

$$Z \sim \int [Dg] e^{\frac{i}{\hbar}\int R\sqrt{-g}} \tag{1.2}$$

The kinematical states of the theory are called "spin-network states" and were firstly introduced by Penrose in a different context. They are quantum states defined over an abstract graph Γ "coloured" with two kind of quantum numbers j_l and i_n: the first one associated to the links ($l = 1, \dots, L$) while the second one to the nodes $n = 1, \dots, N$ of the graph. These states lives in the Hilbert space of an $SU(2)$ lattice Yang-Mills theory where the $SU(2)$ gauge invariance is imposed at each node and by the Peter-Weyl theorem it can be written as

$$\mathcal{H}_\Gamma = L^2[SU(2)^L/SU(2)^N] = \bigoplus_{j_l} \bigotimes_n \mathcal{K}_n^{(0)} \tag{1.3}$$

$$\mathcal{K}_n^{(0)} = \mathrm{Inv}_{SU(2)} \left[\bigotimes_{l\in n} \mathcal{H}_l \right] \qquad\qquad |\Gamma, j_l, i_n\rangle \in \mathcal{H}_\Gamma \tag{1.4}$$

It is possible to provide a geometric interpretation of these states as a collection of quantum polyhedra locally flat, in the same spirit of the Regge calculus, where the continuous manifold is replaced by a discrete *Regge manifold* made by regular 4D polyhedra. We will study this issue in details in Chapter 4.

In the simpler context of 3D Euclidean quantum gravity a definition of Z as a "sum over histories of the gravitational field" was found by Ponzano and Regge [37] in 1968 and it was proven to recover the Regge action, which

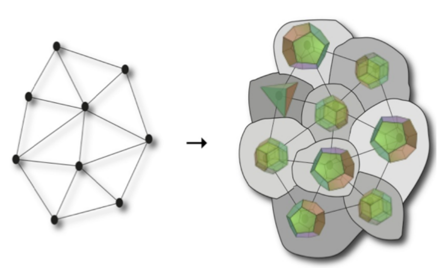


Figure 1.1: Spin-network states $|\Gamma, j_l, i_n\rangle$

is the truncation of the continuous Einstein-Hilbert action over a discrete manifold. The Ponzano-Regge theory is defined by triangulation Δ of the 3D space-time, which is obtained by "chopping" the continuous manifold in tetrahedra. Then, to each segment (bone) one associates semi-integers quantum numbers j_l, standing for $SU(2)$ representations and on tetrahedron is defined an amplitude:

$$Z_{PR} = \sum_{j_f} \prod_f (2j_f + 1) \prod_v \{6j\} \tag{1.5}$$

The index v runs over the tetrahedra and the $\{6j\}$ are the $6-j$ Wigner symbols, nothing but a combination of Clebsh-Gordan coefficients defining the way in which different $SU(2)$ representation living on different bones are contracted at the nodes. The Ponzano-Regge analysis showed that the $\{6j\}$ symbols, in the large spin limit, recover the Regge action which is a truncation to a discrete manifold of General Relativity. All of this was in 3D but it is possible to extend it to the full 4D theory defining the following partition function of Loop Quantum Gravity:

$$Z_{LQG} = \sum_{j_f, i_e} \prod_f (2j_f + 1) \prod_v A_v(j_e, i_v) \tag{1.6}$$

Where now $A_v(j_e, i_v)$ is a suitable 4D generalisation of the $\{6j\}$ symbols and i_e are other $SU(2)$ quantum numbers called intertwiners and they are associated to the 3D polyhedra. The former choice to associate the "spins" quantum number to the edge lengths was motivated by the fact that in 3D they are the geometrical quantity that carries the information on the curvature, whereas in 4D such information is carried by the 2D objects: the faces of the polyhedra showed in Fig. 1.1.

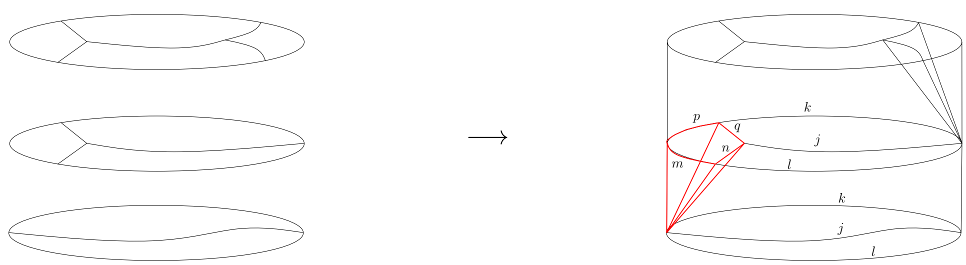
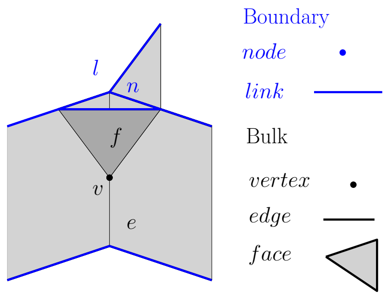


Figure 1.2: On the left side we have three spin-network states, living on the 3D boundary while on the right side it is showed the spin-foam, with vertex interactions, from where they originates

With the help of the intuition provided by the upside figure, it is easier to understand that the spin-networks can be seen as the boundary graph of a more general structure called *two-complex*. In the same way in which one can interpret the spin-networks state as a collection of 3D polyhedra, a two complex can be seen as a collection of 4D polyhedra, which in the case of the Regge action are the 4D extension of the tetrahedron and they are called *four-simplexes*. We will study them in Chapter 4. It is important to stress that the theory is actually defined over a generic two-complex, which is more general then a triangulation. Indeed it is always true that to a 4D triangulation correspond a two-complex but not the converse. Nonetheless it is useful to set up a duality relation connecting a two-complex with its dual triangulation.



Bulk duality	
Triangulation Δ	Two $-$ complex Δ^*
$4-$simplex	vertex
tetrahedron	edge
triangle	face

Boundary duality	
Triangulation $\partial\Delta$	Graph Γ
tetrahedron	node
triangle	link

There are many important results from covariant Loop Quantum Gravity but here we are going to mention only the most important ones. The first one is that $A_v(j_e, i_v)$ recovers the Regge theory in the semiclassical limit and therefore it can be considered as a discretisation of the path-integral for general relativity, moreover the result seems to be very robust since the expression of Z_{LQG} was recovered by different research group who followed very different paths[38, 39, 40, 41, 42]. The theory is free from ultraviolet divergences thanks to the natural cut-off provided by the Planck scale and it is possible to use the formalism of *q-deformations* to introduce the cosmological constant [48, 49] and assure that it is even infrared finite. Furthermore it is possible to couple matter via both Yang-Mills fields and fermions [50].

Usually in Quantum Field Theory after the definition of the partition function one inserts some bulk operator in order to extract physical information but this is a very hard task to achieve, due to the invariance under diffeomorphisms of the theory. The alternative technique is to compute the partition function on a spin-foam with boundary. Suppose for example to work with the spin-foam in the top picture of the former page, the boundary of the two-complex is a disconnected graph made by two spin networks so we are left with the task of computing transition amplitudes between spin-network states.

$$W(\{j_l\}, \{j_n\}) = \sum_{j_f, i_e} \prod_f (2j_f + 1) \prod_v A_v(j_e, i_v) \tag{1.7}$$

Since $\{j_l\}$ and $\{j_n\}$ are the quantum numbers of the relative boundary state, the sum runs only over them and the expression become easier to treat. For example in quantum cosmology one can compute the n-point function for the graviton over a background field or it is possible to proceed with refinement of the foam to improve the approximation, in the same spirit in which in lattice QCD extrapolation one takes the limit for the lattice spacing that goes to zero. Different approximation techniques have been introduced in order to compute approximate transition amplitudes. Among them we mention the graph expansion, in which computations are performed within a fixed graph approximation, for example a single $4-$simplex, this means to truncate the full theory to a finite number of degrees of freedom, as in any lattice version of a quantum field theory; the vertex expansion in which one consider to chop the space-time using an increasing number of 4-simplex and it correspond to refine the approximation.

Finally, it is possible to work within a different approximation: since the theory has an intrinsic scale, one can choose to work with a boundary which is large with respect to the Planck scale and this is usually taken as the semi-classical approximation since it is precisely in such a regime that it is possible to recover the Regge action from the partition function.

Chapter 2

Canonical "Loop" Gravity

In this chapter we will present the main features of the canonical formulation of Loop Gravity. The starting point is the classical Hamiltonian analysis of General Relativity performed by ADM in 1967, which is based on the Palatini's action of General Relativity. Such formalism however is not the best starting point for the purpose of the canonical quantisation and we shall use a different formulation of General Relativity, based on the work of Sören Holst in 1996[57]. The Holst's action is not formulated through the usual metric tensor formalism, it is based on Cartan formalism, proposed in 1922-25 by Elie Cartan [28, 29, 30, 31] and then revised and developed by D. Sciama[32] and T. Kibble [33] around 1960. The work developed in this direction today goes under the name of Einstein-Cartan[1] theory and its main feature is to exploit Palatini's intuition to treat connection as independent variable, relax the condition of zero torsion for the connection so that the formalism is apt to allow the coupling with matter via the torsion tensor. Even if our main purpose is not the coupling with matter, this formalism is still the best approach to General Relativity where space-time quantisation can be achieved.

Before we start the presentation of the canonical study of Loop Gravity, a review of Cartan's formalism is necessary. We will present the theory in a constructive way, nonetheless a basics knowledge of the mathematical tools would help the understanding.

2.1 Differential geometry: basic tools for General Relativity

An n-dimensional manifold $\mathcal{M}$ is a topological space such that for each point p there is a neighbourhood which is homeomorphic to $\mathbb{R}^n$. Although locally a manifold resembles the Euclidean space, it might not be so globally. It is the case of the circle or the sphere. When two homeomorphisms overlap in a region, a transition function is needed to connect them and the manifolds can be classified, depending on the properties these transition functions may have. For our purposes we need the notion of *smooth manifold*, in which the transition functions ϕ are not only differentiable but $\phi \in C^{\infty}$. Thanks to this property we can use calculus over the manifold and defines tangent spaces at each point: $T_p\mathcal{M}$. See Fig. [2.1].

A smooth manifold $\mathcal{M}$, in which each tangent space $T_x\mathcal{M}$ is equipped with non-degenerate bilinear form g is called *pseudo-Riemannian manifold*. In other words, there is a notion of inner product but unlike in a Riemannian

[1]Sometimes it has been called Einstein-Cartan-Sciama-Kibble theory

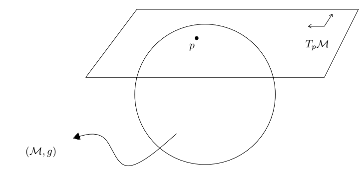


Figure 2.1: A 2-sphere is a 2-dimensional Manifold. Here is a pictorial representation of the Tangent space $T_p\mathcal{M}$ at the point p

manifold, g is not positive definite. This is, indeed, in the case of General Relativity where g has the signature $(-,+,+,+)$ and it describes the gravitational field degrees of freedom:

$$\mathbf{u}, \mathbf{v} \in T_x\mathcal{M} \qquad \text{then} \qquad \langle \mathbf{u}, \mathbf{v} \rangle = g(\mathbf{u}, \mathbf{v}) = u^\mu g_{\mu\nu}(x) v^\nu \tag{2.1}$$

Now, suppose to start with a pseudo-Riemannian Manifold $(\mathcal{M}, g)$. Since calculus is defined on the tangent vectors, in order to allow the possibility to compare vectors in different points one needs a notion of *connection* Γ. Such notion is strictly related to the *covariant derivative* and to the parallel transport operation in which a vector is transported along any curve, remaining parallel to itself.

The covariant derivative and differential can be defined through the following properties. Given two vector fields $\mathbf{X}, \mathbf{Y}$ the covariant derivative $\mathbf{d_X Y}$, satisfying the following properties, is still a vector field:

$$\mathbf{d}_{f\mathbf{X}+g\mathbf{Z}}\mathbf{Y} = f\mathbf{d_X Y} + g\mathbf{d_Z Y} \tag{2.2}$$

$$\mathbf{d_X}\left(\alpha\mathbf{Y} + \beta\mathbf{Z}\right) = \alpha\mathbf{d_X Y} + \beta\mathbf{d_X Z} \qquad \alpha, \beta \in \mathbb{R} \tag{2.3}$$

$$\mathbf{d_X}\left(f\mathbf{Y}\right) = X\left(f\right)\mathbf{Y} + f\mathbf{d_X Y} \tag{2.4}$$

The covariant derivative can be seen as a vector valued 1-form $\mathbf{dY}$ applied to the vector field $\mathbf{X}$:

$$\mathbf{d_X Y} = \langle \mathbf{dY}, \mathbf{X} \rangle = \mathbf{dY}\left(\mathbf{X}\right) \tag{2.5}$$

The covariant differential of the basis vectors $\{\mathbf{e}_a\}$ is another vector whose components define the connection Γ. The definition can be interpreted in both directions: one can assign Γ so defining a notion of covariant differential and derivative:

$$\mathbf{de}_a = \Gamma^a_b \mathbf{e}_b \qquad \Rightarrow \qquad \nabla Y^a = dY^a + \Gamma^a_b Y^b \tag{2.6}$$

Notes that Γ is not a tensor, it does not transform “well” under changes of basis (frame):

$$\Omega \in GL(n, \mathbb{R}) \qquad \mathbf{e}' = \mathbf{e}\Omega \quad \Rightarrow \quad \Gamma' = \Omega^{-1} d\Omega + \Omega^{-1}\Gamma\Omega \tag{2.7}$$

The next step is to write the curvature 2-form R, which is defined as the exterior covariant differential of the connection definition[2]:

$$\mathbf{dde}_a = \mathbf{d}\left(\mathbf{e}_b\Gamma^b_{\ a}\right) = \mathbf{e}_b d\Gamma^b_{\ a} + \mathbf{e}_c\Gamma^c_{\ b}\wedge\Gamma^b_{\ a} = \mathbf{e}_b R^a_b \tag{2.8}$$

$$R = d\Gamma + \Gamma\wedge\Gamma \tag{2.9}$$

The curvature tensor transforms covariantly under changes of basis:

$$\Omega \in GL(n,\mathbb{R}) \qquad \mathbf{e}' = \mathbf{e}\Omega \quad \Rightarrow \quad R' = \Omega^{-1}R\Omega \tag{2.10}$$

Equipped with this structure we can write the Einstein-Hilbert action for General Relativity, with the cosmological term

$$S_{EH}[g] = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\sqrt{-g}\left(\mathcal{R}[g] - 2\Lambda\right) \tag{2.11}$$

where g is the determinant of the metric tensor (or the gravitational field) and $\mathcal{R}$ is the trace of the curvature tensor R. Exploiting the least action principle, one finds that the equations of motion are the Einstein's equation in the case of pure gravity:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \qquad\qquad \text{Einstein Equations} \tag{2.12}$$

Boundary term

Such an action is not correct from the variational point of view. It involves second order derivatives of the metric thus, it is like to write the following action for a free particle

$$S = \int_{q_0}^{q_1} dq\int_{t_0}^{t_1} dt\,\left(-\frac{1}{2}q\ddot{q} - V(q)\right) \tag{2.13}$$

Performing the variation, the boundary condition $\delta q = 0$ is not enough to obtain the correct equation of motions, we need to impose a further boundary condition on the time derivative $\delta\dot{q} = 0$. This problem can be fixed by adding a proper boundary term to the action:

$$S' = S + \frac{1}{2}q_1\dot{q}_1 - \frac{1}{2}q_0\dot{q}_0 \tag{2.14}$$

A similar procedure is necessary for the Einstein-Hilbert action and the proper boundary term is called the Gibbons-Hawking-York term [53, 54]:

[2]Being $\mathbf{d}$ the covariant exterior differential $\mathbf{d}^2 \neq 0$. For the same reason Γ is not a tensor so $\Gamma\wedge\Gamma \neq 0$

$$S_{GR} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\,\sqrt{-g}R + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3y\sqrt{h}K \tag{2.15}$$

Where K is the trace of the extrinsic curvature tensor K_{ij} and h is the determinant of the metric h_{ij} on the boundary $\partial\mathcal{M}$, with coordinates y.

The problem of the boundary term should not be underestimate. In the covariant formulation of Loop Quantum Gravity the object providing the classical counterpart of the quantum theory is not the action but rather the Hamilton principal function and since the bulk term vanishes on the Einstein equations, the Gibbons-Hawking-York term is going to be the only one that lasts in the Hamilton function. Nonetheless, since it is not relevant for our purposes we will drop the term everywhere but where it is essential.

2.2 Cartan formalism: tetrad and spin-connection

As mentioned in the previous section, in the Einstein-Hilbert theory the gravitational degrees of freedom are represented by the metric tensor $g_{\mu\nu}$ since it defines the geometry of the space-time at each point. The Cartan formalism is different because it treats the metric tensor as a derived quantity and the degrees of freedom of the gravitational field are described through the *Tetrads*: Minkowski valued 1-forms[3]:

$$e^I(x) = e^I_\mu(x)dx^\mu \tag{2.16}$$

They provide a local isomorphism between the tangent space at each point of the manifold $\mathcal{M}$ and the Minkowski space $\mathfrak{M}$:

$$\mathfrak{e}_x\;:\;T_x\mathcal{M} \to \mathfrak{M} \tag{2.17}$$

As the fibres can be glued together to give the tangent bundle, the local maps can be collected to give the bundle map. So the tetrads 1-forms provide an isomorphism between the tangent bundle $T\mathcal{M}$ and the $SO(3,1)$-principal bundle[4] with support on the smooth manifold $\mathcal{M}$:

$$\mathfrak{e}\;:\;\bigcup_{x\in\mathcal{M}} T_x\mathcal{M} = T\mathcal{M} \to P\left(\mathcal{M}, SO(3,1)\right) \tag{2.18}$$

For example

$$v \in T_x\mathcal{M} \quad \text{then} \quad \mathfrak{e}_x(v) = e^I_\mu(x)v^\mu = v^I \in \mathfrak{M} \tag{2.19}$$

[3]In the language of differential geometry and principal bundles, these variables are the *solder forms* on the principal bundle $P(\mathcal{M}, SO(3,1))$ over the smooth manifold $\mathcal{M}$. They are differential 1-forms, such that the bundle map from the tangent bundle $T\mathcal{M}$ to the associated bundle $P_{SO(3,1)}\mathcal{M}$ is a bundle isomorphism

[4]The *principle bundle* (P_GX or $P(X,G)$) concept formalise the idea of cartesian product $X \times G$ between a space X and a group G. It has a well defined action of the group G over the elements $x \in X$ and a notion of projection onto the space $(x,g) \to x$. A neat example is the $\mathbb{R}^n$ equipped with every changes of basis $\Omega \in GL(n,\mathbb{R})$

The metric tensor is then a second order quantity given by the pullback of the Minkowski metric η through the tetrads. This should clarify the leading role of the tetrads over the tensor metric as representing and describing the geometrical properties of the space-time and then of the gravitational fields:

$$g_{\mu\nu}(x) = \eta_{IJ} e^I_\mu(x) e^J_\nu(x) \tag{2.20}$$

Fermions in curved space-times. There is a more physical reason that suggests that the tetrads are a more fundamental description of the gravitational field, at least for what concerns quantum physics. As we learn from quantum field theory on curved space-time, if we had only the metric tensor to describe the gravitational field we could not correctly couple it to the fermonic matter. Let us look at the standard Dirac's fermions Lagrangian:

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}\,(i\;\partial\!\!\!/ - m)\,\psi = \bar{\psi}\,(\gamma_\mu \partial^\mu - m)\,\psi = \bar{\psi}\,(\gamma_\mu \eta^{\mu\nu}\partial_\nu - m)\,\psi$$

Where we pointed out that the contraction involve the Minkowski metric. If we want a candidate for the Dirac equation in curved space-time the first thought is to replace the Minkowski metric withe the metric tensor $g_{\mu\nu}$:

$$\mathcal{L}_{\text{Dirac}} \quad \to \quad \bar{\psi}\,(\gamma_\mu g^{\mu\nu}\partial_\nu - m)\,\psi \qquad \text{wrong!}$$

A moment of reflection shows why such a procedure is not correct. The metric tensor g is a solution of the Einstein equation and, as such, it must transform covariantly under diffemorphisms, in this sense its indices are “diffeomorphisms indices” as well as the derivative index $\partial_\mu = \frac{\partial}{\partial x^\mu}$. On the contrary, the index labeling the γ-matrices is a genuine “Lorentz index” since it arises from its very definition as a representation of the Dirac's algebra

$$\{\gamma^\mu\,,\,\gamma^\nu\} = 2\eta^{\mu\nu}$$

For this reason, in order to contract the γ-matrices with partial derivatives, we need an object with “mixed indices”, one transforming under diffeomorphisms and the other under Lorentz transformation. The tetrad field is the object that suits the case

$$\mathcal{L}_{\text{Dirac}} \quad \to \quad \bar{\psi}\left(\gamma_I e^I_\mu \partial^\mu - m\right)\psi$$

and allows to define a quantum field theory for fermions on curved space-times.

In order to understand what these variables are actually doing, it is useful to think about the equivalence principle, which assures that *locally* it is always possible to find an inertial reference frame where there is no gravitational field and the geometry is flat. The tetrad formalism allows to make concrete use of the equivalence principle, providing a set of transformations to locally identify an inertial reference frame. However as we know from the standard courses, given a local reference frame, it is always possible to perform a Lorentz transformation Λ to obtain an equivalent inertial reference frame. So if one would decide to formulate the theory exploiting these variables, there would be an additional gauge symmetry to deal with: the Lorentz $SO(3,1)$ group[5].

[5] Actually in the final version of the theory we will have to deal with a larger group, $SL(2,\mathbb{C})$, which is the double covering of the restricted Lorentz subgroup $\mathcal{L}^\uparrow_+$: $\det\Lambda = 1$ and $\Lambda^0_0 \geq 1$

In facts, given two tetrads e and $\tilde{e}$, that differ only for a Lorentz transformation, at the metric level it is impossible to distinguish between them

$$\tilde{e}^I = \Lambda^I_J e^J \quad \Rightarrow \quad \tilde{g}_{\mu\nu}(x) = \eta_{IJ}\tilde{e}^I_\mu(x)\tilde{e}^J_\nu(x) = \eta_{IJ}\Lambda^I_A e^A_\mu(x)\Lambda^J_B e^B_\nu(x) = \eta_{AB}e^A_\mu(x)e^B_\nu(x) = g_{\mu\nu}(x) \quad (2.21)$$

As for every principal bundle, on $P_{SO(3,1)}\mathcal{M}$ it is possible to define a connection ω, which we will refer to as the *spin-connection*: a 1-form with values on the Lorentz algebra $\mathfrak{so}(3,1)$ and consequently a notion of covariant derivative and parallel transport.

$$(Dv)^I = dv^I + \omega^I_J v^J \quad (2.22)$$

or in the components of the basis of the tangent bundle

$$D_\mu v^I = \partial_\mu v^I(x) + \omega^I_{\mu\ J}(x)v^J(x) \quad (2.23)$$

It is possible to define the covariant derivative for objects with mixed indices, such as the tetrads

$$D_\alpha e^I_\beta = \partial_\alpha e^I_\beta(x) + \omega^I_{\alpha\ J}(x)e^J_\beta(x) - \Gamma^\sigma_{\beta\alpha}(x)e^I_\sigma(x) \quad (2.24)$$

Introducing the exterior calculus notation for the forms, we define the curvature 2-form F (Field strength) associated to the spin-connection

$$F^{IJ}[\omega] = d\omega^{IJ} + \omega^I{}_A \wedge \omega^{AJ} = \frac{1}{2}F^{IJ}{}_{\alpha\beta}[\omega]\left(dx^\alpha \wedge dx^\beta\right) \quad (2.25)$$

$$F^{IJ}{}_{\alpha\beta}[\omega] = \partial_\alpha\omega^{IJ}_\beta - \partial_\beta\omega^{IJ}_\alpha + \omega^I_{A\,\alpha}\omega^{AJ}_\beta - \omega^J_{A\,\alpha}\omega^{AI}_\beta \quad (2.26)$$

Now we have all the ingredients necessary to write an action which will be shown to be equivalent to General Relativity, in the guise of Palatini's action:

$$S_P[e,\omega] = \frac{1}{32\pi G}\varepsilon_{IJAB}\int_{\mathcal{M}} e^I \wedge e^J \wedge F^{AB}[\omega] \quad (2.27)$$

where ε_{IJAB} is the Levi-Civita completely antisymmetric tensor. In order to show the equivalence with the Palatini's action we need to explicitly perform the action variation through the fields (e,ω):

$$\delta S_P[e,\omega] \quad \rightarrow \quad \delta_\omega S_P \quad \text{and} \quad \delta_e S_P \quad (2.28)$$

Consistently with equations (2.2) - (2.4), we introduce the covariant exterior derivative notation $\mathbf{d}_\omega$ and start with the Palatini's identity

$$\delta_\omega S \quad \rightarrow \quad \delta_\omega F[\omega] \qquad\qquad \delta_\omega F[\omega] = \mathbf{d}_\omega(\delta\omega) \quad (2.29)$$

The vanishing of the variations on the action gives, by means of an integration by parts

$$\delta_\omega S_P = 0 \quad \rightarrow \quad \varepsilon_{IJAB} \left(e^I \wedge \mathbf{d}_\omega e^J\right) = 0 \tag{2.30}$$

$$\delta_e S_P = 0 \quad \rightarrow \quad \varepsilon_{IJAB} e^J \wedge F^{AB}[\omega] = 0 \tag{2.31}$$

With the additional hypothesis that the tetrad field must be invertible the equations become simpler:

$$\mathbf{d}_\omega e^I = 0 \qquad\qquad \varepsilon_{IJAB} e^J \wedge F^{AB}[\omega] = 0 \tag{2.32}$$

In the first equation it is possible to read the request that the covariant derivative defined through the spin-connection ω must be *tetrad compatibile*. In the same way in which one ask to the standard covariant derivative $\nabla = \partial + \Gamma$ to be *metric compatible* $\nabla_\alpha g_{\mu\nu} = 0$. The geometrical meaning of such request is that the torsion tensor associated to the spin-connection must be zero and, exactly as in the Palatini's formulation, it gives the proper relation between the spin-connection and the tetrad field:

$$T^I \equiv \mathbf{d}_\omega e^I = T^I_{JK} e^I \wedge e^K = T^I_{JK} e^J_\mu e^K_\nu \left(dx^\mu \wedge dx^\nu\right) = T^I_{\mu\nu} \left(dx^\mu \wedge dx^\nu\right) \tag{2.33a}$$

$$d_\omega e^I = \left(D_\alpha e^I_\beta\right)\left(dx^\alpha \wedge dx^\beta\right) = \left(\partial_\alpha e^I_\beta + \omega^I_{\alpha J} e^J_\beta - \Gamma^\mu_{\alpha\beta} e^I_\mu\right)\left(dx^\alpha \wedge dx^\beta\right) = 0 \tag{2.33b}$$

$$\omega^I_{\mu\ J}(e) = e^I_\alpha \nabla_\mu e^\alpha_J \tag{2.33c}$$

Proof of the solution of the torsionless equation.

$$\left(\partial_\alpha e^I_\beta + \omega^I_{\alpha J} e^J_\beta - \Gamma^\mu_{\alpha\beta} e^I_\mu\right) = 0 \tag{2.34}$$

$$\partial_\alpha e^I_\beta - \Gamma^\mu_{\alpha\beta} e^I_\mu = \nabla_\alpha e^I_\beta = -\omega^I_{\alpha J} e^J_\beta = \omega^{JI}_\alpha e_{J\,\beta} \tag{2.35}$$

We contract with $e^{H\,\beta}$ and obtain:

$$e^{H\,\beta} \nabla_\alpha e^I_\beta = \omega^{JI}_\alpha \delta^H_J = \omega^{HI}_\alpha \tag{2.36}$$

Lowering the index I with an η_{IJ} and changing the position of the contracted indices β we get

$$\omega^H_{\alpha\ J}(e) = e^H_\beta \nabla_\alpha e^\beta_J \tag{2.37}$$

Decomposing the field strength tensor in its components on the chart x^μ, it is possible to show

$$F^{AB} = \frac{1}{2} F^{AB}{}_{\mu\nu} dx^\mu \wedge dx^\nu \qquad\qquad F^{AB}{}_{\mu\nu}[\omega(e)] = e^A_\rho e^B_\sigma R^{\rho\sigma}{}_{\mu\nu}[g(e)] \tag{2.38}$$

where $R^{\rho\sigma}{}_{\mu\nu}[g(e)]$ is the Riemann curvature tensor constructed with the metric induced by e. We are not going to show it here, but we mention that it requires a lot of algebra but the computation is straightforward, from the definition of F plus the solution of the torsionless equation. Multiplying (2.31) by $\wedge e^H$ and performing the contractions we recover the Einstein's equations:

$$\varepsilon_{IJAB}e^J \wedge F^{AB}\left[\omega(e)\right] \wedge e^H = 0 \qquad \Longleftrightarrow \qquad G_I^H = 0 \tag{2.39}$$

Einstein-Hilbert action and Einstein equations. Here we work in units $16\pi G = 1$. First we show that the introduced action (2.27) is equivalent to the Einsten-Hilbert action when the covariant derivative is tetrad-compatible, or $\omega = \omega(e)$:

$$\begin{aligned} S_{\mathrm{P}}\left[\omega(e), e\right] &= \frac{1}{2}\int_{\mathcal{M}} \varepsilon_{IJAB} e^I \wedge e^J \wedge F^{AB}[\omega(e)] = \frac{1}{4}\int_{\mathcal{M}} d^4x\ \varepsilon_{IJAB}\varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu}^{IJ}[\omega(e)] e_\alpha^A e_\beta^B = \\ &= \int_{\mathcal{M}} d^4x\ \det\left(e_\gamma^H\right) e_I^\mu e_J^\nu F_{\mu\nu}^{IJ}\left[\omega(e)\right] \end{aligned} \tag{2.40}$$

We call $e = \det\left(e_\gamma^H\right)$ and the field-strength tensor is related to the Riemann curvature tensor via the tetrad field, so we can use the relation (2.31) here. Moreover, thanks to the definition of the tetrad (2.20) we have $e^2 = -g$. Finally we have

$$\begin{aligned} S_{\mathrm{P}}\left[\omega(e), e\right] &= \int_{\mathcal{M}} d^4x\ \det\left(e_\gamma^H\right) e_I^\mu e_J^\nu F_{\mu\nu}^{IJ}\left[\omega(e)\right] = \int_{\mathcal{M}} d^4x\ e\, e_{\mu\, I} e^{\nu\, I}\ e_J^\alpha e^{\beta\, J} R^\mu_{\alpha\nu\beta} = \\ &= \int_{\mathcal{M}} d^4x\ \sqrt{-g}\, \delta_\mu^\nu R_\nu^\mu = \int_{\mathcal{M}} d^4x\ \sqrt{-g}\, g_{\mu\nu} R^{\mu\nu} = S_{EH}\left[g\right] \end{aligned} \tag{2.41}$$

Beside, it is easy to show that the expression (2.39) actually leads to the Einstein equation:

$$\varepsilon_{IJAB}e^J \wedge F^{AB}\left[\omega(e)\right] \wedge e^H = \frac{1}{2}\varepsilon_{IJAB}\varepsilon^{\mu\alpha\beta\nu} e_\mu^J F_{\alpha\beta}^{AB} e_\nu^H\ d^4x \tag{2.42}$$

We drop the volume element and explicitly write the contraction of two Levi-Civita tensor in terms of the δ tensors. As before, we remember that the field-strength is essentially the Riemann curvature tensor, with to indices contracted with the tetrad field. Computing the contractions we get:

$$\varepsilon_{IJAB}\varepsilon^{\mu\alpha\beta\nu} e_\mu^J F_{\alpha\beta}^{AB} e_\nu^H = 2\left(R\delta_I^H - 2R_I^H\right) = -4G_I^H \qquad \Rightarrow \qquad G_I^H = 0 \tag{2.43}$$

2.3 The Holst's term and the Immirzi parameter

We just showed that the action (2.27) is equivalent to the Palatini's action for General Relativity since it provides the correct field equations. However it is not the most general action we can think of. Since now the gravitational field is expressed via the tetrads, there is another term that can be added to the Lagrangian, with the same symmetries as the first one and the right dimension; we will show that the theory that arise, again is the same as General Relativity. Such term is often called *Holst's term*[6] and the full action, with the introduction of the new term, is called *Holst's action*:

[6]In literature the term is associated to Holst since he showed that it actually reproduces the Ashtekar-Barbero variables, however he was not the one who first introduced the term. In facts, the term was presented in 1980 by Hojman, Mukku, Sayed [114].

$$S_{\text{Holst}}\left[e,\omega\right]=\frac{1}{16\pi G}\left(\frac{1}{2}\varepsilon_{IJAB}+\frac{1}{\gamma}\delta_{I[J}\delta_{A]B}\right)\int e^I\wedge e^J\wedge F^{AB}[\omega] \tag{2.44}$$

There is a similar term in the non-abelian $SU(3)$ Yang-Mills theory for Quantum Chromodynamics called θ-term [97]. It is a topological term since in QCD it is proportional to the topological charge and it does not carry local degrees of freedom. Nonetheless, the arising phenomenology is quite interesting since the term violates the CP symmetry and its coupling constant θ is heavily restricted by the current experimental data to the value $|\theta|\lesssim 10^{-10}$.

Let's see what are the equations of motion for the Holst's action and if they differ from General relativity:

$$\delta_\omega S_{\text{Holst}}=0 \quad\rightarrow\quad \left(\varepsilon_{IJAB}+\frac{1}{\gamma}\delta_{I[J}\delta_{A]B}\right)\left(e^I\wedge d_\omega e^J\right)=0 \tag{2.45}$$

$$\delta_e S_{\text{Holst}}=0 \quad\rightarrow\quad \left(\varepsilon_{IJAB}+\frac{1}{\gamma}\delta_{I[J}\delta_{A]B}\right)\left(e^J\wedge F^{AB}\left[\omega\right]\right)=0 \tag{2.46}$$

Again we suppose to deal with invertible tetrads so the first set of equations is the same as before:

$$\mathbf{d}_\omega e^I=0 \qquad\Rightarrow\qquad \omega^I_{\mu\,J}(e)=e^I_\alpha\nabla_\mu e^\alpha_J \tag{2.47}$$

For what concerns the second set of equations, the first term gives the Einstein equations in the case of pure gravity while the second term vanishes, on the solutions $\omega=\omega(e)$ i.e. when the tetrad compatibility condition holds for the spin-connection:

$$\delta_{I[J}\delta_{A]B}\left(e^I\wedge e^J\wedge F^{AB}\left[\omega(e)\right]\right)=\varepsilon^{\alpha\beta\mu\nu}R_{\alpha\beta\mu\nu}[g(e)]=0 \tag{2.48}$$

One can now understand why the Holst term is not present in the second-order Einstein-Cartan theory. When the metric-compatibility condition holds it is identically zero thanks to the symmetries of the Riemann tensor. Even if the classical theory of pure gravity is completely transparent to the addition of this term, it might not be so for the quantum version. That is exactly what happens in QCD for the θ-term. So the presence of such term is a key ingredient, since we are actually preparing the theory for a canonical quantisation process and neglecting it could mean to forsake some phenomenology that may reveal to be interesting.

On the measurability of the Immirzi parameter

In quantum gravity contexts the coupling constant of the Holst's term γ is called *Immirzi parameter*[100, 101]. It is worth to mention that the role of the Holst's term at the classical level might become important in the Einstein-Cartan version of the theory, in which the matter coupling is achieved through the torsion tensor. In this scheme the metric compatibility condition does not hold anymore, so the coupling with fermions may induce some effective interactions proportional to the Immirzi parameter which become, in principle, a measurable quantity. Without giving any further details we say that in [100] it was showed that in the minimal coupling scheme, the effective action has a four-fermions terms written in term of axial currents $A^I = \bar{\psi}\gamma^5\gamma^I\psi$:

$$S_\gamma[e,\psi] = -\frac{3}{2}\pi G \frac{\gamma^2}{\gamma^2+1}\int_{\mathcal{M}} d^4x A^I A_I \tag{2.49}$$

However it has been argued by Alexandrov[102] that the appearance of such term is only a result of the minimal-coupling scheme. Considering a more general scheme for the coupling of the fermions to the gravitational field, which can be easily reduced to minimal coupling, he showed that the effective action obtained integrating out the degrees of freedom of the spin-connection is independent of γ, even in the minimal coupling limit.

2.4 Canonical analysis of the Holst's action

For the canonical quantum gravity purposes the Holst's action and the self-dual ones are surely the most used formulations of General Relativity. Before the quantisation process can be implemented, we need to present the canonical analysis of the Holst's action in order to identify the pairs of canonically conjugated variables and the phase space in which the Hamiltonian version of the theory is formulated. The self-dual formulation of General Relativity, with its canonical analysis will be the subject of Chapter 5.

It is important to mention that the treatment of tetrads and spin-connection as independent variables complicates the Hamiltonian analysis since it makes the constraints algebra second class, i.e. some constraints have non-zero Poisson bracket. Since the Hamiltonian is a linear combination of the constraints, if the algebra is second class there are going to be consistency conditions, for the stability of the constraint equations through the evolution. These conditions are called *secondary constraints* and in the full canonical analysis of the Holst action, it is possible to show that they implement part of the torsionless equation, that in the Lagrangian formalism emerges as an equation of motion[59].

In what follows we will borrow the main ideas of the canonical analysis originally performed by ADM, in the second-order version of the theory. All these ideas will be integrated within the structure developed by Dirac and Bergmann to study the Hamiltonian constrained systems[110, 111, 112]; moreover, in order to prepare the theory for quantisation, a proper choice of the variables is necessary. The key result that allowed to fully develop the Hamiltonian analysis and eventually led to the *loop representation* of General Relativity[7] is the use of a set of variables, proposed by Abhay Ashtekar in 1986 [6] which today are called *Ashtekar variables*. The choice of these variables makes the algebra first class, because it implements the secondary constraints within the variables (it will be shown explicitly in Chapter 5). Moreover it represents the induced 3D metric variables in terms of $SU(2)$ gauge fields, so simplifying considerably the constraints of General Relativity and enabling one to embed the phase-space of General Relativity into that of an $SU(2)$ Yang-Mills theory.

We set-up the variables, according to the ADM prescript, for the details we refer to the Appendix A.

Given a foliation F^μ_t we chose our coordinates $(t, \vec{x})$ to be the ADM coordinates that suit the foliation. The vector field that generates the "time flow" has the most simple expression:

[7] It is even the reason for the name of the theory: *Loop* quantum gravity

$$t^\mu = \frac{\partial F_t^\mu}{\partial t} = (1,0,0,0) \tag{2.50}$$

A tetrad compatible with the ADM metric can be easily set up. We introduce the lapse function (N) and the shift vector (N^a) and remember that the components of the unit vector normal to the hypersurface Σ are

$$n^\mu = (\frac{1}{N}, -\frac{N^a}{N}) \qquad \mu = 0,1,2,3 \qquad a = 1,2,3 \tag{2.51}$$

Furthermore, in order to simplify the calculations it is customary work in the time-gauge in which the tetrad is "aligned" to the unit vector n^I in the principal bundle. These conditions allow to parametrise the tetrad to give a metric tensor compatible with the ADM parametrisation:

$$\begin{cases} e^I_\alpha t^\alpha = e^I_0 = N n^I + N^a e^I_a \\ e^I_\mu n^\mu = n^I = \delta^I_0 \end{cases} \longrightarrow \begin{cases} e^0_\mu = (N, 0) \\ e^I_0 = \left(N, N^a e^i_a\right) \end{cases} \tag{2.52}$$

As mentioned before, the next step is to introduce the Ashtekar-Barbero[8] variables: the *densitized triad* and the *Ashtekar-Barbero connection*.

$$E^a_i = \frac{1}{2} e e^a_i = \frac{1}{2}\varepsilon_{ijk}\varepsilon^{abc} e^j_b e^k_c \tag{2.53}$$

$$A^i_a = \frac{1}{2}\varepsilon^i_{jk}\omega^{jk}_a + \gamma\omega^{0i}_a = \Gamma^i_a + \gamma K^i_a \tag{2.54}$$

$$E^a = E^a_i \tau^i \in \mathfrak{su}(2) \qquad\qquad A_a = A^i_a \tau_i \in \mathfrak{su}(2) \tag{2.55}$$

Exploiting these variables it is possible to write the Holst's action in such a way that become clear that the introduced variables are canonically conjugated:

$$S_{\text{Holst}} = \frac{1}{\gamma}\int d^3x\, dt\, \left(\dot{A}^i_a E^a_i - A^j_0 \mathcal{G}_j - NH - N^a H_a\right) \tag{2.56}$$

$$H = \frac{F^i_{ab}\varepsilon_{ijk}E^a_j E^b_k}{\det E} + \left(\gamma^2+1\right)\left[\frac{1}{\gamma}G^k \partial_b\left(\frac{E^b_k}{\det E}\right) + \frac{\varepsilon_{lmn}K^m_a K^n_b \varepsilon_{ljk}E^a_j E^b_k}{\det E}\right] \tag{2.57}$$

$$H_a = \frac{1}{\gamma}\left(E^b_j F^j_{ab} - (\gamma^2+1)K^i_a \mathcal{G}_i\right) \tag{2.58}$$

$$\mathcal{G}_j = D_b E^b_j = \partial_b E^b_j + \varepsilon_{jhk} A^h_b E^{b\,k} \tag{2.59}$$

[8]The original variables proposed by Ashtekar were the densitized triad and a complex connection with $\gamma \to i$. This choice would simplify the expression of the diffeomerphisms constraints but the price one have to pay is that, in order to recover General Relativity, reality conditions on the the variables must be imposed.

Thanks to this expression for the Holst's action, the Legendre transform is trivial in the sense that the Lagrangian already has a nice expression from where we can read both the Hamiltonian and the conjugate variables:

$$\int d^4x\,\mathcal{L} = \int d^4x \left(\sum_i p_i \dot{q}_i - \mathcal{H}\right) \quad \Rightarrow \quad \begin{cases} p_i & \longrightarrow \quad A_a^i \in \mathfrak{su}(2) \\ q_i & \longrightarrow \quad E_i^a \in \mathfrak{su}(2) \end{cases} \tag{2.60}$$

Here it is the Hamiltonian structure of General Relativity in the guise of the Holst's action, written in terms of Ashtekar-Barbero variables:

$$\left\{A_a^i(x)\,,\,E_j^b(y)\right\} = \gamma \delta_j^i \delta_b^a \delta^3\left(x-y\right) \tag{2.61}$$

$$\mathcal{H} = \frac{1}{\gamma}\int_{\mathcal{M}} d^3x\,dt\,\left(A_0^j \mathcal{G}_j + NH + N^a H_a\right) \tag{2.62}$$

It is worth noting that the quantities with time derivatives in the action are only the spatial components of the Ashtekar-Barbero connection. A_0^i, lapse and shift do not appear in the same way so their conjugate momenta vanishes and they can be interpreted as Lagrange's multipliers associated to specific constraints equations:

$$\frac{\delta\mathcal{L}}{\delta\dot{A}_0^j} = 0 \qquad \frac{\delta\mathcal{L}}{\delta\dot{N}^a} = 0 \qquad \frac{\delta\mathcal{L}}{\delta\dot{N}} = 0 \tag{2.63}$$

$$\frac{\delta L}{\delta A_0^j} = \mathcal{G}_j = 0 \qquad \frac{\delta L}{\delta N^a} = H_a = 0 \qquad \frac{\delta L}{\delta N} = H = 0 \tag{2.64}$$

The theory of the Hamiltonian constrained system was fully developed by Dirac and Bergmann around 1950; for the reader not familiar with the topic, see the original literature by Dirac [110, 111], the pedagogical review by G. Date [112] and the detailed book on the canonical quantisation of gauge systems by Henneaux and Teitelboim [113]. Furthermore, the canonical analysis of the Holst action has been performed in many guises and following different paths but the results are always the same, we refer the reader to the analysis of Barros e Sa[60] in which the time-gauge is not imposed and to the all-encompassing book by Thiemann [95].

