On the discrete and continuous Miura Chain associated with the Sixth Painlevé Equation

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Abstract

A Miura chain is a (closed) sequence of differential (or difference) equations that are related by Miura or Bäcklund transformations. We describe such a chain for the sixth Painlevé equation ($\mathrm{P_{VI}}$), containing, apart from $\mathrm{P_{VI}}$ itself, a Schwarzian version as well as a second-order second-degree ordinary differential equation (ODE). As a byproduct we derive an auto-Bäcklund transformation, relating two copies of $\mathrm{P_{VI}}$ with different parameters. We also establish the analogous ordinary difference equations in the discrete counterpart of the chain. Such difference equations govern iterations of solutions of $\mathrm{P_{VI}}$ under Bäcklund transformations. Both discrete and continuous equations constitute a larger system which include partial difference equations, differential-difference equations and partial differential equations, all associated with the lattice Korteweg-de Vries equation subject to similarity constraints.

1 Introduction

The six Painlevé equations are second-order nonlinear ordinary differential equations (ODEs) that arose in the classification programme originated by Painlevé [1, 2]. In this paper, we concentrate on the sixth Painlevé equation (first found by R. Fuchs in [3]):

$$\begin{aligned}\frac{d^2w}{dt^2} &= \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right)\left(\frac{dw}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{w-t}\right)\frac{dw}{dt}\\ &+\frac{w(w-1)(w-t)}{8t^2(t-1)^2}\left(\alpha-\beta\frac{t}{w^2}+\gamma\frac{t-1}{(w-1)^2}-(\delta-4)\frac{t(t-1)}{(w-t)^2}\right),\end{aligned} \qquad \mathrm{P_{VI}}$$

and equations related to it by Miura or Bäcklund transformations.

Painlevé's classification programme aims to identify classes of ODEs with the property that all movable singularities of all solutions are poles. He and his school completed the classification work for first-degree second-order ODEs (under some natural assumptions). However, the work for third-order ODEs remains incomplete. The third-order classification was undertaken by Chazy, Garnier, and Bureau [4, 5] and more recently by Cosgrove [6]. However, they restricted their attention to special classes which exclude *Schwarzian* equations [7], i.e. ODEs of the form

$$\Big(\{Z,x\}\Big)^k = R(Z_{xx},Z_x,Z,x), \quad \{Z,x\} := \frac{Z_{xxx}}{Z_x}-\frac{3}{2}\frac{Z_{xx}^2}{Z_x^2}, \tag{1.1}$$

where k is a positive integer and R is a rational function of Z_{xx}, Z_x, and Z with polynomial coefficients. Equations of Schwarzian form appear naturally as similarity reductions of soliton equations (just as the Painlevé equations do [8]). For example, the third-order ODE

$$Z'''-\frac{3}{2}\frac{Z''^2}{Z'} = \mu Z-\frac{1}{3}xZ' \tag{1.2}$$

(where primes represent derivatives w.r.t. x) arises as a reduction of the Schwarzian KdV equation

$$\frac{z_t}{z_x} = \{z,x\}, \tag{1.3}$$

cf. [9, 10]. It is related to the second Painlevé equation

$$V'' = 2V^3-\frac{1}{3}xV-\frac{3\mu-1}{6} \qquad \mathrm{P_{II}}$$

by the Cole-Hopf transformation $V=-Z''/(2Z')$. Even though equations of the type (1.1) are not Möbius invariant (whereas the Schwarzian operator is), the partial differential equations (PDEs) from which they arise as similarity reductions *are* invariant under the action of $PSL_2(\mathbb{C})$. Moreover, they appear to possess some special symmetries that give rise to Bäcklund transformations of the Painlevé equations to which they are related.

In this note we present a generalization of Eq (1.2), namely a four-parameter Schwarzian $\mathrm{P_{VI}}$ equation ($\mathrm{SP_{VI}}$) (see Eq (2.1f) below) and the Miura transformation relating it to $\mathrm{P_{VI}}$. In addition, we give a second-order second-degree equation (see Eq. (2.15) below) which we call a *modified* $\mathrm{P_{VI}}$ ($\mathrm{MP_{VI}}$) which is also related to $\mathrm{P_{VI}}$ by a Miura transformation. As a byproduct, we obtain an auto-Bäcklund transformation for $\mathrm{P_{VI}}$ (i.e. a transformation relating $\mathrm{P_{VI}}$ to a copy of itself with possibly different parameters). These three transformations form what we call the *Miura chain*.

