Continuous second order sliding mode based robust finite time tracking of a fully actuated biped robot

Harshal B. Oza, Yury V. Orlov, Sarah K. Spurgeon, Yannick Aoustin and Christine Chevallereau

***Abstract*— A second order sliding mode controller is modified to form a continuous homogeneous controller. Uniform finite time stability is proved by extending the homogeneity principle of discontinuous systems to the continuous case with uniformly decaying piece-wise continuous nonhomogeneous disturbances. The modified controller is then utilised to track reference trajectories for all the joints of a fully actuated biped robot where the joint torque is modeled as the control input. The modified controller ensures the attainment of a finite settling time between two successive impacts. The main contribution of the paper is to provide straightforward and realizable engineering guidelines for reference trajectory generation and for tuning a robust finite time controller in order to achieve stable gait of a biped in the presence of an external force disturbance. Such a disturbance has destabilising effects in both continuous and impact phases. Numerical simulations of a biped robot are shown to support the theoretical results.**

I. INTRODUCTION

A continuous second order sliding mode state feedback synthesis for achieving finite time convergence of the joint trajectories of a biped is explored. The biped robot under consideration is a fully actuated robot. Gait stability is to be ensured by designing the reference trajectories and ensuring that they are tracked in finite time. Second order sliding mode (SOSM) controllers [1] are recognized as a good candidate for robotics [2] due to their simplicity and underlying robustness properties. There are geometric homogeneity based results [3], [4], [5] for SOSM which highlight finite time convergence in the presence of persisting disturbances.

The main focus of this paper is on finite time tracking of the joint trajectories of a fully actuated biped robot to desired periodic trajectories by using a continuous second order sliding mode controller. Tuning guidelines of the controller parameters are also a focus. The present paper utilizes a robust continuous homogeneous controller while requiring only knowledge of an upper bound on the disturbance to cause the tracking errors to converge to zero in finite time between successive impacts. The finite time stability is substantiated by non-smooth Lyapunov analysis and homogeneity of the discontinuous right hand side. The theoretical motivation to study a new Lyapunov and homogeneity framework for planar continuous homogeneous vector fields is to give *uniform* finite time stability with respect to the initial data and the disturbances. The motivation also lies in studying robustness to discontinuous disturbances which is a stronger property than the existing methods for continuous disturbances which utilise the link between homogeneity [6] [7] and finite time stability [8].

From a practical viewpoint, the motivation stems from the need to propose tuning guidelines for a continuous SOSM control for the presented class of biped robots. Thus, the goal is to provide engineering guidelines for achieving stable walking gait of a biped in finite time.

Since the early results [9], [10], [11], there has been active interest amongst researchers in the study of finite time stability. The homogeneous controller used in this paper is capable of rejecting disturbances of a broader class than those reported in the existing continuous finite time controller literature. Of particular importance is the fact that piece-wise continuous vanishing disturbances can be rejected. Asymptotic stability of a continuous homogeneous controller introduced in [12] was proved in [2] in the presence of vanishing non-smooth disturbances. The proofs of the claims of robust finite time stability presented in the following sections of this paper are rather involved and are being published elsewhere [13]. Similar results without the proof of robustness can also be found in [14].

The literature on control of biped robots is vast (see [15], [16], [17], or [18] for a comprehensive survey). Previous work utilizing SOSM includes [19], [20]. The main contribution from the viewpoint of biped control is that the tuning guidelines for the robust continuous controller are given as a function of the upper bound on the disturbance while solving the tracking problem to attain a pre-specified cyclic walking gait. During this walking gait, which is composed of single support phases and impacts, the biped is always fully actuated. This cyclic gait is the result of an optimization based on a sthenic criterion and on the definition of nonlinear constraints such as no take-off, no rotation and no sliding of the stance foot on the ground. Then the fully actuated biped is stable when the desired cyclic gait is tracked perfectly. The robust controller ensures that the tracking errors always converge to the origin before the subsequent impact event occurs thereby guaranteeing a stable walking biped. The continuous controller is implementable since the joint torque is modeled as the control input. Previous results that utilise a finite time controller without robustness studies for an under-actuated biped can be found in [21]. The remainder of the

This work was partially supported by EPSRC via research grant EP/G053979/1.