The first application to gravity of the theory was the Hamiltonian formulation developed by Arnowitt, Deser and Misner [4] (see Appendix A). The key difference with respect to the ADM formalism is the appearance of a new constraint $\mathcal{G}_i$. This is perfectly reasonable once one keeps in mind the geometrical interpretation of first class constraints as generators of gauge symmetry [111, 113]. In facts, as stated in the beginning of the section, the tetrad formulation for General Relativity forces to deal with an additional gauge symmetry which was not present in the second-order formulation: an SO(3,1) gauge symmetry. The Gauss constraint is here to enforce this symmetry into the solutions of the theory.

However, if one takes a quick look at the Gauss constraint, the $\mathcal{G}_i$ generates $SU(2)$ transformations and not the full $SO(3,1)$ ones, so one could wonder why the symmetry is just an $SU(2)$ rather than the full Lorentz group. The answer is a bit tricky but the reason lies in the choice of the variables and in the "time-gauge" choice. The Ashtekar-Barbero connection is not the pull-back of the space-time connection originated by the Lorentz gauge group, so in this sense A is not a space-time connection but just a spatial connection and we should not apply the same arguments we used for space-time variables. This connection is just a wise choice of variables for recasting the Holst's action in a convenient way, suitable for the purpose of canonical analysis and quantisation.

A more technical point of view could be useful. It is now clear that the actual variables of the theory are not the tetrads but just its spatial components: the *triads*. We remember that such variables implement local isomorphisms towards the Minkowski space and since the $SO(3,1)$ gauge action on this variables can be split into boost and rotations, it is clear that the Lorentz boosts are not a symmetry of our variables anymore since they have been gauge-fixed by the time-gauge choice. So we are left with the symmetry of the spatial part, an $SO(3)$ symmetry, which has the same algebra as $SU(2)$. We conclude that it is not puzzling that the constraint $\mathcal{G}_i$ does not generate the full Lorentz group. All these considerations are going to be more clear when the self-dual canonical analysis will be presented.

Constraints algebra

Defining the smeared version of the constraints

$$\mathcal{G}_i\left[\Lambda^i\right] = \int_\Sigma d^3x \Lambda^i(x)\mathcal{G}_i(x) \tag{2.65}$$

$$H_a[N^a] = \int_\Sigma d^3x N^a(x) H_a(x) \tag{2.66}$$

$$H[N] = \int_\Sigma d^3x N(x) H(x) \tag{2.67}$$

one can compute their action on the canonical variables which reveals their interpretation as generators of gauge symmetries. Furthermore one should compute the algebra of the constraints: the algebra of the spatial diffeomorphisms H_a and scalar H constraints is the same as the in the ADM formulation (we refer to Appendix A) and the introduction of the Gauss constraints does not change the key property of the algebra of being first class. For this reason the constraint equations are preserved under the evolution and we have a well posed Hamiltonian problem which does not require any consistency conditions. We give here the algebra generated by the smeared Gauss constraint while the algebra of the diffeomorphisms and scalar constraints is given in Appendix A.

Algebra of the Gauss constraint.

$$\vec{\mathcal{G}}[\vec{\Lambda}] = \int_\Sigma d^3x\ \mathcal{G}_i(x)\Lambda^i(x) \tag{2.68}$$

$$\left\{\vec{\mathcal{G}}[\vec{\Lambda}]\,,\,A^i_a(y)\right\} = \gamma\partial_a\Lambda^i(y) + \gamma\varepsilon^i_{jk}\Lambda^j(y)A^k_a(y) \qquad \left\{\vec{\mathcal{G}}[\vec{\Lambda}]\,,\,E^a_i(y)\right\} = \gamma\varepsilon_{ijk}\Lambda^j(y)E^{ak}(y) \tag{2.69}$$

$$\begin{aligned}\left\{\vec{\mathcal{G}}[\vec{\Lambda}_1]\,,\,\mathcal{G}_i\right\} &= \left\{\vec{\mathcal{G}}[\vec{\Lambda}_1]\,,\,\partial_a E^a + \varepsilon_{ijk}A^j_a E^{ak}\right\} = \left\{\vec{\mathcal{G}}[\vec{\Lambda}_1]\,,\,\partial_a E^a\right\} + \\ &+ \varepsilon_{ijk}\left\{\vec{\mathcal{G}}[\vec{\Lambda}_1]\,,\,A^j_a\right\}E^{ak} + \varepsilon_{ijk}A^j_a\left\{\vec{\mathcal{G}}[\vec{\Lambda}_1]\,,\,E^{ak}\right\}\end{aligned} \tag{2.70}$$

$$\left\{\vec{\mathcal{G}}[\vec{\Lambda}_1]\,,\,\mathcal{G}_i(y)\right\} = \gamma\varepsilon_{ijk}\Lambda^j(y)\mathcal{G}^k(y) \tag{2.71}$$

which after the smearing over y gives the algebra of the smeared Gauss constraint

$$\left\{\vec{\mathcal{G}}[\vec{\Lambda}_1]\,,\,\vec{\mathcal{G}}[\vec{\Lambda}_2]\right\} = \gamma\vec{\mathcal{G}}\left[\vec{\Lambda}_1 \times \vec{\Lambda}_2\right] \tag{2.72}$$

Once the problem is well posed from the point of view of the Hamiltonian analysis, the next step is to write a regularised version of the algebra generated by the conjugated variables and then compute the algebra of the constraints. In the ADM analysis the smearing is performed at the level of the constraints, but since we are now preparing the theory for quantisation we must perform the smearing directly on the conjugate variables defining the phase space: the pairs $(E^a_i\,, A^j_b)$.

2.5 Holonomy-Flux algebra

The regularisation of the algebra is a step that should not be underestimated. It is an important procedure which requires a lot of attention, since the structure that arises will constitutes the final version of the classical theory on which the Dirac's quantisation procedure will be implemented.

The variables which undergo the quantisation process do not involve directly the connection but rather its holonomy along a path; the resulting *holonomy-flux* algebra is one the pillar on which Loop Quantum Gravity is builded, since it gives rise to the phase space of loop gravity and its quantum counterpart is straightforward.

Holonomy: definition and properties

The holonomy of a field h has well known properties, nonetheless we think it is useful to recall them here. It can be defined as the solution of the following differential equation, which can be formally integrated through the path-order exponential

$$\frac{dh_\gamma(t)}{dt} = h_\gamma(t)A[\gamma(t)] \qquad\qquad h_\gamma = \mathcal{P}\exp\left[\int_0^1 dp\, A_a(x(p))\dot{\gamma}^a(p)\right] \tag{2.73}$$

The fundamental properties we are going to use are: its composition when one composes two or more paths and the action of the gauge transformations of the theory. All of them follows straightforwardly from its definition

$$h_{\gamma_2\circ\gamma_1}[A] = h_{\gamma_2}[A]h_{\gamma_1}[A] \tag{2.74}$$
$$(\mathcal{G} \triangleright h_\gamma) = g_{s(\gamma)}h_\gamma g^{-1}_{t(\gamma)} \tag{2.75}$$
$$h_\gamma[\phi A] = h_{\phi(\gamma)}[A] \tag{2.76}$$

where ϕ is a diffeomorphism and $s(\gamma)$ and $t(\gamma)$ are, respectively the source and the target of the path γ. Moreover, we can compute its functional derivative with respect to the field A:

$$\frac{\delta h_\gamma[A]}{\delta A^i_a(x)} = \frac{\delta}{\delta A^i_a(x)} \left(\mathcal{P} \exp \left[\int_0^1 dp\, A^j_b(x(p)) \dot{\gamma}^b(p) \tau_j \right] \right) =$$
$$= \delta^a_b \delta^j_i \int_0^1 ds\ h_\gamma(0,s) \dot{\gamma}^b(s) \delta^{(3)}(\gamma(s) - x) \tau_j h_\gamma(s,1) = h_\gamma(0,x) \dot{x}^a \tau_i h_\gamma(x,1) \tag{2.77}$$

Smearing the variables

In order to provide a regular version of the Poisson algebra for general relativity we need to smear our variables and their conjugate momenta over suitable test functions. Looking at their respective definitions (2.53) - (2.54) one understand that they have a different tensorial nature: the Ashtekar-Barbero connection is a $1-$form so it is natural to smear it along a 1D path, on the other hand the densitized triad is a $2-$form and it is natural to smear it over a 2D hypersurface.

At first one could think that integrating the connection over a path should work. However such smearing would produce a variable which does not transform covariantly under the $SU(2)$ gauge transformation. Beside it is useful to mention the Giles' result [8] who in 1980 showed that all the gauge invariant information carried by a gauge potential (connection) is stored inside its Wilson Loops. Which in different words means that if one knows all the Wilson loops h_γ, it is possible to reconstruct the gauge connection A from where these loops came from. This result, together with the impulse of the Ashtekar variables (1986) led Rovelli and Smolin to propose the *loop representation of General Relativity* [7] and eventually started the Loop Quantum Gravity program.

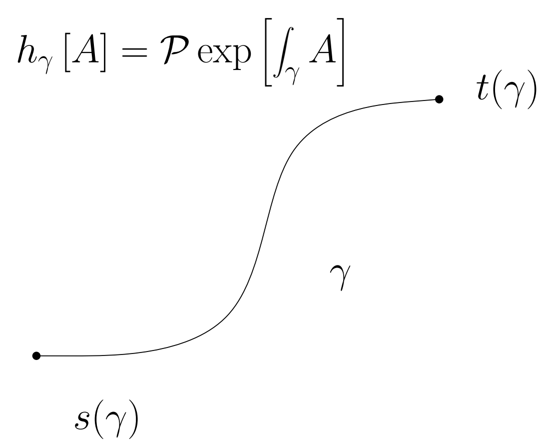
The object that suits our necessities is the holonomy of the Ashtekar-Barbero connection since it is the *path-ordered* line integral of the connection. The A^i_a are the components of the $SU(2)$ connection A_a so calling τ^i the generators we have

$$A_a = A^i_a \tau_i \in \mathfrak{su}(2) \tag{2.78a}$$
$$h_\gamma\,[A] = \mathcal{P} \exp \left[\int_\gamma A \right] = \mathcal{P} \exp \left[\int_0^1 dp\, A_a(x(p)) \dot{\gamma}^a(p) \right] \in SU(2) \tag{2.78b}$$

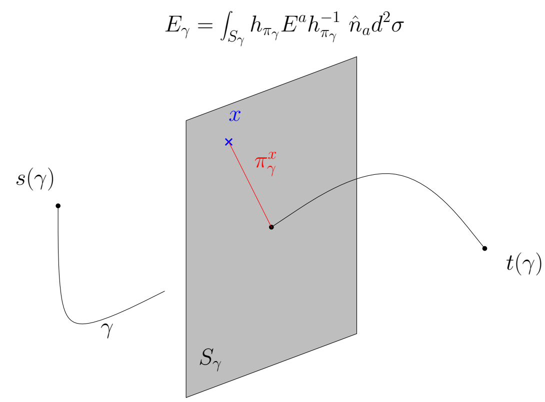
Concerning the smearing of the triad, there is a point which is often overlooked and that we would like to emphasise. Since it is a $2-$form, we choose to smear it across a surface. The first idea that comes into mind is to integrate it across a surface, but again, this would result in a variable which is not covariant under $SU(2)$ transformations. The way around this problem is to insert in the definition of the smeared triad the holonomy previously defined. In order to fully understand the construction, it is convenient to look at Fig. (2.2). Given a 1D path on which we smeared the connection and a point on this path $p \in \gamma$, we can define the 2D hypersurface S_γ orthogonal to the path γ at the point p. For each point in the surface $x \in S_\gamma$ we define a path π^x_γ which starts from the source of γ and ends at x.

$$\forall\, x \in S_\gamma \quad \exists\; \pi^x_\gamma : S_\gamma \times [0,1] \quad \to \quad \Sigma \qquad \text{and} \qquad \pi^x_\gamma(0) = s(\gamma) \quad \pi^x_\gamma(1) = x \tag{2.79}$$

It goes along the path γ, until it reaches the intersection point p and then goes from p to each point on the



(a) Since the A connection is a $1-$form the most natural way to smear it is along a line. We smear it along the path γ and chose its holonomy as variable since it carries all the gauge invariant information.



(b) The triad E is a $2-$form so the natural way to smear it is across a surface. Requesting the covariance under $SU(2)$ transformations for the smeared variables impose to define it through the holonomy h_γ so we choose to smear it across the surface S_γ "dual" to the path γ. The arising variable is the flux of the triad over the surface S_γ.

Figure 2.2: Smearing connection and triad respectively along a path and across a surface dual to the path.

surface $x \in S_\gamma$ keeping a direction tangential to the surface. On each one of this paths π^x_γ we can define the holonomy of the connection $h_{\pi^x_\gamma} = h_{\pi_\gamma}(x)$. We are ready to set up the smearing of the triad:

$$E_\gamma = E(S_\gamma) = \int_S h_{\pi_\gamma}(x) E^a(x) h^{-1}_{\pi_\gamma}(x) \hat{n}_a d^2\sigma \tag{2.80a}$$

$$\hat{n}_a = \varepsilon_{abc} \frac{\partial x^b}{\partial \sigma^1} \frac{\partial x^c}{\partial \sigma^2} \tag{2.80b}$$

$$\sigma^i = \left(\sigma^1\,,\,\sigma^2\right) \quad \text{coordinates over S} \tag{2.80c}$$

So the smeared variables associated to the triad's components are the fluxes across the 2D hypersurface S.

We completed the definition of the *Holonomy-Flux* algebra. It provide the regularisation of the Poisson algebra arising in the Hamiltonian analysis of the Holst's action for General Relativity.

$$A^i_a \quad \longrightarrow \quad h_\gamma = \mathcal{P}\exp\left[\int_\gamma A\right] \in SU(2) \tag{2.81}$$

$$E^a_i \quad \longrightarrow \quad E(S) = \int_S h_{\pi_\gamma} E^a h^{-1}_{\pi_\gamma}\ \hat{n}_a d^2\sigma \in \mathfrak{su}(2) \tag{2.82}$$

Phase-space structure

Thanks to the smearing procedure our theory can be described by a couple of variables (h, E) with values respectively in $SU(2)$ and $\mathfrak{su}(2)$. Since we are going to perform the Hamiltonian analysis it is legitimate to ask what kind of phase space is defined by the Holonomy and Flux variables.

In order to answer, the first set of notions that we need, is the symplectic structure over a Lie group. For all the details we refer to [104, 105]. Briefly, a Lie group G can be seen as a manifold, its tangent space $T_eG \cong \mathfrak{g}$ is the algebra and then its cotangent space is $T^*_eG = G \times \mathfrak{g}^*$, where $\mathfrak{g}^*$ is the dual of the algebra. From now on we will deal only with the group we are interested in $G = SU(2)$.

For $SU(2)$, it is possible to define a linear action that identify $\mathfrak{su}(2)$ with its dual $\mathfrak{su}(2)^*$ which means that we can use elements of $\mathfrak{su}(2)$ to parametrise its dual. Furthermore it is possible to find a symplectic potential that trivialise the $SU(2)$ cotangent bundle in the following way:

$$SU(2) \times \mathfrak{su}(2) \qquad \mapsto \qquad T^*SU(2) \tag{2.83}$$

In other words, thanks to the definition of the Holonomy and Flux variables as elements of $SU(2)$ and $\mathfrak{su}(2)$, respectively, and to the theory of the symplectic structures over the Lie groups we understand that the phase space we were looking for is the $SU(2)$ *cotangent bundle*: $T^*SU(2)$.

$$(h, E) \in T^*SU(2) \tag{2.84}$$

Chapter 3

Canonical Loop Quantum Gravity

The efforts that have been done to understand the Hamiltonian structure of first order formulation of general relativity aimed to the purpose of putting the theory in such a way that it is amenable to quantisation. The quantisation of the second order formulation, with the ADM variables, has already been achieved but it led to inconsistency and ill-defined equations such as the famous Wheeler-deWitt equation. There is a fundamental reason for that: the metric is a full dynamical quantity and when one try to quantise the theory one lacks of a notion of fixed background with respect to which, one can defines the scalar product on the arising space. A proposal to overcome such an obstacle comes from Loop quantum gravity, simply by changing the variables in which the classical theory is formulated. As we are going to show, the internal Lorentz symmetry, the choice of the Ashtekar-Barbero variables and the switch to Holonomy and Flux variables allow to correctly define the space of the states and a scalar product between its elements so defining an Hilbert space where quantisation of the space-time can be achieved.

3.1 Dirac's quantisation procedure

The canonical analysis performed in the previous chapter led to the following formulation of General Relativity, in terms of $SU(2)$ variables, as a *totally constrained Hamiltonian system*:

Canonical Variables
$(E_i^a \, , \, A_b^j)$
$\left\{A_a^i(x) \, , \, E_j^b(y)\right\} = \gamma \delta_j^i \delta_b^a \delta^3 \left(x - y\right)$

Constraints	Gauge symmetry
$\mathcal{G}_i = 0$	$\mathrm{Gauss} - SU(2)$
$H_\mu = 0$	Diffeomorphisms

The explicit expressions of the constraints equations can be found in (2.59) - (3.74) - (2.57).

Both the classical treatment and the quantisation of this kind of systems is mainly due to the work of Dirac and Bergmann [110, 111, 112] and it is far more general then quantum gravity. Its application to gauge theories like QED it is now matter for standard textbooks [136]. The core procedure to quantise a constrained Hamiltonian system can be summarised in the following steps:

$i)$ Promote your canonical variables to operators that satisfy commuting relation analogue to the classical Poisson brackets, with a Schroedinger representation, then consider the space of the functionals of the connection $\mathcal{H} = L^2\left[A\right]$

$$\left[\begin{array}{ccc} A_i^a & \to & \hat{A}_i^a \\ E_b^i & \to & \hat{E}_b^i \end{array}\right] \qquad \text{and} \qquad \{\cdot\,,\,\cdot\} \to \frac{1}{i\hbar}\left[\cdot\,,\,\cdot\right]$$

$$\hat{A}_i^a \psi\left[A\right] = A_i^a \psi\left[A\right] \tag{3.1}$$

$$\hat{E}_a^i \psi\left[A\right] = -i\hbar\gamma \frac{\delta}{\delta A_i^a}\psi\left[A\right] \tag{3.2}$$

This means that each constraint is promoted to a constraint operator

$$\mathcal{G}_i\left[A_i^a, E_a^i\right] \quad \to \quad \hat{\mathcal{G}}_i\left[\hat{A}_i^a, \hat{E}_a^i\right] \qquad\qquad H_\mu\left[A_i^a, E_a^i\right] \quad \to \quad \hat{H}_\mu\left[\hat{A}_i^a, \hat{E}_a^i\right] \tag{3.3}$$

$ii)$ The characterisation of the solutions of the equations of motion as the elements of the hypersurfaces defined by the constraints equations inside the phase space translate in the following manner. Define an auxiliary Hilbert space called $\mathcal{H}_{\rm kin}$ on which all the operators can naturally act. On such define the "physical" Hilbert space as the set of functionals over the connection that annihilate all the constraints operators:

$$\mathcal{H}_{kin} \quad \stackrel{\hat{\mathcal{G}}\psi=0}{\longrightarrow} \quad \mathcal{H}_{kin}^{\mathcal{G}} \quad \stackrel{\hat{H}_a\psi=0}{\longrightarrow} \quad \mathcal{H}_{diff} \quad \stackrel{\hat{H}\psi=0}{\longrightarrow} \quad \mathcal{H}_{phys} \tag{3.4}$$

$$\psi[A] \in \mathcal{H}_{phys} \quad \Longleftrightarrow \quad \hat{\mathcal{G}}_i\psi[A] = \hat{H}_\mu\psi[A] = 0 \tag{3.5}$$

The structure seems straightforward but there are some *caveat*:

- Without a notion of integration measure we lack of the inner product so the space $L^2[A]$ will not be an Hilbert space until we define such measure. The answer to this problem was found by Ashtekar and Lewandoski [61, 62, 63, 64] and it is called the Ashtekar-Lewandoski measure $d\mu_{AL}$. The key notions are the *cylindrical function* and the *Haar measure* over a group.
- The characterisation of the physical states as the ones annihilating the constraint operator require a further step if the algebra of the constraint is second class i.e. when some constraints have non-vanishing Poisson brackets. In this case there are two directions that can be taken, at the classical level:
  1. Explicitly solve the second class constraint and put the solution back in the theory, hopefully ending up with a first class algebra
  2. Replace the standard Poisson brackets with the Dirac brackets [110] and forget about the second class constraints since Dirac brackets are vanishing even for second class constraints. The quantisation can proceed as previously stated since, with respect to Dirac brackets the algebra will be first class.

The key notion that is necessary to achieve the quantisation and to have a well defined notion of scalar product is the notion of *cylindrical functions*.

The kinematical Hilbert space

For our purposes one can consider a cylindrical function as a functional of the connection A, that depends on A only via its value over a certain subset of elements, for example a 1D paths. So if we consider functions of the holonomy $h_\gamma[A]$ these are going to be cylindrical functions of the connections. Consider a *graph* Γ as an ordered collection of piecewise differentiables and oriented paths called *links* $l_i \subset \Sigma$ and $i = 1, \ldots, L \in \mathbb{N}$. The meeting points of links are called *nodes* $v_n \in \Sigma$ and $n = 1, \ldots, N \in \mathbb{V}$.

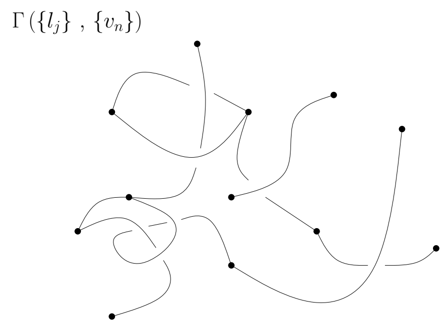


Figure 3.1: A graph Γ is a combinatorial structure made of a collection of *links* meeting at *nodes*

One uses the defined graph to probe the value of the connection and define over each link $l = 1, \ldots, L$ the holonomy of the connection $h_l[A]$. A cylindrical function f over the graph Γ assign to each $L-$tuple of holonomies a complex number $f(h_1, \ldots, h_L)$ and it is characterised by its graph and the function f.

$$f \; : \; SU(2)^L \mapsto \mathbb{C} \tag{3.6}$$
$$\langle A| \; \Gamma, f\rangle = \psi_{\Gamma,f}[A] = f(h_{l_1}[A] \; \ldots \; , h_{l_L}[A]) \tag{3.7}$$

Since ψ has support only on $SU(2)^L$, it is possible to define an integration measure on the space of the cylindrical functions through the notion of measure over a group: the *Haar measure*. We equip the space of the cylindrical functions Cyl_Γ with the scalar product inherited by the Haar measure and this Cyl_Γ into an Hilbert space:

$$\mathcal{H}_\Gamma = (\mathrm{Cyl}_\Gamma, d\mu_H) \qquad |\Gamma, f_1\rangle \; , \; |\Gamma, f_2\rangle \in \mathrm{Cyl}_\Gamma \tag{3.8}$$
$$\langle \Gamma, f_1 | \Gamma, f_2\rangle = \int_{SU(2)} d^L\mu_H \left(\overline{f_1(h_{l_1}[A], \ldots, h_{l_L}[A])} f_2(h_{l_1}[A], \ldots, h_{l_L}[A]) \right)$$

Finally, the kinematical Hilbert space is the direct sum[1] of the Hilbert space on each graph

$$\mathcal{H}_{kin} = \bigoplus_{\Gamma \subset \Sigma} \mathcal{H}_\Gamma \tag{3.9}$$

[1]Strictly speaking it is the projective limit. It is equivalent to a direct sum if one consider abstract graphs and the cylindrical consistency condition holds

The remarkable result achieved by Ashtekar and Lewandoski was to prove that it is possible to interpret $\mathcal{H}_{kin}$ exactly as the space of the square-integrable functional over the connection that one need to consider for quantisation purposes. Over this space the scalar product is defined through the integration measure inherited by the Haar measure and is called *Ashtekar-Lewandoski measure*:

$$\mathcal{H}_{kin} = L^2\left[A, d\mu_{AL}\right] \qquad |\psi_{\Gamma_1,f_1}\rangle\ ,\ |\psi_{\Gamma_2,f_2}\rangle \in \mathcal{H}_{kin} \tag{3.10}$$

$$\langle\psi_{\Gamma_1,f_1}|\psi_{\Gamma_2,f_2}\rangle = \int d\mu_{AL}\overline{\psi_{\Gamma_1,f_1}[A]}\psi_{\Gamma_2,f_2}[A] \tag{3.11}$$

The next steps are to promote our variables to operators, define a Schroedinger representation for connection and densitized triad then find a representation of the Holonomy-Flux algebra on $\mathcal{H}_{kin}$:

$$\hat{h}\psi\left[A\right] = h\psi\left[A\right] \tag{3.12}$$

$$\hat{E}^i_a\psi\left[A\right] = -i\hbar\gamma\frac{\delta}{\delta A^a_i}\psi\left[A\right] \tag{3.13}$$

Mimicking the strategy one usually adopt in quantum mechanics, the best way to solve the problem is to find a basis of the Hilbert space and approach the problem on each element of the basis. Since our variables lives in the group, this procedure should carried out carefully but the solution to the problem is a well known result in the theory of harmonic analysis applied to topological compact groups and goes under the name of *Peter-Weyl theorem*.

Peter-Weyl theorem: a glimpse. The content of the Peter-Weyl theorem is made of three statement. The first part is the analog for compact groups of the Stone-Weierstrass theorem that allow to approximate continuous function with polynomials. The second and the third part are the ones we are going to need since they regard the decomposition of unitary representations in finite-dimensional representations. Indeed, the second statement is a proof of the existence of such a decomposition, for compact groups.

The third statement is the most important for our purpose since it states that over each compact group G there is a natural Hilbert space of square-integrable functions $L^2[G]$. This is always true since the Haar measure on compact groups can always be defined. The theorem also provide an explicit decomposition of the unitary representations, exploiting the first statement. Before we explicitly use this result, it is useful to understand it in the simple case of $G = U(1)$.

$$g \in U(1) \quad \Rightarrow \qquad D^{(n)}(g) = e^{in\alpha} \qquad\qquad \alpha \in [0, 2\pi] \tag{3.14a}$$

$$\int_{U(1)} d\mu_H = 1 \qquad d\mu_H = \frac{d\alpha}{2\pi} \qquad \int_{U(1)} d\mu_H D^{(n)}(g)\bar{D}^{(n')}(g) = \delta^{nn'} \tag{3.14b}$$

$$L^2[U(1), d\mu_H] \cong L^2[\mathcal{S}^1, \frac{d\alpha}{2\pi}] \qquad \int_{U(1)} d\mu_H \quad \mapsto \quad \frac{1}{2\pi}\int_{\mathcal{S}^1} d\alpha \tag{3.14c}$$

Then the Peter-Weyl theorem reads

$$f\ :\ U(1) \quad \mapsto \quad \mathbb{C} \quad \begin{cases} f(g) = \sum_n f_n D^{(n)}(g) \\ \\ f_n = \int_{U(1)} d\mu_H f(g)\bar{D}^{(n)}(g) \end{cases} \tag{3.15}$$

For our purpose the theorem is important since it states that a basis for the Hilbert space of the square integrable functions over a $SU(2)$, $L^2[G, d\mu_{\rm H}]$, is given by the matrix elements of the unitary and irreducible representations: the *Wigner matrices* $D^{(j)}_{mn}(g)$. In this sense, the key property of the Wigner matrices is the following:

$$\int_{SU(2)} d\mu_H \; D^{(j)}_{mn}(g)\bar{D}^{j'}_{m'n'}(g) = \frac{1}{d_j}\delta^{jj'}\delta_{mm'}\delta_{nn'} \qquad d_j = 2j+1 \tag{3.16}$$

Which means that we can always expand any function of the group in terms of them:

$$\begin{array}{cccc} g \in SU(2) & \psi_{\Gamma,f}[A] \in \mathcal{H}_\Gamma & \text{and} & j = 0, \frac{1}{2}, 1, \dots \\ f \in L^2[SU(2), d\mu_H] & f(g) = \sum_j f^{(j)}_{mn} D^{(j)}_{mn}(g) & & m, n = -j, \dots, j \end{array} \tag{3.17}$$

Since $\mathcal{H}_\Gamma = L^2[SU(2), d\mu_H]^L$, a basis on this space is just the tensor product of L representations of $SU(2)$ and each $\psi_{\Gamma,f}[A]$ can be expanded in this basis:

$$\langle A | j_l, m_l, n_l \rangle = D^{(j_1)}_{m_1 n_1}(h_1) D^{(j_2)}_{m_2 n_2}(h_2) \dots D^{(j_L)}_{m_L n_L}(h_L) \tag{3.18}$$

$$\psi_{\Gamma,f}[A] = \sum_{j_l, m_l, n_l} f^{j_1,\dots,j_L}_{m_1 n_1,\dots,m_L,n_L} \; |\Gamma, j_l, m_l, n_l\rangle \tag{3.19}$$

Now the problem to find a representation of the Holonomy-Flux algebra over $\mathcal{H}_\Gamma$ is reduced to give its action over the basis elements and it is possible to define it as a Schroedinger representation in which the Holonomy acts by multiplication and the Flux by derivative. Consider a generic representation $h_e = D^{(j)}(h_e)$:

$$\hat{h}_l h_e[A] = h_l[A] h_e[A] \tag{3.20}$$

$$\hat{E}^i(S) h_e[A] = -i\hbar\gamma \int_S d^2\eta \, n^b \frac{\delta h_e}{\delta A^b_i} = \pm i\hbar\gamma \; h_{e_1}[A] J^i h_{e_1}[A] \tag{3.21}$$

If the intersection between the surface S and the link e is not empty, the action of the Flux across the surface S splits the holonomy in two parts, at the point $p = S \cap e$, h_{e_1} and h_{e_2}, and amount to insert the $SU(2)$ generators J^i among them. If the intersection is empty the action of the flux vanishes since the derivative is null. Finally, the sign is defined by the relative orientation between the surface S and the link e. If the point p is the source or the target of the link we have the following expression

$$\hat{E}^i_{s(e)}(S) h_e = \pm i\hbar\gamma \; J^i h_e \tag{3.22}$$

$$\hat{E}^i_{t(e)}(S) h_e = \pm i\hbar\gamma \; h_e J^i \tag{3.23}$$

With the definition of a Schroedinger representation for the Holonomy-Flux algebra we fulfilled the definition of the kinematical Hilbert space for Loop Quantum Gravity. Following the Dirac's prescription the physical states are characterised as those annihilating the quantum constraints.

3.2 Spin-networks

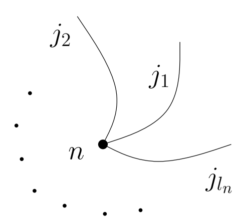
We start characterising the solutions of the quantum Gauss constraint, these are called *spin-networks* and are defined as the $SU(2)$-invariant elements of the kinematical Hilbert space. We start recalling how the $SU(2)$ gauge symmetry acts on the Holonomy:

$$(g \triangleright h_l) = g_{s(l)} h_l g^{-1}_{t(l)} \tag{3.24}$$

Where $s(l)$ and $t(l)$ are the source and the target of the link: $s(l) = \gamma_l(0)$ and $t(l) = \gamma_l(1)$. This means that after the smearing of the algebra, the gauge transformations act only on the nodes of the graph, which is exactly what usually happens in lattice QCD so the gauge group of the smeared theory is $SU(2)^V$ where V is the number of nodes. The invariance of a state under $SU(2)$ can be achieved through the *group averaging* procedure

$$\forall f \in \mathrm{Cyl}_\Gamma \qquad f_0(h_1, \dots, h_L) = \int \prod_n dg_n f(g_{s_1} h_1 g^{-1}_{t_1}, \dots, g_{s_L} h_L g^{-1}_{t_L}) \tag{3.25}$$

and f_0 is clearly invariant under the $SU(2)$ action defined by (3.24). Since the gauge transformations act only on the nodes, it is possible to implement the group average procedure on each graph via the application of a projector $\mathcal{P}$ at each node, its explicit expression is going to depend on the number of links around the node under consideration:



$$\mathcal{P} = \int dg \prod_{l \in n} D^{(j_l)}(g) \tag{3.26}$$

A basis for the Hilbert space is given by the Wigner matrices at each link. Following the combinatorial information on the graph Γ, given a node n we know what representations "meet" at the node n; the insertion of the projector amounts to look at the invariant part of the tensor product among the Wigner matrices defined on the links around the node. We would like to point out that what we are doing is just a sum of angular momenta, indeed the Wigner matrices are linear combinations of the Clebsh-Gordan coefficients:

$$\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} \mathcal{H}_J \tag{3.27}$$

The tensor product of different irreducible representations can be reduced to a sum over the irreducible representations and in this sum we are looking for its *singlet state*:

$$\bigotimes_{l \in n} D^{(j_l)}(h_l) \in \bigotimes_l \mathcal{H}^{(j_l)} = \bigoplus_j \mathcal{H}^{(j)} \quad \xrightarrow{\mathcal{P}} \quad \mathcal{K}^{(0)}_n \tag{3.28}$$

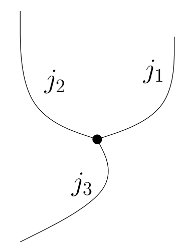
$$\mathcal{K}^{(0)}_n = \left[\mathcal{H}^{(j_1)} \otimes \ldots \otimes \mathcal{H}^{(j_{l_n})}\right]_{SU(2)\mathrm{Inv}} \tag{3.29}$$

The elements i_k a basis of this $\mathcal{H}_n^{(0)}$ and i_k^* its dual are called intertwiners and usually the projector is written in term of these objects

$$\mathcal{P} = \sum_{k=1}^{\dim \mathcal{H}_n^{(0)}} i_k i_k^* \tag{3.30}$$

The dimension of this space clearly depend on the number of Wigner matrices one contracts and the three-valent case is the simplest one since the dimension of the *single state* is 1. However in general this is not going to be a fixed number and it will depend on the details of the graph and for this reason, at the gauge invariant level, we need one more "quantum number" on each node to uniquely identify the quantum state, we call it the *intertwiner* quantum number.

Three-valent node



$$\mathcal{P}^{(3)} = \int dg D^{(j_1)}_{m_1 k_1}(g) D^{(j_2)}_{m_2 k_2}(g) D^{(j_3)}_{m_3 k_3}(g) =$$
$$= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \overline{\begin{pmatrix} j_1 & j_2 & j_3 \\ k_1 & k_2 & k_3 \end{pmatrix}} = i_{m_1 m_2 m_3} i^*_{k_1 k_2 k_3}$$

Where i_{abc} is the $3-$valent *Intertwiner* and it is represented by the Wigner $3j - m$ symbol, which is different from zero only if the triangular inequalities hold:

$$\mathcal{H}^{(j_1)} \otimes \mathcal{H}^{(j_2)} \otimes \mathcal{H}^{(j_3)} = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} \mathcal{H}^{(J)} \otimes \mathcal{H}^{(j_3)} = \bigoplus_{J=|j_1-j_2|}^{j_1+j_2} \bigoplus_{K=|J-j_3|}^{J+j_3} \mathcal{H}^{(K)} \tag{3.31}$$

since we are looking for the singlet

$$K = 0 \quad \Rightarrow \quad J = j_3 \quad \Rightarrow \quad |j_1 - j_2| \leq j_3 \leq j_1 + j_2 \tag{3.32}$$

Which means that either these inequalities hold or the invariant part is not in the decomposition.

$$\mathcal{H}_3^{(0)} = \left[\mathcal{H}^{(j_1)} \otimes \mathcal{H}^{(j_2)} \otimes \mathcal{H}^{(j_3)}\right]_{SU(2)-\mathrm{Inv}} = \mathcal{P}^{(3)} \left[\mathcal{H}^{(j_1)} \otimes \mathcal{H}^{(j_2)} \otimes \mathcal{H}^{(j_3)}\right] \tag{3.33}$$
$$\left[D^{(j_1)}_{m_1 k_1} D^{(j_2)}_{m_2 k_2} D^{(j_3)}_{m_3 k_3}\right]_{SU(2)-\mathrm{Inv}} = \mathcal{P}^{(3)}_{m_1 m_2 m_3, a_1 a_2 a_3} D^{(j_1)}_{a_1 k_1} D^{(j_2)}_{a_2 k_2} D^{(j_3)}_{a_3 k_3} =$$
$$= i_{m_1 m_2 m_3} i^*_{a_1 a_2 a_3} D^{(j_1)}_{a_1 k_1}(h_1) D^{(j_2)}_{a_2 k_2}(h_2) D^{(j_3)}_{a_3 k_3}(h_3)$$

Finally, for a cylindrical function we have that its invariant part is

$$f(h_1,h_2,h_3) = \sum_{j_1,j_2 j_3} f^{j_1,j_2,j_3}_{m_1 m_2 m_3,k_1 k_2 k_3} D^{(j_1)}_{m_1 k_1}(h_1) D^{(j_2)}_{m_2 k_2}(h_2) D^{(j_3)}_{m_3 k_3}(h_3) \tag{3.34}$$

$$f(h_1,h_2,h_3) \quad \overset{\mathcal{P}^{(3)}}{\longmapsto} \quad f_0(h_1,h_2,h_3) = \sum_{j_1,j_2,j_3} f^{j_1,j_2,j_3}_{m_1 m_2 m_3,k_1 k_2 k_3} \left[D^{(j_1)}_{m_1 k_1} D^{(j_2)}_{m_2 k_2} D^{(j_3)}_{m_3 k_3} \right]_{SU(2)-\mathrm{Inv}}$$

$$= \sum_{j_1,j_2,j_3} f^{j_1,j_2,j_3}_{m_1 m_2 m_3,k_1 k_2 k_3} i_{m_1 m_2 m_3} i^*_{a_1 a_2 a_3} D^{(j_1)}_{a_1 k_1}(h_1) D^{(j_2)}_{a_2 k_2}(h_2) D^{(j_3)}_{a_3 k_3}(h_3) =$$

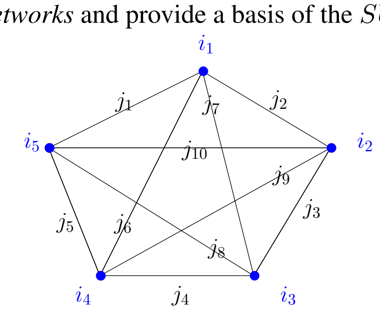
$$= \sum_{j_1,j_2,j_3} f^{j_1,j_2,j_3} \prod_{l \in n} D^{(j_l)}(h_l)\ i_n \tag{3.35}$$

Where the "magnetic" indices are hidden for simplicity. In general, given a graph with L links and V nodes we have

$$f(h_1,\ldots,h_L) \in \mathcal{H}_\Gamma \tag{3.36}$$

$$f_0(h_1,\ldots,h_L) = \sum_{j_1,\ldots,j_L} f^{j_1,\ldots,j_L} \prod_l D^{(j_l)} \prod_n i_n \tag{3.37}$$

The states are characterised by cylindrical functions over a graph Γ, coloured with an irreducible representation j_l of $SU(2)$ on each link and with an element i_n of the space of the intertwiners $\mathcal{K}^{(0)}_n$ at each node are called *spin-networks* and provide a basis of the $SU(2)$-gauge invariant Hilbert space $\mathcal{H}^{\mathcal{G}}_{kin}$:



$$\psi_{\Gamma,j_l,i_n}[A] = \langle A|\Gamma, j_l, i_n\rangle = \bigotimes_l D^{(j_l)}(h_l[A]) \bigotimes_n i_n \tag{3.38}$$

As before, different graphs select orthogonal subspaces so we write the following equations for our gauge-invariant Hilbert space:

$$\mathcal{H}^{\mathcal{G}}_{kin} = \bigoplus_\Gamma \mathcal{H}^{\mathcal{G}}_\Gamma \tag{3.39}$$

$$\mathcal{H}^{\mathcal{G}}_\Gamma = L^2\left[SU(2)^L/SU(2)^V, d\mu_H\right] = \bigoplus_{j_l} \left(\bigotimes_n \mathcal{K}^{(0)}_n \right) \tag{3.40}$$

$$\mathcal{K}^{(0)}_n = \left[\bigotimes_{l \in n} \mathcal{H}^{(j_l)} \right]_{SU(2)-\mathrm{Inv}} \tag{3.41}$$

The spin-network states are kinematical objects since in order to solve the theory we need to find the solutions of the diffeomorphism $\hat{H}_a$ and the Hamiltonian $\hat{H}$ constraint. Nonetheless, the phenomenology that arise is quite rich since they actually achieve the quantisation of space-time and we can now compute the spectra of the geometric operator such as the Area and Volume operators.

3.3 Space-time quantisation

The definition of spin-networks as a basis for the gauge-invariant phase space is one of the most interesting result of Loop Quantum Gravity since it achieves the quantisation of the space. As we will show in this section, to each spin-network states it is possible to attach a notion of discrete geometry in terms of polyhedra that shows that the geometry of the space is discrete at the Planck scale. The arising notion of geometry over a graph defines a *quantum geometry*, whose classical counterpart has been recently understood in terms of *twisted geometries*, where each link is dual to a face with area proportional to the attached representation j_l and each node is dual to a polyhedron with volume proportional to the associated intertwiner quantum number i_n. Before we turn to the dynamics we would like to make this relation explicit, by showing how quanta of area and volume arise from the spin-networks.

3.3.1 Quanta of area

In classical differential geometry the area of a curved surface is defined via the following surface integral, that can be easily written in terms of our variables:

$$A(S) = \int_S d\eta_1 d\eta_2 \sqrt{\det\left[(\partial_\alpha \vec{x}) \cdot (\partial_\beta \vec{x})\right]} = \int_S d\eta_1 d\eta_2 \sqrt{g} |\vec{n}| \tag{3.42}$$

$$\det\left[(\partial_\alpha \vec{x}) \cdot (\partial_\beta \vec{x})\right] = \det\left[g_{ab} \partial_\alpha x^a \partial_\beta x^b\right] = \tag{3.43}$$

$$= g_{ab} g_{cd} \left[\partial_1 x^a \partial_1 x^b \partial_2 x^c \partial_2 x^c - \partial_1 x^a \partial_2 x^b \partial_2 x^c \partial_1 x^c\right] = g\, g^{ab} n_a n_b = E^a_i E^{bi} n_a n_b \tag{3.44}$$

where

$$g = \det\left[g_{ab}\right] \qquad\qquad n_a = \varepsilon_{abc} \partial_1 x^b \partial_2 x^c \tag{3.45}$$

So we have

$$A(S) = \int_S d\eta_1 d\eta_2 \sqrt{E_i^a E^{bi} n_a n_b} \tag{3.46}$$

In order to correctly find the quantum version of this operator the strategy is to triangulate the surface in N two dimensional cells $S_k \quad k = 1, \ldots, N$ and write the integral as the limit of the Riemann sum:

$$A_N(S) = \sum_{k=1}^{N} \sqrt{E_i(S_k) E^i(S_k)} \qquad\qquad A(S) = \lim_{N\to\infty} A_N(S) \tag{3.47}$$

Since $E_i(S_k)$ is the flux of the triad over the cell k we can define the quantum area operator simply by replacing the fluxes with its quantum version:

$$\hat{A}(S) = \lim_{N\to\infty} \hat{A}_N(S) = \lim_{N\to\infty} \sum_{k=1}^{N} \sqrt{\hat{E}^i(S_k)\hat{E}_i(S_k)} \tag{3.48}$$

We also know that the action of the flux operator vanishes if there is no link that intersect the surface S_k, which means that beyond the refinement in which each cell intersect only one link, the value of the operator does not change so the limit can be easily performed.