It should be noted that relationships between $\mathrm{P_{VI}}$ and second-degree equations were known before, cf. [11, 12, 13, 14]. However, our point of view that the latter equation constitutes a modified version of $\mathrm{P_{VI}}$ appears to be new, and is motivated by the connection of these equations to certain discrete equations which forms the central part of the present note. In fact, we present the exact discrete counterparts of the $\mathrm{SP_{VI}}$ and $\mathrm{MP_{VI}}$ equations. These are nonlinear nonautonomous ordinary difference equations that can be considered to be nonlinear superposition formulae for the auto-Bäcklund transformations (auto-BTs) of the corresponding continuous equations. In recent years such difference equations have attracted a great deal of attention (see [15] for a review) as integrable difference equations, possibly defining new transcendental functions by means of discrete rather than continuous equations.

The derivation of the equations is based on the results on the similarity reduction of a family of partial difference equations, namely the KdV family of lattice systems, a programme that was initiated in Ref. [16] and continued in [17]. The present note is a further development of the general framework presented in a recent paper [18], in which equations on the lattice as well as in the continuum are naturally linked together. It was shown there that $\mathrm{P_{VI}}$ (with general parameters α, β, γ, δ) naturally arises from such a framework for the lattice KdV family (without imposing a limit). Moreover, a new discrete Painlevé equation with four free parameters was derived, the general solution of which can be expressed in terms of the transcendental solutions of $\mathrm{P_{VI}}$. We will refer to these continuous and discrete results as forming the "regular" $\mathrm{P_{VI}}$ system.

In the present paper we extend the results of [18]. We derive $\mathrm{SP_{VI}}$, $\mathrm{MP_{VI}}$ as well as their natural discrete analogues, and transformations between them. Thus we establish a Miura chain also for ordinary difference equations (O△Es). As is the case with the regular $\mathrm{P_{VI}}$ system, these O△Es govern the same functions as the corresponding ODEs and are compatible with them. Hence there are common solutions to both the discrete and continuous equations. In our view, these results might shed some new light on the algebraic structures behind solutions of $\mathrm{P_{VI}}$, cf. e.g. [19].

The Miura chain can be schematically described as in the following diagram.

$$\begin{tikzcd}
\mathrm{P_{VI}} \arrow[rr, leftrightarrow, "\mathrm{BT}"] \arrow[dr, "\mathrm{Miura}"'] & & \mathrm{P_{VI}} \arrow[dr, "\mathrm{Miura}"] & \\
& \mathrm{MP_{VI}} \arrow[ur, "\mathrm{Miura}"] \arrow[rr, leftrightarrow, "\mathrm{BT}"] & & \mathrm{MP_{VI}}
\end{tikzcd}$$

Note that the pull-back of the Miura between the modified and regular $\mathrm{P_{VI}}$ equations gives rise to an auto-BT. Behind this diagram there is another one for the corresponding discrete equations regulating the shifts in the parameters n,m, μ and ν of the lattice KdV system (see e.g. Eq (2.5)) and associated similarity constraints (see Eq (2.5b)).

The outline of the paper is as follows: in §2 we present our results on three classes of equations: the Schwarzian $\mathrm{P_{VI}}$ class, the modified $\mathrm{P_{VI}}$ class and the regular $\mathrm{P_{VI}}$ class. We also give the formulae relating the various classes. In §3 we outline the derivation of the equations. Only an outline is given here because more complete explanations of the lattice framework can be found in [18]. In §4 we demonstrate some special parameter limits of the equations, some of which have been given before in the literature. Finally in §5 we give some concluding remarks.

2 The Painlevé VI Miura Chain

In this section, we list the three classes of systems i.e. Schwarzian, Modified, Regular, which are intertwined through the Miura chain. Both continuous and discrete equations are given. An outline of their derivation may be found in §3.

We list them in the order given schematically as

$$\textsf{Schwarzian P}_{\mathrm{VI}} \quad \longrightarrow \quad \textsf{Modified P}_{\mathrm{VI}} \quad \longrightarrow \quad \textsf{P}_{\mathrm{VI}}$$

For each system, there are associated or component systems that can be described as follows:

1. The *lattice equation*, i.e. the partial difference equation governing the dependent variable, e.g. $z_{n,m}$, (where n and m are the lattice variables, whilst there are also lattice *parameters*, e.g. p,q, or alternatively $r = q/p$ appearing in the equation);
2. The *similarity constraints* compatible with the corresponding lattice equation [18], containing additional parameters μ, ν;
3. The *ordinary difference equation* for the dependent variable obtained by using similarity constraints and lattice equations to eliminate shifts in one of m or n (thus leading to O△Es governing a function of one lattice variable e.g. $z_n := z_{n,m}$);
4. The *differential-difference equation* for the dependent variable as a function of a lattice variable, e.g. n, and a continuous parameter, e.g. r;
5. The *ordinary differential equation* that results from the elimination of the shifts in the lattice variables altogether by usage of the similarity contraints in favour of continuous operations only.

It should be noted that no continuum limit is carried out here. All equations govern the same functions as in the original lattice equations, subject to the accompanying similarity constraints.