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paper is organized as follows. Section II presents the problem statement. Section III presents the synthesis of the proposed controller along with a Lyapunov analysis including a sketch of the proof of finite time stability. Section IV presents the model of a fully actuated biped. Numerical simulation results are presented in Section V followed by conclusions in Section VI.

II. Problem Statement

Let the system dynamics be given as follows:

$$\dot{\mathbf{e}}_1 = \mathbf{e}_2 \quad \dot{\mathbf{e}}_2 = \mathbf{u} + \boldsymbol{\omega}(e,t), \tag{1}$$

This system corresponds to the tracking dynamics of a mechanical system (such as a biped) with a computed control law (see section IV.A). Here $\mathbf{e}_1 = \mathbf{x}_1(t) - \mathbf{x}_1^d(t)$ and $\mathbf{e}_2 = \mathbf{x}_2(t) - \mathbf{x}_2^d(t)$ are respectively the error variables in position and in velocity, $\mathbf{u}$ is the control law to be synthesized, and $\boldsymbol{\omega}(e,t) = \begin{bmatrix} \omega_1(e,t) & \omega_2(e,t) & \dots & \omega_6(t) \end{bmatrix}^\top$ is an external disturbance. Let $(\mathbf{x}_1^d(t), \mathbf{x}_2^d(t) = \dot{\mathbf{x}}_1^d(t))$ represent a desired trajectory as functions of time for the position $\mathbf{x}_1$ and the velocity $\mathbf{x}_2$. The following assumptions are made throughout.

Assumption 1: $\operatorname*{ess\,sup}_{t\geq 0} |\omega_j| \leq N|e_{2_j}|^{\frac{\alpha}{2}}(|e_{1_j}|^{\frac{\alpha}{2(2-\alpha)}} + |e_{2_j}|^{\frac{\alpha}{2}})$ where N is an *a priori* known positive scalar and $j = 1,2,\dots,6$.

Assumption 2: The upper bound $\tilde{R}$ on the quantity $\max\{|\mathbf{x}_1(t_0)|, |\mathbf{x}_2(t_0)|, |\mathbf{x}_1^d(t_0)|, |\mathbf{x}_2^d(t_0)|\}$, where t_0 is the initial time, is known *a priori* and finite.

The first assumption appears in the literature [1], [4] and represents a uniform upper-bound on the disturbance. The second assumption dictates that the results presented in the paper are of a local nature[1]. ***The aim of the paper is to utilize (i) a continuous SOSM state feedback synthesis and (ii) the corresponding tuning guidelines for the controller gains to ensure finite time convergence of the states of a fully actuated biped robot to the desired trajectory.*** This synthesis and the tuning guidelines are then utilised for biped control using the equivalence between the error dynamics of (1) and that of each actuated joint of the biped.

III. Continuous SOSM Synthesis

The following control law is proposed in [13] as follows:

$$\begin{aligned} u_j(\mathbf{e}_{1_j}, \mathbf{e}_{2_j}) = &-(\mu_1|e_{2_j}|^{\alpha} + \mu_3|e_{1_j}|^{\frac{\alpha}{2(2-\alpha)}}|e_{2_j}|^{\frac{\alpha}{2}})\operatorname{sign}(e_{2_j}) \\ &-\mu_2|e_{1_j}|^{\frac{\alpha}{2-\alpha}}\operatorname{sign}(e_{1_j}) \end{aligned} \tag{2}$$

where, $j = 1,2,\dots,6$, $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \dots & u_6 \end{bmatrix}^\top$, $\mathbf{e}_1 = \begin{bmatrix} e_{1_1} & e_{1_2} & \dots & e_{1_6} \end{bmatrix}^\top$ and $\mathbf{e}_2 = \begin{bmatrix} e_{2_1} & e_{2_2} & \dots & e_{2_6} \end{bmatrix}^\top$. The proof of finite time stability and the tuning guidelines can be substantiated by a rigorous Lyapunov analysis where the details are not included here due to space constraints. The detailed proof is being published elsewhere [13] and is briefly outlined below.

[1] Such an upper bound is generally known *a priori* for a large class of mechanical systems.