We study the action of the square area operator, over a spin-network, by decomposing the state in its basis and studying the action of the flux on each basis "vector".

$$\hat{E}^i(S)\left(\hat{E}_i(S) D^{(j_l)}(h_l)\right) = \hat{E}^i(S)\left(\pm i\hbar\gamma\ h_{l_1} J_i h_{l_2}\right) = \left(\pm i\hbar\gamma\right)^2\ h_{l_1} J^i J_i h_{l_2} \tag{3.49a}$$

$$J^i J_i = \mathcal{C}^{(2)}_{SU(2)} = -j(j+1) \tag{3.49b}$$

$$\hat{E}^i(S)\hat{E}_i(S) D^{(j_l)}(h_l) = \gamma^2\hbar^2 j_l(j_l+1) \tag{3.49c}$$

$$\hat{A}(S)\psi_{\Gamma,j_l,i_n}[A] = \sqrt{\hat{E}_i(S)\hat{E}^i(S)}\psi_{\Gamma,j_l,i_n}[A] = \sum_{l\,:\,l\cap S\neq\varnothing} 8\pi\gamma L_p^2 \sqrt{j_l(j_l+1)}\psi_{\Gamma,j_l,i_n}[A] \tag{3.50}$$

Where in the last equality we restored all the constants.

From the loop quantisation technique we obtained that the area operator is quantised and it is diagonal in the spin-network basis. Its spectrum is discrete, with minimal excitation proportional to the square of the Planck length $L_p^2 = \frac{\hbar G}{c^3} \sim 10^{-66}\mathrm{cm}^2$. Moreover, since the action of the flux on the spin-network vanishes if there is no-link intersecting the surface we conclude that in Loop Quantum Gravity a surface S acquire an area depending only if it is *punctured*[2] by some link.

[2] A puncture is a link intersecting the surface

Thanks to this analysis we can interpret the flux operator as the quantum counterpart of the "area vector" which is normal to the surface. This is more then just an analogy [69], indeed the gauge symmetry of the theory, after the smearing, is an $SU(2)$ on each node and the constraint generates the rotations around the node so it can be written as the sum of all the $SU(2)$ generators around a node:

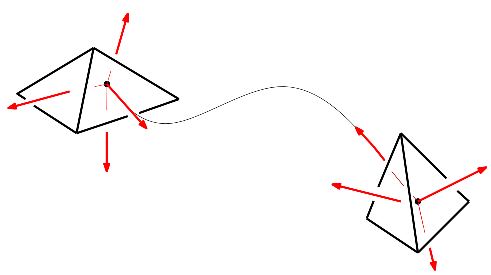
$$\hat{\mathcal{G}}_i^n = \sum_{l\in n} \hat{E}_i^{(l)} = \sum_{l\in n} \hat{E}_i(S_l) \tag{3.51}$$

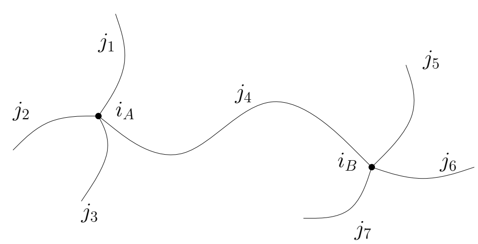
The spin-network states are, by the definitions, invariant under the action the full $SU(2)^V$ gauge generator so, on each node we can write the constraint equation as

$$\sum_{l\in n} \hat{E}_i(S_l)\ |i_n\rangle = 0 \qquad\qquad |i_n\rangle \in \mathcal{K}_n^{(0)} \tag{3.52}$$

Strictly connected to this relation is an old theorem proven by Minkowski in 1897 stating that if you have n non-planar vector that sum to zero, then it is possible to identify them as the normal to the faces of a polyhedron with n faces. Thanks to this relation Bianchi, Donà and Speziale [69] showed that it is possible to bring in a geometrical interpretation of the nodes in a graph as bounded convex polyhedra in $\mathbb{R}^3$: a polyhedron uniquely described by the areas and normals to its faces. This correspondence allows to identify an intertwiner with the state of a quantum polyhedron.

The geometrical interpretation of the spin-network states is now complete and it is expressed setting a duality relation among a graph and its dual picture: each node with n links is dual to a polyhedron with n faces and each link connecting two nodes identify the surface shared by the two polyhedra at each nodes.





3.3.2 Quanta of volume

The volume operator is more involved and a general expression does not exists, nonetheless it is always possible to compute its spectrum (a general algorithm exists) and extract some general properties. In literature there are two main definitions: one is due to Rovelli and Smolin[70] while the other one to Ashtekar and Lewandoski[71]. Both definitions act non-trivially only on the nodes and they agree on the three-valent and on the four-valent node, up to a constant. Here we are going to present only the Rovelli-Smolin operator. From the definition of the triad we have that they account for the spatial part of the 4D metric so it is possible to rewrite the volume element exploiting the densitized triad:

$$h_{ab} = e^i_a e^j_b \delta_{ij} \qquad E^a_i = \sqrt{\det E} e^a_i = \det(e) e^a_i \tag{3.53}$$

$$h = \det\ (h_{ab}) = \det\ (e)^2 \qquad \sqrt{h} = \sqrt{e^2} = \sqrt{|\det E|} \tag{3.54}$$

Thanks to these manipulations it is possible to write the volume as

$$V(\Sigma) = \int_\Sigma \sqrt{h}\, d^3x = \int_\Sigma \sqrt{|\det E|} d^3x \tag{3.55}$$

The strategy is the same as before: one chop the region Σ in three-dimensional cubic cells and refine the decomposition until each cell contain at most one node, thus the integral can be replaced by the Riemann sum over the cells. On each cell C_n it is easy to write the volume in term of the area normals:

$$V_n^2 = \frac{1}{48} \int_{\partial C_n} d^2\eta_1 \int_{\partial C_n} d^2\eta_2 \int_{\partial C_n} d^2\eta_3 |\varepsilon_{ijk} E^a_i(\sigma_1) n_a(\sigma_1) E^b_j(\sigma_2) n_b(\sigma_2) E^c_k(\sigma_3) n_c(\sigma_3)| \tag{3.56}$$

and in the limit in which each cell become a point we have:

$$V_n^2 \overset{\epsilon\to 0}{\longmapsto} \frac{1}{48} \varepsilon^{abc} n_a n_b n_c \det E^a_i(x) \epsilon^6 \simeq \det E^a_i(x) \epsilon^6 \simeq V^2(C_n) \tag{3.57}$$

So we have an expression of the volume in terms of the fluxes and we can turn it into a operator:

$$\hat{V}_{RS} = \lim_{\epsilon\to 0} \sum_n \sqrt{\frac{1}{48} \sum_{A,B,C} \epsilon_{ijk} \hat{E}_i(S_n^A) \hat{E}_j(S_n^B) \hat{E}_k(S_n^C)} \tag{3.58}$$

As previously mentioned the optimal refinement is reached when there is one node inside each cell and the partition induced in the boundary must be refined consistently with the definition of the area operator, i.e. each $S_n = \partial C_n$ has at most one puncture. From such a definition of the volume operator we can see that to have a non-vanishing volume the three fluxes must be different, due to the presence of the ε_{ijk}, which means that the volume operator acts non-trivially only at the nodes, since on the links we have at most the two fluxes associated with the source and the target.

We can now focus on the action of the operator on the three-valent node. From the Gauss law we know that the sum of the fluxes across the surface around a node must be zero:

$$\left(\hat{E}^1_i + \hat{E}^2_i + \hat{E}^3_i\right) |i_n\rangle = 0 \tag{3.59}$$

$$\epsilon_{ijk} \hat{E}_i(S_n^A) \hat{E}_j(S_n^B) \hat{E}_k(S_n^C)\ |i_n\rangle = -\epsilon_{ijk} \left(\hat{E}_i(S_n^B) + \hat{E}_i(S_n^C)\right) \hat{E}_j(S_n^B) \hat{E}_k(S_n^C) = 0 \tag{3.60}$$

Which means that the three-valent node has zero volume. This result agrees with the interpretation of the nodes in terms of polyhedra since there is no-polyhedra with three faces and the only way we can think of it is a tetrahedron with one face that shrinks to zero, which is actually a triangle whose volume is indeed zero.

Now we turn to the four-valent case. The contributions of all the terms are equal, so we gets

$$\left(\hat{E}_i^1 + \hat{E}_i^2 + \hat{E}_i^3 + \hat{E}_i^4\right) |i_n\rangle = 0 \tag{3.61}$$

$$\hat{V}_{RS}^2 \; |i_n\rangle = \varepsilon_{ijk} \hat{E}_i^1 \hat{E}_i^2 \hat{E}_i^3 \; |i_n\rangle = \hbar^3 \gamma^3 \vec{J}_1 \cdot \left(\vec{J}_2 \times \vec{J}_3\right) \; |i_n\rangle \tag{3.62}$$

Where $\vec{J}_l$ are the $SU(2)$ generator in the j_l representation. From this expression it is clear that the spectrum of the volume is discrete and it has minimal excitation proportional to the Planck volume L_p^3.

Eigenvalues of the volume operator

Here we present the computation of the eigenvalues in the easiest case, which does not require further knowledge of the mathematics of the spin-networks: the four-valent node in which all the links have the lowest quantum number $j_l = \frac{1}{2}$.

Eigenvalues of the Volume. In the representation $j = \frac{1}{2}$ the generators are proportional to the Pauli matrices:

$$J_l^i = \frac{8\pi\gamma\hbar G}{c^3} = \alpha \frac{\sigma^i}{2} \tag{3.63}$$

The Hilbert space on which the volume operator act is the tensor product of four Hilbert space of the representation $1/2$ and each of them is represented by a spinor z^A with the index $A = 0, 1$. So the elements of the full space are

$$\mathcal{H} = \mathcal{H}^{\frac{1}{2}} \otimes \mathcal{H}^{\frac{1}{2}} \otimes \mathcal{H}^{\frac{1}{2}} \otimes \mathcal{H}^{\frac{1}{2}} \tag{3.64}$$

$$z^{ABCD} \in \mathcal{H} \qquad A, B, C, D = 0, 1 \tag{3.65}$$

$$\left(\hat{V} z\right) = \left(\frac{\alpha}{2}\right)^3 \varepsilon_{ijk} \left(\sigma_i\right)_{A'}^{A} \left(\sigma_j\right)_{B'}^{B} \left(\sigma_k\right)_{C'}^{C} z^{A'B'C'D} \tag{3.66}$$

The spin-networks are actually basis in the gauge-invariant Hilbert space, so the real space on which the volume operator is acting is its gauge-invariant part

$$\mathcal{K}^{(0)} = \left[\mathcal{H}^{\frac{1}{2}} \otimes \mathcal{H}^{\frac{1}{2}} \otimes \mathcal{H}^{\frac{1}{2}} \otimes \mathcal{H}^{\frac{1}{2}}\right]_{SU(2)-\mathrm{Inv}} \tag{3.67}$$

whose dimension can be computed with standard methods of angular momenta:

$$\left(\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}}\right) \times \left(\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}}\right) = (\mathbf{0} \oplus \mathbf{1}) \otimes (\mathbf{0} \oplus \mathbf{1}) = \mathbf{0} \oplus \mathbf{1} \oplus \mathbf{1} \oplus (\mathbf{0} \oplus \mathbf{1} \oplus \mathbf{2}) \tag{3.68}$$

From this decomposition we can read $\dim \mathcal{K}^{(0)} = 2$ which means that we can construct only two invariant tensors from the contraction of four indices and they are easy to guess

$$\epsilon^{AB} \qquad\qquad \sigma_i^{AB} = \left(\sigma_i\right)_C^A \epsilon^{CB} \tag{3.69}$$

Therefore the gauge-invariant Hilbert space is span by the following basis:

$$\mathbf{z}_1^{ABCD} = \epsilon^{AB}\epsilon^{CD} \tag{3.70}$$
$$\mathbf{z}_2^{ABCD} = (\sigma_i)^{AB}(\sigma_i)^{CD} \tag{3.71}$$

Finally we can compute the matrix element of the Volume operator and find its eigenvalues:

$$\hat{V}^2\mathbf{z}_n = V^2_{nm}\mathbf{z}_m \qquad \begin{cases} \hat{V}^2\mathbf{z}_1 = -i\frac{\alpha^3}{4}\mathbf{z}_2 \\ \\ \hat{V}^2\mathbf{z}_2 = i\frac{3\alpha^3}{4}\mathbf{z}_1 \end{cases} \tag{3.72}$$

$$\hat{V}^2 = -i\frac{\alpha^3}{4}\begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix} \quad \Rightarrow \quad \hat{V}^2_{\pm} = \frac{\sqrt{3}}{4}\alpha^6 \tag{3.73}$$

The sign depends on the fact that the volume square is *oriented* and it depends on the relative orientation of the fluxes in Eq. (3.62).

From this simple computation it becomes clear that the intertwiner quantum numbers associated to the nodes can be seen as “volume quantum numbers” since they pick up one of the elements of the gauge-invariant space $\mathcal{K}^{(0)}$ which in general have $\dim\mathcal{K}^{(0)} > 1$ and intrinsecally specify the “quantum state of the volume”.

3.4 Dynamics: diffeomorphism and Hamiltonian constraint

Dealing with the dynamics in the Hamiltonian formalism is a hard task and it is essentially the reason why the spin-foam formalism today is preferred for dynamical problems. For such a reason in this section we will only briefly present the basic elements of the treatment, without any details. For a pedagogical review on the topic see the review by Ashtekar and Lewandoski [5] and the book by Thiemann [95].

Diffeomorphism constraint

After the imposition of the Gauss constraint, the next step in the Dirac’s procedure is to characterise the solutions of the diffeomorphism constraint $\hat{H}_a\psi_\Gamma = 0$. We remember the expression of the classical smeared constraint

$$H_a[N^a] = \int_\Sigma d^3x \left(N^a E^b_j F^j_{ab} - (\gamma^2+1)N^a K^i_a \mathcal{G}_i\right) \tag{3.74}$$
$$\tag{3.75}$$

these are the diffeomorphism φ generated by the vector field N^a and the quantum counterpart $\hat{\varphi}$ is well defined because the measure $d\mu_{AL}$ is diff-invariant thus the action induced by classical ϕ on $\mathcal{H}$, is unitary. So one have a one-parameter family of $\varphi(s)$ and its quantum counterpart $\hat{\varphi}(s)$. In order to understand what is happening it is important to remember the action of the diffeomorphism on the holonomy:

$$\left(\hat{\phi} \triangleright \hat{h}_l\right) = \hat{h}_{\phi\circ l} \tag{3.76}$$

Since the Hilbert space of the theory is the direct sum of the spaces on each graph, the action of the diffeomorphisms maps Cyl_Γ into $\mathrm{Cyl}_{\phi\circ\Gamma}$, which are orthogonal unless the action of $\hat{\varphi}$ is trivial. For this reason, at the quantum level, the family $\hat{\varphi}(s)$ is not weakly continuous in s, therefore the generators of the diffeomorphisms group do not exist and it is not possible to look at the infinitesimal action. This is not a problem since the theory must be invariant under finite diffeomorphisms so it is still possible to exploit the group average procedure and construct the solution of the constraints but there are some subtleties that must be taken into account.

The first consideration regards the space that we are going to consider. Although one starts with Cyl_Γ the diffeomorphisms are a non-compact group and the request for invariance under their action over the elements of the space $\mathcal{H}^G_{kin}$ do not result in a subspace. For this reason the solutions of the diffeomorphism constraints must be considered in the larger space of the linear functionals: the dual Cyl^*_Γ. The situation is similar to what happens in finite-dimensional constrained systems in which one consider three spaces to find the states solutions of the constraints:

$$\mathcal{S} \subset L^2[\mathbb{R}^n] \subset \mathcal{S}^* \tag{3.77}$$

where, as usual, $\mathcal{S}$ is the space of the test functions and $\mathcal{S}^*$ is the space of the distributions. The solutions obtained by group average typically belong to $\mathcal{S}^*$ rather then to $L^2[\mathbb{R}^n]$ and the case of the diffeomorphisms is exactly the same:

$$\mathrm{Cyl} \subset \mathcal{H} \subset \mathrm{Cyl}^* \tag{3.78}$$

For example, in ordinary quantum mechanics, the request of invariance under translation $p = 0$ gives the constant function, which have an infinite norm so it does not belong to $L^2[\mathbb{R}]$, nevertheless it defines a linear functional over such a space. In the same way we have that the group average procedure for the diffeomorphism constraints will provide states belonging to Cyl^*_Γ.

The second consideration regards the symmetries of the graph. Indeed each graph will have specific symmetries that will make the action of some diffeomorphisms trivial and among them the only ones we need to worry about is the group of the transformations that leave each link untouched and simply shamble the points inside the link. This group of diffeomorphism must be taken out since its presence would ruin the group average procedure[5].

On the space Cyl^*_Γ is possible to implement the group average procedure, by asking for invariance under the action of diffeomorphisms:

$$\eta \in \mathrm{Cyl}^*_\Gamma \quad : \quad \forall \psi \in \mathrm{Cyl}_\Gamma \qquad \eta(\hat{\phi}\psi) = \eta(\psi) \tag{3.79}$$

This will result in the space $\mathcal{H}^*_{diff}$ and the space $\mathcal{H}_{diff}$ is constructed by duality. In the end, this construction provides a general solution of the diffeomorphsim constraints since through the map η, to each $\phi \in \mathrm{Cyl}_\Gamma$ is associated a diff-invariant element $\Psi \in \mathrm{Cyl}^*_\Gamma$. We refer to [5, 66, 67] for all the details and discussions on the topic. Finally, it is possible to define the procedure on the $SU(2)$-invariant space $\mathcal{H}^{\mathcal{G}}_{kin}$ so that we have at our disposal the Hilbert space of the states invariant under spatial diffeomorphisms and $SU(2)$ transformation: $\mathcal{H}^G_{diff}$.

The result of this procedure are the *knotted spin-networks*: ordinary spin-networks defined on equivalence class of graphs under diffeomorphism, called knots. Since the diffeomorphism change the way in which a graph is embedded in Σ, it is possible to interpret these spin-networks as defined on graphs which are *non-embedded* into a manifold and are only combinatorial structures. In this way the graph is not a collection of real path rather a collection of relational information by the means of which one can represents all the graphs equivalent to it but with different embedding on Σ.

Hamiltonian constraint

The last step of the Dirac's procedure is to characterise the solution of the quantum Hamiltonian constraint. The classical constraint is made by two pieces which are both non-linear in the variables and it makes difficult to turn it into an operator. However thanks to the so-called "Thiemann trick" it is possible to map the non-linearity of this operator into poisson bracket and this makes the operator more suitable for quantisation. We are not going through all the details and refer the reader to [88, 95, 5]. The Hamiltonian operator split in two parts

$$H(N) = H^E(N) - 2(1+\gamma^2)T(N) \tag{3.80}$$

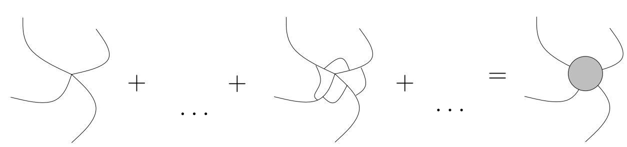
$$H^E(N) = \int d^3x N \varepsilon^{abc}\delta_{ij}F^i_{ab}\left\{A^j_c\,,\,V\right\} \tag{3.81}$$

$$T(N) = \int d^3x \frac{N}{\gamma^2}\varepsilon^{abc}\varepsilon_{ijk}\left\{A^i_a\,,\,\left\{H^E(1)\,,\,V\right\}\right\}\left\{A^j_b\,,\,\left\{H^E(1)\,,\,V\right\}\right\}\left\{A^k_c\,,\,V\right\} \tag{3.82}$$

and the second part is written in terms on the first one. So we give the explicit expression of the "Euclidean term" after quantisation:

$$\hat{H}^E = \lim_{\varepsilon\to 0}\sum_I N_I \varepsilon^{abc}\mathrm{Tr}\left[\left(\hat{h}_{\gamma_{ab}} - \hat{h}^{-1}_{\gamma_{ab}}\right)\hat{h}^{-1}_{e_c}\left\{\hat{h}_{e_c}\,,\,\hat{V}\right\}\right] \tag{3.83}$$

The action of this operator is explicitly known: it acts only on the nodes since it contain the volume operator and it create an *exceptional* link around the node modifying the spin-network. The limit can be safely taken since the $\varepsilon-$dependence is only on the position of the new link and since we are in the space $\mathcal{H}_{diff}$, the position of the new link is irrelevant. In conclusion, the Hamiltonian operator is well-defined and is action is understood. Moreover it is possible to find an infinite number of solution, at least formally, as an infinite linear combination of spin-networks with arbitrary number of exceptional links, defining the concept of *dressed node*; however such a procedure is only formal since it is not known how to control whole sum and we do not have an explicit characterisation of the solutions.



Unfortunately the Hamiltonian operator is affected by some issues, first of all the fact that explicit computations are quite hard to be performed and full spectrum of $\hat{H}$ is not known. Moreover, even if the operator is well-defined and its action is well understood there are a number of ambiguities in its definitions: the first one regard the spin of the exceptional link which in principle can assume any value; the second one is that it exists an infinite number of regularisation that can be used to define the operator [5]. Both the ambiguities and the complexity in performing explicit computations motivates the introduction of the spin-foam formalism [11] (see Chapter 1), which keeps manifest the covariant character of the theory and in this framework it is easier to deal with the dynamics of the theory.

Summary

The loop gravity formulation of General Relativity in terms of $SU(2)$ variables allows to implement the Dirac's quantisation program, in such a scheme the characterisation of the physical state is defined as the states annihilating the quantum version of the classical constraint equation and, as a first step (the *kinematics*), we solved the Gauss constraint. A basis for such Hilbert space was found, that inherits the $SU(2)$ structure introduced by the Asktekar variables: these are the *spin-networks*. These states $\psi_\Gamma(\{j_l\}\ ,\ \{i_n\})$ are described by a collection of *links* and *nodes* arranged in a *graph* $\Gamma(L, V)$, with two sets of quantum numbers $(\{j_l, i_n\})$, associated with each link and each node, respectively. The $j_l \quad l = 1, \ldots, L$ are the "spins", semi-integers quantum numbers associated to each link, standing for the $SU(2)$ representations that one uses to perform parallel transport on that specific link while the $i_n \quad n = 1, \ldots, V$ are the "intertwiners" quantum numbers associated to the nodes. They are related to the Clebsh-Gordan coefficients and define they way in which the $SU(2)$ representations are contracted over the spin-network state. The phenomenology that arise in quite interesting since we showed that the quantisation of the space is accomplished: we computed the general spectrum of the Area operator and provided an example of the spectrum of the Volume operator in a simple case. Those turned out to be both discrete, with minimal excitation proportional to the Planck area and the Planck volume, respectively.

As for the dynamics, a basis for the solutions of the diffeomorphism constraint is made by the knotted spin-network: ordinary spin-networks defined on the equivalence classes of graphs under diffeomorphism. Their invariance under diffeomorphism suggests that one can thought of them as purely combinatorial structure whose non-embedding in a manifold stands for the diff-invariance. In the end, we presented the action of the Hamiltonian operator over a spin-network state, which is a well-defined operator and has a known and finite action over the knotted spin-network states. Nonetheless the canonical program is far from being complete since we still do not have a characterisation of the physical Hilbert space and we still do not have the spectrum of the Hamiltonian operator. Furthermore some ambiguities affect the Hamiltonian operator definition[5], such as: the representation of the exceptional link j is completely arbitrary and this occurrence will have physical effects on the evolution; the regularisation procedure is not unique and for example one can define regularisation on which the action of the operator is non-local in the sense that it acs on different nodes simultaneously.

Thanks to the full development of the kinematics of loop quantum gravity, Bianchi, Donà and Speziale [69] interpreted the gauge-invariant spin-networks as a collection of *quantum polyhedra* where the quantum numbers $\{j_l\}, \{i_n\}$ are the bridge with the geometrical meaning since the latter are "volume quantum numbers" specifying the quantum state of the volumes while the former are the quantum areas of the shared faces between two polyhedra. Thanks to this work we now have a more intuitive picture of the spin-networks as a collection of adjacent polyhedra, which remind us the famous discretisation of General Relativity in terms of simplicial triangulation of space-time: the Regge calculus. In the next chapter we will review the basic elements of the Regge's theory and we will point out a key difference with the geometry arising both in discrete loop gravity and in loop quantum gravity.

Chapter 4

Phase space of discrete General Relativity

We started with the Holst's Lagrangian and introduced the Ashtekar-Barbero variables which turns out to be canonically conjugated. Thank to them we wrote the Holst's action so that the Legendre transformation to find the Hamiltonian was trivial. We regularised the Poisson algebra generated by the Ashtekar-Barbero variables and ended up with the Holonomy-Flux algebra.

We would like to comment on the meaning of the smearing procedure introduced in the last chapter. The theory is now formulated through 3D boundary variables (h, E) with support only on a very specific subset of the space Σ: the graph Γ and its dual triangulation Γ^*. This means that we actually performed a discretisation of the theory. As a matter of fact the smearing of the variables is something more than a way to regularise the algebra, it can be seen as the boundary of a *covariant discretisation* of the space-time that suit the mathematical structure of General Relativity.

In what follows we will present the geometric interpretation after the discretisation of the theory. The arising picture is quite involved and a rigorous treatment of this interpretation is not our purpose, since the topic is quite vast; nevertheless we think that it is possible to catch the main idea without having to look at all the details, indeed we would like to equip the reader with some geometrical intuition that later will help to understand the fuzziness of geometries arising from loop gravity, both at the classical and at the quantum level. For such a purpose in the first section of this chapter we provide a very short review of Regge calculus [73] and its discretisation of the space-time. In the second and third section we will present the recent results [21, 74, 75, 76] on the geometric parametrisation of the loop gravity phase space, both in terms of the twisted geometries and in terms of twistor variables.

4.1 Regge geometries

Tullio Regge is an Italian physicist who around 1960, when he was at Palmer Laboratory, Princeton University, developed a discretisation (and a truncation) of General Relativity which today goes under the name of *Regge calculus*. It is based on the idea to approximate curved manifolds through the triangulation procedure: a consistent gluing of 4D regular polyhedra: the $4-$simplexes. The triangulation technique is valid in any dimension so we will try to keep the notation as general as possible. An $n-$simplex (see Fig. 4.1) is defined as the convex hull of

$n+1$ *vertices* in an n-dimensional manifold. These vertices are connected by $\frac{n(n+1)}{2}$ *segments* and its boundary is made by an $n+1$ number of $n-1$-simplexes. We follows the Regge's notation for which the $n-$dimensional simplex is indicated by T_n.

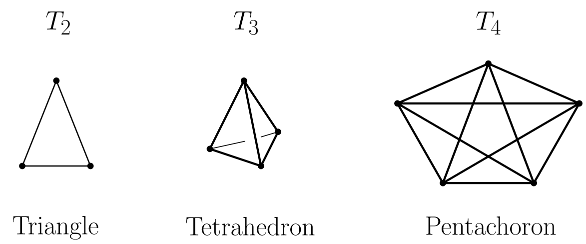


Figure 4.1: The first two simplex are a point T_0 and a segment T_1 and here are the following three $n-$simplexes.

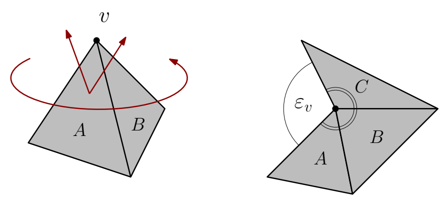
It is quite intuitive that a n-simplex (up to rotations) is completely determined by assigning the lengths of its $\frac{n(n+1)}{2}$ segments. This set of lengths define the way in which distances can be computed on the simplex, so they actually are the discrete counterpart of the metric tensor. Thus we can define a *Regge geometry* as the n-dimensional metric manifold originated by consistent gluing $n-$simplexes through their boundary $n-1$-simplexes and the assignment of the segments lengths. For example a 2D Regge geometry is a surface obtained by gluing triangles along their segments. In the 4D theory there will be 4-simplexes glued by their boundary tetrahedra.

Since these objects are used to triangulate curved space-time, it is mandatory to understand from where the curvature can arise in this scheme. The best way to show it is in two dimensions. The 2-sphere $\mathcal{S}^2$ is a curved 2D manifold and we can ask what is the roughest triangulation that we can use to approximate it: since three triangles cannot close, we need at least four of them and they must be glued by their segments to form the tetrahedron inscribed in $\mathcal{S}^2$, see Fig. (4.2).

Regge showed that, in arbitrary dimensions, gluing n-simplexes can generate curvature on the $n-2$ simplexes and we will call these objects *hinges*. For the $2D$ case (refer to Fig. 4.2) it is clear that the hinges are the vertices of the inscribed tetrahedron and account for the curvature of $\mathcal{S}^2$. In three and four dimensions the objects accounting for curvature will be *segments* and *triangles*, respectively.

Considering that a triangulation can be refined increasing the number of n-simplexes one uses, it is not hard to believe that a manifold $(\mathcal{M}, g)$ can be approximated by a Regge geometry and this approximation can be refined as much as requested, in the following sense: for any $\varepsilon > 0$ there exists a triangulation Δ_ε such that the difference between distances computed in the original manifold and distances computed in the Regge geometry is always less than ε.

The structure presented is a very intuitive way to discretise metric manifolds and the Regge's idea was to exploit these techniques to define a discrete version of General Relativity. However, we still lack of a crucial ingredient before we can define the *Regge action*: it is a notion of curvature in discrete manifolds that will converge to the Riemannian curvature in the continuum limit. Regge found a very elegant way to encode the geometrical meaning of intrinsic curvature in a discrete manifold: through the notion of *deficit angle*. We present the idea in the 2D case but the concept can be extended to hinges of arbitrary dimension.



Around an hinge (in 2D is a vertex) v there are the triangles A, B, C and from the knowledge of the lengths (a_i, b_j, c_k) of the segments we can easily compute the three angles, $\theta_A(a_i), \theta_B(b_j), \theta_C(c_k)$ around the vertex v:

$$\cos\theta_J = \frac{j_1^2 + j_2^2 - j_3^2}{2 j_1 j_2} \qquad \begin{array}{l} J = A, B, C \\ j = a, b, c \end{array} \tag{4.1}$$

If the surface were to be flat the sum of the three angles should be 2π. However, as you can see from the figure above, due to the fact that the faces are part of a triangulation of $\mathcal{S}^2$, they fail to "close" around v and the deficit angle ε_v amount exactly for such a failure, providing a way to quantify the curvature on triangulated surfaces. Since we are going to construct an action using the deficit angle as discrete analog of curvature, we should stress the fact that it is actually a proper function of the set of lengths $\{l_s\}$.

The concept can be straightforwardly extended to vertices with an arbitrary number of faces around:

$$\varepsilon_v(l_s) = 2\pi - \sum_{F\,:\,v\in\partial F} \theta_F(l_s) \tag{4.2}$$

The same logic can be used to define the 4D *deficit angle*. The key difference is that now the hinges are triangles and the angles are dihedral angles between 4D vectors and the scalar product has a different signature $(-1, 1, 1, 1)$. Nonetheless it is possible to extend the idea to Lorentzian manifolds and the concept of deficit angle is still well defined. The geometrical interpretation is the following: take a 4D vector and parallel transport

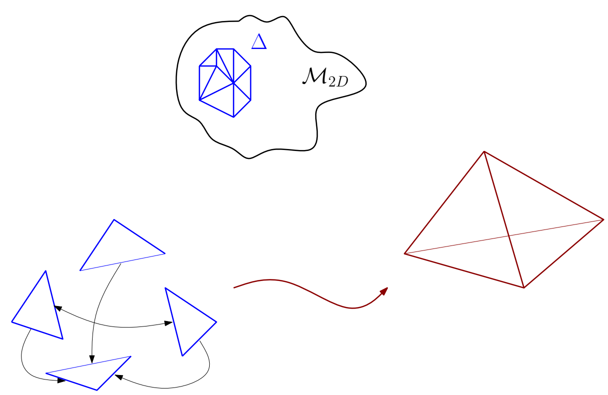


Figure 4.2: The triangulation of a manifold $\mathcal{M}$ is usually indicated with $\Delta(\mathcal{M})$. In order to triangulated $\mathcal{S}^2$ at least four triangles are requested, in order to form a closed non-degenerate surface. The 2-sphere is topologically equivalent to the surface of a tetrahedron, so we can use it as the roughest triangulation. It become evident, from the geometrical point of view that curvature is hidden in the vertices.

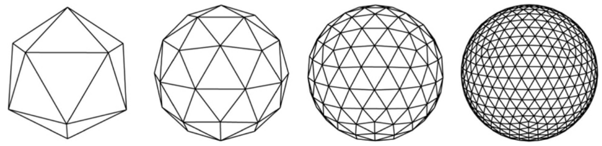


Figure 4.3: The original triangulation of $\mathcal{S}^2$ can be refined in order to increase the accuracy.

it around an hinge, it will come back rotated by the deficit angle δ_h of the hinge around which the loop is performed.

Equipped with these definitions, Regge proposed to study the Einstein-Hilbert action, with support on the skeleton Δ:

$$S_{EH}(g) = \frac{1}{16\pi G}\int_{\mathcal{M}} \mathcal{R}\sqrt{-g}dx^4 \qquad \overset{\mathcal{M}\to\Delta}{\longrightarrow} \qquad S_{\Delta}^{\text{Regge}}[l_s] = \frac{1}{8\pi G}\sum_{h\in\Delta} A_h(l_s)\varepsilon_h(l_s) \tag{4.3}$$

Where the index h run over all the hinges of the triangulation Δ, ε_h are the deficit angles and A_h is the area of the hinges, since in 4D they are triangles.

The remarkable result achieved by Regge was to prove that in the limit in which a Regge geometry (Δ, l_s) approaches the metric manifold (M, g), the Regge action converges to the Einstein-Hilbert action.

$$(\Delta, l_s) \quad\to\quad (\mathcal{M}, g) \qquad\Longrightarrow\qquad S_{\Delta}^{\text{Regge}}[l_s] \quad\to\quad S_{EH}(g) \tag{4.4}$$

Which proves that a Regge geometry, with the Regge action, is a good discretisation of General Relativity.

Varying the action with respect to the lengths gives the equations of motion:

$$\delta S_{\Delta}^{\text{Regge}}[l_s] \overset{!}{=} 0 \qquad\Longleftrightarrow\qquad \sum_{h\in\Delta}\left(\frac{\partial A_h}{\partial l_s}\varepsilon_h + A_h\frac{\partial\varepsilon_h}{\partial l_s}\right) = 0 \tag{4.5}$$

It is possible to show that the variation of the deficit angle vanishes $\delta\varepsilon(l_s) = 0$, for any dimension of the manifold. It is the discrete version of the fact that when one perform the variation of the scalar curvature, the variation of the Ricci tensor vanish. We are left with the following equations:

$$\forall s\in\Delta \qquad \sum_{h\,:\,s\in\partial h}\frac{\partial A_h}{\partial l_s}\varepsilon_h = 0 \tag{4.6}$$

Since ε_h can be interpreted as the Riemannian curvature in discrete manifolds, the quantity that appear in this equation could be considered as a measure of the Ricci tensor in the discrete manifolds. We could check these equations in three space-time dimensions. In this case the hinges are the very same quantity as the independent variables l_s so the variation of the action will produce $\varepsilon_h = 0$ on each hinges, which means flat space-time as in the continuum limit.

We would also like to stress the fact that, if we restrict the theory to a fixed triangulation Δ we will obtain *a truncation* of General Relativity to a finite number of degrees of freedom. From this point of view the Regge theory relates to General Relativity as the lattice formulation of QCD relates to the full theory. However a substantial difference emerge in the way in which the limit is performed. In QCD one must shrink to zero the size of the plaquette of the lattice and send the number of vertices N to infinity. General relativity in this sense is quite different, since to recover the continuum limit, one only need to send the number of vertices to infinity. The reason for such a difference lies in the general covariant character of General relativity. While in the Wilson action for lattice QCD one must rescale the parameter to its critical value in order to reach the continuum limit there is no such a parameter in Regge calculus since the variables of the theory are actually the "lattice spacing" so it does not make any sense to send their size to zero.

Geometric picture

In the original paper of Regge, the definition of a metric on the discrete space-time is achieved by axiomatisation. It is explicitly formulated for Euclidean manifolds and then generalised to Lorentzian manifolds. We are interested in this construction since one of these axioms is responsible for the continuity of the metric, it has a very nice geometrical interpretation in terms of the so-called *shape-matching conditions* that will play a leading role in the next chapters. Here is the axiomatisation of the metric proposed by Regge:

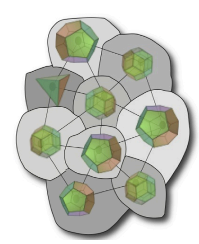
A) The metric in the interior of any T_n is euclidean (lorentzian). This means that we can compute the distance of any points inside T_n, define a cartesian system of co-ordinates, and give the co-ordinates of the points of the boundary of T_n in this frame.

B) In the metric of T_n, the boundary is decomposable into the sum of $n+1$ closed simplexes T_{n-1} and these T_{n-1} are flat.

C) *If a simplex T_{n-1} is common boundary of T_n and T'_n the distance of any two points of T_{n-1} is the same in both frames of T_n and T'_n*

D) If $P \in T_n$ and $P' \in T'_n$ and P, P' are close enough to T_{n-1} we define the distance PP' as the infimum of $PQ + QP'$ for all $Q \in T_{n-1}$

This construction, together with the definition of the Regge's action for General Relativity, define a geometrical interpretation of discrete curved space-time as a dynamical collection of polyhedra, locally flat. The quantity that amounts for curvature is the deficit angle: it can be revealed through the scalar product between a vector and its parallel transported around an hinge and it has support only on 2D faces.



The variables of the theory are the lengths of 1D segments (l_s), since they carry the same kind of information as the metric tensor (g) in the continuum. Thanks to the last consideration and to the axiom C it is obvious that the gluing of T_4 through a boundary T_3 does not present any problem since the assignment of the ten lengths on each $4-$simplex not only specify the 4D geometry but even its 3D boundary: these quantities endow each boundary tetrahedron with the six lengths needed to characterise its geometry. An example of this procedure in 4D is the triangulation of a four-ball $\mathcal{B}^4$, the inside of a $3-$sphere in $4D$, which is showed in Fig. (4.4).

For what concerns the metric, this condition translates in the continuity across the boundary T_3. Suppose to have a tetrahedron in the boundary of two different $4-$simplex. Since each T_4 provide a local Lorentz frame (axiom B) and the tetrahedron is the same, if one can compute any distance in the two different reference frames, these quantities will have to match. The statement can be made rigorous exploiting a set of relations which in literature are called *shape-matching conditions*, because they imply that the shape of the T_3 simplex which is in the boundary of two T_4 must be the same as seen from the two reference frames sharing it.

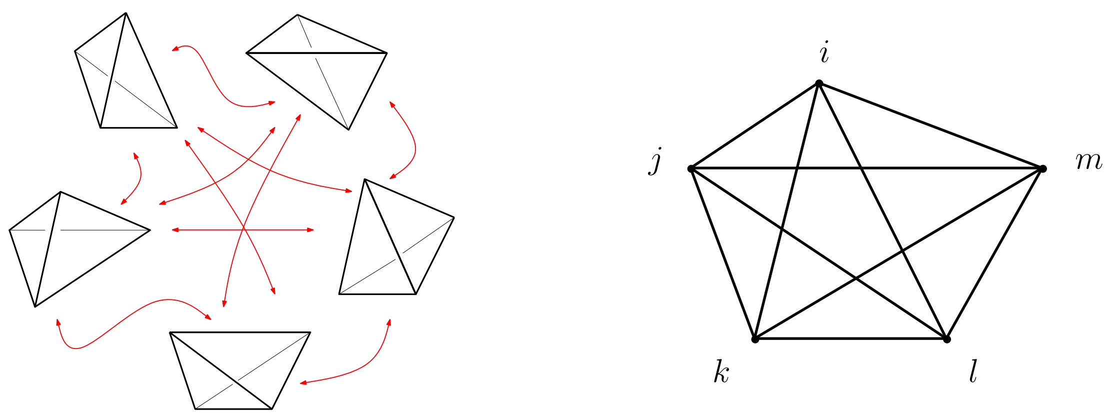


Figure 4.4: The $\mathcal{S}^2$ can be seen as an hypersurface in the ordinary 3D space, and in the same way the $3-$sphere $\mathcal{S}^3$ can be seen as a boundary hypersurface in 4D space-time and its roughest simplicial triangulation is made by one T_4, having five tetrahedra in the boundary.

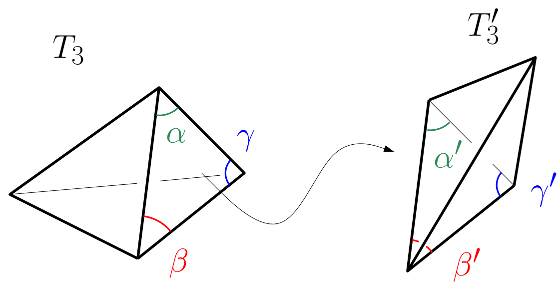
Shape-matching conditions

The name *shape-matching conditions* does not identify a precise set of equations, rather it refers to the concept we expressed in the previous section. These conditions are usually written through the measure of angles between the edges of a triangulations or via the angles between the normal of the hinges. In this section we will show how they work, in the case of one $4-$simplex.

Before we start, it is necessary to fix the notation we are going to use, see the right side on figure 4.4. A $4-$simplex is the four dimensional equivalent of a tetrahedron, its $3D$ boundary is made by 5 tetrahedra, with a total of 10 triangles as 2D boundary and 10 segments connecting 5 vertices. We call the $4-$simplex σ, with its $4-$volume V. The notation will exploit an *elimination method*, for which we define $\sigma(i)$ as the boundary tetrahedron we obtain eliminating the vertex i. In the same way $\sigma(ij)$ is going to be a triangle, $\sigma(ijk)$ a segment and $\sigma(ijkl)$ a vertex. Same method will be used for the volumes, areas and lengths of the respective objects: $V(i), V(ij), V(ijk)$.

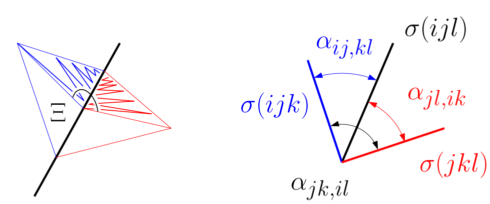
For the angles notation we would like to keep track of the frame in which the angle is computed. θ_{ij} is the 4D dihedral angle between two tetrahedra $\sigma(i)$ and $\sigma(j)$; $\phi_{ij,k}$ is the 3D dihedral angle between the triangles $\sigma(ik)$ and $\sigma(jk)$, meeting at the segment $\sigma(ijk)$ and computed inside the tetrahedron $\sigma(k)$. In the end, the 2D angle between the segments $\sigma(ijk)$ and $\sigma(ijl)$, in the triangle $\sigma(kl)$ will be indicated by $\alpha_{ij,kl}$. Equipped with this notation we now study the formulation of the shape-matching conditions, inside a $4-$simplex.