2.1 Schwarzian Painlevé VI System

The Schwarzian equations stand at the base of the Miura chain and seem to be the most fundamental in the family. We use them as the starting point for the derivations of other members in the $P_{\rm VI}$ family of equations. The relevant equations are listed below.

- Lattice Equation

$$\frac{(z_{n,m}-z_{n+1,m})(z_{n,m+1}-z_{n+1,m+1})}{(z_{n,m}-z_{n,m+1})(z_{n+1,m}-z_{n+1,m+1})}=r^2 \tag{2.1a}$$

see Ref. [20].

- Similarity Constraint # 1

$$n\frac{(z_{n+1,m}-z_{n,m})(z_{n,m}-z_{n-1,m})}{z_{n+1,m}-z_{n-1,m}}+m\frac{(z_{n,m+1}-z_{n,m})(z_{n,m}-z_{n,m-1})}{z_{n,m+1}-z_{n,m-1}}=\bar{\mu}z_{n,m}\ , \tag{2.1b}$$

see Ref. [10], where we use the notation $\bar{\mu}=\mu+\frac{1}{2}$ to remain consistent with that of [18].

- Similarity Constraint # 2

$$\mu-\nu=n\frac{z_{n+1,m}+z_{n-1,m}-2z_{n,m}}{z_{n+1,m}-z_{n-1,m}}+m\frac{z_{n,m+1}+z_{n,m-1}-2z_{n,m}}{z_{n,m+1}-z_{n,m-1}}\ , \tag{2.1c}$$

in which $\nu=\lambda(-1)^{n+m}$ is an alternating coefficient with λ constant.

- Ordinary Difference Equation

$$\begin{aligned}&(r^2-1)(z_{n+1}-z_n)^2=\\&=\left[2r^2\frac{\bar{\mu}z_{n+1}(z_{n+2}-z_n)-(n+1)(z_{n+2}-z_{n+1})(z_{n+1}-z_n)}{(m-\mu-\nu)(z_{n+2}-z_n)+(n+1)(z_{n+2}-2z_{n+1}+z_n)}+z_{n+1}-z_n\right]\times\\&\quad\times\left[2r^2\frac{\bar{\mu}z_n(z_{n+1}-z_{n-1})-n(z_{n+1}-z_n)(z_n-z_{n-1})}{(m-\mu+\nu)(z_{n+1}-z_{n-1})+n(z_{n+1}-2z_n+z_{n-1})}+z_n-z_{n+1}\right]\ ,\end{aligned} \tag{2.1d}$$

i.e. the discrete Schwarzian PVI equation.

- Differential-Difference Equation

$$-t\frac{dz_n}{dt}=n\frac{(z_{n+1}-z_n)(z_n-z_{n-1})}{z_{n+1}-z_{n-1}} \tag{2.1e}$$

- Ordinary Differential Equation

$$\begin{aligned}
\{z,t\} = \Big[&(a-b)^2\bar{\mu}^4 z^4 - 2(a-b)^2(t-2)t\bar{\mu}^3 z^3 z' + \\
&+ (t-1)t^4 z'^3\Big(-[(a-b)^2 - 4(1-t)^2 - 4at + 2abt - 2b^2t - 4ct + 2(a+b)ct + \\
&(b-c)^2t^2 - 16\bar{\mu} + 32t\bar{\mu} - 8t^2\bar{\mu} - 12\bar{\mu}^2 + 12t\bar{\mu}^2 - 4t^2\bar{\mu}^2]z' + \\
&8t(2-3t+t^2)\bar{\mu}z''\Big) + \\
&t^2\bar{\mu}^2 z^2\Big([4 + 6(a-b)^2 - 16t - 4at - 5a^2t + 12abt - \\
&7b^2t - 4ct + 2(a+b)ct + 12t^2 + 4at^2 - 2abt^2 + \\
&b^2t^2 + 4ct^2 - 2act^2 - c^2t^2 + 4(t-1)\bar{\mu}^2]z'^2 + \\
&8t(2-5t+3t^2)z'z'' + 8(t-1)^2t^2z''^2\Big) - \\
&4(t-1)t^3\bar{\mu}zz'\big(((a-b)^2 - 2t - 2at + (a-b)bt - \\
&2ct + act + bct - 4\bar{\mu} + 6t\bar{\mu} - 4\bar{\mu}^2 + 2t\bar{\mu}^2)z'^2 + \\
&2t(2-4t+t^2-2\bar{\mu}+2t\bar{\mu})z'z'' + t^2(t-1)(t-2)z''^2\big)\Big]\Big/ \\
&\Big[8(t-1)^2t^4z'^2(\bar{\mu}z + tz')(-\bar{\mu}z + (t-1)tz')\Big] \ , \qquad (2.1f)
\end{aligned}$$

where the parameters a,b,c are given by

$$a = \frac{1}{2} + \nu + m - n \quad , \quad b = \frac{1}{2} + \nu + m + n \quad , \quad c = \frac{1}{2} + \nu - m + n \ .$$

and the independent variable $t = 1/r^2$. Eq (2.1f) is the Schwarzian P_{VI}.