Theorem 1: Given $\alpha \in (\frac{2}{3}, 1)$, the closed-loop system (1), (2) is globally equiuniformly finite time stable, regardless of the presence of any disturbance $\omega_j(e,t)$, satisfying the condition stated in Assumption 1 with

$$0 < N < \min\{\mu_1, \mu_3\}, \mu_2 > \max\{\mu_1, \mu_3\}. \tag{3}$$

The definitions [4, 2.1-2.10] and [4, lemma 2.11] for the concepts of equiuniform stability and homogeneity are recalled. The key idea is that of proving equiuniform asymptotic stability of the perturbed double integrator (1), (2). Then, equiuniform finite time stability of (1), (2) is proven via homogeneity of the same under a homogeneous parameterisation of the disturbance $\boldsymbol{\omega}(\boldsymbol{e},t)$ thereby enabling the application of the existing result [4, Th. 3.1]. It is important to note that there is no assumption being made about the continuity of $\boldsymbol{\omega}(\boldsymbol{e},t)$. This allows for possibly discontinuous vanishing disturbances thereby calling for Filippov's solution concept [22]. Hence, the controller (2) is robust to a broader class of disturbances than those reported thus far in the continuous finite time stabilisation paradigm.

Proof: The sketch of the proof is provided for one perturbed double integrator out of a total of six in the closed-loop system (1), (2). The proof for one double integrator is enough in the context of biped robots (to be discussed in the next section) since the six double integrators are decoupled.

Proof of global equiuniform stability: The Lyapunov function (proposed in [12] for the unperturbed case and in [2] for the perturbed case)

$$V(e_{1_j}, e_{2_j}) = \mu_2 \frac{2-\alpha}{2}|e_{1_j}|^{\frac{2}{2-\alpha}} + \frac{1}{2}e_{2_j}^2 \tag{4}$$

has a non-positive temporal derivative

$$\dot{V} \quad \leq \quad -(\mu_1 - N)|e_{2_j}|^{\alpha+1} - (\mu_3 - N)|e_{1_j}|^{\frac{\alpha}{2(2-\alpha)}}|e_{2_j}|^{\frac{\alpha}{2}+1}$$

along the trajectory of the closed-loop system (1), (2). The equiuniform stability of the closed-loop system (1), (2) can be concluded by applying the invariance principle [23], [24] since the equilibrium point $e_{1_j} = e_{2_j} = 0$ is the only trajectory of (1), (2) on the invariance manifold $e_{2_j} = 0$ where $\dot{V}(e_{1_j}, e_{2_j}) = 0$.

Proof of global equiuniform asymptotic stability: Undertaking a similar semi-global analysis to that proposed in [4] for the present continuous case, it is assumed that an *a priori* $\tilde{R}$ is known such that $V \leq \tilde{R}$ holds true for all $t \in \mathbb{R}$ (see Assumption 2). The previous step proves that the trajectories initialised within the set $\{(e_{1_j}, e_{2_j}) : V(e_{1_j}, e_{2_j}) \leq \tilde{R}\}$ for arbitrarily chosen $\tilde{R}$ do not leave this set for all t. Then, considering a Lyapunov function

$$V_{\tilde{R}}(e_{1_j}, e_{2_j}) = V + \sum_{i=1}^{4} U_i, \tag{5}$$

where the indefinite functions $U_i, i = 1,2,3,4$ are defined by the expressions

$$\begin{aligned} U_1 &= \kappa_1 e_{1_j} e_{2_j}|e_{2_j}|, \quad U_3 = 2\kappa_1\kappa_2\kappa_3 e_{1_j}^3 e_{2_j}|e_{2_j}|^{\frac{\alpha}{2}} \\ U_2 &= \kappa_1\kappa_2|e_{1_j}|^{\frac{4-\alpha}{2(2-\alpha)}}\operatorname{sign}(e_{1_j})e_{2_j}|e_{2_j}|^{\alpha} \\ U_4 &= \kappa_1\kappa_2\kappa_3\kappa_4 e_{1_j}^5 e_{2_j}, \end{aligned} \tag{6}$$

and its temporal derivative

$$\dot{V}_{\tilde{R}} = \dot{V} + \sum_{i=1}^{4} \dot{U}_i \quad \leq \quad -((\mu_1 - N) - \kappa_1\beta_1)|e_{2_j}|^{\alpha+1} \quad (7)$$
$$-\kappa_1\kappa_2\kappa_3\kappa_4 e_{1_j}^4 |e_{1_j}|^{\frac{2}{2-\alpha}},$$

where $\kappa_i, i = 1,2,3,4$ and β_1 are adequately small positive scalars that can be chosen *a priori* for every given $\tilde{R}$, proves that the closed-loop system (1), (2) is equiuniformly asymptotically stable as per [4, Definition 2.7] within the compact set $\{(e_{1_j}, e_{2_j}) : V(e_{1_j}, e_{2_j}) \leq \tilde{R}\}$ for arbitrarily chosen $\tilde{R}$.