Two glued tetrahedra will share a triangle, so one is allowed to compute the internal angles of this triangle, with the two metrics defined on each tetrahedron. The equalities of these angles computed with different metric (on different tetrahedra), provide one version of the shape-matching conditions.



$$\alpha \stackrel{!}{=} \alpha' \qquad \beta \stackrel{!}{=} \beta' \qquad \gamma \stackrel{!}{=} \gamma' \tag{4.7}$$

The analytic expression of these equalities is based on the *reconstruction formula* that can be used, for example, to evaluate the 3D dihedral angle between two triangles. The figure aside illustrate the idea: the internal 3D dihedral angle $\Xi = \phi_{il,j}$ between the two triangles $\sigma(ij)$ and $\sigma(lj)$ can be computed from the knowledge of the three 2D angles among the three edges $\sigma(ijk)$, $\sigma(ijl)$, $\sigma(jkl)$:



$$\cos\phi_{il,j} = \frac{\cos\alpha_{jk,il} - \cos\alpha_{ij,kl}\cos\alpha_{jl,ik}}{\sin\alpha_{ij,kl}\sin\alpha_{jl,ik}} \tag{4.8}$$

Remarkably, the reconstruction formula can be inverted and it has a very simple expression: from the knowledge of the 3D dihedral angles ϕ one can compute the 2D angles α.

$$\cos\alpha_{ij,kl} = \frac{\cos\phi_{ij,k} + \cos\phi_{il,k}\cos\phi_{jl,k}}{\sin\phi_{il,k}\sin\phi_{jl,k}} \tag{4.9}$$

Now, looking at the figure upside, two tetrahedra share a triangle and since both of them represent a reference frame there are two possibilities for computing the 2D angles, depending on which tetrahedron one choose: $\alpha_{ij,kl}$ and $\alpha_{ij,lk}$. The consistency of these two quantities $\alpha_{ij,kl} = \alpha_{ij,lk}$ gives the explicit expression for the shape-matching conditions, as equalities between 2D angles written in terms of 3D dihedral angles:

$$\cos\alpha_{ij,kl} - \cos\alpha_{ij,lk} = \frac{\cos\phi_{ij,k} + \cos\phi_{il,k}\cos\phi_{jl,k}}{\sin\phi_{il,k}\sin\phi_{jl,k}} - \frac{\cos\phi_{ij,l} + \cos\phi_{ik,l}\cos\phi_{jk,l}}{\sin\phi_{ik,l}\sin\phi_{jk,l}} = 0 \tag{4.10}$$

There are three relations on each triangle and as previously stated there are ten triangles inside a four-simplex, so we have 30 relations, of which only 20 are independent. Moreover, thanks to the high degree of symmetry of the $4-$simplex, equivalent relations can be found, using the 3D and 4D dihedral angles:

$$\cos\theta_{ij}^{(k)} = \frac{\cos\phi_{ij,k} - \cos\phi_{ik,j}\cos\phi_{jk,i}}{\sin\phi_{ik,j}\sin\phi_{jk,i}} \tag{4.11}$$

$$\cos\theta_{ij}^{(k)} = \cos\theta_{ij}^{(l)} = \cos\theta_{ij}^{(m)} \tag{4.12}$$

A lot of emphasis has been put on these relations. They are indeed the key to understand the difference between the Regge discretisation of General Relativity and the phase space of discrete loop gravity. As we will see in the next section, the latter has a structure which is more general than the former. In this sense the Regge geometries are "too rigid" to fully represent the Holonomy-Flux algebra. At the geometrical level it is possible to visualise the difference between them, through the gluing conditions among polyhedra which do hold in the Regge case but do not in the phase space of loop gravity. In the next section we will study in detail the phase space that originates from the Holonomy-Flux algebra and we will present its geometrical interpretation in terms of new objects: the *Twisted geometries*.

4.2 The phase space of loop gravity: Twisted geometry

Thanks to the Ashtekar-Barbero variables in Chapter 2 we were able to find the phase space of loop gravity, whose algebra is plagued by the presence of $\delta^{(3)}(x, y)$ distributions which makes the theory impossible to quantise. For this reason, an important step toward the quantisation was the regularisation of the algebra and for such a task we introduced a graph, embedded in the spatial manifold $\Gamma \subset \Sigma$, and replaced the continuous variables with their smeared counterpart: the holonomies of the connection over the links $h_l = \mathcal{P}\exp\left(-\int_l A\right)$ and the fluxes of the triad over the surfaces orthogonal to the links $E(S_l) = \int_{S_l} hEh^{-1}d^2\eta$, this gave rise to the $T^*SU(2)^L$ discrete phase space of loop gravity. This is the classical phase space which has undergone the Dirac's quantisation program and constitutes the classical counterpart of the kinematical Hilbert space.

From the very definition of the new variables it is clear that they are distributional versions of the continuous ones, with support only on the graph Γ and therefore capturing only a finite number of degrees of freedom. Nevertheless, the theory is not defined over the Hilbert space $\mathcal{H}_\Gamma$ but rather on the sum over all the possible graphs (see eq. 3.39), which is unfortunately intractable, so we need to explore some truncation of the theory that will lead to an approximate dynamics. As far as we know, there are essentially three truncations [89] explored in loop quantum gravity but here we present only one of them since it is the one we will use: the *fixed graph truncation*.

Consider the component $\mathcal{H}_\Gamma$ of the full Hilbert space and truncate the sum to just this term. The first remarkable consideration is that, due to the diffeomorphism invariant character, all the spaces relative to a subgraph $\Gamma' \subset \Gamma$ are taken into account, therefore truncating the theory to Γ means only to forsake those states that need a richer graph to be defined. We stress the fact that such a truncation, in an ordinary quantum field theory, correspond to a truncation of the full Fock space to a certain number of particles and any perturbative calculation is actually performed in such a way. The matter about the effective usefulness of such an approximation is still under debate but the hope is that the truncation may be sufficient to capture the physics in appropriate regimes since it is essentially a truncation to the low modes of an expansion of the gravitational field in compact space.

So within this approximation we are working with a truncation of the full theory to a finite number of degrees of freedom and as we know, for example from lattice QCD, a discretisation is exactly a truncation of the full SU(3) gauge theory to a finite number of degrees of freedom. Relying on this idea one might wonder if there exists interpretation for the classical counterpart of $\mathcal{H}_\Gamma$ in terms of a *discrete geometry*. Indeed, as suggested by Immirzi in 1996 [55, 56] it could be possible to equip the classical phase space $T^*SU(2)$ with a geometrical meaning, thus suggesting a relation with the only consistent discretisation of General Relativity known: the Regge calculus.

The geometrical parametrisation of $T^*SU(2)$ was achieved in 2010 by Speziale and Freidel [21] and it defines a new notion of discrete geometry, namely a *twisted geometry*. As anticipated in the previous section, they are a collection of locally flat polyhedra but differ from Regge calculus since the local patches lack of a consistent gluing. There is a fundamental reason for this occurrence and it lies in the definition of the two discretisations: from one hand in Regge calculus the discrete metric is provided by a set of axioms which includes the continuity; on the other hand in loop gravity the metric tensor is not a fundamental quantity and it can be derived from the actual discrete variables so "a priori" the metric is allowed to be discontinuous, in this sense a twisted geometry is a generalisation of the Regge geometry.

Before we present the twisted geometries we would like to mention that it is possible to generalise them to parametrise the $T^*SL(2,\mathbb{C})$ phase space [74] arising in the self-dual formulation of General Relativity. Furthermore a fascinating relation between twisted geometries and *twistors* was found [75] which eventually led to the introduction of the *twistor networks* and the *covariant twisted twisted geometries*. We will present these topics in the next Chapters since we are going to use them in last chapter.

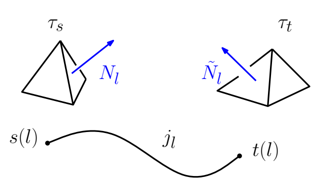
Twisted geometries

We start assuming that the theory is smeared over a graph Γ dual to a simplicial triangulation Δ but the results hold for a general graph. Each node is dual to a tetrahedron while each link is dual to a face, shared by two tetrahedra. The phase space of a *twisted geometry* $\mathcal{P}_l$ is associated to each link of a graph, and the definition extends straightforwardly to the whole graph:

$$\mathcal{P}_l \equiv \mathcal{S}^2 \times \mathcal{S}^2 \times T^*\mathcal{S}^1 \cong \mathcal{P}_l \equiv \mathcal{S}^2 \times \mathcal{S}^2 \times \mathbb{R} \times \mathcal{S}^1 \tag{4.13a}$$

$$\left(N_l, \tilde{N}_l, j_l, \xi_l\right) \in \mathcal{P}_l \qquad\qquad \mathcal{P}_\Gamma = \bigoplus_{l\in\Gamma} \mathcal{P}_l \tag{4.13b}$$

To address the question we focus on the phase space on each link $\mathcal{P}_l$ which in the triangulation correspond to a pair of tetrahedra sharing a triangle. The real quantity j_l defines the area of the triangle dual to the link l. N_l and $\tilde{N}_l$ are two unit vector that parametrise the $two2-$spheres $\mathcal{S}^2$ and are associated, respectively to the node $s(l)$ "source" and $t(l)$ "target" connected by the link l. They can be interpreted as the normals to the triangle dual to the link, as seen from the two tetrahedra sharing it: $\tau_{s(l)}$ and $\tau_{t(l)}$.



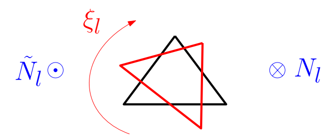
Thanks to the presence of the two normals it is possible to define the h_l as the group element rotating N_l into $\underset{\sim}{N}_l$ and the rotation matrix acts as usual via the adjoint representation:

$$h_l \in SU(2) \,:\, \underset{\sim}{N}_l = (h_l \triangleright N_l) = R(h_l) N_l \tag{4.14}$$

Thus h_l defines a notion of parallel transport along the link, rotating the frame of τ_s into that of τ_l. Given the two normals we can solve the equation (4.14) for the SU(2) element $h_l = h_l(N_l, \underset{\sim}{N}_l)$ but since $(N_l, \underset{\sim}{N}_l)$ are unit vectors such a relation does not uniquely define the parallel transport because it provides only two independent conditions. Indeed given $h_l(N_l, \underset{\sim}{N}_l)$ solving the equation (4.14) we have a one parameter class of equivalent solutions that can be generated by a rotation around the direction given by N_l:

$$h_l \in SU(2) \,:\, \underset{\sim}{N}_l = R(h_l)N_l \quad \text{and} \quad h_l^{(\xi)} = h_l e^{\xi_l N_l^i \tau_i} \quad \Rightarrow \quad \underset{\sim}{N}_l = R(h_l^{(\xi)})N_l \tag{4.15}$$

The most general solution for the group element can be constructed by hand, defining two group elements n_l and $\tilde{n}_l$ rotating the "z-axis" (τ_3), respectively in N_l and $\tilde{N}_l$ and then the most general solution has the following expression



$$h_l(N_l, \tilde{N}_l, \xi_l) = n_l e^{\xi_l \tau_3} \tilde{n}_l^{-1} \begin{cases} n_l(N_l) \,:\, R(n_l) = N_l^i \tau_i \\ \tilde{n}_l(\tilde{N}_l) \,:\, R(\tilde{n}_l) = \tilde{N}_l^i \tau_i \end{cases} \tag{4.16}$$

We can now understand where does the mismatch come from. On each triangle we have a ξ_l angle responsible for a rotation around the axis of the normal N_l and there are four triangles around a tetrahedron, so it become clear that the same triangle, as seen from the two different reference frames sharing it, has different shapes.

This construction was proposed in [21] and the remarkable result accomplished was to prove that exists a global symplectomorphism[1] between the space of twisted geometries $\mathcal{P}$ and the phase space of loop quantum gravity $T^*SU(2)^L$.

$$\mathcal{P} = \bigotimes_{l=1}^{L} \mathcal{P}_l \cong T^*SU(2)^L = \mathcal{S}_\Gamma \tag{4.17}$$

Here we simply show the definition of the map. Given $n(N)$ and $\tilde{n}(\tilde{N})$ such that $N = n\tau_3 n^{-1}$ and $\tilde{N} = \tilde{n}\tau_3\tilde{n}^{-1}$ the map is defined as follows:

$$\left(N, \tilde{N}, j, \xi\right) \quad \mapsto \quad (E, h) \,:\, \begin{cases} E = jn\tau_3 n^{-1} \\ h = ne^{\xi\tau_3}\tilde{n}^{-1} \\ \tilde{E} = j\tilde{n}\tau_3\tilde{n}^{-1} = -g^{-1}Eg = -j\tilde{N} \end{cases} \tag{4.18}$$

It is possible to invert[2] the map so that one can actually reconstruct the geometrical data from the holonomy and flux variables:

$$j = |E| \qquad N = \frac{E}{|E|} \qquad \tilde{N} = \frac{g^{-1}Eg}{|E|} \qquad \xi = \mathrm{Tr}\left[\tau_3 \log(n^{-1}g\tilde{n})\right] \tag{4.19}$$

[1] A symplectomorphism is an isomorphism preserving the Poisson brackets structure

[2] Since the maps does not distinguish among $\left(N, \tilde{N}, j, \xi\right)$ and $\left(-N, -\tilde{N}, -j, -\xi\right)$, the maps is $2-1$ and it can be inverted in the two branches. However one identify the two configurations by a symplectic reduction of the space $\mathcal{P}_l /\!/ \mathbb{Z}_2$ and this makes the map invertible, indeed the actual isomorphism is with this space. One should pay attention to the $E = 0$ but it has been checked that the two spaces coincide at the origin

All the details and the proof that the isomorphism is actually symplectomorphism are in [21].

Closed twisted geometries

This parametrisation of the phase space is particularly useful at the gauge invariant level and indeed it provides a noteworthy implementation of such a reduction. The $SU(2)$ gauge invariant phase space can be reached from $T^*SU(2)^L$ by symplectic reduction through the Hamiltonian flow generated by the Gauss constraint

$$S_\Gamma^{\mathcal{G}} = T^*SU(2)^L /\!\!/ SU(2)^N \tag{4.20a}$$

$$\mathcal{G}_n = \sum_{l\in n} E_l = 0 \qquad n = 1,\dots,N \tag{4.20b}$$

where N is the total number of nodes of the graph. The imposition of the Gauss law on $T^*SU(2)^L$ can be tricky since the kinematical space is constructed over an *oriented* graph, in which one specifies the direction used to "walk through" the link, i.e. which node is the source and which one is the target. This imposes that under the reversal of the orientation of the link $h_{-l} = h_l^{-1}$ but $E_{-l} = -h_l E_l h_l^{-1}$. From the point of view of the group action on $T^*SU(2)$ this can be easily understood because E_l is the left-invariant vector field but the action of the right-invariant one is defined as well so, reversing the orientation of the link we obtain the right-invariant vector field.

This relation can be used to split the Gauss law, on a node, in two pieces[3], incoming and outcoming links:

$$\mathcal{G}_n = \sum_{l\in n\,:\,s(l)=n} E_l \; + \sum_{l\in n\,:\,t(l)=n} \tilde{E}_l = 0 \tag{4.21}$$

Actually performing the quotient in eq. (4.20a) is not an easy task since the structure is non-local in the nodes because the space factorise over the links, involving pairs of nodes $s(l)$ and $t(l)$, and it is not obvious how to implement the Gauss law on such a space. However, the twisted geometries parametrisation completely reduce the problem since it assigns two normals to each link, one referred to the source, the other to the target, making it possible to implement the Gauss law separately at each node and actually perform the symplectic reduction of the phase space. Moreover through the symplectomorphism it is possible to give a precise geometric meaning to the Gauss law, thanks to the following theorem due to Minkowski [72].

The Minkowski theorem. This theorem was proved by Minkowski back in 1897 and it consist of two statements:

First - Takes K non-planar unit vectors $\vec{n}_k$ and k positive real numbers A_k such that they fulfil the following *closure condition*

$$C \equiv \sum_{l=1}^{K} A_l \vec{n}_l = 0 \tag{4.22}$$

[3]Nothing is changed, we are adjusting the notation to keep track of the orientation of the link

then it exist a convex polyhedron with K faces, in which the outward unit normals to the faces are given by the K vectors $\vec{n}_k$ and the areas of the faces are given by the A_k.

Second - If each face of the defined polyhedron has the same area and the same normal of a second polyhedron, then they are equal up to a translation

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The phase space that originates can be understood in terms of the *space of the shapes* of a polyhedron defined by Kapovich and Millson in [94] and extensively studied by Bianchi, Donà and Speziale in [69].

Kapovich and Millson phase space. Consider K unit vector $\vec{n}_k$ in the ordinary 3D space (planar vectors are allowed and they correspond to degenerate configurations) with K real numbers such that they satisfy the closure constraint (4.22). The space of such vectors up to rotations has a natural symplectic structure and it is called *Kapovich and Millson symplectic manifold*:

$$\mathcal{S}_K = \left\{ \vec{n}_l \in \bigotimes_{i=1}^{K} \mathcal{S}_i^2 \ : \ \sum_{i=1}^{K} A_i \vec{n}_i = 0 \right\} \Big/ SO(3)$$
$$\dim \mathcal{S}_K = 2(K-3)$$

This is the phase space of the *shapes of a polyhedron at fixed areas*, the symplectic structure is naturally inherited from the symplectic structure on each two-sphere $\mathcal{S}^2$ and thanks to the Minkowski's theorem, to each point in $\mathcal{S}_K$ we can associate a unique polyhedron, modulo rotations.

Both the results mentioned contributes to the understanding of the phase space of twisted geometries. Thanks to the result of Kapovich and Millson it is possible to factorise the whole space $\mathcal{P}_\Gamma$ both over the links and over the nodes.

$$\mathcal{P}_\Gamma = \bigotimes_l \mathcal{S}_l^2 \times \mathcal{S}_l^2 \times T^*\mathcal{S}_l^1 = \bigotimes_l T^*\mathcal{S}_l^1 \bigotimes_n \left[\bigotimes_{l\in n} \mathcal{S}_l^2 \right] \tag{4.23}$$

Thanks to the Minkowski theorem the Gauss constraint was identified with the Minkowski closure constraint[4]

$$\mathcal{G}_n = \sum_{l\in n\,:\,s(l)=n} E_l + \sum_{l\in n\,:\,t(l)=n} \tilde{E}_l = \sum_{l\in n\,:\,s(l)=n} j_l N_l - \sum_{l\in n\,:\,t(l)=n} j_l \tilde{N}_l = C_n \tag{4.24}$$

so that it is actually possible to take the symplectic quotient at each vertex. This is equivalent to impose the closure relation and divide by the gauge orbits generated by the Hamiltonian flux of $\mathcal{C}$ and, by construction, the resulting space is isomorphic to the gauge-invariant phase space of loop quantum gravity (4.20a). Calling $F(n)$ the generic valence of a node we have

$$\mathcal{C} = \prod_{n\in\Gamma} C_n \qquad \longleftrightarrow \qquad \mathcal{G} = \prod_n \mathcal{G}_n \tag{4.25}$$

$$\mathcal{P}_\Gamma^{\mathcal{C}} = \mathcal{P}/\!\!/\mathcal{C} = \bigotimes_l T^*\mathcal{S}^1 \bigotimes_n \mathcal{S}_{F(n)} \cong T^*SU(2)^L /\!\!/ SU(2)^N = S_\Gamma^{\mathcal{G}} \tag{4.26}$$

[4]The change of sign is simply due to a different choice in the convention for the definition of the variables N and $\tilde{N}$ in the symplectomorphism (4.19) and indeed, the actual flux of a "tilde" link is $\tilde{E}_l = -j_l \tilde{N}_l$.

4.3 Twistorial parametrisation

In this last section we present the result [75] where it has been shown that it is possible to parametrise the discrete phase space of loop quantum gravity in terms of twistors, by means of the map (4.19). Indeed, in the authors' words, the name *twisted* geometries is meant both to emphasise the arising of discontinuity in the metric and to suggest a relations with Penrose's twistorial formalisms.

The starting point is the *Twistor space* $\mathbb{T}$ with coordinates z_A and $\underset{\sim}{z}_A$, which carries a natural symplectic form, with the standard Poisson algebra[5]:

$$\mathbb{T} \equiv \mathbb{C}^2 \oplus \bar{\mathbb{C}}^{2*} \qquad Z_\alpha = (z_A, \underset{\sim}{z}_A) \in \mathbb{T} \quad A = 0, 1 \tag{4.27a}$$

$$\{z_A\, ,\, \bar{z}_B\} = -i\delta_{AB} \qquad \left\{\underset{\sim}{z}_A\, ,\, \bar{\underset{\sim}{z}}_B\right\} = i\delta_{AB} \tag{4.27b}$$

On each $\mathbb{C}^2$ we can introduce two spinors:

$$|z\rangle = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \qquad |z] = \begin{pmatrix} -\bar{z}_1 \\ \bar{z}_0 \end{pmatrix} \tag{4.28}$$

and construct a future-pointing null vector X^μ exploiting the fact that $(\mathbb{I}, \sigma_i)$ are a basis for the square matrix of dimension two. Instead of the standard matrices we use the linear combinations $\sigma^\pm = \sigma_1 \pm i\sigma_2$.

$$X^\mu = (X^0, X^i) \tag{4.29a}$$

$$|z\rangle\langle z| = X^0\mathbb{I} + X^i\sigma_i \tag{4.29b}$$

Where the components can be found via the scalar product induced by the trace

---

Scalar product. Due to the use of the matrix $\sigma^\pm$, the scalar product and the raising of the indices have the following rules:

$$\sigma^- = \frac{1}{2}\sigma_+ \qquad \sigma^+ = \frac{1}{2}\sigma_- \qquad \sigma^3 = \sigma_3 \tag{4.30a}$$

$$\mathrm{Tr}\,[XY] = \frac{1}{2}X^+Y^- + \frac{1}{2}X^-Y^+ + X^3Y^3 \tag{4.30b}$$

$$X^i = \mathrm{Tr}\left[X\sigma^i\right] \qquad X^0 = \mathrm{Tr}\,[X] \tag{4.31a}$$

$$X^+ = \bar{z}_0 z_1 \quad X^- = z_0\bar{z}_1 \quad X^3 = \frac{1}{2}\left(|z_0|^2 - |z_1|^2\right) \qquad X^0 = \frac{1}{2}\left(|z_0|^2 + |z_1|^2\right) \equiv \frac{1}{2}\langle z|z\rangle \tag{4.31b}$$

The construction for the "tilded" quantities is the same:

$$\underset{\sim}{X}^i = \mathrm{Tr}\left[\underset{\sim}{X}\sigma^i\right] \qquad \underset{\sim}{X}^0 = \mathrm{Tr}\left[\underset{\sim}{X}\right] \tag{4.32a}$$

$$\underset{\sim}{X}^+ = \bar{\underset{\sim}{z}}_0\underset{\sim}{z}_1 \quad \underset{\sim}{X}^- = \underset{\sim}{z}_0\bar{\underset{\sim}{z}}_1 \quad \underset{\sim}{X}^3 = \frac{1}{2}\left(|\underset{\sim}{z}_0|^2 - |\underset{\sim}{z}_1|^2\right) \qquad X^0 = \frac{1}{2}\left(|\underset{\sim}{z}_0|^2 + |\underset{\sim}{z}_1|^2\right) \equiv \frac{1}{2}\langle\underset{\sim}{z}|\underset{\sim}{z}\rangle \tag{4.32b}$$

[5]Note the change of the sign in the definition of the algebra. It is the difference that is present in the twisted parametrisation, but it is now clear that we could have chosen differently. We will continue to keep track of this aspect of the formalism.

We then parametrise $\mathbb{C}^2/\left\{\langle z|z\rangle = 0\right\}$ in terms of X^μ and a phase $\varphi \equiv \arg(z_0) + \arg(z_1)$. Indeed the number of degrees of freedom in $\mathbb{C}^2$ is the same as $\mathbb{R}^4$ and it can be expressed in terms of four real quantities. X^μ carries three of them since it is a null vector, so we need one more variable to fully parametrise the space, namely φ.

Starting with the Poisson structure defined in 4.27b we can deduce the algebra of the new variables:

$$\left\{X^i\,,\,X^j\right\} = \varepsilon^{ij}{}_k X^k \qquad \left\{X^\pm\,,\,\varphi\right\} = \frac{X^0}{X^\mp} \qquad \left\{X^0\,,\,\varphi\right\} = \left\{X^3\,,\,\varphi\right\} = 0 \tag{4.33a}$$

$$\left\{\utilde{X}^i\,,\,\utilde{X}^j\right\} = -\varepsilon^{ij}{}_k \utilde{X}^k \qquad \left\{\utilde{X}^\pm\,,\,\utilde{\varphi}\right\} = -\frac{\utilde{X}^0}{\utilde{X}^\mp} \qquad \left\{\utilde{X}^0\,,\,\utilde{\varphi}\right\} = \left\{\utilde{X}^3\,,\,\utilde{\varphi}\right\} = 0 \tag{4.33b}$$

Here is an example of the computation:

Induced algebra of the new variables. Here the computation of the first Poisson brackets

$$\left\{X^+\,,\,X^-\right\} = \left\{\bar{z}_0 z_1\,,\,z_0\bar{z}_1\right\} = z_1\left\{\bar{z}_0\,,\,z_0\right\}\bar{z}_1 + z_0\left\{z_1\,,\,\bar{z}_1\right\}\bar{z}_0 = |z_1|^2(i) + |z_0|^2(-i) = -2iX^3$$
$$\left\{X^+\,,\,X^-\right\} = \left\{X^1 + iX^2\,,\,X^1 - iX^2\right\} = -2i\left\{X^1\,,\,X^2\right\}$$
$$\left\{X^1\,,\,X^2\right\} = X^3$$

and here a computation of the second Poisson brackets, for the "tilded" variables. We define

$$\utilde{\varphi} = \utilde{\varphi}_0 + \utilde{\varphi}_1 \qquad \utilde{\varphi}_0 = \arg(\utilde{z}_0) \qquad \utilde{\varphi}_1 = \arg(\utilde{z}_1)$$

and we compute the two terms separately

$$\left\{\utilde{X}^+\,,\,e^{i\utilde{\varphi}_0}\right\} = ie^{i\utilde{\varphi}_0}\left\{\utilde{X}^+\,,\,\utilde{\varphi}_0\right\} \tag{4.34}$$

$$\left\{\utilde{X}^+\,,\,e^{i\utilde{\varphi}_0}\right\} = \left\{\bar{\utilde{z}}_0\utilde{z}_1\,,\,\frac{\utilde{z}_0}{|\utilde{z}_0|}\right\} = \utilde{z}_1\left\{\bar{\utilde{z}}_0\,,\,\utilde{z}_0\right\}\frac{1}{|\utilde{z}_0|} + \utilde{z}_1\left\{\bar{\utilde{z}}_0\,,\,\frac{1}{|\utilde{z}_0|}\right\}\utilde{z}_0 =$$
$$-i\frac{\utilde{z}_1}{|\utilde{z}_0|} - \frac{\utilde{z}_1\utilde{z}_0}{2|\utilde{z}_0|}\left\{\bar{\utilde{z}}_0\,,\,\utilde{z}_0\right\}\bar{\utilde{z}}_0 = -i\left(\frac{\utilde{z}_1}{|\utilde{z}_0|} - \frac{1}{2}\frac{\utilde{z}_1}{|\utilde{z}_0|}\right) = -\frac{i}{2}\frac{\utilde{z}_1}{\utilde{z}_0} \tag{4.35}$$

$$\left\{\utilde{X}^+\,,\,\utilde{\varphi}_0\right\} = -ie^{-i\utilde{\varphi}_0}\left\{\utilde{X}^+\,,\,e^{i\utilde{\varphi}_0}\right\} = (-i)\frac{|\utilde{z}_0|}{\utilde{z}_0}\left(-\frac{i}{2}\frac{\utilde{z}_1}{|\utilde{z}_0|}\right) = -\frac{1}{2}\frac{\utilde{z}_1}{\utilde{z}_0} = -\frac{1}{2}\frac{|\utilde{z}_1|^2}{\utilde{X}^-} \tag{4.36}$$

In the same way we get $\left\{\utilde{X}^+\,,\,\utilde{\varphi}_1\right\} = -\frac{1}{2}\frac{|\utilde{z}_0|^2}{\utilde{X}^-}$ and finally

$$\left\{\utilde{X}^+\,,\,\utilde{\varphi}\right\} = -\frac{1}{2}\frac{|\utilde{z}_0|^2}{\utilde{X}^-} - \frac{1}{2}\frac{|\utilde{z}_1|^2}{\utilde{X}^-} = -\frac{\utilde{X}^0}{\utilde{X}^-} \tag{4.37}$$

The definition of the new variables provides a parametrisation of the twistor space $\mathbb{T}_*$, in which the origin has been removed:

$$\mathbb{T}_* = \mathbb{C}^2_* \times \mathbb{C}^2_*\left\{\left(X^i, \utilde{X}^i, \varphi, \utilde{\varphi}\right)\right\} \tag{4.38}$$

We now define a constraint C, imposing that the two spatial vectors X^i and $\underset{\sim}{X}^i$ have the same norm and study its hamiltonian flow

$$C \equiv X_0 - \underset{\sim}{X}_0 = 0 \tag{4.39}$$

$$\{C\,,\,z_A\} = \frac{i}{2} z_A \qquad \{C\,,\,\underset{\sim}{z}_A\} = \frac{i}{2}\underset{\sim}{z}_A \tag{4.40}$$

$$\mathfrak{C} \triangleright \left(|z\rangle\,,\,|\underset{\sim}{z}\rangle\right) = \left(e^{i\frac{\theta}{2}}\;|z\rangle\,, e^{i\frac{\theta}{2}}\;|\underset{\sim}{z}\rangle\right) \tag{4.41}$$

While X^i and $\underset{\sim}{X}^i$ do not change, it generates $U(1)$ transformation on the spinors translating the angles $\varphi \mapsto \varphi+\theta$ and $\underset{\sim}{\varphi} \mapsto \underset{\sim}{\varphi}+\theta$. The result achieved in [75] is that the symplectic reduction of the space $\mathbb{T}_*$ via the constraint C gives the space of the twisted geometries on a link, where the origin has been removed, i.e. $\mathcal{P}_*$. The map can be extended to the origin taking the completion of the space, in this one obtain a symplectomorphism with the whole space of twisted geometries[6].

$$\mathbb{T}/\!\!/C \cong \mathcal{S}^2_j \times \mathcal{S}^2_j \times T^*\mathcal{S}^1 = \mathcal{P}_j \tag{4.42}$$

Here is the map:

$$\begin{array}{c}\left(X^i, \underset{\sim}{X}^i, \varphi, \underset{\sim}{\varphi}\right) \\ \updownarrow \\ (z_A, \underset{\sim}{z}_A)\end{array} \quad \mapsto \quad \left(N, \tilde{N}, j, \xi\right) \;:\; \begin{cases} j = \frac{1}{2}\left(X^0 + \underset{\sim}{X}^0\right) \\ N^i = \frac{X^i}{j} \\ \tilde{N}^i = \frac{\underset{\sim}{X}^i}{j} \\ \xi = i\left(\ln \frac{z_0}{\bar{z}_0} - \ln \frac{\underset{\sim}{z}_0}{\bar{\underset{\sim}{z}}_0}\right) \end{cases} \tag{4.43}$$

The proof that such a map is a symplectomorphism is in [75]. We need to mention that a different choice is allowed for the variable ξ but they are related by a canonical transformation so we simply chose one of them

$$\xi_0 = i\left(\ln \frac{z_0}{\bar{z}_0} - \ln \frac{\underset{\sim}{z}_0}{\bar{\underset{\sim}{z}}_0}\right) \qquad\qquad \xi_1 = i\left(\ln \frac{z_1}{\bar{z}_1} - \ln \frac{\underset{\sim}{z}_1}{\bar{\underset{\sim}{z}}_1}\right) \tag{4.44}$$

Twistorial parametrisation of the Holonomy-Flux algebra

Thanks to the twistorial parametrisation of the twisted geometries and to the symplectomorphism between the latter and $T^*SU(2)$, showed in the previous section, it is possible to obtain the twistorial parametrisation of $T^*SU(2)$ combining the previous two parametrisations. Here we give the map

$$E^i = X^i \qquad\qquad \tilde{E}^i = -hEh^{-1} = \underset{\sim}{X}^i \qquad\qquad h(z_A, \underset{\sim}{z}_A) = \frac{|z\rangle\,\langle\underset{\sim}{z}| - [z|\,[\underset{\sim}{z}|}{\sqrt{\langle z|z\rangle\,\langle\underset{\sim}{z}|\underset{\sim}{z}\rangle}} \tag{4.45}$$

[6] We remember the note 2 at page 52 where we state that the actual isomorphism is with $\mathcal{P}/\!\!/\mathbb{Z}_2$ that identify the two configurations $(N, \tilde{N}, j, \xi) \leftrightarrow (-N, -\tilde{N}, -j, -\xi)$.

which provide the Holonomy-Flux algebra where X^i and $\undertilde{X}^i$ are respectively the right-invariant and the left-invariant vector field:

$$\left\{X^i\,,\,X^j\right\} = \varepsilon^{ij}{}_k X^k \qquad \left\{\undertilde{X}^i\,,\,\undertilde{X}^j\right\} = -\varepsilon^{ij}{}_k \undertilde{X}^k \tag{4.46a}$$

$$\left\{X^i\,,\,g\right\} = -\tau^i g \qquad \left\{\undertilde{X}^i\,,\,g\right\} = g\tau^i \tag{4.46b}$$

Area-matching constraint

The space of a twistor gave the opportunity to parametrise both twisted geometries and $T^*SU(2)$ in a simple way. Before we continue and show the extension of such a picture to $SL(2,\mathbb{C})$ variables, it is important to understand the geometrical meaning of the constraint C imposed to reduce the twistor space down to the twisted geometries. For such a purpose one has to look at the twistor space from the twisted geometries perspective.

Twisted geometries assign two unit vector $\vec{N}_l$ and $\vec{\tilde{N}}_l$ to each link, representing the normals to the face dual to the link l, as sees from the two polyhedra sharing it. However the area of such a face has the same value, by hypothesis, since it is represented by the variable j. The two vectors $j\vec{N}$ and $j\vec{\tilde{N}}$ in the twistorial parametrisation are represented by X^i and $\undertilde{X}^i$ which have the same norm since we reduced the $\mathbb{T}$ space through the constraint C imposing exactly this conditions. We can now reverse the logic and understand that working with the full twistor space, from the geometrical point of view, means to relax the uniqueness condition on the area $j \to (j, \undertilde{j})$ since in the unconstrained space the two area vectors have different norms and then the faces have different areas:

$$C \equiv X_0 - \undertilde{X}_0 = ||\vec{X}|| - ||\vec{\undertilde{X}}|| = ||z||^2 - ||\undertilde{z}||^2 = j - \undertilde{j} = 0 \tag{4.47}$$

Summary

In the first section of this chapter we presented the Regge calculus, a discretisation of the Einstein-Hilbert action that exploits the simplicial triangulations of a manifold to discretise the space-time through $4-$simplexes, the $4-$dimensional equivalent of a tetrahedron. An interesting aspect we emphasised is the concept of *deficit angle* which provide a notion of curvature on discrete manifold: it has support only on the 2D triangles and reduce to the Riemannian curvature in the continuum limit. Thanks to this notion we defined the Regge action as the sum over the triangulation of the products “area × deficit angle” which has the remarkable property to recover the Einstein-Hilbert action in the continuum limit. At the end of the section we focused on two important aspect of a Regge geometry: the properties of the discrete metric and the shape-matching conditions. We argued that the continuity imposed by the Regge axiomatisation guarantee that the shape-matching conditions are fulfilled. The geometrical picture that emerges from a *Regge geometry* is a collection of polyhedra, locally flat, with peculiar conditions that specify how these polyhedra are glued together.

In the second section we came back to loop quantum gravity and presented the recent developments in the comprehension of the classical picture behind the quantum spin-network states. According to [76], the definition of the theory over a single graph is a truncation of the full theory to a finite number of degrees of freedom so in this sense it can be seen as a discretisation and it could be interpreted as a discrete geometry. The full parametrisation of the phase space of loop gravity $T^*SU(2)$ provide a notion of discrete geometry which is far more general then the

Regge scheme: it is a *twisted geometries*[21]. We presented the map connecting the Holonomy and Flux variables with the twisted geometries variables: to each face shared by two polyhedra they assign two unit vector $\vec{N}_l - \tilde{\vec{N}}_l$ and a real number j_l, respectively standing for the normal to the face, as seen from the two tetrahedra sharing it, and the area of the face. There is another variable ξ_l, which is an angle representing the freedom to rotate the face around the direction $\vec{N}_l$ and it is responsible for the lack of the gluing conditions among the polyhedra. The picture that emerges and can be compared with a Regge geometry is the following: a *twisted geometry* is again a collection of locally flat polyhedra but it lacks of the shape-matching conditions among polyhedra, thus it allows the presence of discontinuous metrics, forbidden in Regge calculus.

In the last section we summarised the result achieved in [75]. We introduced the twistor space $\mathbb{T}$, its spinorial coordinates and their symplectic structure. On such a space it is possible to define a constraint C, to study its Hamiltonian flow and the symplectic reduction of the full space with respect to it. The resulting space is symplectomorphic to the twisted geometries space and thus to the phase space of loop quantum gravity on a link: it constitutes their *twistorial parametrisation*. In the end, we focused on the geometrical meaning of the constraint C understanding that it is possible to work in the full space $\mathbb{T}$ and it corresponds to relax the uniqueness condition on the area of the surface dual to the link.

Chapter 5

$SL(2,\mathbb{C})$ Hamiltonian General Relativity

Before Ashtekar proposed to treat General Relativity exploiting complex $SU(2)$ variables, it was impossible to proceed with the canonical quantisation since the structure of the constraints was highly non-linear, in fact, not even polynomial. In this sense the Ashtekar work was a turning point since, as written in these variables the constraints have a nice polynomial structure and the canonical quantisation was achieved. However the theory became complex and further reality conditions had to be imposed in order to recover General Relativity until, for different technical reasons, these variables were abandoned and replaced by the real Ashtekar-Barbero ones. Indeed, in the meanwhile, thank to the works of Barbero [10] and Immirzi [77, 78] an entire family of real $SU(2)$ variables was found, depending on the Immirzi parameter[1] and it was proved that the two set of variables were related by setting $\gamma = i$.

In this variables the Hamiltonian constraint assumed again a non-polinomial expression and the simplification was lost. Later, Speziale and Freidel introduced the twistor framework to deal with the nonlinear character of the theory, as showed in the previous chapter, but only in the case of the $SU(2)$ Ashtekar-Barbero variables. The extension of the twistorial formalism to $SL(2,\mathbb{C})$ variables, was recently achieved by Speziale, Wieland, Livine and Tambornino [79, 80, 81] and it has essentially two advantages:

- it allows to deal with the nonlinearity inside the theory exploiting the twistor variables, while working in a linear space, with Darboux coordinates
- re-introduce in the picture the complex Ashtekar variables to deal with the non-polinomial expression of the Hamiltonian constraint but it keeps the Immirzi parameter real and unspecified

The twistorial structure of Loop Quantum Gravity has recently proven to be interesting and it is gaining attention from the community in the last years. It originates from the $SL(2,\mathbb{C})$ Hamiltonian formulation of General Relativity within the Holst action and it is the framework in which the original part of this dissertation is formulated so we think it is mandatory to present here the basis of the canonical analysis[82, 83, 84, 85, 86]. In the first part we will present the standard theory, written in terms of $SL(2,\mathbb{C})$ variables and in the second part we

[1] This is actually the historical reason for the appearance of the γ parameter in the theory. Barbero found a real counterpart of Ashtekar variables and Immirzi showed that there was a one-parameter family of real variables analogue to the Barbero's one. So the parameter was called Immirzi or Barbero-Immirzi parameter.

will review the canonical analysis, focusing on the emergence of the torsionless equation as a secondary constraint.

5.1 Covariant variables

We start with the definition of the self-dual projector and its orthogonal complex conjugate:

$$P^{AB}{}_{IJ} = \frac{1}{2}\left(\delta^A_{[I}\delta^B_{J]} - \frac{i}{2}\varepsilon^{AB}{}_{IJ}\right) \qquad \bar{P}^{AB}{}_{IJ} = \frac{1}{2}\left(\delta^A_{[I}\delta^B_{J]} + \frac{i}{2}\varepsilon^{AB}{}_{IJ}\right) \qquad P^{AB}{}_{IJ}\bar{P}^{IJ}{}_{CD} = 0 \tag{5.1}$$

Replacing all contractions with the insertion of the self-dual projector we obtain the self-dual part of the Holst action. Adding the anti-self-dual part one recovers the whole Holst's action:

$$S_{\mathbb{C}}\left[e,\omega\right] = \int_{\mathcal{M}} \Sigma^A_B\left[e\right] \wedge \mathcal{F}^B_A\left[\omega\right] \tag{5.2a}$$

$$S_{\text{Holst}}\left[e,\omega\right] = -\frac{\hbar}{l_p^2}\frac{\gamma+i}{i\gamma}S_{\mathbb{C}} + \frac{\hbar}{l_p^2}\frac{\gamma-i}{i\gamma}\bar{S}_{\mathbb{C}} \tag{5.2b}$$

$$\mathcal{F}^A{}_B\left[\omega\right] = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B \ \in\ \mathfrak{so}(3,1) \qquad\qquad \Sigma^A{}_B[e] = e^{A\bar{C}} \wedge e_{B\bar{C}} \ \in\ \mathfrak{so}(3,1) \tag{5.3}$$

Where we introduced the Plebanski 2−form Σ. Since $SL(2,\mathbb{C})$ is the double cover of $SO(3,1)$ it is possible to set up an isomorphism between the two algebras and use elements of $\mathfrak{sl}(2,\mathbb{C})$ instead. The details of the isomorphism are in Appendix B. Introducing the anti-hermitian generators of the algebra we have

$$\tau_i = \frac{1}{2i}\sigma_i \qquad\qquad \sigma_i \quad \rightarrow \quad \text{Pauli matrices} \tag{5.4a}$$

$$\Sigma \ \in\ \mathfrak{sl}(2,\mathbb{C}) \ :\ \Sigma = \Sigma^i\tau_i \qquad \Sigma^i = \frac{1}{2}\varepsilon^i_{jk}\Sigma^{jk} + i\Sigma^i_0 \tag{5.4b}$$

$$\omega \ \in\ \mathfrak{sl}(2,\mathbb{C}) \ :\ \omega = \omega^i\tau_i \qquad \omega^i = \frac{1}{2}\varepsilon^i_{jk}\omega^{jk} + i\omega^i_0 \tag{5.4c}$$

Furthermore, since $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,1)$ the curvature of the connection ω is the the self-dual projection of the curvature:

$$\mathcal{F}\left[P^{IJ}_{AB}\omega^{AB}\right] = P^{IJ}_{AB}\mathcal{F}^{AB}\left[\omega\right] \tag{5.5a}$$

$$\mathcal{F} \ \in\ \mathfrak{sl}(2,\mathbb{C}) \ :\ \mathcal{F} = \mathcal{F}^i\tau_i \qquad \mathcal{F}^i = d\omega^i + \frac{1}{2}\varepsilon^i_{jk}\omega^j \wedge \omega^k \tag{5.5b}$$

Which gives the complex action for General Relativity in term of the covariant $SL(2,\mathbb{C})$ variables:

$$S_{\mathbb{C}}\left[\Sigma,\omega\right] = \int_{\mathcal{M}} \Sigma_i[e] \wedge \mathcal{F}^i[\omega] \qquad \Sigma,\omega \in \mathfrak{sl}(2,\mathbb{C}) \tag{5.6a}$$

$$S_{\text{Holst}} = -\frac{\hbar}{l_p^2}\frac{\gamma+i}{i\gamma}S_{\mathbb{C}} + \frac{\hbar}{l_p^2}\frac{\gamma-i}{i\gamma}\bar{S}_{\mathbb{C}} \tag{5.6b}$$

Notice that we have just performed the chiral splitting of the Lorentz algebra that one usually exploits to study Dirac's bi-spinors, Weyl spinors and in general to classify the representation of the Lorentz's algebra in terms of two indices labelling $SU(2)$ representations. It is standard matters for quantum field theory. An illuminating introduction to Lorentz and Poincarè Lie algbras is the Chapter 2 of [120]

Three-plus-one split

Following the ADM formalism A, in order to perform the Hamiltonian analysis we start assuming the topology of the manifold split in the usual form "three-plus-one" $\mathcal{M} = \Sigma \times \mathbb{R}$ so we can foliate our space-time in three dimensional surfaces and split the variables according to the topology.