2.2 Painlevé VI System

We now move on to the "regular" P_{VI} equations, i.e. the equations that give rise directly to P_{VI} itself. These are given in terms of the intermediate variable $v_{n,m}$ which is related to the Schwarzian variable $z_{n,m}$ via the (discrete Cole-Hopf type) relations

$$\begin{aligned}
p(z_{n,m} - z_{n+1,m}) &= v_{n+1,m}v_{n,m} \ , \qquad (2.2a)\\
q(z_{n,m} - z_{n,m+1}) &= v_{n,m+1}v_{n,m} \ . \qquad (2.2b)
\end{aligned}$$

The main objects of interests, however, are combinations of the variables $v_{n,m}$ given by

$$x_{n,m} \equiv \frac{v_{n,m}}{v_{n+1,m+1}} \quad , \quad y_{n,m} \equiv \frac{v_{n+1,m}}{v_{n,m+1}} \ , \qquad (2.3)$$

which are related through the fractional linear transformation

$$x_{n,m} = \frac{y_{n,m} - r}{1 - ry_{n,m}} \quad \Leftrightarrow \quad y_{n,m} = \frac{x_{n,m} + r}{1 + rx_{n,m}} \ . \qquad (2.4)$$

The following equations arise from the results of [18]:

- Lattice Equation

$$\frac{v_{n,m+1}}{v_{n+1,m+1}} - \frac{v_{n+1,m}}{v_{n,m}} = r\left(\frac{v_{n+1,m}}{v_{n+1,m+1}} - \frac{v_{n,m+1}}{v_{n,m}}\right) \ , \tag{2.5a}$$

- Similarity Constraint

$$n\frac{v_{n+1,m} - v_{n-1,m}}{v_{n+1,m} + v_{n-1,m}} + m\frac{v_{n,m+1} - v_{n,m-1}}{v_{n,m+1} + v_{n,m-1}} = \mu - \nu \ , \tag{2.5b}$$

cf. [16].

- Ordinary Difference Equation

$$\begin{aligned}(n+1)(r+x_n)(1+rx_n)&\frac{x_{n+1} - x_n + r(1 - x_n x_{n+1})}{x_{n+1} + x_n + r(1 + x_n x_{n+1})} \\ - n(1-r^2)x_n&\frac{x_n - x_{n-1} + r(1 - x_n x_{n-1})}{x_n + x_{n-1} + r(1 + x_n x_{n-1})} \\ = \mu r(1 + 2rx_n + x_n^2) &+ \nu(r + 2x_n + rx_n^2) - mr(1 - x_n^2),\end{aligned} \tag{2.5c}$$

cf. [17, 18]. The relation between the variable x_n and $v_{n,m}$ is explained in section 3.

- Differential-Difference Equation

$$-2t\frac{d}{dt}\log v_n = n\frac{v_{n+1} - v_{n-1}}{v_{n+1} + v_{n-1}} \tag{2.5d}$$

- Ordinary Differential Equation

$$\begin{aligned}&(1-r^2)^2 y(ry-1)(y-r)\left[\frac{2}{r}\frac{\partial y}{\partial r} - \frac{\partial^2 y}{\partial r^2}\right] = \\ &= \frac{1}{2}(1-r^2)^2\left[r(3y^2+1) - 2(1+r^2)y\right]\left(\frac{\partial y}{\partial r}\right)^2 + \\ &\quad + (1-r^2)\frac{1}{r}\left[2y(y-r)(ry-1) + (1-r^2)^2 y^2\right]\frac{\partial y}{\partial r} \\ &\quad + \frac{1}{2r}\Big[(\alpha y^2 - \beta)(y-r)^2(ry-1)^2 \\ &\qquad + (1-r^2)y^2\left((\gamma-1)(ry-1)^2 - (\delta-1)(y-r)^2\right)\Big] \ ,\end{aligned} \tag{2.5e}$$

which is P_{VI} with the identifications $t = r^{-2}$, $w(t) = y/r$ and identifying the parameters

$$\begin{aligned}\alpha &= (\mu - \nu + m - n)^2 \quad , \quad \beta = (\mu - \nu - m + n)^2 \ , \\ \gamma &= (\mu + \nu - m - n - 1)^2 \quad , \quad \delta = (\mu + \nu + m + n + 1)^2 \ .\end{aligned}$$

Note that the variables n, m need not be integers: we only require that they *shift by units*. In (2.5e) the variable y is fractionally linearly related to x_n as is explained in section 3.