Proof of global equiuniform finite time stability: The following parameterisation $\omega(e,t) \triangleq \omega^c(e,t)$ of the disturbance

$$\omega^c(e_{1_j}, e_{2_j}, t) = c^{q+r_2}\omega(c^{-r_1}e_{1_j}, c^{-r_2}e_{2_j}, c^q t) \quad (8)$$

is valid in the sense of the upper bound stated in Assumption 1 since the expression

$$\begin{aligned} |\omega^c(e_{1_j}, e_{2_j}, t)| &\leq c^{q+r_2-r_2\alpha} N(|e_{1_j}|^{\frac{\alpha}{2(2-\alpha)}} |e_{2_j}|^{\frac{\alpha}{2}} + |e_{2_j}|^{\alpha}) \\ &\leq N(|e_{1_j}|^{\frac{\alpha}{2(2-\alpha)}} |e_{2_j}|^{\frac{\alpha}{2}} + |e_{2_j}|^{\alpha}) \end{aligned} \quad (9)$$

holds true due to the fact that the expression $c^{q+r_2-\alpha r_2} \leq 1$ holds true, where c is homogeneity parameter [4], $r_1 = \frac{2-\alpha}{1-\alpha}, r_2 = \frac{1}{1-\alpha}$ are the dilations and $q = -1$ is the homogeneity degree. The closed-loop system (1), (2) is a globally homogeneous system with negative homogeneity degree with respect to dilations r_1, r_2 under the parameterisation (8). Combining this with equiuniform asymptotic stability proves global finite time stability by applying [4, Th. 3.1]. □

IV. Biped Model

The bipedal robot considered in this section is walking on a rigid and horizontal surface. It is modeled as a planar biped, which consists of a torso, hips, two legs with knees and feet (see Fig. 1). The walking gait takes place in the sagittal plane and is composed of single support phases and impacts. The complete model of the biped robot consists of two parts: the differential equations describing the dynamics of the robot during the swing phase, and an impulse model of the contact event (the impact between the swing leg and the ground is modeled as a contact between two rigid bodies [25]). In the single support phase, the dynamic model, considering an implicit contact of the stance foot with the ground (*i.e.* there is no take-off, no rotation and no sliding during the single support phase), can be written as follows:

$$\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) = \Gamma \quad (10)$$

with $\mathbf{q} = (q_1\ q_2\ q_3\ q_4\ q_5\ q_6)^\top \in \mathbb{R}^6$ the vector of the generalized coordinates (see Fig. 1), $\Gamma = (\Gamma_1\ \Gamma_2\ \Gamma_3\ \Gamma_4\ \Gamma_5\ \Gamma_6)^\top \in \mathbb{R}^6$ is the vector of joint torques [2], $\mathbf{D}$ is the symmetric, positive definite 6×6 inertia matrix. As the kinetic energy of the biped is invariant under rotation of the world frame, q_1 defines the orientation of the biped in the world frame. The terms $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ and $\mathbf{G}(\mathbf{q})$ are the 6×1 matrices of the centrifugal, coriolis and gravity forces respectively. From (10), the state-space form can be written as follows:

$$\begin{aligned} \dot{\mathbf{x}} = \begin{pmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{pmatrix} &= \begin{pmatrix} \mathbf{x}_2 \\ \mathbf{D}^{-1}(-\mathbf{C} - \mathbf{G} + \mathbf{\Gamma}) \end{pmatrix} \\ &= f(\mathbf{x}) + g(\mathbf{x}_1) \cdot \mathbf{\Gamma} \end{aligned} \quad (11)$$