The foliation is defined as the one-parameter embedding of Σ in $\mathcal{M}$:

$$X_t : \Sigma \mapsto \Sigma_t \subset \mathcal{M} \tag{5.7}$$

that allows to define the pullback of the 4D objects into Σ. Given any $\mathfrak{sl}(2,\mathbb{C})$ $q-$form γ^i on $\mathcal{M}$ its pullback $X_t^*(\gamma^i)$ is a q-form on Σ. The notation that is generally adopted uses the same letters for both the objects, so that one we will speak about the "spatial part" of an object, what we really mean is the 3D pull-back of the spatial part of the 4D object:

$$\gamma^i = X_t^*(\gamma^i) \tag{5.8}$$

We still need to identify the generator of the "time-changing" diffeomorphisms that map the Σ_t into $\Sigma_{t+\delta t}$. Since the theory is covariant under diffeomorphisms there is no unique way to do it so we choose the time flow to be generated by the future pointing unit vector:

$$\mathbf{t}^\mu = \frac{d}{dt} X_t^\mu = (1,0,0,0) \tag{5.9}$$

We can exploit the $\mathbf{t}$ vector field to identify the pullback on Σ of the time components of any $q-$form:

$$\gamma^0 = X_t^*(\mathbf{t}\gamma^i) \qquad\qquad \mathbf{t}\gamma^i(V_2,\ldots,V_q) = \gamma^i(\mathbf{t},V_2,\ldots,V_q) \tag{5.10}$$

Thanks to this definitions we can define the 3D pullback of curvature and, consistently, of the covariant derivative and the connection:

$$A^i = X_t^*(\omega^i) \qquad\qquad Df^i = X_t^*(\mathcal{D}f^i) = df^i + \varepsilon^i_{jk} A^j \wedge f^k \tag{5.11}$$

$$D^2 f^i = \varepsilon^i_{jk} F^j \wedge f^k \qquad\qquad \mathcal{D}^2 f^i = \varepsilon^i_{jk} \mathcal{F}^j \wedge f^k \qquad\qquad F^i = X_t^*(\mathcal{F}^i) \tag{5.12}$$

The densitized triads are just the pull-back of the Plebanski two-form

$$E_i = -X_t^*(\Sigma_i) = \frac{1}{2}\varepsilon_{ijk} e^j \wedge e^k \tag{5.13}$$

and we will work in the “time-gauge” which simplify considerably the calculations, see eq. (2.52):

$$\begin{cases} e^0_\mu = (N, 0) \\ e^I_0 = \left(N, N^a e^i_a\right) \end{cases} \tag{5.14}$$

Moreover, it is important to mention that the Ashtekar connection A, i.e. the spatial part of the $\mathfrak{sl}(2,\mathbb{C})$ connection ω, has an imaginary and a real part which carries very different geometrical information since the former is the Levi-Civita connection and the latter is the extrinsic curvature:

$$A^i = X^*_t(\omega^i) = \frac{1}{2}\varepsilon^{ij}_{\ \ k} X^*_t(\omega^k_j) + i X^*_t(\omega^i_0) = \Gamma^i + iK^i \tag{5.15}$$

In such a way the Holst action split into an integral over space and time and in the new variables it has the following expression:

$$S_{\mathbb{C}} = \int dt \int_\Sigma \left(-E^a_i (A^a_i - D_a \Lambda^i) + N^a F^i_{ab} E^b_i + \frac{i}{2} \underset{\sim}{N} \varepsilon_i^{\ lm} E^a_l E^b_m F^i_{ab} \right) \tag{5.16}$$

$$S_{\text{Holst}}\,[e,\omega] = -\frac{\hbar}{l_p^2}\frac{\gamma+i}{i\gamma} S_{\mathbb{C}} + \frac{\hbar}{l_p^2}\frac{\gamma-i}{i\gamma}\bar{S}_{\mathbb{C}} \tag{5.17}$$

Where $\Lambda^i = X^*_t(\mathfrak{t}\omega^i)$ and $\dot{A}^i - D\Lambda^i = X^*_t(\mathfrak{t}\mathcal{F}^i)$ is the pullback of the time components of the curvature and $\underset{\sim}{N}$ is the densitised lapse function defined as the densitised triad: $\underset{\sim}{N} = N \det e$

5.2 Hamiltonian analysis, with $\mathfrak{sl}(2,\mathbb{C})$ variables

We first introduce the following shorthand notation

$$\Pi^a_i = \frac{\hbar}{L_p^2}\frac{\gamma+i}{i\gamma} E^a_i \in \mathfrak{sl}(2,\mathbb{C}) \qquad \bar{\Pi}^a_i = -\frac{\hbar}{L_p^2}\frac{\gamma-i}{i\gamma} E^a_i \in \mathfrak{sl}(2,\mathbb{C}) \tag{5.18}$$

The Holst Lagrangian (2.56), written in terms of the covariant variables has the following expression, where the notation “c.c.” stands for “complex conjugate” of all the previous terms:

$$S_{\text{Holst}} = \int_{\mathcal{M}} d^4x \,\left(\dot{A}^i_a \Pi^a_i + \dot{\bar{A}}^i_a \bar{\Pi}^a_i - H \right) \qquad H = G_i[\Lambda^i] + H_a[N^a] + H[N] \tag{5.19}$$

$$G_i[\Lambda^i] = \int_\Sigma \Lambda^i D_a \Pi^a_i + \text{c.c.} \tag{5.20a}$$

$$H_a[N^a] = \int_\Sigma N^a F^i_{ab} \Pi^b_i + \text{c.c.} \tag{5.20b}$$

$$H[N] = -\frac{L_p^2}{\hbar}\int_\Sigma N \frac{\gamma}{\gamma+i} \varepsilon_i^{\ lm} F^i_{ab} \Pi^a_l \Pi^b_m + \text{c.c.} \tag{5.20c}$$

Looking at the system of the constraints, as a functionals of the variables $\left(\Pi_i^a, A_b^j\right)$ and $\left(\bar{\Pi}_i^a, \bar{A}_b^j\right)$, one realise that they are well defined and so it is possible to replace the $\left(E_i^a, A_b^j\right)$ with the complex variables, provided one add the following reality conditions, accounting for the relations (5.18):

$$C_i^a = \frac{L_p^2}{i\hbar}\left(\frac{i\gamma}{i+\gamma}\Pi_i^a + \frac{i\gamma}{i-\gamma}\bar{\Pi}_i^a\right) = 0 \tag{5.21}$$

Obviously this is not a step that one should take so easily. However, even if there are some subtleties, it has been proven rigorously that the Lagrangian and the Hamiltonian system presented here are actually equivalent. Through the real and imaginary part of the generators Π we can write the reality condition in the following way:

$$\hbar L_i^a = \Pi_i^a + \bar{\Pi}_i^a \qquad\qquad i\hbar K_i^a = \Pi_i^a - \bar{\Pi}_i^a \tag{5.22}$$

$$C_i^a = L_P^2 \frac{\gamma}{\gamma^2+1}\left(K_i^a + \gamma L_i^a\right) \qquad \Rightarrow \qquad \left(K_i^a + \gamma L_i^a\right) = 0 \tag{5.23}$$

These *reality conditions* are usually called *simplicity constraints* since it has been proven by Wieland [86] that they can also be interpreted as the relations imposing that the Plebanski two-form is "simple" in the sense that it arise from the wedge product among two one-forms (see eq. (5.3)):

$$\Sigma^{AB} = e^A \wedge e^B \tag{5.24}$$

The simplicity constraints arise in the context of the Plebanski theory [83, 85, 87], where General Relativity can be seen as a constrained BF-topological theory and the constraints projecting the theory onto General Relativity are the simplicity constraints. Moreover since the spin-foam formulation of Loop Quantum Gravity [11, 12, 13, 14] exploits such a formulation as a classic starting point, the appearance of these constraints in the canonical formulation makes us hope that canonical and covariant formulation will soon converge towards a unified formulation.

We are now ready to look at the Hamiltonian formalism of the theory. The symplectic structure can be read from the action (5.19) and it is defined on an infinite-dimensional auxiliary phase space $\mathcal{P}_{aux}$ constructed by pairs of field configurations $\left(\Pi_i^a, A_a^i\right)$ where A_a^i is the complex Ashtekar connection, while its conjugate momentum, Π_a^i is an $\mathfrak{sl}(2,\mathbb{C})$-valued two-form. The only non-vanishing Poisson brackets of the elementary variables are the following:

$$\left\{\Pi_i^a(x)\,,\, A_b^j(y)\right\} = \delta_b^a \delta_i^j \delta^{(3)}(x,y) \qquad\qquad \left\{\bar{\Pi}_i^a(x)\,,\, \bar{A}_b^j(y)\right\} = \delta_b^a \delta_i^j \delta^{(3)}(x,y) \tag{5.25}$$

The Hamiltonian constraint generating the flow is a linear combination of the constraint (5.20a) - (5.20b) - (5.20c), plus the simplicity constraints (5.23) with their respective Lagrange's multiplier:

$$H^* = H + C_i^a[V_a^i] \tag{5.26}$$

$$\frac{d}{dt}X = \{H^*\,,\, X\} \tag{5.27}$$

We discuss here the constraint algebra and check the stability of the constraint equations. As we will see, the reality conditions are preserved under the Hamiltonian flow only if two additional constraints are satisfied.

The first class constraints are those that form a closed algebra since they are the generators of the gauge transformations of the theory. All the constraints but the simplicity ones are first class:

$$\Big\{G_i[\Lambda_1^i]\,,\,G_j[\Lambda_2^j]\Big\} = G_i\Big[\varepsilon^{ijk}\Lambda_1^j\Lambda_2^k\Big] \approx 0 \tag{5.28a}$$

$$\big\{G_i[\Lambda^i]\,,\,H_a[N^a]\big\} = \big\{G_i[\Lambda^i]\,,\,H[\underset{\sim}{N}]\big\} = 0 \tag{5.28b}$$

$$\big\{H_a[N_1^a]\,,\,H_b[N_2^b]\big\} = H_a\Big[N_1^b\partial_b N_2^a - N_2^b\partial_b N_1^a\Big] - G_i[F^i_{ab}N_1^a N_2^b] \approx 0 \tag{5.28c}$$

$$\big\{H_a[N^a]\,,\,H[\underset{\sim}{N}]\big\} = -H[N^a\partial_a\underset{\sim}{N} - \underset{\sim}{N}\partial_a N^a] - G_i\left[\frac{\delta H}{\delta \Pi^a_i}V^a\right] \approx 0 \tag{5.28d}$$

$$\big\{H[\underset{\sim}{N}_1]\,,\,H[\underset{\sim}{N}_2]\big\}\big|_{C=0} = -H_a\left(E^a_j E^{jb}(\underset{\sim}{N}_1\partial_b\underset{\sim}{N}_2 - \underset{\sim}{N}_2\partial_b\underset{\sim}{N}_1)\right) \approx 0 \tag{5.28e}$$

The only constraints that are second class are the simplicity constraints, whose stability under the Hamiltonian flow requests further conditions to be imposed. Following Dirac's nomenclature the new conditions are called *secondary constraints*:

$$\dot{C}^i_a = \big\{C^i_a\,,\,H\big\} \approx -\varepsilon_{il}^{\;\;m}\left(\frac{1}{2i}\left(\Lambda^l - \bar{\Lambda}^l\right) - N^b K^l_b - e^{lb}\partial_b N\right)E^a_m + N\varepsilon^{abc}\frac{1}{2}\left(D_a + \bar{D}_a\right)e_{ic} \tag{5.29}$$

The first part of this equation can be fulfilled by fixing the imaginary part of the Lagrange multiplier $\Lambda^l = \phi^i + i\xi^i$ to satisfy:

$$\frac{1}{2i}\left(\Lambda^l - \bar{\Lambda}^l\right) = \xi^l = N^b K^l_b + e^{lb}\partial_b N \tag{5.30}$$

Inserting this solution inside the former equation the last term must vanish independently:

$$N\varepsilon^{abc}\frac{1}{2}\left(D_a + \bar{D}_a\right)e_{ic} = N\varepsilon^{abc}\nabla_a e_{ic} = 0 \tag{5.31}$$

Where $\nabla_a = \partial_a + [\Gamma_a, \cdot]$ is equal to the covariant derivative defined with the spatial Levi-Civita connection up to a difference tensor Δ^i_a. Moreover, one the surface $C = G = 0$ it is possible to rewrite the full equation in the following way:

$$\begin{aligned} -\frac{1}{2i}\left(\Lambda^l - \bar{\Lambda}^l\right) + N^b K^l_b + e^{lb}\partial_b N &\approx -\frac{N}{2}(e^{ic}e^{jb} - e^{ib}e^{jc})\mathcal{D}_a e_{ic} \\ &\approx \frac{N}{2}\varepsilon^{\;j}_{il}\Delta^l_b e^{ib} = 0 \end{aligned} \tag{5.32}$$

$$\Delta^i_a = \frac{1}{2}\left(A^i_a + \bar{A}^i_a\right) - \Gamma^i_a[E] = 0 \tag{5.33}$$

Where $\Gamma^i_a[E]$ is the three dimensional Levi-Civita connection constructed by the metric induced by the triad E. Since the difference tensor vanishes on the hyper-surface of the solutions of both the simplicity constraint and the Gauss constraint we conclude that the covariant derivative ∇_a equals the covariant derivative induced by the spatial Levi-Civita connection and the condition (5.32) imposes that such a derivative must be compatible with the triad, i.e. the spatial part of the torsionless equation. In facts, the last equation is highly non-polynomial as pointed out in [95] but we can write it in the following equivalent way

$$2T^i_{ab} = D_a e^i_b + \bar{D}_a e^i_b = 2\nabla_a e^i_b = 0 \tag{5.34}$$

which has exactly the same geometrical meaning since it impose that the spatial components of the four-dimensional torsion tensor must vanish. These are the *secondary constraints* and we must add them to the Hamiltonian for the procedure to be consistent. Moreover we should check their stability under the Hamiltonian flow.

$$H' = H + T^i_{ab}[M^{ab}_i] \tag{5.35a}$$

$$\dot{T}^i_{ab} = \left\{T^i_{ab}\,,\, H'\right\} \approx -\frac{L_p^2}{\hbar}\frac{2\gamma^2}{\gamma^2+1}\varepsilon^i_{\ lm}V^l_a e^m_b = 0 \quad \Rightarrow \quad V^a_i = 0 \tag{5.35b}$$

We conclude that the stability of the secondary constraints does not give rise to additional constraints since it can be fulfilled by setting to zero the value of the Lagrange multipliers of the simplicity constraints.

The canonical analysis highlight the presence of secondary constraints whose solutions are well known since they impose that the spatial part of the torsion tensor must vanish, moreover some of the Lagrange's multiplier have been fixed. We can put the solutions of the secondary constraints back in the Hamiltonian H' and fix the Lagrange's multiplier to the requested values thus we obtain again the Hamiltonian H but the residual gauge symmetry of the system are different. Since we gauge-fixed the imaginary part of the Gauss Lagrange's multipliers, we gauge-fixed the boost in the time direction generated by $SL(2,\mathbb{C})$ which means that we are left with an $SU(2)$ symmetry inducing spatial rotations on Σ.

On the vanishing of the torsion tensor

We would like to conclude this analysis pointing out how the other terms of the torsionless equations arise.

Torsionless equation. The 4D torsion tensor has the following expression:

$$T^I \equiv \mathcal{D}e^I \tag{5.36a}$$

$$T^I_{\mu\nu} = \partial_\mu e^I_\nu - \partial_\nu e^I_\mu + \omega^{IJ}_\mu e_{J\nu} - \omega^{IJ}_\nu e_{J\mu} \tag{5.36b}$$

The spatial components are defined via the embedding map X_t and we need to remember that $A^i = X^*_t(\omega^i)$. We already proved that the spatial part vanishes due to the secondary constraints:

$$\dot{C} \approx 0 \qquad \Rightarrow \qquad T^i_{ab} = 0 \tag{5.37}$$

Now we deal with the remaining components:

$$T^0_{ab} = X^*_t(\mathcal{D}e^0) \tag{5.38a}$$

$$T^0_{ab} = \partial_a e^0_b - \partial_b e^0_a + A^{0i}_a e_{ib} - A^{0i}_b e_{ja} = A^{0i}_a e_{ib} - A^{0i}_b e_{ja} = K_{ab} - K_{ba} \tag{5.38b}$$

Thanks to the Gauss and simplicity constraints we have

$$D_a\Pi^a_i\Big|_{C=0} \approx D_aE^a_i\Big|_{G=0} \approx 0 \tag{5.39}$$

$$\mathrm{Im}\,[D_aE^a_i] = \varepsilon_i^{\ lm}K_{lb}E^b_m = \det(e)\,\varepsilon_i^{\ lm}K_{lb}e^b_m = \det(e)\,\varepsilon_i^{\ lm}K_{lm} = 0 \tag{5.40}$$

$$K_{lm} = K_{ml} \quad \Rightarrow \quad T^0_{ab} = 0 \tag{5.41}$$

Now we turn our attention to T^0_{0a}:

$$T^0_{0a} = X^*_t(\mathbf{t}\mathcal{D}e^i) \tag{5.42a}$$

$$T^0_{0a} = \partial_0 e^0_a - \partial_a e^0_0 + A^{0i}_0 e_{ia} - K^i_a e_{i0} = -\partial_a N + A^{0i}_0 e_{ia} - K^i_a e_{i0} \tag{5.42b}$$

Comparing this equation with eq. (5.30) contracted with e_{ia} we see that it is fulfilled thanks to the gauge-fixing condition on ξ^i, see eq. (5.30), indeed

$$-\partial_a N + \xi^i e_{ia} - N^b K_{ab} = 0 \qquad \Leftrightarrow \qquad T^0_{0a} = 0 \tag{5.43}$$

Finally, it is possible to show that the last torsionless condition comes from the equation of motion of the triad, which is natural since it explicitly involves the time derivative of the triad:

$$T^i_{0a} = \partial_0 e^i_a - e^i_b\partial_a N^b + \omega^{ij}_0 e^j_a - \omega^{ij}_a e^j_0 + \omega^{i0}_0 e^0_a - \omega^{i0}_a e^0_0 \tag{5.44}$$

we are not going through all the details and show that this equation is indeed the last torsionless condition here because it is just a long computation that requires algebraic manipulation and it has no future utility for our purpose. For all the details we refer to [60].

$$\partial_0 e^i_a = \Big\{H\,,\,e^i_a\Big\} \qquad \Leftrightarrow \qquad T^i_{0a} = 0 \tag{5.45}$$

Dirac brackets and the Ashtekar-Barbero variables

We now have an Hamiltonian system where first class constraints are the real part of the $\mathfrak{sl}(2,\mathbb{C})$-Gauss constraints that generates rotations plus the spatial diffeomorphism and the Hamiltonian constraints. The second class constraints are the reality conditions (or simplicity constraints) (5.23) and the secondary constraints (5.33). Here we give the algebra of the second class constraints

$$\big\{C^a_i(x)\,,\,C^b_j(y)\big\} = 0 \tag{5.46}$$

$$\Big\{\Delta^i_a(x)\,,\,\Delta^j_b(y)\Big\} = \frac{L_p^2}{\hbar}\frac{\gamma}{\gamma^2+1}\left(\frac{\delta\Gamma^j_b(y)}{\delta E^a_i(x)} - \frac{\delta\Gamma^i_a(x)}{\delta E^b_j(y)}\right) \tag{5.47}$$

$$\Big\{C^a_i(x)\,,\,\Delta^j_b(y)\Big\} = \frac{L_p^2}{\hbar}\frac{\gamma^2}{\gamma^2+1}\delta^j_i\delta^a_b\delta^{(3)}(x,y)$$

Second class constraint algebra. Here we compute the only non-vanishing Poisson bracket among the constraints.

$$\left\{C_i^a(x)\,,\,\Delta_b^j(y)\right\} = \frac{L_p^2}{2\hbar}\left\{\frac{\gamma}{\gamma+i}\Pi_i^a + \frac{\gamma}{\gamma-i}\bar{\Pi}_i^a\,,\,A_b^j+\bar{A}_b^j\right\} = \frac{L_p^2}{2\hbar}\left(\frac{\gamma}{\gamma+i}+\frac{\gamma}{\gamma-i}\right)\delta_i^j\delta_b^a\delta^{(3)}(x,y) =$$
$$= \frac{L_p^2}{\hbar}\frac{\gamma^2}{\gamma^2+1}\delta_i^j\delta_b^a\delta^{(3)}(x,y)$$

$$\left\{\Delta_a^i(x)\,,\,\Delta_b^j(y)\right\} = -\frac{1}{2}\left\{A_a^i+\bar{A}_a^i\,,\,\Gamma_b^j\right\} - \frac{1}{2}\left\{\Gamma_i^a\,,\,A_b^j+\bar{A}_b^j\right\} = \frac{1}{2}\frac{\delta\Gamma_b^j}{\delta\Pi_j^a} + \frac{1}{2}\frac{\delta\Gamma_b^j}{\delta\bar{\Pi}_j^a} - \frac{1}{2}\frac{\delta\Gamma_a^i}{\delta\Pi_j^b} - \frac{1}{2}\frac{\delta\Gamma_a^i}{\delta\Pi_j^b} =$$
$$= \frac{L_p^2}{2\hbar}\left(\frac{i\gamma}{\gamma+i}-\frac{i\gamma}{\gamma-i}\right)\frac{\delta\Gamma_b^j}{\delta E_j^a} - \frac{L_p^2}{2\hbar}\left(\frac{i\gamma}{\gamma+i}-\frac{i\gamma}{\gamma-i}\right)\frac{\delta\Gamma_a^i}{\delta E_j^b} = \frac{L_p^2}{\hbar}\frac{\gamma}{\gamma^2+1}\left(\frac{\delta\Gamma_b^j(y)}{\delta E_i^a(x)} - \frac{\delta\Gamma_a^i(x)}{\delta E_j^b(y)}\right)$$

In full generality Dirac proved [110, 111, 112] that the hyper-surface of the solutions of the second class constraints $\mathcal{P}$ carries a natural symplectic structure that can be extended as a degenerate symplectic form on the full phase space $\mathcal{P}_{aux}$. The explicit expression of the symplectic form is given in terms of the Dirac brackets that have the remarkable property of making the algebra of the constraints first class:

$$\{A\,,\,B\}_D = \{A\,,\,B\} - \frac{\hbar}{L_p^2}\frac{\gamma^2+1}{\gamma}\int_{x\in\Sigma}\{C_i^a(x)\,,\,A\}\left\{\Delta_a^i(x)\,,\,B\right\} + \tag{5.48a}$$
$$+\frac{\hbar}{L_p^2}\frac{\gamma^2+1}{\gamma}\int_{x\in\Sigma}\{C_i^a(x)\,,\,B\}\left\{\Delta_a^i(x)\,,\,A\right\} \tag{5.48b}$$

Thanks to the Dirac brackets we have a constrained Hamiltonian system with a first class algebra. The noteworthy result is that such a symplectic structure, on the space $\mathcal{P}$ is diagonalised by the $SU(2)$ Ashtekar-Barbero connection $A^{(\gamma)}{}^i_a = \Gamma_a^i + \gamma K_a^i$ together with the real densitized triad E_i^a

$$\left\{E_i^a(x)\,,\,E_j^b(y)\right\}_D \approx \left\{A^{(\gamma)}{}^i_a(x)\,,\,A^{(\gamma)}{}^j_b(y)\right\}_D \approx 0 \tag{5.49a}$$
$$\left\{E_i^a(x)\,,\,A^{(\gamma)}{}^j_b(y)\right\}_D \approx \gamma\frac{L_p^2}{\hbar}\delta_b^a\delta_i^j\delta^{(3)}(x,y) \tag{5.49b}$$
$$\{C_i^a\,,\,\cdot\}_D \approx \left\{\Delta_b^j\,,\,\cdot\right\}_D \approx 0 \tag{5.49c}$$

Since they already are solutions of the second class constraints in the sense that they identify a point on the space $\mathcal{P}$ we can use them as coordinate on this space and thus as canonical conjugate variables. In this way we recovered the standard Hamiltonian formulation in terms of $SU(2)$ variables, presented in section 2.4.

Summary

In this chapter we presented a review of the canonical analysis for the Hamiltonian formulation of General Relativity written in terms of the covariant $SL(2,\mathbb{C})$ variables. Starting with the Holst action written in terms of tetrad one introduces the self-dual and anti self-dual projectors and perform the three-plus-one split, working with the complex Ashtekar variables, while keeping the Immirzi parameter real and untouched. In facts, the complex variables are a better choice essentially for two reasons: the real Ashtekar-Barbero connection does not transform well under Lorentz transformation[108, 109], while the complex Ashtekar connection transform linearly; furthermore written in this variables the structure of the constraints is far more simple, especially the Hamiltonian constraint which now has a nice polynomial form.

We showed the algebra of the constraints. The first class constraints generates the gauge symmetries of the theory, there is the "complex" Gauss law enforcing the $SL(2,\mathbb{C})$ symmetry and the usual diffeomorphism constraints that split into spatial diffeomorphisms and Hamiltonian constraint generating the "time" evolution. With respect to the $SU(2)$ case the new feature is the presence of the simplicity constraints, the reality conditions on the spatial triad, which make the algebra second class since they have a non-vanishing Poisson brackets with the Hamiltonian. We reviewed the canonical analysis in which, in order to assure the stability under the Hamiltonian flow of the simplicity constraints, further conditions must be added: the secondary constraints. Their geometrical interpretation is neat since they are the spatial components of the four-dimensional torsionless equation so results agree with the ordinary Lagrangian formulation. Moreover there is a restriction on those components of the Lagrange multiplier that generate boosts along the time direction which, together with the Gauss law $D_a \Pi_i^a = 0$ and the equations of motions for the triad, amounts to set the four-dimensional torsion to zero. The stability of the secondary constraints does not provide any auxiliary condition since it is fulfilled by simply fixing the Lagrange's multipliers of the simplicity constraints.

Finally, on the space of the solutions of the second class constraints $\mathcal{P}$ there is a natural symplectic structure provided by the Dirac's bracket. This structure is diagonal if expressed in terms of the Ashtekar-Barbero variables so that one can choose them as coordinates on the phase space $\mathcal{P}$ and recover the $SU(2)$ Hamiltonian formulation presented in Chapter 3. It is worth to mention that the original version of the theory exploited the complex $SU(2)$ Ashtekar variables and it was believed that the two formulations were related only by fixing the Immirzi parameter to an imaginary value $\gamma = i$. Even if this is a possibility, it is not the only one. Thanks to the covariant formalism it is possible to look at the $SU(2)$ variables from the wider perspective offered by $SL(2,\mathbb{C})$ connection and fluxes and a key role is played by the secondary constraints which actually provide a clear answer to the problem of relating the two formalisms without having to fix the Immirzi parameter.

Furthermore in the formulations both with the complex Ashtekar variables and with the real Ashtekar-Barbero variables, the covariance of the theory under Lorentz transformations was not manifest since they exploit $SU(2)$ variables, breaking the Lorentz transformations in boosts and rotations and gauge-fixing the boost part. Both choices are not natural as seen form the point of view of the symmetries of the theory, which actually requires to work with $SO(3,1)$ variables, on the contrary the Hamiltonian covariant formulation is much more intuitive since it uses the "covariant variables" of the double covering of the Lorentz group $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,1)$. At the same time it keep manifest the Lorentz covariance of the theory and shed new light on the relation between the complex Ashtekar connection, the Ashtekar-Barbero connection and the role of the Immirzi parameter.

Chapter 6

Twistorial structure of loop gravity

In the last chapter we presented the canonical analysis of the $SL(2,\mathbb{C})$ canonical formulation of General Relativity and explicitly stated the importantance to work within such a formalism since, with respect to the standard formulation, it provides a wider point of view on the theory and allows us to better understand some obscure issues in the Hamiltonian formulation, like the $\gamma = i$ fixing. However the canonical analysis presented concerns the standard General Relativity, which is formulated in continuous space-time and since the main purpose of Loop Quantum Gravity is to understand the quantum property of space-time, we now turn our attention to the discretisation of the action.

The smearing procedure was introduced in section 2.5 for the Ashtekar-Barbero variables and, in Loop Quantum Gravity, it is a mandatory step to achieve the quantisation of the theory. We presented the classical phase space structure that emerge after this discretisation in Chapter 4 and its recent geometric interpretation in terms of a new notion of discrete geometry called *twisted geometry*, the classical counterpart of the quantum spin-network states. Furthermore, we showed the fascinating relation among loop quantum gravity and twistors proposed in [75], which accomplish the twistorial parametrisation of the $T^*SU(2)$ phase space of loop gravity, generated by the real Ashtekar-Barbero variables. On this ground, in this chapter we will first present the discrete phase space that arise after imposing the smearing procedure on the Holst action, written in terms of the covariant variables, which is the *covariant phase space* of loop gravity $T^*SL(2,\mathbb{C})$. In the second and third section we will review the landscape of the remarkable work by Speziale, Wieland, Livine and Tambornino [79, 80, 81] in which the twistorial formalisms can be pushed further to describe the covariant discrete phase space $T^*SL(2,\mathbb{C})$ giving rise to the twistorial phase space for loop quantum gravity, together with the notion of *twistor networks* and *covariant twisted geometries*.

6.1 Discrete covariant phase space

In this first section we present the covariant phase space arising after the smearing procedure of the covariant Hamiltonian General Relativity. In order to read-off the symplectic structure the $(3+1)$ split was performed and led to the "canonical form" of the Holst action. The switching to the $SL(2,\mathbb{C})$ variables was performed and led to the introduction of the simplicity constraints thus we were left with the following Hamiltonian system which we studied

$$H^{*}=G_{i}[\Lambda^{i}]+H_{a}[N^{a}]+H[N]+C_{i}^{a}[V_{a}^{i}] \tag{6.1}$$

$$\left\{\Pi_{i}^{a}(x)\,,\,A_{b}^{j}(y)\right\}=\left\{\bar{\Pi}_{i}^{a}(x)\,,\,\bar{A}_{b}^{j}(y)\right\}=\delta_{b}^{a}\delta_{i}^{j}\delta^{(3)}(x,y) \tag{6.2}$$

The explicit expression of the constraint is in equations (5.20a), (5.20b), (5.20c), (5.21).

As we did in Chapter 2 for the $SU(2)$ variables, we introduce an oriented graph Γ and smear the variables in the usual way, see (2.78a) - (2.80c): the Plebanski two form Σ is smeared over the surfaces S_l dual to the links l while the $\mathfrak{sl}(2,\mathbb{C})$ connection is smeared along the link. The smeared variables are the $SL(2,\mathbb{C})$ Holonomy and the $\mathfrak{sl}(2,\mathbb{C})$ fluxes

$$h_{l}=\mathcal{P}\exp\left[-\int_{l}A\right]\;\in\;SL(2,\mathbb{C}) \qquad\qquad \Pi_{l}=\int_{S}h_{\pi_{\gamma}}\Pi^{a}h_{\pi_{\gamma}}^{-1}\;\hat{n}_{a}d^{2}\sigma\;\in\;\mathfrak{sl}(2,\mathbb{C}) \tag{6.3}$$

Each link carries an orientation so we can defined the elements with reversed orientation l^{-1}:

$$h_{l^{-1}}=h_{l}^{-1} \qquad\qquad \Pi_{l^{-1}}\equiv\underset{\sim}{\Pi}_{l}=-h_{l}\Pi_{l}h_{l}^{-1} \tag{6.4}$$

Again, thanks to the theory of Poisson structures over groups[105], $SL(2,\mathbb{C})$ can be seen as a manifold and we can trivialise the cotangent bundle with its right-invariant vector field Π. The smeared phase space is thus the direct sum of one $T^{*}SL(2,\mathbb{C})$ over the links and on each one of them we have the Poisson structure generated by the symplectic two-form:

$$\mathcal{P}=\bigoplus_{l\in\Gamma}T^{*}SL(2,\mathbb{C}) \tag{6.5a}$$

$$\left(\Pi_{l},\underset{\sim}{\Pi}_{l},h_{l}\right)\in T^{*}SL(2,\mathbb{C}) \tag{6.5b}$$

$$\left\{\Pi_{l}\,,\,\underset{\sim}{\Pi}_{l'}\right\}=0 \qquad\qquad \left\{h_{l}\,,\,h_{l'}\right\}=0 \tag{6.6a}$$

$$\left\{(\Pi_{l})_{j}\,,\,h_{l'}\right\}=-h_{l}\tau_{j}\delta_{ll'} \qquad\qquad \left\{(\underset{\sim}{\Pi}_{l})_{j}\,,\,h_{l'}\right\}=\tau_{i}h_{l}\delta_{ll'} \tag{6.6b}$$

$$\left\{(\Pi_{l})^{i}\,,\,(\Pi_{l'})^{j}\right\}=-\varepsilon^{ij}_{\ \ k}\,(\Pi_{l})^{k}\,\delta_{ll'} \qquad\qquad \left\{\left(\underset{\sim}{\Pi}_{l}\right)^{i}\,,\,\left(\underset{\sim}{\Pi}_{l'}\right)^{j}\right\}=-\varepsilon^{ij}_{\ \ k}\,\left(\underset{\sim}{\Pi}_{l}\right)^{k}\delta_{ll'} \tag{6.6c}$$

In full analogy with the theory formulated in terms of Ashtekar-Barbero variables, the smeared phase space is the direct sum of $T^{*}SL(2,\mathbb{C})$ on each link. The Lorentz invariance is imposed by the smeared Gauss constraint which amounts to impose gauge invariance, locally at each node in the graph, the only difference is that now we have an $SL(2,\mathbb{C})$ gauge group, instead of an $SU(2)$ one. However, as in the continuous case, we are going to use a partial gauge-fixing of the Lorentz symmetry since we will use the time-gauge, which means that the left symmetry is going to be the $SU(2)$ rotations subgroup of $SL(2,\mathbb{C})$ and consequently we will be left with and ordinary $SU(2)$ Gauss law.

6.2 Twistor networks

Thanks to Penrose's work [91, 92] it is known that the twistor space carries a representation of $T^*SL(2,\mathbb{C})$ space so it is possible to parametrise the discrete phase space of general relativity in terms of twistorial variables [80, 81]. We begin with the introduction of the oriented graph Γ which we colour with a couple of twistor on each link $(Z_l, \underset{\sim}{Z}_l)$ associated respectively to the source and the target of the link l. Each twistor comes with its space and the whole structure define a *twistor network*:

$$(Z_l, \underset{\sim}{Z}_l) \in \mathbb{T}_l \oplus \underset{\sim}{\mathbb{T}}_l \qquad S_\Gamma = \bigoplus_{l\in\Gamma} \left(\mathbb{T}_l \oplus \underset{\sim}{\mathbb{T}}_l\right) \tag{6.7}$$

$$\mathbb{T}_l = \underset{\sim}{\mathbb{T}}_l \equiv \mathbb{C}^2_l \oplus \bar{\mathbb{C}}^{2*}_l \qquad \left((\omega_l)^A, (\bar{\pi}_l)_{\bar{A}}\right) \in \mathbb{T}_l \tag{6.8}$$

From now on, we will focus on the single space $\mathbb{T}^2_l = \left(\mathbb{T}_l \oplus \underset{\sim}{\mathbb{T}}_l\right)$ on each link and we will drop the index l. The space can be endowed with an $SL(2,\mathbb{C})$ invariant symplectic structure which we give through the induced Poisson brackets:

$$\left\{\pi_A\,,\,\omega^B\right\} = \delta^B_A \qquad \left\{\bar{\pi}_{\bar{A}}\,,\,\bar{\omega}^{\bar{B}}\right\} = \delta^{\bar{B}}_{\bar{A}} \tag{6.9}$$

$$\left\{\underset{\sim}{\omega}_A\,,\,\underset{\sim}{\pi}^B\right\} = \delta^B_A \qquad \left\{\underset{\sim}{\bar{\omega}}_{\bar{A}}\,,\,\underset{\sim}{\bar{\pi}}^{\bar{B}}\right\} = \delta^{\bar{B}}_{\bar{A}} \tag{6.10}$$

Coherently with the symplectic structure, it is possible to define the hamiltonian vector fields $\left(\Pi, \bar{\Pi}\right)$ and $\left(\underset{\sim}{\Pi}, \underset{\sim}{\bar{\Pi}}\right)$ that generate the $SL(2,\mathbb{C})$ action on $\left(Z; \underset{\sim}{Z}\right)$ preserving the symplectic structure. The left-invariant vector field $SL(2,\mathbb{C})$ generators with their complex-conjugated counterparts, on each one of the two copies $\mathbb{T}$ and $\underset{\sim}{\mathbb{T}}$, are the bispinors

$$\Pi^{AB} = -\frac{1}{2}\pi^{(A}\omega^{B)} \qquad \underset{\sim}{\Pi}^{AB} = \frac{1}{2}\underset{\sim}{\pi}^{(A}\underset{\sim}{\omega}^{B)} \tag{6.11a}$$

$$\bar{\Pi}^{\bar{A}\bar{B}} = -\frac{1}{2}\bar{\pi}^{(\bar{A}}\bar{\omega}^{\bar{B})} \qquad \underset{\sim}{\bar{\Pi}}^{\bar{A}\bar{B}} = \frac{1}{2}\underset{\sim}{\bar{\pi}}^{(\bar{A}}\underset{\sim}{\bar{\omega}}^{\bar{B})} \tag{6.11b}$$

where we introduced the notation of round bracket standing for the symmetrisation of the indices. In order reach the covariant phase space, additional $SL(2,\mathbb{C})$ structures on $\mathbb{T}^2$ need to be defined:

- the algebric duals of the spinors, that is the mapping between covariant and contravariant spinors;
- the generators of the algebra $\mathfrak{sl}(2,\mathbb{C})$ on which decompose the chiral generators. They contain both rotational and boost's information, meaning that later they will be associate to the fluxes in the *holonomy-flux* algebra.

The mapping between covariant and contravariant vectors is defined through the Levi-Civita antisimmetric tensor ϵ^{AB} in $\mathbb{C}^2$ and $\epsilon^{\bar{A}\bar{B}}$ in $\bar{\mathbb{C}}^{2*}$ and it maps each vector in its algebraic dual:

$$\omega_A = \omega^B\epsilon_{BA}\,, \qquad \bar{\pi}^{\bar{A}} = \epsilon^{\bar{A}\bar{B}}\bar{\pi}_{\bar{B}}\,, \qquad \epsilon^{AC}\epsilon_{BC} = \delta^A_B$$

The anti-hermitian generator of the algebra $\mathfrak{sl}(2,\mathbb{C})$ (τ_i) are related to the Pauli's matrices through $2i\tau_i = \sigma_i$ and we find here the decomposition of the hamiltonian vector field generators on them, through the scalar product induced by the trace.

$SL(2,\mathbb{C})$ generator components.

$$\Pi \in \mathfrak{sl}(2,\mathbb{C}) \quad : \quad \Pi^A_B = \Pi^i \, (\tau_i)^A_B \quad \Pi^i \in \mathbb{C}^3 \tag{6.12}$$

$$\mathrm{Tr}\left[\tau^i \tau^j\right] = \frac{1}{(2i)^2}\mathrm{Tr}\left[\sigma^i \sigma^j\right] = -\frac{1}{2}\delta^{ij} \implies \Pi^k = -2\mathrm{Tr}\left[\Pi \tau^k\right] = \omega^A \pi_B \left(\tau^k\right)^B_A \tag{6.13}$$

The poisson structure (6.10) defined above induce the following relations:

$$\left\{\Pi^i \,,\, \Pi^j\right\} = -\epsilon^{ij}{}_k \Pi^k \qquad \left\{\underset{\sim}{\Pi}^i \,,\, \Pi^j\right\} = 0 \qquad \left\{\underset{\sim}{\Pi}^i \,,\, \underset{\sim}{\Pi}^j\right\} = -\epsilon^{ij}{}_k \underset{\sim}{\Pi}^k \tag{6.14a}$$

$$\left\{\bar{\Pi}^i \,,\, \bar{\Pi}^j\right\} = -\epsilon^{ij}{}_k \bar{\Pi}^k \qquad \left\{\underset{\sim}{\bar{\Pi}}^i \,,\, \bar{\Pi}^j\right\} = 0 \qquad \left\{\underset{\sim}{\bar{\Pi}}^i \,,\, \underset{\sim}{\bar{\Pi}}^j\right\} = -\epsilon^{ij}{}_k \underset{\sim}{\bar{\Pi}}^k \tag{6.14b}$$

Thanks to these relations the generators are interpreted as the chiral complex generators, splitting the algebra in $\mathfrak{su}(2) \times \mathfrak{su}(2)$ with real (L) and imaginary (K) part standing respectively for rotations and boosts:

$$\Pi^i = \frac{1}{2}\left(L^i + iK^i\right) \qquad\qquad \underset{\sim}{\Pi}^i = \frac{1}{2}\left(\underset{\sim}{L}^i + i\underset{\sim}{K}^i\right) \tag{6.15}$$

$$L^i = \Pi^i + \bar{\Pi}^i \qquad K^i = -i\left(\Pi^i - \bar{\Pi}^i\right) \qquad\qquad \underset{\sim}{L}^i = \underset{\sim}{\Pi}^i + \underset{\sim}{\bar{\Pi}}^i \qquad \underset{\sim}{K}^i = -i\left(\underset{\sim}{\Pi}^i - \underset{\sim}{\bar{\Pi}}^i\right) \tag{6.16}$$

Lorentz generator algebra. The interpretation of L and K as respectively rotations and boosts arise from their algebra, which we show here. We stress the fact that since we chose the anti-hermitian generators, the structure constants lack of an i and a sign with respect to the standard definitions.

$$\left\{L^i \,,\, L^j\right\} = \left\{\Pi^i + \bar{\Pi}^i \,,\, \Pi^j + \bar{\Pi}^j\right\} = \left\{\Pi^i \,,\, \Pi^j\right\} + \left\{\bar{\Pi}^i \,,\, \bar{\Pi}^j\right\} = -\varepsilon^{ij}_k \Pi^k - \varepsilon^{ij}_k \bar{\Pi}^k = -\varepsilon^{ij}_k L^k \tag{6.17}$$

$$\left\{L^i \,,\, K^j\right\} = -i\left\{\Pi^i + \bar{\Pi}^i \,,\, \Pi^j - \bar{\Pi}^j\right\} = -i\left\{\Pi^i \,,\, \Pi^j\right\} + i\left\{\bar{\Pi}^i \,,\, \bar{\Pi}^j\right\} = i\varepsilon^{ij}_k \Pi^k - i\varepsilon^{ij}_k \bar{\Pi}^k = -\varepsilon^{ij}_k K^k \tag{6.18}$$

$$\left\{K^i \,,\, K^j\right\} = -\left\{\Pi^i - \bar{\Pi}^i \,,\, \Pi^j - \bar{\Pi}^j\right\} = -\left\{\Pi^i \,,\, \Pi^j\right\} - \left\{\bar{\Pi}^i \,,\, \bar{\Pi}^j\right\} = \varepsilon^{ij}_k \Pi^k + \varepsilon^{ij}_k \bar{\Pi}^k = \varepsilon^{ij}_k L^k \tag{6.19}$$

The foliation Σ_t carries the time normal n^a and it allows to define a metric on $\mathbb{C}^2$. As usual, we work In the time-gauge, so this normal is represented (through the isomorphism among $\mathfrak{so}(3,1)$ and $\mathfrak{sl}(2,\mathbb{C})$) by the identity matrix. See Appendix B for the details of the isomorphism.

$$n^{A\bar{A}} = (\sigma^a)^{A\bar{A}} n_a = \frac{i}{\sqrt{2}}\delta^{A\bar{A}} \qquad\qquad \delta^{0\bar{0}} = \delta^{1\bar{1}} = 1 \quad \delta^{0\bar{1}} = \delta^{1\bar{0}} = 0 \tag{6.20}$$

As always done the foliation is chosen to be space-like hypersurface and this conditions reflects on the twistor variables imposing $\Pi_{AB}\Pi^{AB} \neq 0$. This condition is particularly helpful because it implies the linear independence of the spinors on both $\mathbb{C}^2_l$ and $\underset{\sim}{\mathbb{C}}^2_l$ and it allows to use them as basis on the respective spaces:

$$\pi\omega \equiv \epsilon_{AB}\pi^A\omega^B = \pi_A\omega^A \neq 0 \qquad\qquad \underset{\sim}{\pi}\underset{\sim}{\omega} \equiv \epsilon_{AB}\underset{\sim}{\pi}^A\underset{\sim}{\omega}^B = \underset{\sim}{\pi}_A\underset{\sim}{\omega}^A \neq 0 \tag{6.21}$$

$$\left(\pi^A,\omega^B\right) \in \mathbb{C}^2 \qquad\qquad \left(\underset{\sim}{\pi}^A,\underset{\sim}{\omega}^B\right) \in \underset{\sim}{\mathbb{C}}^2 \tag{6.22}$$

Since the purpose is to parametrise the phase space $T^*SL(2,\mathbb{C})$, a linear mapping g from Z to $\underset{\sim}{Z}$ can be defined, through its action over the basis elements:

$$g \in GL(2,\mathbb{C}) \quad : \quad g^A_B\pi^B = \underset{\sim}{\pi}^A, \;\; g^A_B\omega^B = \underset{\sim}{\omega}^A \tag{6.23}$$

$$g^A_B = \frac{\underset{\sim}{\omega}^A\pi_B - \underset{\sim}{\pi}^A\omega_B}{\pi\omega} \qquad \text{and} \qquad \left(g^{-1}\right)^A_B = \frac{\omega^A\underset{\sim}{\pi}_B - \pi^A\underset{\sim}{\omega}_B}{\underset{\sim}{\pi}\underset{\sim}{\omega}} \tag{6.24}$$

However the holonomy needed to be represented is an $SL(2,\mathbb{C})$ element so we can ask for unimodularity, forcing g to be $SL(2,\mathbb{C})$. Such a condition translates into the request that g must preserve the bilinear generated by ε^{AB} and it assumes the expression of the *area-matching* constraint (4.47) studied at the end of Chapter 4 to parametrise the $T^*SU(2)$ phase space in term of twistors.

$$C \equiv \pi\omega - \underset{\sim}{\pi}\underset{\sim}{\omega} = 0 \tag{6.25}$$

$$h^A_B = \frac{\underset{\sim}{\omega}^A\pi_B - \underset{\sim}{\pi}^A\omega_B}{\sqrt{\pi\omega}\sqrt{\underset{\sim}{\pi}\underset{\sim}{\omega}}} \qquad\qquad \left(h^{-1}\right)^A_B = \frac{\omega^A\underset{\sim}{\pi}_B - \pi^A\underset{\sim}{\omega}_B}{\sqrt{\pi\omega}\sqrt{\underset{\sim}{\pi}\underset{\sim}{\omega}}} \tag{6.26}$$

$$h^\dagger = \left(\bar{h}\right)^T \quad \Rightarrow \quad \left(h^\dagger\right)^A_B = \bar{h}^{\dot{B}}_{\dot{A}}\delta^{\dot{A}A}\delta_{\dot{B}B} = \frac{\bar{\underset{\sim}{\omega}}^{\dot{B}}\bar{\pi}_{\dot{A}} - \bar{\underset{\sim}{\pi}}^{\dot{B}}\bar{\omega}_{\dot{A}}}{\sqrt{\bar{\pi}\bar{\omega}}\sqrt{\bar{\underset{\sim}{\pi}}\bar{\underset{\sim}{\omega}}}}\delta^{\dot{A}A}\delta_{\dot{B}B} \tag{6.27}$$

A straightforward computation reveals that, on the solution of the area-matching constraint, the algebra of the defined objects $\left(\Pi,\underset{\sim}{\Pi},h\right)$ reproduce the $SL(2,\mathbb{C})$ Holonomy-Flux algebra.

$$\left\{\Pi_i\,,\,h^A_B\right\} = -\left(h\tau_i\right)^A_B = -h^A_C\left(\tau_i\right)^C_B \tag{6.28a}$$

$$\left\{\underset{\sim}{\Pi}_i\,,\,h^A_B\right\} = \left(\tau_i h\right)^A_B = \left(\tau_i\right)^A_C h^C_B \tag{6.28b}$$

$$\left\{h^A_B\,,\,h^C_D\right\} = -\frac{2C}{\left(\pi\omega\right)\left(\underset{\sim}{\pi}\underset{\sim}{\omega}\right)}\left[\epsilon^{AC}\Pi_{BD} + \epsilon_{BD}\Pi^{AC}\right]\Bigg|_{C=0} \approx 0 \tag{6.28c}$$

It is quite clear that all these three objects are not completely independent, indeed the following relation holds:

$$\underset{\sim}{\Pi} = -\frac{\pi\omega}{\underset{\sim}{\pi}\underset{\sim}{\omega}}h\Pi h^{-1} \overset{C=0}{\approx} -h\Pi h^{-1} \tag{6.29}$$

which reveal that $\underset{\sim}{\Pi}$ are the fluxes, with reversed orientations of the surface dual to the link, exactly as in (6.4).