2.3 Bäcklund Transformation

It should be noted that the Schwarzian P_{VI} can be written in second-order form, namely:

$$2t\frac{d}{dt}\left\{\frac{t(1-t)W'-\frac{1}{2}[(a-b)+(c+b)t]W+\frac{1}{2}\bar{\mu}t(c+b)}{(W-\bar{\mu})[(t-1)W-\bar{\mu}t]}\right\}$$
$$=\frac{cbt}{(1-t)W}-W-\frac{(c+b)\bar{\mu}t[t(1-t)W'-\frac{1}{2}[(a-b)+(c+b)t]W+\frac{1}{2}\bar{\mu}t(c+b)]}{(1-t)W(W-\bar{\mu})[(t-1)W-\bar{\mu}t]}$$
$$+\left(W+\frac{\bar{\mu}^2t}{(1-t)W}\right)\frac{\left[t(1-t)W'-\frac{1}{2}[(a-b)+(c+b)t]W+\frac{1}{2}\bar{\mu}t(c+b)\right]^2}{(W-\bar{\mu})^2[(t-1)W-\bar{\mu}t]^2}\ , \quad (2.6)$$

by using the transformation

$$W(t)=\bar{\mu}+t\frac{z'(t)}{z(t)}\ . \quad (2.7)$$

Since we know that P_{VI} is the only representative in the relevant class of the Painlevé-Gambier classification [21], this second–order first-degree ODE must be equivalent to P_{VI} up to a Möbius transformation. Indeed, eq. (2.6) is the P_{VI} equation with the identifications:

$$W(t)=\bar{\mu}\bar{w}(\bar{t})\quad ,\quad \bar{t}=\frac{t}{t-1}\ , \quad (2.8)$$

where $\bar{w}(\bar{t})$ solves P_{VI} with new parameters

$$\overline{\alpha}=4\bar{\mu}^2\quad ,\quad \overline{\beta}=(b-c)^2=4m^2$$
$$\overline{\gamma}=(a-b)^2=4n^2\quad ,\quad \overline{\delta}=(a+c-2)^2=(2\nu-1)^2\ .$$

The connection with the modified P_{VI} equation takes place via a Miura type transformation giving rise to an auto-BT of the form:

$$w=t-W^{-1}\left(\bar{\mu}+(b-\bar{\mu})\left[1-\frac{t(1-t)W'-\frac{1}{2}[(a-b)+(c+b)t]W+\frac{1}{2}\bar{\mu}t(c+b)}{(W-\bar{\mu})[(t-1)W-\bar{\mu}t]}\right]^{-1}\right), \quad (2.9)$$

for P_{VI} itself. Concrete auto-BT's for P_{VI} were given in e.g. [22, 12, 13]; however, this one seems to be different from those.

The explicit relation between the Schwarzian equation (2.1f) and P_{VI} is given by (2.9) whereas the inverse relation is

$$W=\left(1-\frac{w}{t}\right)^{-1}[\bar{\mu}+(b-\bar{\mu})(1-h)^{-1}], \quad (2.10)$$

where in the above $h=h(w',w,t)$ is given by the formula (2.19) below. Analogous connections also exist on the level of the OΔEs. In fact, as is explained in section 3, the relations between x, y on the one hand and z on the other hand can be written in the form

$$r y_{n,m} = \frac{z_{n+1,m} - z_{n,m}}{z_{n,m+1} - z_{n,m}} \quad , \quad x_{n,m} = r \frac{z_{n,m+1} - z_{n,m}}{z_{n+1,m+1} - z_{n,m+1}} \ , \tag{2.11}$$

which reduces to a discrete Cole-Hopf transformation between the OΔEs (2.1d) and (2.5c) if we eliminate the shifts $z_{n,m} - z_{n,m+1}$ using the relation

$$z_{n,m} - z_{n,m-1} = 2\frac{\bar{\mu} z_{n,m} - n A_{n,m}}{\mu - \nu + m - n D_{n,m}} \quad , \quad z_{n,m+1} - z_{n,m} = 2\frac{\bar{\mu} z_{n,m} - n A_{n,m}}{m - \mu + \nu + n D_{n,m}} \ , \tag{2.12}$$

in which

$$A_{n,m} \equiv \frac{(z_{n+1,m} - z_{n,m})(z_{n,m} - z_{n-1,m})}{(z_{n+1,m} - z_{n-1,m})} \quad , \quad D_{n,m} \equiv \frac{z_{n+1,m} + z_{n-1,m} - 2z_{n,m}}{z_{n+1,m} - z_{n-1,m}} .$$

Note that the latter objects only involve shifts in n.