with $\mathbf{x} = \begin{pmatrix} \mathbf{q}^\top & \dot{\mathbf{q}}^\top \end{pmatrix}^\top = \begin{pmatrix} \mathbf{x}_1^\top & \mathbf{x}_2^\top \end{pmatrix}^\top$. The state space is chosen such that $\mathbf{x} \in \mathscr{X} \subset \mathbb{R}^{12} = \{\mathbf{x} = [\mathbf{x}_1^\top\ \mathbf{x}_2^\top]^\top \mid \mathbf{x}_1 \in \mathscr{N},\ \mathbf{x}_2 \in \mathscr{M}\}$, where $\mathscr{N} = (-\pi,\pi)^6$ and $\mathscr{M} = \{\ \mathbf{x}_2 \in \mathbb{R}^6 \mid |\mathbf{x}_2| < M < \infty\}$ such that $M > 0$ is a positive scalar. One of the difficulties of biped control is ensuring that the contact with the ground is the expected one in the presence of perturbation. To correctly check the behavior of the controller in simulation, a general model able to deal with any condition of contact is used for simulation. The model (11) is used to define the control with the assumption that a flat contact occurs between the stance foot and the ground. This simulation model includes a unilateral contact between the foot and the ground with contact points, at both the heel and the toe, of each foot. Various solutions exist to determine the contact of each corner of the foot with the ground. The contact forces between the foot and the ground reaction are calculated using a constraint-based approach. This approach belongs to the family of time-stepping approaches. Let the vector $\mathbf{R} \in \mathbb{R}^8$ be the reaction force vector, which is obtained by stacking the reaction force vectors of the two corners of each foot. Vector $\mathbf{R}_k$ at $t = t_k$ is expressed at each sampling period as a function of the generalized position vector $\mathbf{q^k} \in \mathbb{R}^9$ composed of the variable orientation of each link and the Cartesian coordinates $x_h,\ y_h$ of the hips, the associate velocity vector $\mathbf{q_v^k} \in \mathbb{R}^9$ for the biped and Γ^k with an algebraic equation

$$G(\mathbf{R}_k, \mathbf{q^k}, \mathbf{q_v^k}, \Gamma^k) = 0 \quad (12)$$

Let vector $\mathbf{v}_{in}^{k+1}$ be the Cartesian velocities of the corners in contact with the ground at $t = t_{k+1}$. The normal components must be non negative to avoid interpenetration. The identity $\mathbf{v}_{in}^{k+1} = 0$ means that the contact remains and the inequality $\mathbf{v}_{in}^{k+1} > 0$ means that the contact vanishes. The normal components $\mathbf{r}_{in}^k > 0$ of $\mathbf{R}_k$, when contact occurs, are also subject to non negative constraints. These components can avoid interpenetration but they cannot avoid the stance foot take-off. It is clear that the variables $\mathbf{v}_{in}^{k+1}$ and $\mathbf{r}_{in}^k$ are complementarity quantities:

$$\mathbf{v}_{in}^{k+1} \geq 0 \quad \perp \quad \mathbf{r}_{in}^k \geq 0 \quad (13)$$

Furthermore, the variables $\mathbf{v}_{in}^{k+1}$ and $\mathbf{r}_{in}^k$ are subject to constraints imposed by friction which leads to a linear complementarity condition. The valid cases of contact for each corner are determined using constrained optimization [26].

A. Pre-feedback and Reference Trajectory

The cyclic walking gait, which is composed of single support phases and impacts, has been defined by $\mathbf{x}_1^d(t)$, $\mathbf{x}_2^d(t)$ and $\dot{\mathbf{x}}_2^d(t)$ satisfying the conditions of contact using off-line optimization [27]. This means that if the tracking of

[2] Leg 1 is the stance one, leg 2 the swing one.

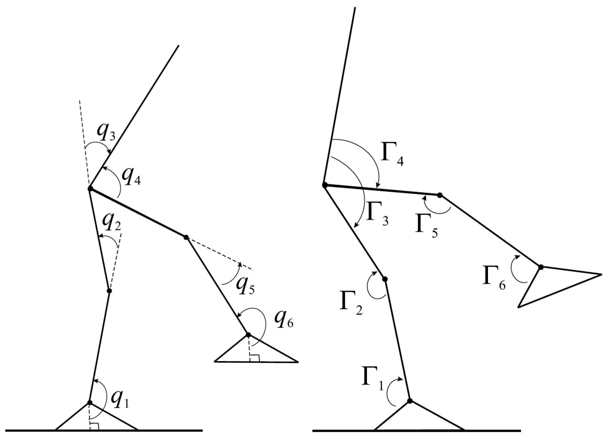

Fig. 1. Seven-link bipedal robot.