At this point it should be clear that the structure implemented, on the $C = 0$ surface, is equivalent to the $SL(2,\mathbb{C})$ holonomy-flux algebra[1]. Indeed, this is exactly what one obtains, performing the symplectic reduction:

$$\mathbb{T}/\!\!/C \simeq SL(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C}) \simeq T^*SL(2,\mathbb{C}) \tag{6.30}$$

Π, $\underset{\sim}{\Pi}$, h are invariant under the hamiltonian vector field $\mathfrak{H}_C = \{C\,,\,\cdot\}$ so it is possible to use them as coordinates on the space obtained by symplectic reduction of $\mathbb{T}^2$, i.e. $T^*SL(2,\mathbb{C})$. The orbits generated by $\mathfrak{H}_C$ are simply those generated by a complex $U(1)$ group:

$$\exp\left[z\mathfrak{H}_C + \bar{z}\mathfrak{H}_{\bar{C}}\right] \;\; : \left(\omega, \pi, \underset{\sim}{\omega}, \underset{\sim}{\pi}\right) \;\mapsto\; \left(e^z\omega, e^{-z}\pi, e^z\underset{\sim}{\omega}, e^{-z}\underset{\sim}{\pi}\right) \tag{6.31}$$

In order to complete the parametrisation we should mention that, as in the case of $SU(2)$ variables, the maps is two-to-one since it does not distinguish between the following configurations:

$$\left(\pi, \omega, \underset{\sim}{\pi}, \underset{\sim}{\omega}\right) \leftrightarrow \left(\omega, \pi, \underset{\sim}{\omega}, \underset{\sim}{\pi}\right) \tag{6.32}$$

This is the covariant version of the $\mathbb{Z}_2$ symmetry mentioned in the footnote 2 at page 52 but it does not cause any problem since it is possible to divide by it and identify these configurations just as a coherent change in the names of the variables.

A consideration is mandatory at this point, concerning the complex *area-matching* constraint C which was firstly introduced in section 4.3 in the context of $SU(2)$ variables. The twistorial parametrisation of $T^*SL(2,\mathbb{C})$ offer a different point of view which at the same time completes and clarifies the one offered at the end of the Chapter 4. This constraint arises from the explicit request that the phase space pertain $SL(2,\mathbb{C})$ and not the full $GL(2,\mathbb{C})$ since the holonomy introduced g correctly parametrise the phase space of discrete General Relativity only if the unimodularity request is fulfilled $g \mapsto h$.

We end this section summarising the result [81] of the *twistorial parametrisation of the covariant phase space of loop gravity* $T^*SL(2,\mathbb{C})$:

$$\left(\Pi_l, h_l\right) \;\in\; T^*SL(2,\mathbb{C}) \quad\Longleftrightarrow\quad \left(\mathbb{T}_l \oplus \underset{\sim}{\mathbb{T}}_l\right)/\!\!/C_l \;\ni\; \left(\pi_l^A, \omega_l^B, \underset{\sim}{\omega}_l^A, \underset{\sim}{\pi}_l^B\right) \tag{6.33}$$

$$\Pi_l^{AB} = -\frac{1}{2}\pi_l^{(A}\omega_l^{B)} \qquad\qquad (h_l)_B^{\;A} = \frac{\underset{\sim}{\omega}_l^A(\pi_l)_B - \underset{\sim}{\pi}_l^A(\omega_l)_B}{\sqrt{\pi_l\omega_l}\sqrt{\underset{\sim}{\pi}_l\underset{\sim}{\omega}_l}} \tag{6.34}$$

6.3 Covariant Twisted geometries parametrisation

In Chapter 4 we presented both the twistorial and the twisted geometries parametrisation of the $T^*SU(2)$ phase space that originates from discrete general relativity in terms of Ashtekar-Barbero variables. In this section we will

[1] Actually we have $T^*SL(2,\mathbb{C})$ removed of its degenerate configurations $\pi\omega = \underset{\sim}{\pi}\underset{\sim}{\omega} = 0$ but it is possible to extend the definition to include them

complete the same path exploiting the results presented in [74] where the geometric interpretation of the covariant phase space is achieved, in terms of the $SL(2,\mathbb{C})$ generalisation of $SU(2)$ twisted geometries. In literature they are usually called *covariant twisted geometries*. In this brief section we will also assume to be on-shell of the area-matching constraint, i.e. $\pi\omega = \underset{\sim}{\pi}\underset{\sim}{\omega}$.

We start writing the holonomy in the following way

$$g(\pi,\omega) \equiv \frac{1}{\sqrt{\pi\omega}} \begin{pmatrix} \omega^0 & \pi^0 \\ \omega^1 & \pi^1 \end{pmatrix} \qquad h(\pi,\omega) = g(\underset{\sim}{\pi},\underset{\sim}{\omega})g^{-1}(\pi,\omega) \tag{6.35}$$

and then exploit the Iwasawa decomposition of $SL(2,\mathbb{C})$ elements

$$g = n(\zeta)T_\alpha e^{\Phi\tau_3} \qquad n(\zeta) = \frac{1}{\sqrt{1+|\zeta|^2}} \begin{pmatrix} 1 & \zeta \\ -\bar{\zeta} & 1 \end{pmatrix} \qquad n(\zeta) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \tag{6.36}$$

where $(\zeta,\alpha,\Phi) \in \mathbb{C}^2$. From a straightforward comparison with (6.35) we get

$$-\bar{\zeta}^{-1} = \frac{\omega^0}{\omega^1} \qquad \Phi = -2\mathrm{Arg}(\omega^0) + i\ln\frac{||\omega||^2}{\pi\omega} \qquad \alpha = \frac{\pi^A \delta_{A\bar{A}}\bar{\omega}^{\bar{A}}}{\pi\omega} e^{2i\mathrm{Arg}(\omega^0)} \tag{6.37}$$

We define the auxiliary angle ξ which will be related to the class angle in $SU(2)$ twisted geometries

$$\xi = 2\mathrm{Arg}(\underset{\sim}{\omega}^0) - 2\mathrm{Arg}(\omega^0) + \gamma\Xi \qquad \Xi = 2\ln\left(\frac{||\omega||}{||\underset{\sim}{\omega}||}\right) \tag{6.38}$$

and finally obtain the *covariant twisted geometries* parametrisation of $T^*SL(2,\mathbb{C})$:

$$\Pi = -\frac{i}{2}\pi\omega n(\zeta)T_\alpha\tau_3T_\alpha^{-1}n^{-1}(\zeta) \qquad \underset{\sim}{\Pi} = \frac{i}{2}\pi\omega n(\underset{\sim}{\zeta})T_{\underset{\sim}{\alpha}}\tau_3T_{\underset{\sim}{\alpha}}^{-1}n^{-1}(\underset{\sim}{\zeta}) \tag{6.39}$$

We are not going to use this parametrisation explicitly, rather we will use a sort of combination of the twistor variable and covariant twisted geometries. The only two quantities that we will use are the two angles ξ_l and Ξ_l which has been shown to be, respectively, the $SL(2,\mathbb{C})$ counterpart of the "mismatch angle" and the 4D dihedral angle between the normals of two adjacent tetrahedra, sharing the triangle dual to the link l.

6.4 Simplicity constraints

So far the phase space of covariant loop quantum gravity is the direct sum over the links of $T^*SL(2,\mathbb{C})$ spaces and it can be derived from a twistor network, imposing the area-matching constraint. The Lorentz invariance is enforced by the Gauss $SL(2,\mathbb{C})$ constraint which, as in lattice gauge theory, amounts to impose local invariance at each node of the graph. On the other hand the simplicity constraints necessity is two-fold: from one hand they can be seen as reality conditions on the spatial triad field, on the other hand they force the Plebanski two-form to be simple and guarantee that a metric tensor can arise from such a two form. In [79] the simplicity constraints have been solved and the full space of solutions has been characterised in terms of $SU(2)$ spinors, this eventually led to recover the $T^*SU(2)$ phase space. However, in order to reach $T^*SU(2)$ a peculiar condition on the dihedral angle Ξ must be imposed. As we learned from the analysis in the continuum, in order to recover the reduced phase space a non-trivial embedding is expected to appear through the imposition of secondary constraints, relating the $SL(2,\mathbb{C})$ connection A and the Ashtekar-Barbero connection $A^{(\gamma)} = \Gamma + \gamma K$, with the Immirzi parameter playing the role of "relative strength" between the Levi-Civita connection and extrinsic curvature. It is not known whether this actually happens after the discretisation so, in order to recover $T^*SU(2)$ Speziale and Wieland set $\Xi = 0$, and call it the "trivial section". However, in the very same work, Ξ was proven to be the discrete counterpart of the extrinsic curvature so if one would pursue dynamical purposes, any condition on Ξ can not be imposed by hand, rather it should arise from a full dynamical treatment. Nevertheless, on the trivial section it is possible to recover the $T^*SU(2)$ phase space and this give us the opportunity to present the last preliminary ingredient, before we present our dynamical treatment in the last chapter, furthermore the reduction has been performed on a single link so we accept such a condition for the moment while we will show explicitly that it does not hold in our dynamical treatment.

From twistors to the Ashtekar-Barbero variables

We start manipulating the continuous simplicity constraint (5.23) equation to reach the following expression

$$\frac{1}{\gamma+i}\Pi_i + \frac{1}{\gamma-i}\bar{\Pi}_i = 0 \quad \Longleftrightarrow \quad K_i + \gamma L_i = 0 \quad \Longleftrightarrow \quad \Pi_i = -\frac{\gamma+i}{\gamma-i}\bar{\Pi}_i = -e^{i\vartheta}\bar{\Pi}_i \tag{6.40}$$

which has a straightforward smeared version, factorised over the links[2] of the graph

$$(\Pi_l)^i = e^{i\vartheta}(\Pi_l^\dagger)^i \qquad (\underset{\sim}{\Pi}_l)^i = e^{i\vartheta}(\underset{\sim}{\Pi}_l^\dagger)^i \tag{6.41}$$

$$(\Pi^\dagger)^A_B = \delta^{A\bar{A}}\delta_{B\bar{B}}\bar{\Pi}^{\bar{A}}_{\bar{B}} \qquad (\underset{\sim}{\Pi}^\dagger)^A_B = \delta^{A\bar{A}}\delta_{B\bar{B}}\bar{\underset{\sim}{\Pi}}^{\bar{A}}_{\bar{B}} \tag{6.42}$$

$$\omega_{(A}\pi_{B)} = -e^{i\vartheta}\delta_{A\bar{A}}\delta_{B\bar{B}}\bar{\omega}^{(\bar{A}}\bar{\pi}^{\bar{B})} \qquad \underset{\sim}{\omega}_{(A}\underset{\sim}{\pi}_{B)} = -e^{i\vartheta}\delta_{A\bar{A}}\delta_{B\bar{B}}\bar{\underset{\sim}{\omega}}^{(\bar{A}}\bar{\underset{\sim}{\pi}}^{\bar{B})} \tag{6.43}$$

The solution of the simplicity constraint can be found after one realise that the last of these equation seems actually to say that the generators Π_{AB} have two equivalent decompositions in terms of (π,ω) and $(\bar{\pi},\bar{\omega})$. Since it is well known that a decomposition of symmetric bispinors is unique, up to exchange of variables and complex rescaling one conclude that the two must be linearly related. Moreover, the phase of the rescaling is already fixed by the angle ϑ so we are left with a real rescaling. The full solution can be parametrised in term of the real quantity $r \neq 0$:

$$\pi_A = re^{i\vartheta/2}\delta_{A\bar{A}}\bar{\omega}^{\bar{A}} \qquad \omega_A = -\frac{1}{r}e^{i\vartheta/2}\delta_{A\bar{A}}\bar{\pi}^{\bar{A}} \tag{6.44}$$

$$\underset{\sim}{\pi}_A = \underset{\sim}{r}e^{i\vartheta/2}\delta_{A\bar{A}}\bar{\underset{\sim}{\omega}}^{\bar{A}} \qquad \underset{\sim}{\omega}_A = -\frac{1}{\underset{\sim}{r}}e^{i\vartheta/2}\delta_{A\bar{A}}\bar{\underset{\sim}{\pi}}^{\bar{A}} \tag{6.45}$$

[2]The minus sign arise from the choice of anti-hermitian generator for $SL(2,\mathbb{C})$

Contracting equation (6.43) with ω and π, it splits in two parts

$$F^{(1)} = \frac{i}{\gamma+i}\pi\omega - \frac{i}{\gamma-i}\bar{\pi}\bar{\omega} = 0 \qquad F^{(2)} = \frac{i}{\sqrt{2}}\delta^{A\bar{A}}\pi_A\bar{\omega}_{\bar{A}} = 0 \tag{6.46a}$$

$$\underset{\sim}{F}^{(1)} = \frac{i}{\gamma+i}\underset{\sim}{\pi}\underset{\sim}{\omega} - \frac{i}{\gamma-i}\underset{\sim}{\bar{\pi}}\underset{\sim}{\bar{\omega}} = 0 \qquad \underset{\sim}{F}^{(2)} = \frac{i}{\sqrt{2}}\delta^{A\bar{A}}\underset{\sim}{\pi}_A\underset{\sim}{\bar{\omega}}_{\bar{A}} = 0 \tag{6.46b}$$

Following the standard literature we will refer to them as the *diagonal* (F_1) and *off-diagonal* (F_2) simplicity constraints. F_1 is a real equation, Lorentz-invariant since it is builded through the $SL(2,\mathbb{C})$ invariant $\pi\omega$. The off-diagonal constraint is a complex equation giving two real conditions and it is only $SU(2)$ invariant. Together with the complex area-matching constraint C

$$C = \pi\omega - \underset{\sim}{\pi}\underset{\sim}{\omega} \qquad \bar{C} = \bar{\pi}\bar{\omega} - \underset{\sim}{\bar{\pi}}\underset{\sim}{\bar{\omega}} \tag{6.47}$$

they form a system of constraint with the following algebra

$$\left\{F^{(1)}\,,\,F^{(2)}\right\} = -\frac{2i\gamma}{\gamma^2+1}F^{(2)} \qquad \left\{\underset{\sim}{F}^{(1)}\,,\,\underset{\sim}{F}^{(2)}\right\} = \frac{2i\gamma}{\gamma^2+1}\underset{\sim}{F}^{(2)} \tag{6.48a}$$

$$\left\{F^{(2)}\,,\,\bar{F}^{(2)}\right\} = i\mathrm{Im}\left(\pi\omega\right) \qquad \left\{\underset{\sim}{F}^{(2)}\,,\,\underset{\sim}{\bar{F}}^{(2)}\right\} = -i\mathrm{Im}\left(\underset{\sim}{\pi}\underset{\sim}{\omega}\right) \tag{6.48b}$$

$$\left\{F^{(2)}\,,\,C\right\} = \left\{\bar{C}\,,\,F^{(2)}\right\} = F^{(2)} \qquad \left\{\underset{\sim}{F}^{(2)}\,,\,C\right\} = \left\{\bar{C}\,,\,\underset{\sim}{F}^{(2)}\right\} = \underset{\sim}{F}^{(2)} \tag{6.48c}$$

$$\left\{F^{(i)}\,,\,\underset{\sim}{F}^{(j)}\right\} = \left\{C\,,\,F^{(1)}\right\} = 0 \qquad \left\{C\,,\,\underset{\sim}{F}^{(1)}\right\} = \left\{\bar{C}\,,\,F^{(1)}\right\} = \left\{\bar{C}\,,\,\underset{\sim}{F}^{(1)}\right\} = 0 \tag{6.48d}$$

Neglecting possible secondary constraints we conclude that $F^{(1)}$ and $\underset{\sim}{F}^{(1)}$, together with C and $\bar{C}$ are first class constraints generating gauge transformations whereas $F^{(2)}$ and $\underset{\sim}{F}^{(2)}$ are second class. The orbits generated by the diagonal simplicity constraints can be found from the action of the constraints over the variables and they generate complex $U(1)$ action

$$\left\{F^{(1)}\,,\,\pi^A\right\} = -\frac{i}{\gamma+i}\pi^A \qquad \left\{F^{(1)}\,,\,\omega^A\right\} = \frac{i}{\gamma+i}\omega^A \tag{6.49a}$$

$$\left\{\underset{\sim}{F}^{(1)}\,,\,\underset{\sim}{\pi}^A\right\} = \frac{i}{\gamma+i}\underset{\sim}{\pi}^A \qquad \left\{\underset{\sim}{F}^{(1)}\,,\,\underset{\sim}{\omega}^A\right\} = -\frac{i}{\gamma+i}\underset{\sim}{\omega}^A \tag{6.49b}$$

Gauge orbits of the Hamiltonian vector field generated by diagonal simplicity constraints. The action of the Hamiltonian vector field is revealed by the Poisson brackets of the constraints with the element of the basis (π,ω) and $(\underset{\sim}{\pi},\underset{\sim}{\omega})$.

$$\left(e^{z\mathfrak{h}_F}\triangleright\cdot\right) = e^{\left\{F^{(1)}\,,\,\cdot\right\}} \qquad \left(e^{z\mathfrak{h}_{\underset{\sim}{F}}}\triangleright\cdot\right) = e^{\left\{\underset{\sim}{F}^{(1)}\,,\,\cdot\right\}} \tag{6.50}$$

$$\left(e^{z\mathfrak{h}_F}\triangleright\pi^A\right) = e^{-\frac{i}{\gamma+i}z}\pi^A \quad \left(e^{z\mathfrak{h}_F}\triangleright\omega^A\right) = e^{\frac{i}{\gamma+i}z} \quad \left(e^{z\mathfrak{h}_{\underset{\sim}{F}}}\triangleright\underset{\sim}{\pi}^A\right) = e^{\frac{i}{\gamma+i}z}\underset{\sim}{\pi}^A \quad \left(e^{z\mathfrak{h}_{\underset{\sim}{F}}}\triangleright\underset{\sim}{\omega}^A\right) = e^{-\frac{i}{\gamma+i}z}\underset{\sim}{\omega}^A$$

At this point two choices can be made, in order to reduce the space $\mathbb{T}^2$ down to $T^*SU(2)$: one can solve the simplicity constraint first and then the area-matching or the other way around. Since the two procedure commutes it is irrelevant and we chose to solve the simplicity constraints first. Before we start it is important to mention that all these constraints are not linearly independent because we can reduce the system to the four first class constraints $F^{(1)}$, $\utilde{F}^{(1)}$, C, $\bar{C}$ to three independent constraints. Indeed an easy manipulation shows that one can obtain a combination of C and $\bar{C}$ through $F^{(1)}$ and $\utilde{F}^{(1)}$ so the only independent combination is called D:

$$F^{(1)} - \utilde{F}^{(1)} = \frac{i}{\gamma+i}\left(\pi\omega - \utilde{\pi}\utilde{\omega}\right) - \frac{i}{\gamma-i}\left(\bar{\pi}\bar{\omega} - \bar{\utilde{\pi}}\bar{\utilde{\omega}}\right) = \frac{i}{\gamma+i}C - \frac{i}{\gamma-i}\bar{C} \tag{6.51}$$

$$D \equiv F^{(1)} + \utilde{F}^{(1)} \tag{6.52}$$

We already have the solutions of the simplicity constraints (both $F^{(1)}$ and $F^{(2)}$) in equations (6.46a) and (6.46b) but this unfortunately are not invariant under the orbits generated by $F^{(1)}$ so it is not possible to use them to parametrise the space after the symplectic reduction so. The solution of the simplicity constraints, as expected, introduce an Hermitian metric so that one can compute scalar products and in particular the norm $||\cdot||$ which is $F^{(1)}$ invariant: $||\pi||^2 = \pi^A \delta_{A\bar{A}} \bar{\pi}^{\bar{A}}$. Thanks to such a product the following quantity $\mathcal{J}$ is a real $F^{(1)}$-invariant quantity and the solutions can be written in terms of these quantities:

$$\mathcal{J} = \frac{||\omega||^2}{\sqrt{1+\gamma^2}} r \ \in \ \mathbb{R} \qquad \pi\omega = \mathcal{J}(\gamma+i) \qquad \pi_A = \delta_{A\bar{A}}\bar{\omega}^{\bar{A}}\left(\gamma+i\right)\frac{\mathcal{J}}{||\omega||^2} \tag{6.53}$$

It is possible to choose the following spinors, which are a $F^{(1)}$-gauge invariant quantity and thus can be used to parametrise the reduced space together with its "tilded" counterpart living on the other half of the link.

$$z^A = \sqrt{2\mathcal{J}}\frac{\omega^A}{||\omega||^{1+i\gamma}} \qquad \utilde{z}^A = \sqrt{2\utilde{\mathcal{J}}}\frac{\utilde{\omega}^A}{||\utilde{\omega}||^{1+i\gamma}} \tag{6.54}$$

We show their invariance here.

$F^{(1)}$-gauge invariance of the reduced spinor.. We compute explicitly the action of the constraint $F^{(1)}$ on the reduced spinor z^A and show that it is invariant, so that we can choose it as coordinate on the reduced space $\mathbb{T}/\!\!/C$.

$$\left\{F^{(1)}, z^A\right\} = \left\{F^{(1)}, \sqrt{2\mathcal{J}}\frac{\omega^A}{||\omega||^{1+i\gamma}}\right\} = \frac{\omega^A}{||\omega||^{1+i\gamma}}\left\{F^{(1)}, \sqrt{2\mathcal{J}}\right\} + \\ + \frac{\sqrt{2\mathcal{J}}}{||\omega||^{1+i\gamma}}\left\{F^{(1)}, \omega^A\right\} + \sqrt{2\mathcal{J}}\omega^A\left\{F^{(1)}, \frac{1}{||\omega||^{1+i\gamma}}\right\} \tag{6.55}$$

We compute each term separately:

$$\left\{F^{(1)}, \omega^A\right\} = \frac{i}{\gamma+i}\omega^A \qquad \left\{F^{(1)}, \sqrt{2\mathcal{J}}\right\} = \frac{1}{2\sqrt{2\mathcal{J}}}\left\{F^{(1)}, 2\mathcal{J}\right\} = \frac{\gamma+i}{\sqrt{2\mathcal{J}}}\left\{F^{(1)}, \pi\omega\right\} = 0 \tag{6.56}$$

$$\left\{F^{(1)}\,,\,\frac{1}{||\omega||^{1+i\gamma}}\right\} = -\frac{1}{||\omega||^{2+2i\gamma}}\left\{F^{(1)}\,,\,||\omega||^{1+i\gamma}\right\} = -\frac{1+i\gamma}{||\omega||^{2+2i\gamma}}||\omega||^{i\gamma}\left\{F^{(1)}\,,\,||\omega||\right\} =$$
$$= -(1+i\gamma)||\omega||^{i\gamma-2i\gamma-2}\frac{1}{2||\omega||}\left\{F^{(1)}\,,\,\omega^A\delta_{A\bar{A}}\bar{\omega}^{\bar{A}}\right\} = -\frac{1+i\gamma}{2||\omega||^{3+i\gamma}}\left[\left\{F^{(1)}\,,\,\omega^A\right\}\delta_{A\bar{A}}\bar{\omega}^{\bar{A}}+\right.$$
$$\left.\left\{F^{(1)}\,,\,\bar{\omega}^{\bar{A}}\right\}\delta_{A\bar{A}}\omega^A+\right] = -\frac{1+i\gamma}{2||\omega||^{3+i\gamma}}\left[\frac{i}{\gamma+i}||\omega||^2 - \frac{i}{\gamma-i}||\omega||^2\right] = -\frac{1+i\gamma}{\gamma^2+1}\frac{1}{||\omega||^{1+i\gamma}} =$$
$$-\frac{i(\gamma-i)}{\gamma^2+1}\frac{1}{||\omega||^{1+i\gamma}} = -\frac{i}{\gamma+i}\frac{1}{||\omega||^{1+i\gamma}} \tag{6.57}$$

We put the non-vanishing terms together

$$\left\{F^{(1)}\,,\,z^A\right\} = \frac{\sqrt{2\mathcal{J}}}{||\omega||^{1+i\gamma}}\left\{F^{(1)}\,,\,\omega^A\right\} + \sqrt{2\mathcal{J}}\omega^A\left\{F^{(1)}\,,\,\frac{1}{||\omega||^{1+i\gamma}}\right\} =$$
$$= \frac{\sqrt{2\mathcal{J}}}{||\omega||^{1+i\gamma}}\frac{i}{\gamma+i}\omega^A - \sqrt{2\mathcal{J}}\omega^A\frac{i}{\gamma+i}\frac{1}{||\omega||^{1+i\gamma}} = 0 \tag{6.58}$$

The computation for the spinor $\underset{\sim}{z}^A$ follows the same path.

Thanks to the definition of the spinors z^A and $\underset{\sim}{z}^A$ we have the embedding $\mathscr{I}$ of the $F = 0$ hypersurface in the full space $\mathbb{T}^2$ and it is possible to explicitly compute the pullback $\mathscr{I}^*$ of the symplectic potential Θ and obtain the reduced Poisson brackets on the hypersurface of the solutions. From the pullback of the symplectic potential we can compute the reduced symplectic two form as $\omega_D = -d\mathscr{I}^*(\Theta)$ and obtain the Poisson structure of the variables, which is nothing but the algebra of four harmonic oscillators:

$$\left\{z^A\,,\,\bar{z}^{\bar{A}}\right\} = i\delta^{A\bar{A}} \qquad\qquad \left\{\underset{\sim}{z}^A\,,\,\bar{\underset{\sim}{z}}^{\bar{A}}\right\} = -i\delta^{A\bar{A}} \tag{6.59}$$

As showed before, the implementation of the simplicity constraint already account for solving the imaginary part of the area-matching constraint thus on the hypersurface $F = 0$ we have just one real equation which is the reduced form of the area-matching constraints:

$$C_{``F=0''} = ||z||^2 - ||\underset{\sim}{z}||^2 = 0 \tag{6.60}$$

This is exactly the same situation we found when we presented the twistorial parametrisation of $T^*SU(2)$ holonomy-flux algebra, see the discussion at the end of section 4.3 and precisely the equation (4.47), thus we know that the orbits generated by C are real $U(1)$ transformation

$$\left(e^{\phi\mathfrak{h}_C} \triangleright z\right) = e^{i\phi}z \tag{6.61}$$

and it was proven in [75] that reduction of $\mathbb{C}^2 \oplus \bar{\mathbb{C}}^{2*}$ via the constraint C will provide exactly $T^*SU(2)$:

$$\mathbb{T} = \mathbb{C}^2 \oplus \bar{\mathbb{C}}^{2*} \quad \Rightarrow \quad \mathbb{T}/\!\!/C_{``F=0''} \cong T^*SU(2) \tag{6.62}$$

parametrised with the holonomy and flux variables as in equation (4.45)

$$h^A_B(z, \underset{\sim}{z}) = \frac{\underset{\sim}{z}^A \bar{z}^{\bar{B}} \delta_{\bar{B}B} + \bar{\underset{\sim}{z}}_{\bar{A}} \delta^{\bar{A}A} z_B}{||z||\,||\underset{\sim}{z}||} \qquad \Pi_{AB} = \frac{\gamma L_p^2}{\hbar} \frac{i}{2} z_{(A} \delta_{B)\bar{B}} \bar{z}^{\bar{B}} \tag{6.63}$$

6.5 On the dihedral (boost) angle

Before the reduction is implemented one can identifies the variable, canonically conjugated to Ξ, defined in (6.38):

$$\{\gamma \mathcal{J}\,,\, \Xi\} \overset{C=F=0}{\approx} 1 \tag{6.64}$$

This pair of variables can be identified as the oriented area of face dual to the link l and the boost dihedral angle relating the two normals of the tetrahedra sharing the face dual to the link l. This can be seen explicitly computing the area and the scalar produce among the normals:

$$A_l^2 = \delta^i_j \left(\Sigma_l\right)_i \left(\Sigma_l\right)^j = \frac{L_p^4}{\hbar^2} \gamma^2 \mathcal{J} \tag{6.65}$$

$$(\underset{\sim}{n}_l)_I \Lambda^I_J(h_l) n_l^J = (n_l)_{A\bar{A}} (h_l)^A_B (\bar{h}_l)^{\bar{A}}_{\bar{B}} n^{B\bar{B}} = -\frac{1}{2} \left(\frac{||\underset{\sim}{\omega}_l||^2}{||\omega_l||^2} + \frac{||\omega_l||^2}{||\underset{\sim}{\omega}_l||^2} \right) = -\cosh \Xi_l \tag{6.66}$$

We want to highlight the expression of the $SL(2,\mathbb{C})$ holonomy in which the dihedral angle appear clearly. This can be done through the dihedral angle since, after the imposition of the simplicity and area-matching constraints, the spinors z and $\underset{\sim}{z}$ are invariant under the action generated by D but

$$\{D\,,\, \Xi\} = \frac{4}{\gamma^2 + 1} \tag{6.67}$$

then it is possible to label the orbits of D with Ξ. Thanks to this consideration one define the space $T_\Xi = T^*SU(2) \times \mathbb{R}$ in which the solutions of the simplicity and of the area-matching constraints are imposed but the reduction of the orbits of D is not performed. On such a space the holonomy is still a full $SL(2,\mathbb{C})$ element and the dihedral angle appear in a neat way

$$h^A_B \bigg|_{F=C=0} = \frac{e^{-\frac{1}{2}(1+i\gamma)\Xi} \underset{\sim}{z}^A \bar{z}^{\bar{B}} \delta_{B\bar{B}} + e^{\frac{1}{2}(1+i\gamma)\Xi} \bar{\underset{\sim}{z}}_{\bar{A}} \delta^{\bar{A}A} z_B \delta_{B\bar{B}}}{||z||\,||\underset{\sim}{z}||} \tag{6.68}$$

and on the “trivial section” $\Xi = 0$ the $SU(2)$ holonomy is recovered.

Summary

In the previous chapter we introduced the $SL(2,\mathbb{C})$ variables for General Relativity as preliminary analysis in the continuum. In this chapter we presented its discrete counterpart, arising after the imposition of the smearing procedure on the theory, to regularise the algebra: as for the case of $T^*SU(2)$, the phase space factorise over the links of the graph therefore we studied the symplectic structure of the $T^*SL(2,\mathbb{C})$ phase space at each link.

In the second section we presented the recent progress [80, 81] in understanding the connection between discrete loop gravity and twistors. The covariant phase space of discrete loop gravity, over a link, can be reached via symplectic reduction from the space of a pair of twistors $\mathbb{T}^2$, imposing the *complex area-matching constraints*. The full map connecting twistor variables to covariant twisted geometries has been shown in section three and particular emphasis has been put on the variable Ξ_l (in the last section), representing the (boost) dihedral angle between the 4D normals to the two tetrahedra $\tau_{s(l)}$ and $\tau_{t(l)}$ sharing the face dual to the link l. Thanks to these maps, we can now look at the whole phase space of discrete loop gravity over a graph Γ, as a *twistor network* where, to each link, is associated a couple of twistors $(Z_l, \underset{\sim}{Z}_l)$, respectively to the source and the target of the link. At last, in the fourth section we showed how to recover the $SU(2)$ spinorial formalism presented in 4.2 and 4.3 through the symplectic reduction of the full twistor space via the area-matching and the simplicity constraints.

The whole machinery developed until now, both conceptual and technical, is the basis to understand the next chapter, in which we will present the problem studied and the analysis performed to investigate it.

Chapter 7

Twisted geometries and secondary constraints

The most important reason why we introduced the covariant variables in the last chapter is related to the covariant formulation of loop quantum gravity which in facts is written in terms of $SL(2,\mathbb{C})$ variables. After around 30 years of development, loop quantum gravity today can be formulated in two versions, the canonical and the covariant *spin-foam gravity*[11], but unfortunately, up today, they relate to each other only through the kinematics. The formulation of the canonical theory in terms of $SL(2,\mathbb{C})$ variables is actually a step in the direction of merging the two formulations, an important task which would constitute maybe the most important check of the self-coherence for the whole theory.

Since the spin-foam gravity uses the covariant formulation in this framework people have to deal with the equation of the simplicity constraints which assure that the Plebanski two form is "simple" $\Sigma = e \wedge e$; on the other hand from the canonical analysis of the last chapter we learned that the introduction of the simplicity constraints unravel the presence of secondary constraints, imposing that the spatial part of the torsionless equation must be satisfied.

Sadly all these considerations pertain only the continuous formulation, what happens after the smearing is not clear since there is no discrete analogue for the torsion tensor and it is still matter of debate whether or not secondary constraints should arise and, if any, what kind of geometrical information they could possibly carry since, again, we do not know what could be the discrete counterpart of the torsionless equation. Only the full dynamics will have the last word but unfortunately that it is still lacking, so usually in spin-foam gravity the secondary constraints are overlooked and the hope is that the imposition of the primary constraints in some "consistent" way will assure that they are preserved under the evolution. We are against such a point of view since from the canonical analysis (in the continuum) we understand that the secondary constraints arise exactly from the fact that the simplicity constraints are second class so the Hamiltonian flow does not preserve the constraint equation and further conditions must be imposed. Moreover, studying the connection among Lagrangian and Hamiltonian path integral Henneaux and Slavnov [136] found that the presence of second class constraints, from whose stability the secondary constraints arise, actually modify the integration measure in the Lagrangian path integral, which means that the possible presence of secondary constraints may indicate that a modification of the path integral measure is necessary in the spin foam formulation of Loop Quantum Gravity.

Unfortunately, as stressed before, the emergence of secondary constraints is actually a dynamical matter and since the full dynamics is still lacking it is actually very hard to investigate the problem, nonetheless some consideration can be made, trying to see the problem from a different point of view. Concerning the full dynamical problem of loop quantum gravity, it is actually impossible to work within the full Hilbert space $\mathcal{H}$

$$\mathcal{H} = \bigoplus_{\Gamma \subset \Sigma} \mathcal{H}_\Gamma \tag{7.1}$$

so one usually considers truncations of the full Hilbert space where, for example approximate transition amplitudes can be computed. Note that this is *always* done, for example in QED, where computation are performed with a finite number of particles so the Fock space is always truncated; the *fixed graph truncation* heuristically relate to such a kind of approximation since it cut the full Hilbert space to a finite number of degrees of freedom and the quantum states of the theory are represented by all the spin-network Ψ_Γ defined over the same graph Γ. The major implication of this truncation is that, in the semiclassical limit, we can not expect to recover full General Relativity anymore but rather some kind of its truncation down to a finite number of degrees of freedom, which can be interpreted as a discretisation of the full theory, exactly as lattice QCD provide a truncation of the $SU(3)$ gauge theory. The only discretisation known of General Relativity, up today, is Regge calculus which we presented in Chapter 4, and that actually works very well for classical purposes, thus we conclude that within the fixed graph truncation in the semiclassical limit we would expect to recover Regge calculus, which is what happens in the covariant approach to Loop Quantum Gravity [38, 39].

From another perspective, the recent developments [21, 74, 75, 76] presented in Chapter 4 and 6 showed that the quantum spin-networks states already have a classical counterpart but it is not possible to interpret it in term of Regge calculus, it is a *twisted geometry* or a *twistor network* so, what should we expect to recover in the classical limit? the correct question to be asked is *do twisted geometries have a dynamical role?* This is an important question for the canonical theory since they are the complete parametrisation of the phase space and we must understand whether they are full dynamical objects or it exists some dynamical mechanism naturally selecting the Regge geometries from the whole phase space and the dynamics that one can recover in the semiclassical limit will be Regge calculus.

Provided with this insight we come back to the problem of secondary constraints since a similar concern about the presence of secondary constraints is shared by many people in the field, especially from the "canonical side" and an interesting proposal came two years ago by Dittrich and Ryan [47]. Investigating the role of the Immirzi parameter in discrete classical gravity they proposed a discretisation of the second class simplicity constraints that emerge from the analysis in the continuum and obtain a set of constraint for the phase space in the discrete which revealed to be equivalent to the *shape-matching* conditions needed to reduce twisted geometries to Regge geometries and here become clear why the two problem could be entangled. There is the possibility that the advocated mechanism that could pick out the Regge geometries from twisted geometries is the dynamical emergence of the secondary constraints, due to the stability request of the second class simplicity constraints. The underlying geometric picture suggested by the procedure was that the "mismatch" that is responsible for the difference among twisted geometries and Regge geometries could encode the same geometrical information that in the continuum is carried by the torsion tensor and since the torsionless equation is the secondary constraint in the continuum then its discrete counterpart should be the condition reducing twisted geometries to Regge geometries: the *shape-matching* conditions

$$\text{Dittrich} - \text{Ryan} \quad \text{proposal} \qquad\qquad \text{Mismatch} \quad \Longleftrightarrow \quad T \neq 0$$

However the result is not very strong since it is far from obvious that discretisation and dynamical evolution are commuting processes or, in other words, it is not true in principle that the dynamics of the discretised theory looks the same as the discretisation of the solutions of the dynamical problem in the continuum. The correct procedure actually is not the one proposed by Dittrich and Ryan since what underwent the smearing procedure is the whole action and only after that one should face the dynamical problem of the discrete theory.

Such an interpretation of the secondary constraints raised another question on the twisted geometries, about the possibility that they could carry torsion degrees of freedom. The problem was directly faced by the Marseille research group and two years after, a counterargument arose from the work by Haggard, Rovelli, Vidotto and Wieland [46]. The viewpoint emerge from the interpretation in continuum: it is well known that the torsionless equation is an equation for the spin-connection and in principle it has nothing to do with the geometry or even with the *shape-matching* conditions. It turns out that the intuition is correct, they indeed found an explicit solution of the torsionless equation for the spin-connection of the twisted geometries.

$$\text{Counterargument} \qquad\qquad \omega_{\text{Twisted}} \; : \; T(\omega_{\text{Twisted}}) = 0$$

The contradiction between the two results is evident, nonetheless both of them seems to be quite convincing arguments from the respective point of view. The real problem is that both of them are actually neglecting the most important feature of the secondary constraints from where some discrete-torsionless equation may or may not arise: their intrinsic dynamical emergence as stability conditions of the second class simplicity constraints under the Hamiltonian evolution. With such a consideration we conclude that the approach followed by Dittrich and Ryan is flawed since the secondary constraint should not be put "by hand" but rather derived from a consistent canonical analysis of the discretised theory. Furthermore, for the same reason we are not allowed to give dynamical meaning to the counterargument since the solution is found by smoothing out the discontinuity in twisted geometries and solving the torsionless equation in the continuum.

It become clear that the problem is quite hard to solve in general since only the full dynamics will provide the final answer and that is still lacking, nevertheless we think that the problem is intrinsically dynamical and thus only a dynamical treatment can be trustworthy. A problem arise at this point, the full dynamical Hamiltonian clashes against the fixed graph truncation since in general it changes the graph by adding a link and, generally speaking, the smearing procedure breaks the diffeomorphism invariance, there is no Hamiltonian constraint driving the evolution and, as pointed out in [23], the dynamics is implemented trough a "pseudo-constraint". For this reason we decided to face the only preliminary situation that could have been studied, the case of a flat space-time, in which the diffeomorphisms invariance is restored so the evolution is given by a full Hamiltonian constraint and we can study the dynamics by the introduction of a reasonable toy-Hamiltonian imposing flatness, focusing on the potential emergence of secondary constraints. In the rest of the chapter we will present the key-point of our analysis and the results. All the conclusions that can be drawn from the result and its consequences for the whole theory will be drawn in the last chapter.

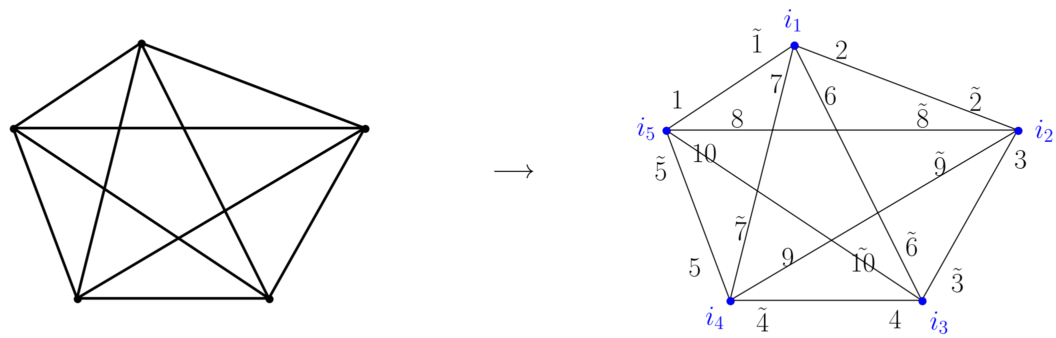

Figure 7.1: On the left picture there is the 2D representation of a 4-simplex while on the right there is its Voronoi-dual graph. As one can see, the two pictures look the same, due to the high degree of symmetry of the 4-simplex. From the duality relation we have that each node is dual to a tetrahedron, each link to a triangle and each triangular face to an edge (a face is a closed loop of links). The 4-simplex is bounded by five 3D tetrahedra (the nodes i_n) and it has ten triangles (the links l) and ten edges (corresponding to the ten triangular faces of the graph).