2.4 Modified Painlevé VI System

The modified equations are formulated in terms of the canonical variables

$$h_{n,m} \equiv \frac{z_{n+1,m} + z_{n,m}}{z_{n+1,m} - z_{n,m}} \quad , \quad k_{n,m} \equiv \frac{z_{n,m+1} + z_{n,m}}{z_{n,m+1} - z_{n,m}} \ . \tag{2.13}$$

Using these expressions and rewriting the various relations in terms of the variables h and k, we obtain the following set of equations.

- Lattice Equations

$$k_{n+1,m} - k_{n,m} = r^2 (h_{n,m+1} - h_{n,m}) \ , \tag{2.14a}$$

$$k_{n+1,m} k_{n,m} - 1 = r^2 (h_{n,m+1} h_{n,m} - 1) \ , \tag{2.14b}$$

- Similarity Constraint # 1

$$\frac{2n}{h_{n-1,m} + h_{n,m}} + \frac{2m}{k_{n,m-1} + k_{n,m}} = \bar{\mu} \ , \tag{2.14c}$$

- Similarity Constraint # 2

$$\frac{2n h_{n,m}}{h_{n-1,m} + h_{n,m}} + \frac{2m k_{n,m}}{k_{n,m-1} + k_{n,m}} = \frac{1}{2} + \nu + n + m \ , \tag{2.14d}$$

- Ordinary Difference Equation

$$\begin{aligned} &(1 - r^2)\left(\bar{\mu} - \frac{2(n+1)}{h_{n+1} + h_n}\right)\left(\bar{\mu} - \frac{2n}{h_{n-1} + h_n}\right) \\ &\quad = \frac{\left[\bar{\mu} h_n - (\frac{1}{2} + \nu + n - m)\right]\left[\bar{\mu} h_n - (\frac{1}{2} + \nu + n + m)\right]}{h_n^2 - 1} \ . \end{aligned} \tag{2.14e}$$

- <u>Differential-Difference Equation</u>

$$2t\frac{dh_n}{dt} = \left(\frac{2(n+1)}{h_{n+1}+h_n} - \frac{2n}{h_{n-1}+h_n}\right)(h_n^2-1)\ . \tag{2.14f}$$

- <u>Ordinary Differential Equation</u> The modified $P_{\rm VI}$ equation can be written as the coupled system of first-order ODE's:

$$2t\frac{dk}{dt} = \frac{a-\bar{\mu}k}{1-t}(h-k) - \frac{b-\bar{\mu}k}{h-k}(k^2-1)\ , \tag{2.14g}$$

$$2t\frac{dh}{dt} = \frac{c-\bar{\mu}h}{1-t^{-1}}(h-k) - \frac{b-\bar{\mu}h}{h-k}(h^2-1)\ , \tag{2.14h}$$

with the identifications of a, b, c as before. Eliminating the variable $k(t)$, we are led to the following second-order second-degree equation for $h(t)$:

$$\begin{aligned}
&4t^2(t-1)^2h''\Big[t(t-1)h'' + (2t-1)h' + 4\bar{\mu}^2h^3 - 3\bar{\mu}(b+c)h^2 + 2(bc-\bar{\mu}^2)h + \bar{\mu}(b+c)\Big]\\
&= t(t-1){h'}^2[a-b+(b+c-2)t+2\bar{\mu}(1-2t)h]\times\\
&\qquad\qquad \times[a-b-2+(b+c+2)t+2\bar{\mu}(1-2t)h]\\
&\quad -2t(t-1)(2t-1)h'\Big[4\bar{\mu}^2h^3 - 3\bar{\mu}h^2(b+c) + 2(bc-\bar{\mu}^2)h + (b+c)\bar{\mu}\Big]\\
&\quad +4\bar{\mu}^4h^6 - 4\bar{\mu}^3[1-a+c+2b+(2a+c-b-2)t]h^5\\
&\quad +\bar{\mu}^2\Big[(1-a)^2 + 6b(1-a) + 5b^2 + 4(1-a)c + 8bc + 5(2a-2+c-b)(b+c)t - 4\bar{\mu}^2\Big]h^4\\
&\quad +\bar{\mu}\Big[-(1-a)^2(b+c) + (1-a)\Big(4\bar{\mu}^2(1-2t) + 2t(b+c)^2 - 2b^2 + 2bc(4t-3)\Big)\\
&\qquad\qquad +(4\bar{\mu}^2+bc)(b+c)t^2 - b^3(t-1)^2 - c^3t^2 + 8\bar{\mu}^2b(1-t) + 4c\bar{\mu}^2\\
&\qquad\qquad +(4t-5)b^2c - 6bc^2t\Big]h^3\\
&\quad +\Big[4\bar{\mu}^4t(1-t) + 10(1-a)(b+c)\bar{\mu}^2t - 14\bar{\mu}^2bct(t-1) - \bar{\mu}^2(6b+4c)(1-a) - 8bc\bar{\mu}^2\\
&\qquad\qquad +(1-a)^2(bc-\bar{\mu}^2) - (3t-5)(t-1)\bar{\mu}^2b^2 - (3t+2)\bar{\mu}^2tc^2 + bc^3t^2\\
&\qquad\qquad +2b^2c(1-a) - 2bc^2t(1-a) + b^3c(1-t)^2 + 2b^2c^2t(1-t) - 2cb^2t(1-a)\Big]h^2\\
&\quad +\bar{\mu}\Big[b^3(1-2t) + (c+b)^3t^2 + 6bc(1-a)(1-2t) + 2b^2(1-t) - 2ab^2(1-t) - 2c^2t(1-a)\\
&\qquad\qquad +\Big((1-a)^2 + 4bct^2 - 4\bar{\mu}^2t(1-t)\Big)(b+c) + (5-12t)b^2c - 2b^2ct\Big]h\\
&\quad +t(1-t)\Big(\bar{\mu}^2(b+c)^2 + 2b^2c^2\Big) - (1-a)^2bc - (1-t)^2b^3c\\
&\qquad\qquad -t^2bc^3 + 2bc^2t(1-a) - 2b^2c(1-t)(1-a)\ .
\end{aligned} \tag{2.15}$$