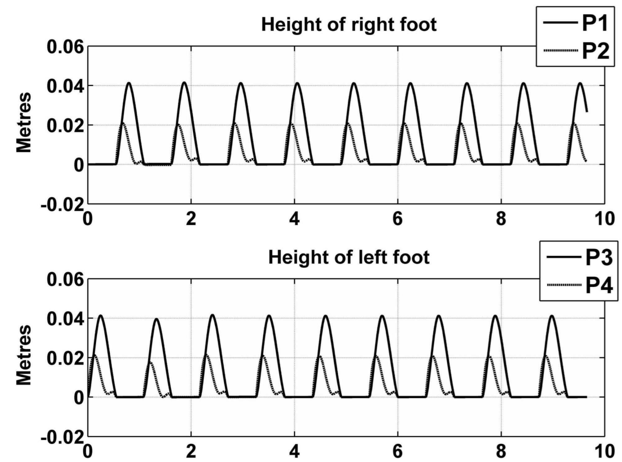

Fig. 2. Foot height in the walking gait with 0.5 s settling time

this reference trajectory is perfect, the stance foot neither rotates nor takes off since the ZMP of the biped lies within the interior of the support polygon defined by the foot geometry. Furthermore there is no slipping of the stance foot on the ground. The torques and velocities of the actuators are bounded by given values. As a consequence, when the fully actuated biped adopts this cyclic walking gait, no zero dynamics appear. The reference walking minimizes the integral of the norm of the torque vector for a given distance. The walking velocity is selected to be 0.5 m/s. The duration of one step is 0.53 s. Since the impact is instantaneous and passive, the control law is defined only during the single support phase. The objective of the control is that each joint angle follows its reference trajectory to track cyclic walking gait. The torque vector $\mathbf{\Gamma}$ is defined based on the dynamic model (10) as follows:

$$\Gamma = \mathbf{D}(\mathbf{x}_1)(\dot{\mathbf{x}}_2^d(t) + \mathbf{u}) + \mathbf{C}(\mathbf{x}_1, \mathbf{x}_2) + \mathbf{G}(\mathbf{x}_1) \tag{14}$$

where $\mathbf{u}$ is defined by (2). The pre-feedback (14) enables the system (10) to be transformed into the form (1), thereby rendering the tuning rules (3) applicable to the biped problem.

V. Numerical Simulation

Numerical results for a fully actuated biped are included in this section.

A. Robust Walking Cycles

The model (11) is utilized in this section to show numerical simulations of a stable walking gait by achieving a finite settling time via the tuning rules (3). The desired convergence time for tracking the reference trajectories is defined to be 0.5 s.

The robustness of the tracking control (14) is verified by introducing a resultant disturbance force $\mathbf{F}_\omega$ on the hip joint of the biped with projections $F_{x\omega} = 50\ N$ and $F_{y\omega} = 2.5\ N$ in the horizontal and vertical planes respectively. Such a force is used for the duration of 0.07 s to simulate a disturbance effect. The effect of $F_{x\omega}$ represents a disturbance in the continuous phase of the dynamics (10) as it starts from 1.08 s in the first cycle of the biped which belongs to the continuous phase of the trajectory.

The effect of the aforementioned disturbance force on the hip joint can be studied via the principle of virtual work as follows. Let a disturbance force $\mathbf{F}_\omega$ be applied as mentioned above. Let the effect of the disturbance force $\mathbf{F}_\omega$ on the dynamics of the generalized coordinates $\mathbf{q}$ be denoted by $\mathbf{\Gamma}_\omega = \mathbf{J}^\top \mathbf{F}_\omega$ where $\mathbf{J}$ is such that $\mathbf{J}\ \mathbf{x}_2$ is the velocity of the hip, the point where the force is applied, where

$$\mathbf{J}^\top = \begin{pmatrix} l_1 \cos q_1 - l_2 \cos(q_1+q_2) & l_1 \sin q_1 + l_2 \sin(q_1+q_2) \\ -l_2 \cos(q_1+q_2) & l_2 \sin(q_1+q_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{15}$$

Here l_1 and l_2 are the lengths respectively of the shin and the thigh. Hence, the biped model (10) can be revised as follows:

$$\ddot{\mathbf{q}} = \mathbf{D}^{-1}\left(\mathbf{\Gamma} + \mathbf{\Gamma}_\omega - \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) - \mathbf{G}(\mathbf{q})\right) \tag{16}$$