7.1 The model

The model we used to face the problem differs from General Relativity only for the Hamiltonian constraints, which is a toy-Hamiltonian imposing flatness and coherently with this decision the graph we chose to perform the discretisation is the one dual to a single 4-simplex, which in the Regge spirit is a single "chop" of space-time and as such, it is indeed flat.

We start with the Holst's action, written in terms of $SL(2,\mathbb{C})$ variables, and perform the smearing over the $4-$simplex graph. Thanks to the huge machinery developed in the last chapter, we can exploit the twistorial parametrisation and work with a twistor network defined over the 4-simplex where each link is coloured with a couple of twistors $(Z_l, \underset{\sim}{Z}_l)$. For the sake of notation sometimes we will change notation, put the "tilde" over the link index $\underset{\sim}{Z}_l = Z_{\underset{\sim}{l}}$ so indicating the semi-link and thus the sum over the links will become a sum over the interested semi-link. Before we present the model, we remember that the analysis showed in the last chapter was performed in the time-gauge, so the whole $SL(2,\mathbb{C})$ gauge symmetry has been partially gauge-fixed then the Gauss constraint will account for an $SU(2)$ Gauss law and not the full $SL(2,\mathbb{C})$ group.

The symplectic structure of a twistor network was given in the previous chapter and it factorise over the links hence, neglecting the Hamiltonian constraint we are left with the following total Hamiltonian which is a linear combination of the constraints studied in the last chapter, each one with its own Lagrange's multiplier:

$$\mathcal{H}' = \underbrace{\sum_l a_l C_l}_{Area\ Matching} + \underbrace{\sum_l \lambda_l D_l + b_l F_l^{(2)} + \underset{\sim}{b}_l \underset{\sim}{F}_l^{(2)}}_{Simplicity} + \underbrace{\sum_k \vec{g}_k \cdot \vec{\mathcal{G}}_k}_{Gauss} \tag{7.2}$$

Complex area-matching

$$\begin{aligned} C_l &\equiv \pi_l\omega_l - \underset{\sim}{\pi}_l\underset{\sim}{\omega}_l \\ \bar{C}_l &\equiv \bar{\pi}_l\bar{\omega}_l - \bar{\underset{\sim}{\pi}}_l\bar{\underset{\sim}{\omega}}_l \end{aligned} \qquad l = 1, ..., 10 \tag{7.3}$$

Gauss constraint

$$\mathcal{G}^i_k \equiv \sum_{l \in i_k} L^i_l = 0 \qquad k = 1, ..., 6 \qquad (\mathrm{l = semi-link}) \tag{7.4}$$

Off-diagonal simplicity constraints

$$\begin{aligned} F^{(2)}_l &= n_{A\bar{A}}\pi^A_l\bar{\omega}^{\bar{A}}_l & \bar{F}^{(2)}_l &= n_{A\bar{A}}\bar{\pi}^{\bar{A}}_l\omega^A_l \\ \underset{\sim}{F}^{(2)}_l &= n_{A\bar{A}}\underset{\sim}{\pi}^A_l\bar{\underset{\sim}{\omega}}^{\bar{A}}_l & \bar{\underset{\sim}{F}}^{(2)}_l &= n_{A\bar{A}}\bar{\underset{\sim}{\pi}}^{\bar{A}}_l\underset{\sim}{\omega}^A_l \end{aligned} \qquad l = 1, ..., 10 \tag{7.5}$$

Diagonal simplicity constraints

$$\begin{aligned} D_l &\equiv F^{(1)}_l + \underset{\sim}{F}^{(1)}_l \\ F^{(1)}_l &= \tfrac{i}{\gamma+i}\left(\pi_l\omega_l\right) - \tfrac{i}{\gamma-i}\left(\bar{\pi}_l\bar{\omega}_l\right) \end{aligned} \qquad l = 1, ..., 10 \tag{7.6}$$

$$D_l \equiv F^{(1)}_l + \underset{\sim}{F}^{(1)}_l \qquad l = 1, ..., 10 \tag{7.7}$$

$$F^{(1)}_l = \frac{i}{\gamma+i}\left(\pi_l\omega_l\right) - \frac{i}{\gamma-i}\left(\bar{\pi}_l\bar{\omega}_l\right) \tag{7.8}$$

In order to understand whether or not secondary constraints arise from this simple model we need to compute the algebra generated by the constraints. Part of this algebra has already been presented in Chapter 6 but we recall it here. Moreover, we stress that once the canonical action has been computed, one can use the constraints equations to evaluate the outcome over the hypersurface defined by the constraints equations and the solutions of the simplicity constraints plays a leading role in this sense so we recall it here:

$$(\pi_l)_A = r_l e^{i\vartheta/2}\delta_{A\bar{A}}\bar{\omega}^{\bar{A}}_l \qquad (\omega_l)_A = -\frac{1}{r_l}e^{i\vartheta/2}\delta_{A\bar{A}}\bar{\pi}^{\bar{A}}_l \tag{7.9}$$

$$(\underset{\sim}{\pi}_l)_A = \underset{\sim}{r}_l e^{i\vartheta/2}\delta_{A\bar{A}}\bar{\underset{\sim}{\omega}}^{\bar{A}}_l \qquad (\underset{\sim}{\omega}_l)_A = -\frac{1}{\underset{\sim}{r}_l}e^{i\vartheta/2}\delta_{A\bar{A}}\bar{\underset{\sim}{\pi}}^{\bar{A}}_l \tag{7.10}$$

Algebra of the constraints

The algebra generated by the Gauss constraint is trivial since it is the generator of $SU(2)$ gauge symmetry:

$$\{\mathcal{G}_n\,,\,\mathcal{G}_k\} = \quad \{C_l\,,\,\mathcal{G}_k\} = \quad \{D_l\,,\,\mathcal{G}_k\} = \quad \left\{\mathcal{G}_n\,,\,F_2^{(l)}\right\} = 0 \tag{7.11}$$

and here is the algebra of the simplicity constraints and area-matching, given in the last chapter:

$$\left\{F_l^{(1)}\,,\,F_t^{(2)}\right\} = -\frac{2i\gamma}{\gamma^2+1}F_l^{(2)}\delta_{lt} \qquad \left\{\underset{\sim}{F}_l^{(1)}\,,\,\underset{\sim}{F}_t^{(2)}\right\} = \frac{2i\gamma}{\gamma^2+1}\underset{\sim}{F}_t^{(2)}\delta_{lt} \tag{7.12a}$$

$$\left\{F_l^{(2)}\,,\,\bar{F}_t^{(2)}\right\} = i\mathrm{Im}\left(\pi_l\omega_l\right)\delta_{lt} \qquad \left\{\underset{\sim}{F}_l^{(2)}\,,\,\underset{\sim}{\bar{F}}_t^{(2)}\right\} = -i\mathrm{Im}\left(\underset{\sim}{\pi}_l\underset{\sim}{\omega}_l\right)\delta_{lt} \tag{7.12b}$$

$$\left\{F_t^{(2)}\,,\,C_l\right\} = \left\{\bar{C}_l\,,\,F_t^{(2)}\right\} = F_t^{(2)}\delta_{lt} \qquad \left\{\underset{\sim}{F}_t^{(2)}\,,\,C_l\right\} = \left\{\bar{C}_l\,,\,\underset{\sim}{F}_t^{(2)}\right\} = \underset{\sim}{F}_t^{(2)}\delta_{lt} \tag{7.12c}$$

$$\left\{F_l^{(i)}\,,\,\underset{\sim}{F}_t^{(j)}\right\} = \left\{C_l\,,\,F_t^{(1)}\right\} = 0 \qquad \left\{C_l\,,\,\underset{\sim}{F}_t^{(1)}\right\} = \left\{\bar{C}_l\,,\,F_t^{(1)}\right\} = \left\{\bar{C}_l\,,\,\underset{\sim}{F}_t^{(1)}\right\} = 0 \tag{7.12d}$$

7.2 The Hamiltonian

We now have to face the hard task to find a reasonable toy-Hamiltonian which impose flatness and our choice is based on an elementary property of the holonomy which we are going to show here. The triangulation of the 4D bulk induce a triangulation on the boundary, which is made by tetrahedra, for example in the boundary of the 4-simplex we have five tetrahedra. We can take the loop α_{ab} around a face of one tetrahedron lying in the plane ab and expand the holonomy around the identity:

$$h_{\alpha_{ab}} = \mathbb{I} + \frac{1}{2}\epsilon^2 F^i_{ab}\tau_i + \mathcal{O}(\epsilon^4) \tag{7.13}$$

where ϵ^2 is the area of the face. Since we want to impose flatness, we can consider a simple combination of h and h^{-1} and obtain the curvature tensor:

$$h_{\alpha_{ab}} - h^{-1}_{\alpha_{ab}} = \epsilon^2 F^i_{ab}\tau_i + \mathcal{O}(\epsilon^4) \tag{7.14}$$

We are now using $SL(2,\mathbb{C})$ variables and the Hamiltonian constraint we want to construct must be real so we replace h^{-1} with $h^\dagger$ so that after the reduction to $SU(2)$ variables via the simplicity constraint we will recover exactly the equation (7.14). Basing our intuition on this property we obtain a real scalar constraint by simply taking the real part of the $SL(2,\mathbb{C})$ Wilson loop and imposing that this quantity must be equal to the flat case, i.e. the trace of the identity matrix.

$$H_f = \Re\left[\mathrm{Tr}\left[h_f - \mathbb{I}\right]\right] = 0 \tag{7.15}$$

The holonomy entering in the Hamiltonian constraint h_f is actually the Holonomy of the *faces* of the graph, the closed loops. In the $4-$simplex the "independent loops" are triangular in the sense that a non-triangular face

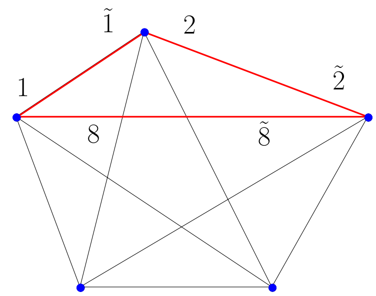

Figure 7.2: In the $4-$simplex the faces are only triangular so we can label them with triple of link on which the path decompose. In the picture the face 128 is indicated in red and the holonomy is h_{128}.

can always be decomposed as a sum of triangular faces so the number of independent loops is the same as the number of triangular faces in a $4-$simplex, which is $F = 10$. This means that we can consider only the triangular loops and identify the holonomy by its decomposition over the links composing the path, see Fig. 7.2.

Equipped with such an Hamiltonian constraint we have the full system of constraints:

$$\mathcal{H} = \underbrace{\sum_l a_l C_l}_{Area\ Matching} + \underbrace{\sum_l \lambda_l D_l + b_l F_l^{(2)} + \underset{\sim}{b}_l \underset{\sim}{F}_l^{(2)}}_{Simplicity} + \underbrace{\sum_k \vec{g}_k \cdot \vec{\mathcal{G}}_k}_{Gauss} + \underbrace{\sum_f N_f H_f}_{Hamiltonian} \tag{7.16}$$

In order to study the arising of secondary constraints we first need to compute the algebra of the Hamiltonian constraints.

Algebra of the Hamiltonian constraint

We start with the Poisson bracket of the Hamiltonian with itself, which is trivial:

$$\left\{(h_l)^A_B\,,\,(h_l)^C_D\right\} \overset{\text{“}C=0\text{”}}{\approx} 0 \quad \Rightarrow \quad \left\{H_f\,,\,H_{f'}\right\} \overset{\text{“}C=0\text{”}}{\approx} 0 \quad \forall\, f, f^{'} \tag{7.17}$$

Before we turn our attention toward the other constraints we should clarify what can happen. The Hamiltonian H_f factorise over the faces f while the simplicity and area-matching constraints factorise over the link, furthermore the Gauss constraint factorise over the nodes and each node involve a sum over the links around the node. However the phase space structure factorise over the links, which means that we will obtain non-vanishing Poisson brackets only among quantity living on the same link. For this reason when we compute the Poisson brackets among the Hamiltonian constraint and constraints living on the links we will use the notation $\delta_{l\partial f}$ to indicate that the quantity is non-vanishing only if the link l is in the boundary of the face ∂f.

Poisson bracket with the area-matching constraint. The Poisson bracket is non-vanishing only if the link t is in the boundary of the face ∂f:

$$\{H_f\,,\,C_t\} \propto \delta_{\partial f,t}\,\{h_t\,,\,C_t\} \tag{7.18}$$

$$\begin{aligned}&\left\{(h_l)^A_B\,,\,C_t\right\} = \delta_{lt}\left\{(h_l)^A_B\,,\,(\pi_t\omega_t)\right\} - \delta_{lt}\left\{(h_l)^A_B\,,\,(\underset{\sim}{\pi}_t\underset{\sim}{\omega}_t)\right\} = \\ &\frac{\delta_{lt}}{\sqrt{\pi_l\omega_l}\sqrt{\underset{\sim}{\pi}_l\underset{\sim}{\omega}_l}}\left(\underset{\sim}{\omega}^A\pi_B + \underset{\sim}{\pi}^A\omega_B\right) - \frac{\delta_{lt}}{\sqrt{\pi_l\omega_l}\sqrt{\underset{\sim}{\pi}_l\underset{\sim}{\omega}_l}}\left(\underset{\sim}{\omega}^A\pi_B + \underset{\sim}{\pi}^A\omega_B\right) = 0\end{aligned} \tag{7.19}$$

Which means

$$\{H_f\,,\,C_t\} = 0 \tag{7.20}$$

The algebra with the Gauss constraint is trivial since the holonomy is covariant under the action of the Gauss constraint.

Poisson bracket with the Gauss constraint. We take the holonomy over the loop f which starts and end at the node n

$$h_l \;\rightarrow\; \Lambda_{s(l)}h_l\Lambda^{-1}_{t(l)} \quad\Rightarrow\quad h^{(n)}_f \;\rightarrow\; \Lambda_n h^{(n)}_f\Lambda^{-1}_n \tag{7.21}$$

and its trace is the Wilson's Loop which is manifestly invariant.

$$\mathrm{Tr}\left[h^{(n)}_f\right] \quad\Rightarrow\quad \mathrm{Tr}\left[\Lambda_n h^{(n)}_f\Lambda^{-1}_n\right] = \mathrm{Tr}\left[h^{(n)}_f\right] \tag{7.22}$$

$$\left\{\vec{\mathcal{G}}_n\,,\,H_f\right\} = 0 \tag{7.23}$$

In the analysis of the continuous formulation, presented in Chapter 5, the arising of secondary constraints was due to the stability of the simplicity constraints which were second class, for this reason we start computing the algebra of the off-diagonal simplicity constraints that are second-class by themselves, focusing on the possible emergence of secondary constraints.

Algebra of the off-diagonal simplicity constraints. The Hamiltonian constraint H_f factorise over the face while the simplicity constraints are factorised over the links $F^{(2)}_l$, this means that the action is non-trivial only when the link l lies in the boundary of the face f. Moreover, since the Hamiltonian involve both h_l and $\bar{h}_l$ there are going to be two non-vanishing terms because $F^{(2)}_l$ involve combination of π and $\bar{\omega}$.

$$\left\{H_f\,,\,F^{(2)}_l\right\} \propto \delta_{l\partial f}\left\{(h_l)^A_B\,,\,F^{(2)}_l\right\} \tag{7.24}$$

$$\left\{h^A_B\,,\,F^{(2)}_l\right\} = -\frac{\underset{\sim}{\pi}^A\bar{\omega}^{\bar{B}}\delta_{\bar{B}B}}{\sqrt{\pi\omega}\sqrt{\underset{\sim}{\pi}\underset{\sim}{\omega}}} + \frac{F_2\,h^A_B}{2(\pi\omega)} \approx -\frac{\underset{\sim}{\pi}^A\bar{\omega}^{\bar{B}}\delta_{\bar{B}B}}{\sqrt{\pi\omega}\sqrt{\underset{\sim}{\pi}\underset{\sim}{\omega}}} \tag{7.25}$$

$$\left\{\bar{h}^{\bar{A}}_{\bar{B}}\,,\,F^{(2)}_l\right\} = -\frac{F_2\,\bar{h}^{\bar{A}}_{\bar{B}}}{2(\bar{\pi}\bar{\omega})} + \frac{\bar{\underset{\sim}{\omega}}^{\bar{A}}\pi^B\delta_{B\bar{B}}}{\sqrt{\bar{\pi}\bar{\omega}}\sqrt{\bar{\underset{\sim}{\pi}}\bar{\underset{\sim}{\omega}}}} \approx \frac{\bar{\underset{\sim}{\omega}}^{\bar{A}}\pi^B\delta_{B\bar{B}}}{\sqrt{\bar{\pi}\bar{\omega}}\sqrt{\bar{\underset{\sim}{\pi}}\bar{\underset{\sim}{\omega}}}} \tag{7.26}$$

To obtain the action of the full Hamiltonian over $F_l^{(2)}$ we need to decompose the holonomy over the three links in the boundary of the face, $1_f, 2_f, 3_f$, and it is useful to define the following notation $h_f^{(l)}$ for the holonomy h_f from which the link l has been singled out. Thanks to the fact that the hamiltonian is the trace over the holonomy, using the cyclic property of the trace we can always consider the case in which the link that has been singled out is at the end of the path:

$$h_f = h_{3_f} h_{2_f} h_{1_f} \qquad \mathrm{Tr}\left[h_{3_f} T h_{1_f}\right] = \mathrm{Tr}\left[h_{1_f} h_{3_f} T\right] = \mathrm{Tr}\left[h_f^{(2_f)} T\right] = \left(h_f^{(2_f)}\right)^A_{\ B} T^B_A \tag{7.27}$$

Exploiting this notation it is simpler to express the action of the Hamiltonian constraint over the off-diagonal simplicity constraints, moreover after the Hamiltonian action has been computed we can use the constraint equation to simplify the expression:

$$\left\{H_f\ ,\ F_l^{(2)}\right\} \overset{F=0}{\approx} \delta_{l\partial f} \frac{\left(\bar{h}_f^{(l)}\right)^{\bar{B}}_{\ \bar{A}} \bar{\omega}_l^{\bar{A}} \pi_l^B \delta_{B\bar{B}}}{\sqrt{\bar{\pi}_l \bar{\omega}_l}\sqrt{\bar{\utilde{\pi}}_l \bar{\utilde{\omega}}_l}} - \delta_{l\partial f} \frac{\left(h_f^{(l)}\right)^{B}_{\ A} \pi_l^{A} \bar{\omega}_l^{\bar{B}} \delta_{\bar{B}B}}{\sqrt{\pi_l \omega_l}\sqrt{\utilde{\pi}_l \utilde{\omega}_l}} \tag{7.28}$$

$$\left\{H_f\ ,\ \utilde{F}_l^{(2)}\right\} \overset{F=0}{\approx} \delta_{l\partial f} \frac{\left(h_f^{(l)}\right)^{B}_{\ A} \bar{\utilde{\omega}}_{\bar{A}} \delta^{\bar{A}A} \utilde{\pi}_B}{\sqrt{\pi_l \omega_l}\sqrt{\utilde{\pi}_l \utilde{\omega}_l}} - \delta_{l\partial f} \frac{\left(\bar{h}_f^{(l)}\right)^{\bar{B}}_{\ \bar{A}} \utilde{\pi}_A \delta^{A\bar{A}} \bar{\utilde{\omega}}_{\bar{B}}}{\sqrt{\bar{\pi}_l \bar{\omega}_l}\sqrt{\bar{\utilde{\pi}}_l \bar{\utilde{\omega}}_l}} \tag{7.29}$$

We now turn our attention to the last constraint equations, the diagonal simplicity constraints.

Poisson bracket of the Diagonal simplicity constraint with the Hamiltonian.

$$\{H_f\ ,\ D_t\} \propto A\ \delta_{\partial f,t} \{h_l\ ,\ D_t\} + B\ \delta_{\partial f,t} \left\{\bar{h}_l\ ,\ D_t\right\} \tag{7.30}$$

$$D_t = \frac{i}{\gamma+i}\left(\pi_t\omega_t\right) - \frac{i}{\gamma-i}\left(\bar{\pi}_t\bar{\omega}_t\right) + \frac{i}{\gamma+i}\left(\utilde{\pi}_t\utilde{\omega}_t\right) - \frac{i}{\gamma-i}\left(\bar{\utilde{\pi}}_t\bar{\utilde{\omega}}_t\right) \tag{7.31}$$

We compute

$$\left\{(h_t)^A_B\ ,\ \pi_t\omega_t\right\} = \left\{(h_t)^A_B\ ,\ \utilde{\pi}_t\utilde{\omega}_t\right\} = \frac{\utilde{\omega}_t^A(\pi_t)_B + \utilde{\pi}_t^A(\omega_t)_B}{\sqrt{\pi_t\omega_t}\sqrt{\utilde{\pi}_t\utilde{\omega}_t}} \equiv \left(\widehat{h}_t\right)^A_{\ B} \tag{7.32}$$

and put it back in the whole Hamiltonian constraint. Again, thanks to the cyclic property of the trace we can work as if the link l is at the end of the path defined by f, and get

$$\{H_f\ ,\ D_t\} = \delta_{t,\partial f}\frac{2i}{\gamma+i}\mathrm{Tr}\left[h_{3_f}h_{1_f}\hat{h}_{2_f}\right] + \delta_{t,\partial f}\frac{2i}{\gamma-i}\overline{\mathrm{Tr}\left[h_{3_f}h_{1_f}\hat{h}_{2_f}\right]} \tag{7.33}$$

We are now ready to perform the canonical analysis of the model. We will implement the Dirac-Bergman algorithm in order to study the stability under the Hamiltonian flow.

7.3 Dirac-Bergman algorithm - Stability

The constraint equations must be compatible with the evolution given by the whole Hamiltonian. The Dirac-Bergman algorithm[110, 111, 112] provides a systematic way to study the stability which is the same procedure that has been performed for the continuous theory, in Chapter 5. The computation of the algebra previously performed, will allow us to understand if some additional condition could emerge.

The area-matching and the Gauss constraint equations are trivially preserved since on the solution of the simplicity constraints they have vanishing Poisson brackets with all the constraints:

$$\dot{C}_t = \{\mathcal{H}\,,\, C_t\} \overset{\text{“}F=0\text{”}}{\approx} 0 \tag{7.34}$$

$$\dot{\mathcal{G}}^i_k = \left\{\mathcal{H}\,,\, \mathcal{G}^i_k\right\} = 0 \tag{7.35}$$

The stability of the off-diagonal simplicity constraints turns out to be an equation for its Lagrange's multiplier and it amount to fix a particular linear combination of them:

$$\dot{F}^{(2)}_t = \left\{\mathcal{H}\,,\, F^{(2)}_t\right\} = \sum_l \bar{b}_l \left\{\bar{F}^{(2)}_l\,,\, F^{(2)}_t\right\} + \sum_f N_f \left\{H_f\,,\, F^{(2)}_t\right\} \equiv \sum_l \bar{b}_l A_{lt} + \sum_f N_f B_{ft} = 0 \tag{7.36}$$

This is a well defined linear system that can be solved for $\bar{b}_l$ since the matrix A_{lt} is invertible. The definition of B_{fl} is in eq. (7.28):

$$A_{lt} = -i\mathrm{Im}\left(\pi_l \omega_l\right)\delta_{lt} \tag{7.37}$$

$$\bar{b}_t = -\sum_{f,l} N_f B_{fl} (A^{-1})_{lt} \tag{7.38}$$

The same situation occur for the other off-diagonal simplicity constraints $\bar{F}^{(2)}, \underset{\sim}{F}^{(2)}_l, \bar{\underset{\sim}{F}}^{(2)}$. We conclude that their stability under the evolution amounts only to fix the value of their Lagrange multipliers and it does not give rise to any secondary constraint:

$$\dot{F}^{(2)}_t = \left\{\mathcal{H}\,,\, F^{(2)}_t\right\} = 0 \qquad \dot{\bar{F}}^{(2)}_t = \left\{\mathcal{H}\,,\, \bar{F}^{(2)}_t\right\} = 0 \tag{7.39}$$

$$\dot{\underset{\sim}{F}}^{(2)}_t = \left\{\mathcal{H}\,,\, \underset{\sim}{F}^{(2)}_t\right\} = 0 \qquad \dot{\bar{\underset{\sim}{F}}}^{(2)}_t = \left\{\mathcal{H}\,,\, \bar{\underset{\sim}{F}}^{(2)}_t\right\} = 0 \tag{7.40}$$

We are left only with the stability of the diagonal simplicity constraints. Off-shell, the only non-vanishing Poisson bracket for D_l are those with the Hamiltonian and with $F^{(2)}$. However the Poisson bracket with the off-diagonal simplicity constraints is proportional to $F^{(2)}$ itself so after the computation we evaluate the obtained quantity on the surface of the simplicity constraints and obtain

$$\dot{D}_t = \{\mathcal{H}\,,\, D_t\} \overset{F=0}{\approx} \sum_f N_f \{H_f\,,\, D_t\} \tag{7.41}$$

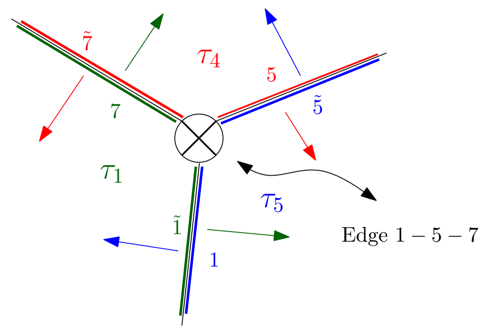

Figure 7.3: From the duality relation we have that each node of the graph i_k is dual to a tetrahedron τ_k, each face f is dual to an edge e and each link l to a triangle t. Here we show the three dihedral angles involved in the sum for the defect angle relative to the edge $1-5-7$ and the figure lies in the plane orthogonal to the edge $(1-5-7)$ around which we are computing the holonomy

This equation has the trivial solution of $N_f = 0$ which however we reject since its counterpart in the continuum is the Lapse function and it does not make any sense to fix its value to zero. We deduce that secondary constraints arise from the stability of the simplicity constraints:

$$S_t = \sum_f N_f \{H_f \, , \, D_t\} = 0 \qquad \Longleftrightarrow \qquad S_{ft} \equiv \{H_f \, , \, D_t\} \overset{!}{=} 0 \tag{7.42}$$

7.4 Secondary constraints

In this section we present the result of the study conducted to solve the secondary constraints and understand their geometrical meaning. To extract the correct geometric interpretation is not an easy task and we need to manipulate the expression in many different way. Before we present the results it is important to mention that the geometric meaning of the Hamiltonian constraint has been clearly understood and it is a generalisation of the Regge deficit angle to the twisted geometries and indeed we recover its original expression when we implement the reduction to the Regge geometry. We provide the details of this computation, in Appendix C.

$$H_f \overset{Regge-SU(2)}{\approx} \mathrm{Tr}\,[\mathbb{I}] - \Re\Big\{ \mathrm{Tr}\left[h_{3_f} h_{2_f} h_{1_f} \right] \Big\} = 2\left(1 - \cos\left(\varepsilon_f\right)\right) = 4\left(\sin\frac{\varepsilon_f}{2}\right)^2 \tag{7.43a}$$

$$\varepsilon_f \equiv \sum_{l \in \partial f} \phi_{int} - 2\pi \tag{7.43b}$$

The secondary constraints emerge from the stability of the simplicity constraints which are factorised over the links, however the Hamiltonian factorise over the faces and since we have three links in the boundary of a face this give rise to three independent equations. In order to extract the geometric meaning of these relation it is useful to focus on a fixed face, which we choose to be the 157. On such a face the constraints equations are:

$$\frac{2i}{\gamma+i}\mathrm{Tr}\left[h_5 h_7 \widehat{h}_1\right]+\frac{2i}{\gamma-i}\overline{\mathrm{Tr}\left[h_5 h_7 \widehat{h}_1\right]}=0 \tag{7.44a}$$

$$\frac{2i}{\gamma+i}\mathrm{Tr}\left[h_5 \widehat{h}_7 h_1\right]+\frac{2i}{\gamma-i}\overline{\mathrm{Tr}\left[h_5 \widehat{h}_7 h_1\right]}=0 \tag{7.44b}$$

$$\frac{2i}{\gamma+i}\mathrm{Tr}\left[\widehat{h}_5 h_7 h_1\right]+\frac{2i}{\gamma-i}\overline{\mathrm{Tr}\left[\widehat{h}_5 h_7 h_1\right]}=0 \tag{7.44c}$$

Our geometrical interpretation is based on the following construction, which we propose with additional details in Appendix C. Suppose to fix a direction ($\vec{F}_i$) in the face i and call α^i_j the 2D dihedral angle between the edges shared by the faces i and j and the fixed direction on the face, $\vec{F}_i$. It has been shown[137] that the scalar product among the spinors z can be geometrically interpreted via the following relations, where and Ξ_l is the 4D dihedral angle among the normal to the tetrahedra that share the link l and θ_{ij} is the 3D dihedral angle between the normal to the triangles i and j:

$$(h_l)^A_B\Big|_{F=0}=\frac{e^{-\alpha_l}\,|\underset{\sim}{z}_l\rangle\langle z_l|+e^{\alpha_l}\,|\underset{\sim}{z}_l]\,[z_l|}{\sqrt{\langle z_l|z_l\rangle}\sqrt{\langle \underset{\sim}{z}_l|\underset{\sim}{z}_l\rangle}} \qquad (\widehat{h}_l)^A_B\Big|_{F=0}=\frac{e^{-\alpha_l}\,|\underset{\sim}{z}_l\rangle\langle z_l|-e^{\alpha_l}\,|\underset{\sim}{z}_l]\,[z_l|}{\sqrt{\langle z_l|z_l\rangle}\sqrt{\langle \underset{\sim}{z}_l|\underset{\sim}{z}_l\rangle}} \tag{7.45}$$

$$\alpha_l=\frac{(1+i\gamma)}{2}\Xi_l \tag{7.46}$$

and

$$[z_i|z_j\rangle=\varepsilon_{ij}\sqrt{\langle z_i|z_i\rangle\langle z_j|z_j\rangle}\sin\frac{\theta_{ij}}{2}e^{\frac{i}{2}\left(\alpha^i_j+\alpha^j_i\right)}=\varepsilon_{ij}||z_i||||z_j||\sin\frac{\theta_{ij}}{2}e^{\frac{i}{2}\left(\alpha^i_j+\alpha^j_i\right)} \tag{7.47a}$$

$$[z_i|z_j]=\sqrt{\langle z_i|z_i\rangle\langle z_j|z_j\rangle}\cos\frac{\theta_{ij}}{2}e^{\frac{i}{2}\left(\alpha^i_j-\alpha^j_i\right)}=||z_i||||z_j||\cos\frac{\theta_{ij}}{2}e^{\frac{i}{2}\left(\alpha^i_j-\alpha^j_i\right)} \tag{7.47b}$$

One should insert these expressions in the secondary constraints and try to solve it for the dihedral angle Ξ_l as a function of the spinors living on each semi-links. However this procedure is highly non trivial and we show here only the final stage of the manipulation

$$\cosh\left(2\alpha_1+i\xi_1\right)=\frac{\cos\theta_{5\underset{\sim}{7}}+\cos\theta_{7\underset{\sim}{1}}\cos\theta_{1\underset{\sim}{5}}}{\sin\theta_{7\underset{\sim}{1}}\sin\theta_{1\underset{\sim}{5}}} \tag{7.48a}$$

$$\cosh\left(2i\alpha_5+i\xi_5\right)=\frac{\cos\theta_{7\underset{\sim}{1}}+\cos\theta_{1\underset{\sim}{5}}\cos\theta_{5\underset{\sim}{7}}}{\sin\theta_{1\underset{\sim}{5}}\sin\theta_{5\underset{\sim}{7}}} \tag{7.48b}$$

$$\cosh\left(2\alpha_7+i\xi_7\right)=\frac{\cos\theta_{1\underset{\sim}{5}}+\cos\theta_{7\underset{\sim}{1}}\cos\theta_{5\underset{\sim}{7}}}{\sin\theta_{7\underset{\sim}{1}}\sin\theta_{5\underset{\sim}{7}}} \tag{7.48c}$$

We mention that the hyperbolic cosine arise from the scalar product among 4D dimensional normals and the dihedral angle is the Lorentzian *rapidity* usually indicated by η.

This expression is the key for the geometrical interpretation of the secondary constraint since it achieves the separation of the 3D geometric data (θ_{ij}) from those 2D (ξ_i) and 4D (α_i). The right-hand is real since it has been computed on the space-like 3D manifold so it must be so for the left-hand side, these equations provide a consistent solution for the 4D dihedral angle Ξ_i *if and only if* it is real

$$2\alpha_l + i\xi_l = \Xi_l + i\left(\xi_l + \gamma\Xi_l\right) \in \mathbb{R} \tag{7.49}$$

$$\Xi_l \in \mathbb{R} \qquad \Longleftrightarrow \qquad \xi_l + \gamma\Xi_l = 0 \qquad\qquad \forall\, l \tag{7.50}$$

Now we have a real solution for the $4D$ dihedral angles between the normals to the tetrahedra (Ξ_i) sharing the link l. It has a neat geometrical interpretation because these relations have already presented in Chapter 4: they are the reconstruction formulas of the 4D dihedral angles in terms of the 3D dihedral angles θ_{lt} between the normal to the triangles

$$\cosh\Xi_1 = \frac{\cos\theta_{5\underset{\sim}{7}} + \cos\theta_{7\underset{\sim}{1}}\cos\theta_{1\underset{\sim}{5}}}{\sin\theta_{7\underset{\sim}{1}}\sin\theta_{1\underset{\sim}{5}}} \tag{7.51}$$

$$\cosh\Xi_5 = \frac{\cos\theta_{7\underset{\sim}{1}} + \cos\theta_{1\underset{\sim}{5}}\cos\theta_{5\underset{\sim}{7}}}{\sin\theta_{1\underset{\sim}{5}}\sin\theta_{5\underset{\sim}{7}}} \tag{7.52}$$

$$\cosh\Xi_7 = \frac{\cos\theta_{1\underset{\sim}{5}} + \cos\theta_{7\underset{\sim}{1}}\cos\theta_{5\underset{\sim}{7}}}{\sin\theta_{7\underset{\sim}{1}}\sin\theta_{5\underset{\sim}{7}}} \tag{7.53}$$

Shape-matching conditions

From these relations we obtain the solution of the secondary constraints arising from the face $1-5-7$ and we must solve the same conditions on each one of the ten faces on the graph. However only ten of them can be independent since we have only ten independent faces so our solutions are required to be consistent with the solutions coming from the other 9 independent loops in the $4-$Simplex. A different way to understand the situation is that each link l is in the boundary of three faces so we have three independent solutions for each Ξ_l and they are requested to be consistent with each other. These consistency conditions are exactly the *shape-matching* conditions previously studied (see chapter 4). Here we write them just for the link 1:

$$\cosh\Xi_1^{(A)} = \frac{\cos\theta_{5\underset{\sim}{7}} + \cos\theta_{7\underset{\sim}{1}}\cos\theta_{1\underset{\sim}{5}}}{\sin\theta_{7\underset{\sim}{1}}\sin\theta_{1\underset{\sim}{5}}} \tag{7.54a}$$

$$\cosh\Xi_1^{(B)} = \frac{\cos\theta_{\underset{\sim}{106}} + \cos\theta_{6\underset{\sim}{1}}\cos\theta_{1,10}}{\sin\theta_{6\underset{\sim}{1}}\sin\theta_{1,10}} \tag{7.54b}$$

$$\cosh\Xi_1^{(C)} = \frac{\cos\theta_{\underset{\sim}{28}} + \cos\theta_{2\underset{\sim}{1}}\cos\theta_{18}}{\sin\theta_{2\underset{\sim}{1}}\sin\theta_{18}} \tag{7.54c}$$

$$\boxed{\Xi_l^{(A)} = \Xi_l^{(B)} = \Xi_l^{(C)}} \quad \Longleftrightarrow \quad \text{Shape} - \text{matching conditions} \tag{7.55}$$

We conclude that, as correctly argued by Haggard, Rovelli, Vidotto and Wieland[46] the secondary constraints impose conditions on the holonomy of the connection, which in principle have nothing to do with the shape-matching conditions. Remarkably, when we solved the constraints to understand the underlying geometry, we found some consistency conditions for the solutions and, as advocated by Dittrich and Ryan [47], these conditions can be geometrically interpreted as the shape-matching conditions reducing a twisted geometry to a Regge geometry.

The result can be seen from another point of view. We previously mentioned (see section) that the orbits of the diagonal simplicity constraints are labeled by Ξ and one can looks at them in the space T_Ξ but, due to the lack of secondary constraints, we previously set $\Xi = 0$ to recover $T^*SU(2)$ from T_Ξ. However, as stressed in Chapter 6 a condition put on Ξ "by hand" can not be accepted from a dynamical point of view, rather it should be consistently derived from a canonical analysis and most likely replaced by an expression for the dihedral angles Ξ_l as a function of the boundary geometric data, i.e. the $SU(2)$ spinors z_l. This is exactly the relation that our solution provides and the result has an immediate twofold implication: from one hand it allow us to understand the geometric meaning of the secondary constraints as a fixing for the gauge orbits of the simplicity constraints $\Xi_l = \Xi_l\left(\{z_t\}\right)$, from the other hand it is interesting that the mechanism giving rise to the secondary constraints is the same as in the continuum, because they emerge from the stability request on the simplicity constraints. The parallel with the continuum can be pushed further: as explicitly shown in section 5.2, the secondary constraints (in the continuum they are the spatial part of the torsionless equation) arise since the simplicity constraints are second class with the Hamiltonian hence, via the Dirac's brackets, they provide the embedding of the $SU(2)$ Ashtekar-Barbero variables into the covariant theory formulated with $SL(2,\mathbb{C})$ variables. The same mechanism is realised at the discrete level by our model: the $T^*SU(2)$ variables can be seen from the wider perspective offered by $T^*SL(2,\mathbb{C})$, thanks to the relations $\Xi = \Xi(\{z_l\})$ which, in facts, provide the embedding of $T^*SU(2)$ into $T_\Xi \subset T^*SL(2,\mathbb{C})$.

We will present the general conclusions that can be drawn from the results of our analysis in the following conclusive chapter.

Chapter 8

Conclusion

The spin foam formalism for the dynamics of Loop Quantum Gravity is based on the Plebanski action for General Relativity. A key feature of this action is the presence of the simplicity constraints which assure that the Plebanski two-form arise from a tetrad field, that is $\Sigma = e \wedge e$. The recent progress in spin foams has led to the EPRL model, with a proper implementation of the constraints, resolving the previous difficulties with an over-restricted Hilbert space, and the impossibility of adding a free Immirzi parameter and the twisting in phase space it introduces. Such model has the key features of providing transition amplitudes for all spin network states, reducing in the large spin limit to exponentials of the Regge action on a given 4-simplex, and be UV finite, and also IR finite in the version based on the quantum group $SU(2)_q$. On the other hand, an important limitation still remains: the model only imposes the primary simplicity constraints, and not also the secondary ones which appear in the canonical analysis. The logic in doing so is that one imposes them 'at all times', and therefore their 'time-preservation' should be granted. However, this logic is based on classical considerations and specifically restricted to using the 4-simplex as a fundamental vertex amplitude, and has been criticised in a recent review by Alxandrov and Roche [22]. In fact, the problem is also likely to be related to other difficulties with the EPRL model that have subsequently emerged, such as the fact that on extended triangulations it may only admit flat solutions, and the unknown large spin behaviour on vertices of valency higher than the 4-simplex.

The logic of ignoring the secondary constraints is fundamentally motivated by the fact that on a single foam, diffeomorphism invariance is broken, and therefore there is no more Hamiltonian constraint with respect to which one can demand the conservation of the primary constraints, so that the issue is ultimately referred to the problem of controlling the full summation and restoring thus the continuum, a research program in full development through the study of radiative corrections [138] and tensor models [139]. However, what was advocated for in [22] is that one could still try to discretise the continuum secondary constraints, and add them to the model. This viewpoint was pursued by Dittrich and Ryan in [47], who argued that the secondary constraints ultimately result in the shape matching conditions. They support their claim using a discretisation procedure that follows the logic of Regge calculus also in the general case when the shapes do not match, encoding the possible shape mismatch in the fact that quantities usually depending only on the triangles, such as dihedral angles, now depend also on a choice of edge of the triangle. This allows them to define a notion of Levi-Civita connection, and to prove that the original covariant connection can be equal to it only if the shapes match. A counter argument was proposed by the Marseille group, who argued that the torsionless equation is actually an equation for the connection, so it should have nothing to do with shape-matching with is a property of the intrinsic geometry. They support their claim discretising Cartan's structural equation and showing that it admits a unique solution also when the shapes do not match. From a mathematical viewpoint, the two procedures, and thus apparently contradicting results, are based on different ways of discretising the torsion.

In this thesis I argued that discretisation of the continuum secondary constraints is a ambiguous procedure, and potentially inconsistent. To really gain control over the issue and clarify the above controversy, one has to derive the discrete secondary constraints in a consistent dynamical way. To do so, we focused on a simple case where the Hamiltonian exists also on a fixed graph, and the problem can be given a definite answer. This is the case of a flat dynamics, imposed ab initio by an appropriate Hamiltonian constraint. First, we showed that secondary constraints do arise, and turn out to exactly reproduce what happens in the continuum: they turn the first class part of the primary into second class, thus their solution provides a non-trivial gauge fixing to the primary's orbits. These orbits in turn allow to embed the reduced SU(2) holonomy into the Lorentzian phase space, thus allowing this auxiliary SU(2) group to probe the boost degrees of freedom. In other words, precisely the logic of the Ashtekar-Barbero connection, restored at the discrete level, a result of importance for the understanding of the covariant geometric interpretation of spin networks. Furthermore, it matches the expectations of the Marseille group: the secondary constraints are conditions on the connection, and have as such nothing to do with properties of the intrinsic geometry such as shape matching. However, there is a catch due to the discrete nature of the system, in which intrinsic and extrinsic geometry are more entangled than in the continuum [21]. In fact, our second result is to show that a consistent solution to the secondary constraints in terms of extrinsic geometry only exists if certain consistency conditions are satisfied. Such consistency conditions arise because of the connectivity of the graph, and an assignment of faces to it made by the Hamiltonian constraint, and turn out to be precisely the shape matching conditions. Therefore, the shape matching conditions do come into the picture, but not directly as secondary constraints, but rather as additional conditions needed to interpret the solution of the constraints in terms of dihedral angles.

The result clarifies the controversy. Furthermore, it shows that one can be more ambitious than what argued for in [22], and study the secondary constraints as a dynamical problem. The key step to achieve this is to be able to extend our analysis to a curved dynamics. We believe this to be possible using the set-up of pseudo-constraints introduced in [23]: the Hamiltonian constraint is lost, as anticipated above, however one can still make sense of a transfer matrix whose smallest eigenvalues approach smoothly zero in the continuum limit. Such object can be used to study the stability, or better pseudo-stability of the primary constraints. Provided the Jacobi identity is shown to hold, this procedure will allow to define the secondary constraints in the curved case. Finally, once extended in this way, our results will show if and how the current EPRL spin foam model should be improved to properly give a dynamics to loop quantum gravity, free of the flatness problem. Summarising, the research presented in this thesis has solved a puzzle recently appeared in the literature, and proposes a research direction that can importantly affect the current study of the dynamics of loop quantum gravity.