Second-order second-degree equations related to $P_{\rm VI}$ were given in e.g. [11, 12, 13, 14], cf. also [19], however we have not established the precise connection between (2.15) and those equations, even though in principle it should fit into the classification of [11, 14].

The various Miura relations that link the modified $P_{\rm VI}$ equations to the regular ones are given as follows.

- Discrete Miura

$$
\begin{aligned}
x_n &= r^{-1}\frac{(1-r^2)+r^2h_n-k_n}{k_n-h_n} \\
&= \frac{\left[(\mu-\nu-n-m)+\bar{\mu}r^2(h_n-1)\right](h_{n-1}+h_n)+2n(1-r^2)(h_n-1)}{r[n+m+\nu+\frac{1}{2}-\bar{\mu}h_n](h_n+h_{n-1})}
\end{aligned} \tag{2.16}
$$

- Discrete Inverse Miura

$$
\frac{h_{n-1}+1}{h_n-1}=\frac{x_n+r}{(1+rx_n)x_{n-1}} \tag{2.17}
$$

- Continuous Miura

$$
\begin{aligned}
&\left[t(t-1)h'-(c-\bar{\mu}h)(1-h)(w-t)\right]^2= \\
&\quad = t^2(t-1)^2h'^2+t(t-1)(c-\bar{\mu}h)(b-\bar{\mu}h)(h^2-1)
\end{aligned} \tag{2.18}
$$

- Continuous Inverse Miura

$$
\begin{aligned}
2\bar{\mu}h &= \frac{1}{1-w}\Big[-a+(2+b)(1-t)-2t(1-t)\frac{w'}{w}+\bar{\mu}t(2-\frac{w}{t}) \\
&\qquad -c(w-t)+(a-\bar{\mu})\frac{t}{w}\Big]
\end{aligned} \tag{2.19}
$$

This completes the list of the various continuous and discrete equations in the Miura chain associated with the $\mathrm{P_{VI}}$ equation. What is manifestly absent in the above list in each of the three classes is the corresponding partial differential equation (PDE). It is important to realise that the full PDE analogue of e.g. (2.1a) can*not* be simply the Schwarzian KdV (1.3), since the latter equation is not rich enough to yield full $\mathrm{P_{VI}}$ as a similarity reduction. In fact, it turns out that there does indeed exist a PDE that is associated with the $\mathrm{P_{VI}}$ system, but this PDE being a very rich equation in its own right, we believe it merits a separate publication, [23].

3 Remarks on the Derivations

In this section, we outline the derivation of the results given in §2. Full details of the general framework within which we work were presented in Ref. [18]. We will not go into the details of this structure here. However, it is worthwhile to note that there is a subtle duality between the set of lattice *parameters* $p, q, r, \ldots$ and the lattice *variables* n, m in this framework. The words "parameters", "variables" follow naturally from the roles they play in the discrete framework. However, in the continuous equations we derive, the parameters play the role of independent variables, whereas the lattice variables play the role of parameters. So the shifts in the lattice variables have the interpretation of Bäcklund transformations for the continuous equations.

The similarity constraints form the starting point for the derivation of results in §2. In particular, the ones for the Schwarzian system play an important role. In Ref. [18], the simlarity constraint for the variables v, i.e. eq. (2.5b), were derived from one for the variable z, i.e. (2.1b). (This involves taking first a difference in the discrete variables n or m and then using the PΔE (2.1a) to rewrite it in a form that can integrated once, thus leading to a new alternating parameter ν.) The resulting constraint for v again in terms of z led to the second similarity constraint (2.1c). Both the Schwarzian constraints are used to derive the ones for h and k, i.e. Eqs (2.14c) and (2.14d). The combination of the two constraints (2.1b) and (2.1c) lead to the relations (2.12). The latter, together with the P$\triangle$E used to eliminate shifts in m, gives rise to the O$\triangle$E (2.1d).