It can be seen from the above that the quantity $\mathbf{\Gamma}_\omega$ appears as a disturbance in the $\ddot{\mathbf{q}}$ dynamics. In the following, an *a priori* known upper bound

$$N \triangleq \sup_{t\geq 0} |\mathbf{D}^{-1}\mathbf{\Gamma}_\omega| = 19.2 \tag{17}$$

is utilized by the tuning rules (3) to cover the worst effect produced by the disturbance force F_ω while tuning the gains μ_1, μ_2 for each joint. Thus, the modeling information utilised for the control synthesis (14) lies in the use of model matrices $\mathbf{D}, \mathbf{C}, \mathbf{G}$ and the *a priori* known upper bound (17).

Next, tuning rules (3) are used to produce a continuous controller with gains $\mu_1 = 20$, $\mu_2 = 25, \mu_3 = \mu_1$ and with the scalar $\alpha = 0.7$. Figure 2 shows the heights of the feet for eight consecutive steps with the selected gains. The corresponding velocities of the feet in the horizontal and

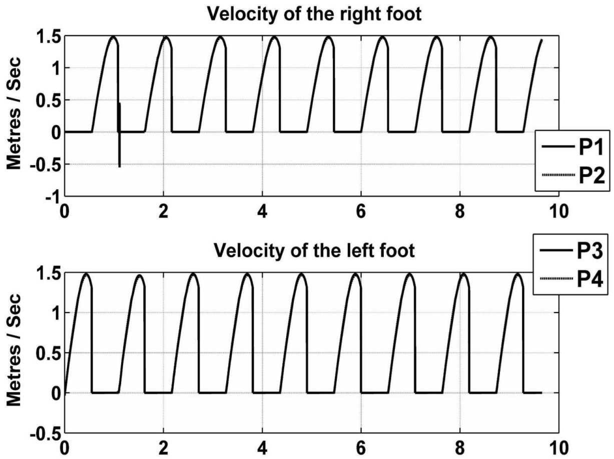

Fig. 3. Foot velocity in the horizontal direction in the walking gait with 0.5 s settling time

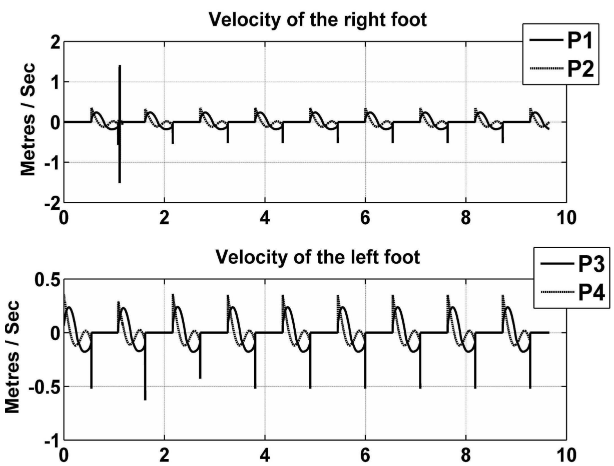

Fig. 4. Foot velocity in the vertical direction in the walking gait with 0.5 s settling time

vertical direction can be seen in Figs. 3 and 4 respectively. Legends 'P1' and 'P3' represent the 'toe' of the right and left foot respectively. Similarly, 'P2' and 'P4' represent the 'heel' of the right and left foot respectively. The control torques produced by SOSM synthesis are shown in Fig. 7.

Periodic orbits in joints 1 and 2 are depicted in terms of phase-plane plots of $q_i, \dot{q}_i, i = 1,2$ in Fig. 5. It can be seen that each joint velocity undergoes a jump at the time of collision of the feet with the walking surface and that the actual trajectory follows the reference trajectory closely due to the robust SOSM control.