Appendix A

ADM formulation

Here we briefly review the ADM formalism, developed around 1959 by Arnowitt, Deser and Misner [4] as the first Hamiltonian formulation of general relativity. One start assuming the space-time $\mathcal{M}$ is globally hyperbolic. Thanks to the Geroch's splitting theorem [51, 52] the hyperbolicity condition assure that the topology of the manifold split into a tensor product of two sub-manifolds with dimensions 3 and 1, respectively:

$$\mathcal{M} \simeq \Sigma \times \mathbb{R} \tag{A.1}$$

The splitting of the topology allows to "foliates" $\mathcal{M}$ into a one-parameter family of space-like embedding $\Sigma_t = F_t(\Sigma)$ so we can identify the time variable with the evolving parameter labelling the foliation. However, we stress the fact that this does not means a breaking of diffeomorphism invariance of the theory. Indeed, there is no prescription on the embedding and nothing prevent us to change it, in full agreement with the diffeomorphism invariance. This means that we can always work choosing a specific foliation but the diffeomorphism invariance of the theory translate in the fact that there is not a unique way to choose such foliation and guarantee that they are all equivalent[1].

We can define the vector field τ^μ as the generator of diffeomorphisms in the direction orthogonal to the foliated surface, mapping the surface Σ_t into $\Sigma_{t+\delta t}$. Since the foliation is space-like, all the vectors in the tangent space of Σ_t are space-like and the normal to the hyper-surface n^μ is time-like. Moreover it is possible to set up a particular coordinate system, called the ADM system $(t, [F_t^{-1}(p)]^j)$ which means that the x^j variables do not change and the vector field has the simplest expression:

$$\tau^\mu = \frac{\partial F_t^\mu}{\partial t} = (1, 0, 0, 0) \tag{A.2}$$

[1] It is easier to think about the rotational invariance. A theory which has rotations as gauge group does not have a preferred direction along which a reference frame can be oriented, so nothing prevents to choose a random direction an call the reference frame S. This does not mean that we are breaking such invariance, because you can always apply a rotation to the chosen reference frame and this will result in a rotation of our S into an S' axes. The rotations invariance just guarantees that the physical quantities do not depend on which one you choice S or S' as reference frame, so you can always pick up one of them.

Again, this is simply a choice and the vector chosen has nothing to do with the usual notion of "time". This historically led to the *problem of time* [88, 90] but it is simply the manifestation of the general covariant character of the theory. We split the vector field into orthogonal and tangential part to Σ:

$$\tau^\mu = N^\mu + N n^\mu = N^\mu - n^\mu \tau^\nu n_\nu \tag{A.3}$$

Since N^μ is the orthogonal part and N is the tangential one we have

$$N^\mu = (0, N^i) \qquad\qquad n_\mu = (-N, 0, 0, 0) \tag{A.4}$$

which lead to the following parametrisation of the time-like normal to the hypersurface n^μ

$$n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right) \tag{A.5}$$

and eventually to the ADM parametrisation of the metric tensor:

$$g_{\mu\nu} = \begin{pmatrix} N_j N^j - N^2 & N_i \\ N_j & g_{ij} \end{pmatrix} \tag{A.6}$$

The spatial part of the metric tensor g_{ij} is not the intrinsic metric on Σ_t, rather it is given by $h_{\mu\nu}$ and via the action of the projector h^μ_ν one is allowed to define calculus over Σ_t from the one defined in $\mathcal{M}$:

$$h_{\mu\nu} \equiv g_{\mu\nu} - n_\mu n_\nu \qquad\qquad h^\mu_\nu = g^{\mu\alpha} h_{\alpha\nu} \tag{A.7}$$

The projection onto Σ_t of the covariant derivative of the normal n is the *extrinsic curvature* tensor

$$K_{\mu\nu} = h^\alpha_\mu h^\beta_\nu \nabla_\alpha n_\beta \tag{A.8}$$

and it is related to the Lie derivative of the intrinsic metric in the direction given by n:

$$\begin{aligned} \mathcal{L}_{\vec{n}} h_{ij} &= 2h_{i(k}\partial_{j)} + n^k \partial_k h_{ij} = 2h_{i(k}\nabla_{j)} + n^k \nabla_k h_{ij} = \\ &= 2n^k n_{(i}\nabla_k n_{j)} + 2\nabla_{(i} n_{j)} = 2K_{ij} \end{aligned} \tag{A.9}$$

For our purposes the importance of the extrinsic curvature lies in the fact that it enters in the Gauss-Codazzi equation (the proof is at the end of the appendix), which in turn allows to rewrite the Einstein-Hilbert action relating three important quantities: the extrinsic curvature K of the foliation, the $4D$ intrinsic curvature that we call R and the 3D intrinsic curvature $\mathcal{R}$.

$$\mathcal{R}^\mu_{\nu\rho\sigma} = h^\mu_\alpha h^\beta_\nu h^\gamma_\rho h^\delta_\sigma R^\alpha_{\beta\gamma\delta} - 2K_{\nu(\sigma} K^\mu_{\rho)} \tag{A.10}$$

$$S_{EH} = \int_t dt \int_\Sigma d^3x \sqrt{h} N \left(\mathcal{R} + \mathrm{Tr}\left[K^2\right] - (\mathrm{Tr}\left[K\right])^2\right) \tag{A.11}$$

From this expression one understand that the Lapse function and the Shift vector do not appear with time derivative in the Lagrangian, so their conjugate momenta are null. They can be interpreted as Lagrange's multiplier and the only dynamical degrees of freedom are inside the spatial metric h_{ab}. In order to switch toward the Hamiltonian analysis the momentum conjugate to h_{ab} is computed and it is remarkably connected to the extrinsic curvature:

$$\frac{\delta \mathcal{L}}{\delta \dot{h}_{ij}} = \pi^{ij} = \sqrt{h}K^{ij} - \sqrt{h}Kh^{ij} \tag{A.12}$$

In such a way it is possible to rewrite the action so that one can read off the symplectic structure without actually having to perform the Legendre transform:

$$S_{EH} = \int_t dt \int_\Sigma d^3x \left(\pi^{ij}\dot{h}_{ij} - N^i H_i - NH\right) \tag{A.13}$$

$$H_i \equiv -2\sqrt{h}\nabla_i \left(\frac{\pi^i_j}{\sqrt{q}}\right) \tag{A.14}$$

$$H \equiv G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{q}\mathcal{R} \tag{A.15}$$

And the tensor G_{ijkl} is called the deWitt metric

$$G_{ijkl} \equiv \frac{1}{\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right) \tag{A.16}$$

It is important to say that the dependence on the Lapse function and the Shift vector has been singled out and we deduce that their equations of motion impose to the quantities just defined, to vanish:

$$\frac{\delta \mathcal{L}}{\delta N} = H = 0 \qquad\qquad \frac{\delta \mathcal{L}}{\delta N^i} = H_i = 0 \tag{A.17}$$

This leads to the interpretation of H and H_i as constraint equations. Furthermore, we can straightforwardly read the total Hamiltonian from equation A.13 which is a linear combination of the constraints:

$$\mathcal{H} \equiv H_i[N^i] + H[N] = \int dt \int_\Sigma \left(N^i H_i + NH\right) d^3x \tag{A.18}$$

The vanishing of the total Hamiltonian is a well known consequence of the general covariant character of the theory. Indeed, as explicitly stated in the beginning of this appendix, there is no unique way to chose the vector field generating the evolution so, in this sense, the t variable has no physical meaning and indeed there is no physical evolution with respect to the t parameter [88]. This does not mean that there is no physical evolution, at all. According to the Dirac's theory [110, 111, 113] of the Hamiltonian constrained systems it just means that t is a "partial observable" and the real evolution is hidden inside its relation with the spatial variables. For a discussion on the topic see [90, 110, 111].

The action written in terms of the ADM variable gives the opportunity to read the symplectic structure and study the phase space of General Relativity, generated by the pair of canonically conjugated variables $\left(h_{ij}, \pi^{ij}\right)$:

$$\left\{\pi^{ij}(q)\,,\,h_{kl}(p)\right\} = \delta^{i}_{(k}\delta^{j}_{l)}\delta^{(3)}(q,p) \tag{A.19}$$

Thanks to the symplectic structure the algebra of the constraints can be computed

$$\left\{H_i[N^i]\,,\,H_j[N^j]\right\} = H_k\Big[(\big[N^i\,,\,N^j\big])^k\Big] \tag{A.20}$$

$$\left\{H_i[N^i]\,,\,H[N]\right\} = H\Big[\mathcal{L}_{N^i}N\Big] \tag{A.21}$$

$$\left\{H[N_1]\,,\,H[N_2]\right\} = H_i[h^{ij}\left(N_1D_iN_2 - N_2D_iN_1\right)] \tag{A.22}$$

The algebra is closed, which means that on-shell the Poisson brackets are all vanishing and the constraint equations are preserved under the evolution generated by the total Hamiltonian:

$$\dot{X} = \{X\,,\,\mathcal{H}\} \tag{A.23}$$

Moreover, the computation of the canonical action of the constraints over the variables (π^{ij}, h_{ij}) reveals the geometric interpretation of the constraints as generators of the diffeomorphism constraints, which split into spatial diffeomorphisms H_i and "time" translations H:

$$\left\{H_i[N^i]\,,\,h_{mn}\right\} = \mathcal{L}_{N^i}h_{mn} \qquad \left\{H[N]\,,\,h_{mn}\right\} = \mathcal{L}_{Nn^i}h_{mn} \qquad \left\{H_i[N^i]\,,\,\pi^{mn}\right\} = \mathcal{L}_{N^i}\pi^{mn} \tag{A.24}$$

The only non-trivial Poisson bracket is the last one but it is easy to see that, on the hypersurface defined by the constraint equations it generates the canonical action of "time" translations:

$$\left\{H[N]\,,\,\pi^{mn}\right\} = \mathcal{L}_{n^iN}\pi^{mn} - 2N\sqrt{h}h^{c[m}h^{n]d}R_{cd} + \frac{1}{2}h^{mn}NH \approx \mathcal{L}_{n^iN}\pi^{mn} \tag{A.25}$$

Where we used the Dirac notation in which $\approx$ means "equality that holds over the surface defined by the constraint equation". Here we sketch the proof of the first Gauss-Codazzi relation:

First Gauss-Codazzi equation. From the definition of the covariant derivative and then of the Riemann tensor, over the 3D sub-manifold Σ we have:

$$D_iT^j_{mn} \equiv h^j_{j'}h^{i'}_i h^{m'}_m h^{n'}_n \nabla_{i'}T^{j'}_{m'n'} \tag{A.26}$$

$$\left(D_iD_j - D_jD_i\right)v^k = \mathcal{R}^k_{nij}v^n \tag{A.27}$$

The key result necessary to prove the first Gauss-Codazzi relation is the following relation:

$$h^{i'}_i h^{j'}_j \nabla_{i'}h^k_{j'} = h^{i'}_i h^{j'}_j \nabla_{i'}n^k n_{j'} = n^k K_{ij} \tag{A.28}$$

Then from equation A.27 one obtain

$$-\mathcal{R}^{i}_{jkl} = -h^{i}_{i'}h^{j'}_{j}h^{k'}_{k}h^{l'}_{l}R^{i'}_{j'k'l'} - K_{kj}K^{i}_{l} + K_{lj}K^{i}_{k} \tag{A.29}$$

Now if we take the trace of the two side and remember the symmetry of the Riemann tensor we get:

$$\mathcal{R} + K^2 - \mathrm{Tr}\,[KK] = 2n^i n^j G_{ij} \tag{A.30}$$

where G_{ij} is the spatial part of the Einstein tensor. It is important to note that, if the first Gauss-Codazzi relation holds in any space-like hypersurface (or any time-like n^i) then it is equivalent to the whole dynamical content of General Relativity.

The outcome of the theory is a well defined constrained Hamiltonian system, whose constraints algebra is first class:

$$\mathcal{H} = H_a[N^a] + H[N] \tag{A.31}$$

$$\dot{X}\,(h,\pi) = \{\mathcal{H}\,,\,X(h,\pi)\} \tag{A.32}$$

Appendix B

Lorentz group and the spinors

B.1 Lorentz group and its Lie algebra

Here are recalled some elements of the Lorentz group and its Lie algebra[120, 140]. The ordinary space $\mathbb{R}^4$ is equipped with the Minkoswki metric $\eta_{\mu\nu} = \mathrm{diag}\,(-1,1,1,1)$ and the Lorentz group $O(3,1)$ is defined as the group of elements for which the metric is an invariant:

$$\Lambda \in O(1,3) \qquad \Longleftrightarrow \qquad \eta_{\mu\nu} = \Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu}\eta_{\alpha\beta} \tag{B.1}$$

The topology is defined via the metric induced by the trace

$$\mathrm{Tr}\left[\left(\Lambda_1 - \Lambda_2\right)^T\left(\Lambda_1 - \Lambda_2\right)\right] \tag{B.2}$$

and the group has four disconnected regions that can be classified on the basis of the sign of the determinant and of the time-time component:

$$\left\{\begin{array}{lll} \det\Lambda = \pm 1 & \rightarrow & (+,-) \\ \Lambda^0_0 \gtrless 0 & \rightarrow & (\uparrow,\downarrow) \end{array}\right\} \Longrightarrow \left(\mathcal{L}^{\uparrow}_{+} \quad \mathcal{L}^{\uparrow}_{-} \quad \mathcal{L}^{\downarrow}_{+} \quad \mathcal{L}^{\downarrow}_{-}\right) \tag{B.3}$$

Among them the only subgroup is the proper orthochronous group $\mathcal{L}^{\uparrow}_{+}$ nevertheless, each one of the disconnected components can be reached from $\mathcal{L}^{\uparrow}_{+}$ applying the following discrete transformations: parity I_s reflection I_t and strong reflections I_sI_t.

The Lie algebra of the Lorentz group is obtained by looking at the vectors in the tangent space to the identity thus we derive with respect to the continuous parameter λ labeling a one-parametr family of continuous Lorentz transformation which goes through the identity at $\lambda = 0$:

$$\omega^{\alpha}_{\beta} = \left.\frac{d(\Lambda_\lambda)^{\alpha}_{\beta}}{d\lambda}\right|_{\lambda=0} \tag{B.4}$$

Considering an infinitesimal transformation

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \tag{B.5}$$

the equation B.1 implies that $\omega_{\mu\nu}$ is antisymmetric

$$\mathfrak{so}(1,3) = \left\{ \omega^{\alpha}{}_{\beta} \in \mathbb{R}^4 \otimes \left(\mathbb{R}^4\right)^* \ : \ \omega_{(\alpha\beta)} = 0 \right\} \tag{B.6}$$

A basis for such a space is given by the set of $\Sigma_{\alpha\beta}$ matrices defined via the following relation, with their commutator defining the structure constants

$$(\Sigma_{\alpha\beta})^{\mu}_{\nu} = 2i\delta^{[\mu}_{\alpha}\delta^{\nu]}_{\beta} \tag{B.7}$$

$$[\Sigma_{\mu\nu}\,,\,\Sigma_{\rho\sigma}] = 2i\left(\eta_{[\nu\rho}\Sigma_{\mu]\sigma} - \eta_{[\nu\sigma}\Sigma_{\mu]\rho}\right) \tag{B.8}$$

so that a general element of the algebra ω can be decomposed as $\omega = \frac{1}{2}\Sigma_{\alpha\beta}\omega^{\alpha\beta}$. The set of generators $\Sigma_{\alpha\beta}$ can be conveniently arranged in two vectors, generating respectively rotations and boosts:

$$L^i = \frac{1}{2}\varepsilon^{i}_{jk}\Sigma^{jk} \qquad\qquad K^i = \Sigma^{i0} \tag{B.9}$$

$$\left[L^i, L^j\right] = i\varepsilon^{ij}_{k}L^k \qquad \left[J^i, K^j\right] = i\varepsilon^{ij}_{k}K^k \qquad \left[K^i, K^j\right] = -i\varepsilon^{ij}_{k}J^k \tag{B.10}$$

Furthermore, the algebra can be diagonalised introducing the complex generators Π that achieve the chiral splitting of the algebra in two copies of $\mathfrak{su}(2)$:

$$\Pi^i = \frac{1}{2}\left(L^i + iK^i\right) \qquad\qquad \bar{\Pi}^i = \frac{1}{2}\left(L^i - iK^i\right) \tag{B.11}$$

$$\left[\Pi^i\,,\,\Pi^j\right] = i\varepsilon^{ij}_{k}\Pi^k \qquad \left[\bar{\Pi}^i\,,\,\bar{\Pi}^j\right] = i\varepsilon^{ij}_{k}\bar{\Pi}^k \qquad \left[\Pi^i\,,\,\bar{\Pi}^j\right] = 0 \tag{B.12}$$

This, together with the fact that an arbitrary element ω can be decomposed as a sum over Π^i and $\bar{\Pi}^i$

$$\omega \in \mathfrak{sl}(2,\mathbb{C}) \qquad \omega = \frac{1}{2}\Sigma_{\alpha\beta}\omega^{\alpha\beta} = -i\Pi^i\omega^i - i\bar{\Pi}^i\bar{\omega}^i \tag{B.13}$$

$$\omega^i = \frac{1}{2}\varepsilon^{i}_{jk}\omega^{jk} + i\omega^{i0} \tag{B.14}$$

explicitly shows that $\mathfrak{so}(1,3) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$.

In the end we give the explicit expression of the two Casimirs, in terms of the complex generators

$$\frac{1}{4}\varepsilon^{\alpha\beta\rho\sigma}\Sigma_{\alpha\beta}\Sigma_{\rho\sigma} = 2L_iK^i = 4\mathrm{Im}\left(\Pi_i\Pi^i\right) \tag{B.15}$$

$$\frac{1}{2}\Sigma_{\alpha\beta}\Sigma^{\alpha\beta} = L_iL^i - K_iK^i = -4\mathrm{Re}\left(\Pi_i\Pi^i\right) \tag{B.16}$$

B.2 Spinors and the Lorentz group

We now turn our attention to $SL(2,\mathbb{C})$, the double covering of $\mathcal{L}_+^\uparrow$, and define an isomorphism between them which is quite important since we make heavy use of it in the body of the dissertation. The definitions exploits the anti-Hermitian matrices of dimension 2 so we start saying that in the vector space of these matrices we call with capital latin indices $A, B, \ldots \in \{0,1\}$ the row indices, while with its "conjugate indices" $A, B, \ldots = \{\bar{0}, \bar{1}\}$ we indicate the column indices. A basis in the space of Hermitian matrices is given by the Pauli matrices plus the identity matrix:

$$\sigma_I^{A\bar{A}} \equiv \Big((\sigma_0)^{A\bar{A}} = \delta^{A\bar{A}} \ , \ \sigma_i^{A\bar{A}} = (\sigma_i)^A_B \delta^{B\bar{A}} \Big) \tag{B.17}$$

We introduce the components $v^I \in \mathbb{R}^4$ of $v^{A\bar{A}}$ in term of the basis of the anti-Hermitian matrices:

$$v^I \quad \longleftrightarrow \quad v^{A\bar{A}} \qquad\qquad v^{A\bar{A}} = \frac{i}{\sqrt{2}} \sigma_I^{A\bar{A}} v^I \qquad\qquad v^{A\bar{A}} \in \mathbb{C}^2 \oplus \bar{\mathbb{C}}^2 \tag{B.18}$$

Thanks to this isomorphism, to each Minkowski index I we can associate pairs of spinorial indices:

$$M^{IJ\ldots} \qquad \longleftrightarrow \qquad M^{A\bar{A}B\bar{B}\ldots} \tag{B.19}$$

Furthermore we can introduce the Levi-Civita invariant tensor ε:

$$\varepsilon^{00} = \varepsilon^{11} = 0 \qquad\qquad \varepsilon^{01} = -\varepsilon^{10} = 1 \qquad\qquad \varepsilon_{01} = \varepsilon^{01} = 1 \tag{B.20}$$

$$\varepsilon^{AB} = -\varepsilon^{BA} \qquad\qquad \varepsilon_{AB} = -\varepsilon_{BA} \qquad\qquad \varepsilon^{AC}\varepsilon_{BC} = \varepsilon^A_B = \delta^A_B \tag{B.21}$$

which is used to map the spinors into their algebraic duals:

$$\pi^A \in \mathbb{C}^2 \qquad\qquad \pi_A \in \left(\mathbb{C}^2\right)^* \qquad\qquad \pi^A = \varepsilon^{AB}\pi_B \qquad\qquad \pi_A = \pi^B \varepsilon_{BA} \tag{B.22}$$

It is important to note that we will always follow the *south-east* contraction rule $\searrow$. This is very important since the opposite rule differ for a sign:

$$(\pi\omega) \equiv \pi^A \varepsilon_{AB} \omega^B = \pi_A \omega^A = -\pi^A \omega_A = -\omega^A \varepsilon_{AB} \pi^B = -(\omega\pi) \tag{B.23}$$

We are going to see in a moment that the Levi-Civita tensor, for the spinors, plays the role of "metric tensor" since we will see that it is invariant under $SL(2,\mathbb{C})$ action. The relations are the same for the complex conjugate spinors $\bar{\pi}^{\bar{A}}$ and $\bar{\omega}^{\bar{A}}$ and we can use the two Levi-Civita tensors defined ε^{AB} and $\bar{\varepsilon}^{\bar{A}\bar{B}}$ to raise and lower indices in the matrices and realises that the isomorphism defined preserves the scalar product:

$$X^{A\bar{A}} Y_{A\bar{A}} = \frac{1}{2} \sigma_I^{A\bar{A}} X^I \sigma^J_{B\bar{B}} X_J = \frac{1}{2} \sigma_I^{A\bar{A}} \sigma^J_{B\bar{B}} X^I X_J = X^I Y_I \tag{B.24}$$

We can now look at action of the Lorentz transformation over the spinorial indices. The following definition

$$g \in SL(2,\mathbb{C}) \quad : (\Lambda(g)X)^{A\bar{A}} = g^A_B X^{B\bar{B}} \bar{g}^{\bar{A}}_{\bar{B}} \tag{B.25}$$

or

$$(g \triangleright X) = gXg^{\dagger} \tag{B.26}$$

which induces a linear mapping Λ on the components X^I

$$g \in SL(2,\mathbb{C}) \quad : \quad g \mapsto \Lambda(g) \ \in \ \mathcal{L}^{\uparrow}_{+} \quad \rightarrow \qquad g^A_B \, \bar{g}^{\bar{A}}_{\bar{B}} \, \sigma^{B\bar{B}}_I = \Lambda(g)^J_I \sigma^{A\bar{A}}{}_J \tag{B.27}$$

It is possible to see that the isomorphism defined in such a way maps g towards $\Lambda(g) \in \mathcal{L}^{\uparrow}_{+}$. The first condition that needs to be checked is

$$\eta_{AB}\Lambda^A_I(g)X^I\Lambda^B_J(g)X^J = X_I X^I \tag{B.28}$$

The proof of this relation is straightforward once one realises that:

$$2\,\mathrm{det}X^{A\bar{A}} = X_I X^I \tag{B.29}$$

which implies

$$\eta_{AB}\Lambda^A_I(g)X^I\Lambda^B_J(g)X^J = 2\mathrm{det}\left(gXg^{\dagger}\right) = 2\mathrm{det}(g)\mathrm{det}(X)\mathrm{det}(g^{\dagger}) = 2\mathrm{det}X = X_I X^I \tag{B.30}$$

so it must be a Lorentz transformation. Moreover, due to the continuity of the map we can reach the identity from $\Lambda(g)$ since $SL(2,\mathbb{C})$ is simply connected, thus we conclude that $\Lambda \in \mathcal{L}^{\uparrow}_{+}$. In the end, we can see that $SL(2,\mathbb{C})$ is the double covering of $\mathcal{L}^{\uparrow}_{+}$ since the map does not distinguishes among g and $-g$, indeed $\Lambda(g) = \Lambda(-g)$.

We can now check explicitly that the Levi-Civita tensor is invariant under $SL(2,\mathbb{C})$ and it corresponds to the η_{IJ} metric in the Minkowski indices. The first property is

$$(g \triangleright \varepsilon) = \varepsilon_{IJ} g^I_A g^J_B = \mathrm{det}(g)\varepsilon_{AB} = \varepsilon \tag{B.31}$$

the second one is a straightforward implication of the isomorphism B.19

$$\eta^{A\bar{A}B\bar{B}} = -\frac{1}{2}\sigma^{A\bar{A}}_I\sigma^{B\bar{B}}_J\eta^{IJ} = \frac{1}{2}\left(\sigma^{A\bar{A}}_0\sigma^{B\bar{B}}_0 - \sigma^{A\bar{A}}_i\sigma^{B\bar{B}}_i\right) = \frac{1}{2}\left(\delta^{A\bar{A}}\delta^{B\bar{B}} - 2\delta^{A\bar{B}}\delta^{B\bar{A}}\right) = \varepsilon^{AB}\varepsilon^{\bar{A}\bar{B}} \tag{B.32}$$

We can now write the relation between $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(3,1)$. Exploiting the intertwining matrices we can define a basis in $\mathfrak{sl}(2,\mathbb{C})$:

$$\Sigma^{A}{}_{B\ IJ} \equiv -\frac{1}{2}\sigma^{A\bar{C}}_{[I}\bar{\sigma}_{\bar{C}B\ J]} \tag{B.33}$$

and it can be seen that they generates the Lorentz algebra since they provide the correct commutation relations. They indeed correspond to the generator of the self-dual sector of the Lorentz algebra: forgetting about the spinorial indices A, B we have

$$P^{MN}_{IJ} = \frac{1}{2}\left(\delta^{[M}_{I}\delta^{N]}_{J} - \frac{i}{2}\varepsilon^{MN}_{IJ}\right) \qquad\qquad P^{MN}_{IJ}\Sigma_{MN} = \Sigma^{IJ} \tag{B.34}$$

Using this basis we can exploit the push-forward of the map (B.27)

$$\Lambda_* :\ \mathfrak{sl}(2,\mathbb{C})\ \ni \Omega^A_B \qquad \Omega^A_B = \frac{1}{2}(\Sigma_{IJ})^A_B\Omega^{IJ} \qquad \Omega^I_J \in \mathfrak{so}(3,1) \tag{B.35}$$

Furthermore one can introduce the anti-Hermitian generators of $\mathfrak{sl}(2,\mathbb{C})$, τ_i and find the decomposition of Ω^I_J:

$$2i\,(\tau_i)^A_B = (\sigma_i)^A_B \qquad\qquad \frac{1}{2}\Omega^{IJ}(\Sigma_{IJ})^A_B = (\tau_i)^A_B\left(\frac{1}{2}\varepsilon^i_{lm}\Omega^{lm} + i\Omega^i_0\right) = (\tau_i)^A_B\Omega^i \tag{B.36}$$

The generators are actually the self-dual generators of the Lorentz algebra so we have that Ω^i are the self-dual components of Ω^I_J. This corresponds to choose complex variables to parametrise $\mathfrak{sl}(2,\mathbb{C})$ and simplifies the calculations because we achieved the splitting of the algebra in two copies of $\mathfrak{su}(2)$, see the previous subsection. Applying the map Λ_* to the generators τ_i one obtains the complex generators introduced previously (B.11):

$$\Lambda_*\tau_i = -i\Pi_i \qquad\qquad \Lambda_*\bar{\tau}_i = -i\bar{\Pi}_i \tag{B.37}$$

B.3 Index-free notation

In the body of the dissertation we used both the notation with explicit indices and without thus here we explicitly states the relation among them. If we suppose to take our foliation of the space-time parallel to a time-like normal n^μ, we can define an Hermitian scalar product, thanks to the isomorphism previously stated:

$$\delta^{A\bar{A}} = \sigma^{A\bar{A}}_{\mu}n^\mu \tag{B.38}$$

Now, with respect to such a metric we can perform Hermitian conjugate operations:

$$|\omega\rangle = \omega^A \qquad\qquad \langle\omega|\, = |\omega\rangle^\dagger = \delta_{A\bar{A}}\bar{\omega}^{\bar{A}} \qquad\qquad ||\omega||^2 = \langle\omega|\omega\rangle \tag{B.39}$$

We would like to stress that such a norm is $SU(2)$ invariant but not $SL(2,\mathbb{C})$. Moreover it is useful to define another $SU(2)$ structure, the J map that allows to introduce the spinors $(J\omega)^A = |\omega]$:

$$|\omega] = -\varepsilon\, |\bar{\omega}\rangle = -\delta^{A\bar{B}}\bar{\varepsilon}_{\bar{B}\bar{A}}\bar{\omega}^{\bar{A}} \qquad [\omega| = |\omega]^\dagger = \omega_A \qquad [\pi|\omega\rangle = \pi_A\omega^A \tag{B.40}$$

Thanks to the time-like normal we can define the metric on the spatial sub-manifold, the 3D Levi-Civita tensor

$$h_{\mu\nu} = \eta_{\mu\nu} + n_\mu n_\nu \qquad \varepsilon_{\alpha\beta\mu} = \varepsilon_{\alpha\beta\mu\nu}n^\nu \tag{B.41}$$

and define the Pauli matrices living on the 3D sub-manifold

$$(\sigma_\mu)^A_B = -2(\Sigma_{\mu\nu})^A_B n^\nu \qquad (\sigma_\mu)^A_B\, n^\mu = 0 \tag{B.42}$$

In this notation the $SL(2,\mathbb{C})$ Holonomy and Flux variables are

$$\Pi^i = [\omega|\,\tau^i\,|\pi\rangle \qquad h = \frac{|\underset{\sim}{\omega}\rangle\,[\pi| - |\underset{\sim}{\pi}\rangle\,[\omega|}{\sqrt{[\pi|\omega\rangle}\sqrt{[\underset{\sim}{\pi}|\underset{\sim}{\omega}\rangle}} \tag{B.43}$$

Appendix C

Geometric interpretation of the Hamiltonian constraint

In this appendix we give a glimpse of the computations that have been performed in order to solve the secondary constraints, showing the geometric interpretation of the Hamiltonian proposed in Chapter 7, in the sub-case of Regge geometry and $SU(2)$ variables. The graph we chose for the smearing is the four-simplex

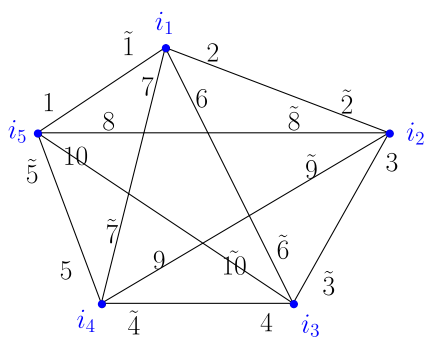

Figure C.1: The graph of the 4-simplex is the same as the 2D picture of the 4-simplex itself. Nodes are dual to tetrahedra, links to triangles shared by couple of tetrahedra while faces are dual to edges.

We focus on the face $1-5-7$, which is dual to an edge in the triangulation. Here is the expression of the Hamiltonian:

$$H_{157} = \Re\left[\mathrm{Tr}\left[h_5 h_7 h_1 - \mathbb{I}\right]\right] = 0 \tag{C.1}$$

To extract the correct geometric interpretation is not an easy task and we need to manipulate the expression in many different way. The first step that needs to be taken is to exploit the formalism developed in the last chapter and write their expression on the surface of the simplicity and area-matching constraints. We use the index-free notation, defined in Appendix B, the reduced spinors z and the expression of the holonomy presented in equation (6.68):

$$h^A_B\Big|_{F=0} = \frac{e^{-\frac{(1+i\gamma)}{2}\Xi}\,\left|\underline{z}\right\rangle\langle z| + e^{\frac{(1+i\gamma)}{2}\Xi}\,\left|\underline{z}\right]\,[z|}{\sqrt{\langle z|z\rangle}\sqrt{\left\langle \underline{z}|\underline{z}\right\rangle}} \tag{C.2}$$

We use the following short-hand notation $\alpha_l \equiv \frac{1+i\gamma}{2}\Xi_l$

$$H_{157} = \mathrm{Tr}\,[\mathbb{I}] - \Re\Big\{\mathrm{Tr}\,[h_5h_7h_1]\Big\} \tag{C.3}$$

$$\begin{aligned}\mathrm{Tr}\,[h_5h_7h_1] = &\left(A\;e^{-(\alpha_1+\alpha_5+\alpha_7)} + \bar{A}\;e^{(\alpha_1+\alpha_5+\alpha_7)}\right)+\\ &+\left(B\;e^{-(\alpha_5+\alpha_7-\alpha_1)} + \bar{B}\;e^{(\alpha_5+\alpha_7-\alpha_1)}\right) + \left(C\;e^{-(\alpha_5+\alpha_1-\alpha_7)} + \bar{C}\;e^{(\alpha_5+\alpha_1-\alpha_7)}\right)+\\ &\left(D\;e^{-(\alpha_5-\alpha_7-\alpha_1)} + \bar{D}\;e^{(\alpha_5-\alpha_7-\alpha_1)}\right)\end{aligned} \tag{C.4}$$

Where the coefficients are:

$$A = \frac{\left\langle z_5|\underline{z}_7\right\rangle\left\langle z_7|\underline{z}_1\right\rangle\left\langle z_1|\underline{z}_5\right\rangle}{||z_5||||\underline{z}_5||\;||z_7||||\underline{z}_7||\;||z_1||||\underline{z}_1||} \tag{C.5}$$

$$B = \frac{\left\langle z_5|\underline{z}_7\right\rangle\left\langle z_7|\underline{z}_1\right]\left[z_1|\underline{z}_5\right\rangle}{||z_5||||\underline{z}_5||\;||z_7||||\underline{z}_7||\;||z_1||||\underline{z}_1||} \tag{C.6}$$

$$C = \frac{\left\langle z_5|\underline{z}_7\right]\left[z_7|\underline{z}_1\right\rangle\left\langle z_1|\underline{z}_5\right\rangle}{||z_5||||\underline{z}_5||\;||z_7||||\underline{z}_7||\;||z_1||||\underline{z}_1||} \tag{C.7}$$

$$D = \frac{\left\langle z_5|\underline{z}_7\right]\left[z_7|\underline{z}_1\right]\left[z_1|\underline{z}_5\right\rangle}{||z_5||||\underline{z}_5||\;||z_7||||\underline{z}_7||\;||z_1||||\underline{z}_1||} \tag{C.8}$$

The geometric interpretation is based on the following construction. We fix a direction $(\vec{F}_i)$ in the face i and call α^i_j the 2D dihedral angle between the edges shared by the faces i and j, and the fixed direction on the i face, $\vec{F}_i$. Than it has been shown [137] that the scalar product inside the coefficients can be geometrically interpreted via the following relations:

$$[z_i|z_j\rangle = \varepsilon_{ij}\sqrt{\langle z_i|z_i\rangle\,\langle z_j|z_j\rangle}\,\sin\frac{\theta_{ij}}{2}e^{\frac{i}{2}(\alpha^i_j+\alpha^j_i)} = \varepsilon_{ij}||z_i||||z_j||\sin\frac{\theta_{ij}}{2}e^{\frac{i}{2}(\alpha^i_j+\alpha^j_i)} \tag{C.9}$$

Where θ_{ij} is the 3D external dihedral angle between the faces i and j or, the angle between their outward normals $\vec{n}_i$ and $\vec{n}_j$ and the sign $\varepsilon_{ij} = \pm 1$ depend on possibility for this vector to be time-like or space-like. Using these relations we can write the coefficients as:

$$A = \cos\frac{\theta_{5\underset{\sim}{7}}}{2}\cos\frac{\theta_{7\underset{\sim}{1}}}{2}\cos\frac{\theta_{1\underset{\sim}{5}}}{2}e^{-\frac{i}{2}\left(\alpha^{5}_{\underset{\sim}{7}}-\alpha^{\underset{\sim}{5}}_{1}\right)}e^{-\frac{i}{2}\left(\alpha^{7}_{\underset{\sim}{1}}-\alpha^{\underset{\sim}{7}}_{5}\right)}e^{-\frac{i}{2}\left(\alpha^{1}_{\underset{\sim}{5}}-\alpha^{\underset{\sim}{1}}_{7}\right)} \tag{C.10}$$

$$B = -\cos\frac{\theta_{5\underset{\sim}{7}}}{2}\sin\frac{\theta_{7\underset{\sim}{1}}}{2}\sin\frac{\theta_{1\underset{\sim}{5}}}{2}e^{-\frac{i}{2}\left(\alpha^{5}_{\underset{\sim}{7}}-\alpha^{\underset{\sim}{5}}_{1}\right)}e^{-\frac{i}{2}\left(\alpha^{7}_{\underset{\sim}{1}}-\alpha^{\underset{\sim}{7}}_{5}\right)}e^{+\frac{i}{2}\left(\alpha^{1}_{\underset{\sim}{5}}-\alpha^{\underset{\sim}{1}}_{7}\right)} \tag{C.11}$$

$$C = -\sin\frac{\theta_{5\underset{\sim}{7}}}{2}\sin\frac{\theta_{7\underset{\sim}{1}}}{2}\cos\frac{\theta_{1\underset{\sim}{5}}}{2}e^{-\frac{i}{2}\left(\alpha^{5}_{\underset{\sim}{7}}-\alpha^{\underset{\sim}{5}}_{1}\right)}e^{+\frac{i}{2}\left(\alpha^{7}_{\underset{\sim}{1}}-\alpha^{\underset{\sim}{7}}_{5}\right)}e^{-\frac{i}{2}\left(\alpha^{1}_{\underset{\sim}{5}}-\alpha^{\underset{\sim}{1}}_{7}\right)} \tag{C.12}$$

$$D = -\sin\frac{\theta_{5\underset{\sim}{7}}}{2}\cos\frac{\theta_{7\underset{\sim}{1}}}{2}\sin\frac{\theta_{1\underset{\sim}{5}}}{2}e^{-\frac{i}{2}\left(\alpha^{5}_{\underset{\sim}{7}}-\alpha^{\underset{\sim}{5}}_{1}\right)}e^{+\frac{i}{2}\left(\alpha^{7}_{\underset{\sim}{1}}-\alpha^{\underset{\sim}{7}}_{5}\right)}e^{+\frac{i}{2}\left(\alpha^{1}_{\underset{\sim}{5}}-\alpha^{\underset{\sim}{1}}_{7}\right)} \tag{C.13}$$

Note that the dependence on the type of the vector disappear in the coefficients thanks to the combination of the faces involved, that are pairwise coupled: $\left(\underset{\sim}{\vec{n}}_1, \vec{n}_1\right) - \left(\underset{\sim}{\vec{n}}_5, \vec{n}_5\right) - \left(\underset{\sim}{\vec{n}}_7, \vec{n}_7\right)$

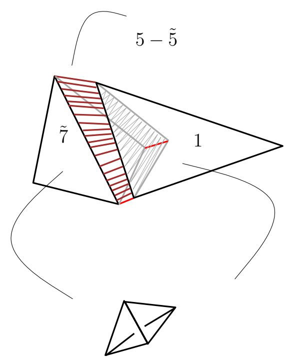

Figure C.2: The edges to which the angles α^i_j refer in (C.10) - (C.13) is the same but the angles are computed in the 3 faces adjacent to it.

Now remember that the geometric information we are considering here is around just one edge, dual to the face $1-5-7$. Following the combinatorial information on the graph we can say that the edges to which the angles $\alpha^{5}_{\underset{\sim}{7}}$ and $\alpha^{\underset{\sim}{5}}_{1}$ refer, are exactly the same, but seen from different tetrahedra. Due to the fact that we are dealing with twisted geometries, we lack of the gluing conditions and all these angles may not be the same, this aspect is fully caught by the differences in the phases of (C.10) - (C.13) that generally are not zero. However we are going to understand the geometric meaning of the Hamiltonian constrain, in the Regge sub-case, so the phases of the coefficient are zero since the two-dimensional dihedral angles have the same value.

$$\mathrm{Tr}\left[h_5 h_7 h_1\right] \overset{Regge}{\approx} 2\left[A\cosh\left(\alpha_1+\alpha_5+\alpha_7\right) + B\cosh\left(\alpha_5+\alpha_7-\alpha_1\right) + \right. \tag{C.14}$$

$$\left. + C\cosh\left(\alpha_5+\alpha_1-\alpha_7\right) + D\cosh\left(\alpha_5-\alpha_7-\alpha_1\right)\right] \tag{C.15}$$

Moreover on the trivial section $\Xi = 0$ we recover the $SU(2)$ structure, with the induced Dirac brackets giving the symplectic structure. In this case the scalar constraint has the following expression:

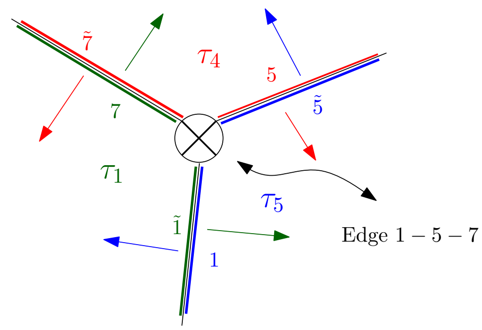

Figure C.3: The three dihedral angles involved in the sum (C.16). The picture lies in the plane orthogonal to the edge $(1-5-7)$ around which we are computing the holonomy

$$\frac{1}{2}\mathrm{Tr}\left[h_5 h_7 h_1\right] \overset{Regge-SU(2)}{\approx} = A+B+C+D =$$
$$= \cos\frac{\theta_{5\underset{\sim}{7}}}{2}\cos\frac{\theta_{7\underset{\sim}{1}}}{2}\cos\frac{\theta_{1\underset{\sim}{5}}}{2} - \cos\frac{\theta_{5\underset{\sim}{7}}}{2}\sin\frac{\theta_{7\underset{\sim}{1}}}{2}\sin\frac{\theta_{1\underset{\sim}{5}}}{2} +$$
$$- \sin\frac{\theta_{5\underset{\sim}{7}}}{2}\sin\frac{\theta_{7\underset{\sim}{1}}}{2}\cos\frac{\theta_{1\underset{\sim}{5}}}{2} - \sin\frac{\theta_{5\underset{\sim}{7}}}{2}\cos\frac{\theta_{7\underset{\sim}{1}}}{2}\sin\frac{\theta_{1\underset{\sim}{5}}}{2} \tag{C.16}$$

Two consideration should be made at this point:

- We are working in a reference frame orthogonal to the time-like normal. This means that all the angles around the edge, account for rotations and not for boosts. So the faces shared by the three tetrahedra i_1, i_4, i_5 are in $\mathbb{R}^3$ and the angles are euclidean;
- The "spinors" we are using are defined as the fundamental representation of $SU(2)$, which differ from the fundamental representation of $SO(3)$ in which the standard geometry is developed. From the point of view of the angles this means that the period of an angle is 4π and not 2π and we have to deal with some non-trivialities in the interpretation of the equation .

The definition of the defect angle as the difference between the sum of the angles and 2π works in $SO(3)$ so we have to write it in the fundamental representation of $SU(2)$:

$$\varepsilon_e^{SO(3)} \equiv \sum \theta^{SO(3)} - 2\pi \qquad\qquad \theta^{SU(2)} = 2\theta^{SO(3)} \tag{C.17}$$

$$\varepsilon_e^{SU(2)} \equiv \frac{1}{2}\sum \theta^{SU(2)} - 2\pi \tag{C.18}$$

$$\frac{\theta_{5\underset{\sim}{7}}^{SU(2)} + \theta_{7\underset{\sim}{1}}^{SU(2)} + \theta_{1\underset{\sim}{5}}^{SU(2)}}{2} = \theta_{5\underset{\sim}{7}}^{SO(3)} + \theta_{7\underset{\sim}{1}}^{SO(3)} + \theta_{1\underset{\sim}{5}}^{SO(3)} \tag{C.19}$$

We use the standard trigonometric identities to put the expression (C) in a more compact form and realise that it is actually as the cosine of the sum of the 3D dihedral angles around the edge $1-5-7$. If the 3D space were to be flat, the sum would have been 2π, but in general it is exactly the defect angle:

$$\frac{1}{2}\mathrm{Tr}\left[h_5 h_7 h_1\right] \overset{Regge-SU(2)}{\approx} \cos\left(\frac{\theta_{5\underset{\sim}{7}} + \theta_{7\underset{\sim}{1}} + \theta_{1\underset{\sim}{5}}}{2}\right) = \cos\left(\varepsilon_e + 2\pi\right) = \cos\left(\varepsilon_e\right) \tag{C.20}$$

Which means

$$H_f \overset{Regge-SU(2)}{\approx} \mathrm{Tr}\left[\mathbb{I}\right] - \Re\left\{\mathrm{Tr}\left[h_5 h_7 h_1\right]\right\} = 2\left(1 - \cos\left(\varepsilon_f\right)\right) = 4\left(\sin\frac{\varepsilon_f}{2}\right)^2 \tag{C.21}$$

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