B. Robustness Analysis

The effect of the disturbance force can be seen in several plots. For example, the velocity in the vertical direction for the right foot is severely affected by the disturbance as can be seen from the high amplitude impulse like change at the end of the first step (see Fig. 4). This is the result of the combination of the disturbance forces $F_{x\omega}$ and $F_{y\omega}$ The effect

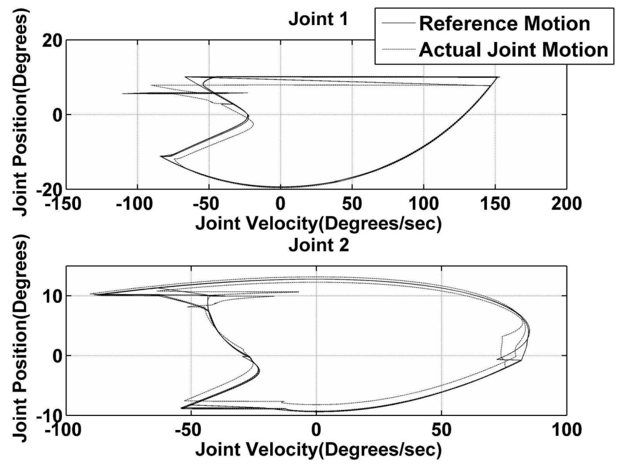

Fig. 5. Periodic orbit in joint 1 and 2 in walking gait

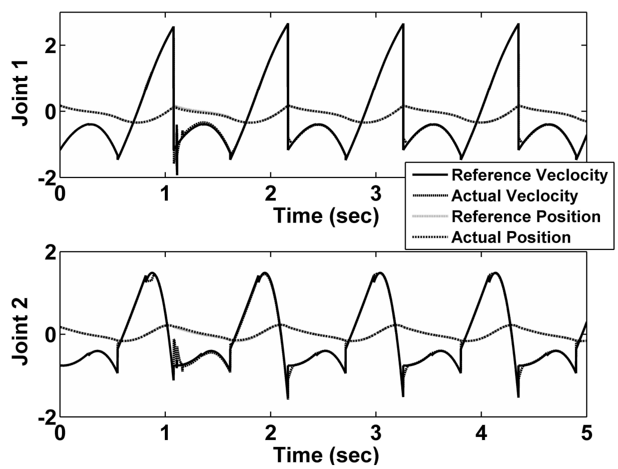

Fig. 6. Periodic orbit in joint 1 and 2 as a function of time

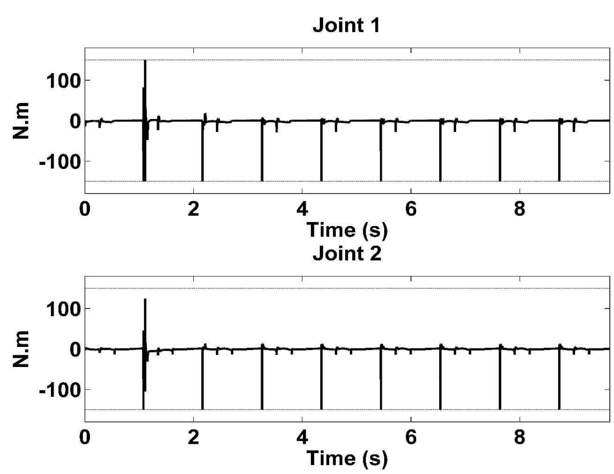

Fig. 7. Torques in joints 1 and 2 with their saturation limits

on the biped is a destabilizing one in the continuous phase also. This can be seen in the plot of velocity of the left foot in the vertical direction as it gets affected in its flight in the next step as shown by an abnormal impulse (or stumbling) at the end of the step Fig. 4. This undesired behaviour disappears due to the robustness of the control and the biped returns to its nominal desired gait as can be seen from Figs. 2, 4 and subsequent orbits in Figs. 5 and 7. The orbital trajectories against time are shown in Fig. 6 where it can be clearly seen that the joint velocity and position return to the nominal trajectory after the disturbance disappears.

VI. Conclusions And Future Work

A robust continuous second order sliding mode controller is utilised along with corresponding tuning guidelines to achieve a finite settling time for the tracking error dynamics of a fully actuated biped robot. The results give straight-forward engineering guidelines to achieve stable walking of a biped. Each joint follows its reference trajectory in finite time before the next impact occurs with the ground, thereby producing a stable periodic orbit in this non-linear system. Numerical results are presented to show reference trajectory planning, robustness of the non-linear synthesis and ease of tuning. If a premature disturbance such as that caused by a severely uneven or slippery surface, which cannot be rejected, occurs before the control reaches this reference trajectory there is no guarantee that the biped does not fall down. Potential future directions include the study of theoretical conditions of stability of the system in the presence of parasitic dynamics and attainment of similar results for under-actuated bipeds.

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