The derivation of P_{VI} in [18] uses two additional intermediate variables:

$$a_{n,m} \equiv \frac{v_{n+1,m} - v_{n-1,m}}{v_{n+1,m} + v_{n-1,m}} \quad , \quad b_{n,m} \equiv \frac{v_{n,m+1} - v_{n,m-1}}{v_{n,m+1} + v_{n,m-1}} \ , \tag{3.1}$$

which are related to the variables x and y via

$$a_{n,m} = \frac{y_{n,m} - x_{n-1,m}}{y_{n,m} + x_{n-1,m}} \quad , \quad b_{n,m} = \frac{1 - x_{n,m-1}y_{n,m}}{1 + x_{n,m-1}y_{n,m}} \ . \tag{3.2}$$

By using the P$\triangle$E (2.5a), we then get

$$(1 + rx_{n,m})(1 + a_{n,m+1}) \ = \ 2 - (1 - r/y_{n,m})(1 + a_{n,m}) \ , \tag{3.3a}$$
$$(x_{n,m} + r)(1 + b_{n+1,m}) \ = \ 2r - (r - y_{n,m})(1 + b_{n,m}) \ , \tag{3.3b}$$

The ODE (2.5e) (equivalent to P_{VI}) is obtained from the differential-difference equation (2.5d) and the second relation (3.3b) together with the similarity constraint (2.5b). More details can be found in Ref. [18].

Finally, to obtain the discrete and continuous modified P_{VI} equations in terms of the variable h we solve the system of constraints (2.14c), (2.14d) as follows

$$\frac{2n}{h_{n-1,m} + h_{n,m}} \ = \ \frac{1}{h_{n,m} - k_{n,m}} \left(\frac{1}{2} + \nu + n + m - \bar{\mu} k_{n,m} \right) \ , \tag{3.4a}$$
$$\frac{2m}{k_{n,m-1} + k_{n,m}} \ = \ \frac{1}{k_{n,m} - h_{n,m}} \left(\frac{1}{2} + \nu + n + m - \bar{\mu} h_{n,m} \right) \ . \tag{3.4b}$$

These relations allow us to express k entirely in terms of h via

$$k = \frac{\frac{1}{2} + \nu + n + m - \frac{2nh_{n,m}}{h_{n-1,m}+h_{n,m}}}{\bar{\mu} - \frac{2n}{h_{n-1,m}+h_{n,m}}} \ , \tag{3.5}$$

involving only shifts in the independent variable n. Furthermore, using the coupled P$\triangle$E's (2.14a) we obtain

$$(h_{n,m} + k_{n+1,m})(h_{n,m} - k_{n,m}) = (1 - r^2)(h_{n,m}^2 - 1) \ , \tag{3.6}$$

which is quadratic in the variable h. Inserting (3.5) into (3.6) we obtain (2.14e). Alternatively, we obtain a scond-order second-degree PΔE for k by solving h from (3.6) and inserting it into (2.14a).

Another relation we need is the m-shifted object

$$\frac{2n}{h_{n,m+1}+h_{n-1,m+1}}=\frac{r^2(k_{n,m}-h_{n,m})[\frac{1}{2}+\nu+m-n-\bar{\mu}k_{n,m}]}{(1-r^2)(k_{n,m}^2-1)}\ , \tag{3.7}$$

which follows from the m-shifted constraints (2.14c) and (2.14d) together with (2.14a). Similarly, we have

$$\frac{2(n+1)}{h_{n+1,m}+h_{n,m}}=\frac{\frac{1}{2}+\nu+n-m+\bar{\mu}k_{n+1,m}}{h_{n,m}+k_{n+1,m}}\ , \tag{3.8}$$

where $k_{n+1,m}$ can be eliminated using (2.14a).

Finally, the continuous Schwarzian P_{VI} equation (2.1f) can be obtained from the coupled system of ODEs for h and k (2.14g) and (2.14h) using the differential-difference relation (2.1e) which leads to

$$-t\frac{\partial}{\partial t}\log z_n=\frac{2n}{h_n+h_{n-1}}\ , \tag{3.9}$$

and consequently

$$k-h=\frac{\frac{1}{2}+\nu+n+m-\bar{\mu}h}{\bar{\mu}+tz'/z}\ . \tag{3.10}$$

Furthermore, the Miura relations (2.18) and (2.19) between P_{VI} in terms of w and the MP_{VI} equation in terms of h follow from the relation

$$w(t)=t\frac{1-k}{1-h} \tag{3.11}$$

where one can either eliminate k by solving $h-k$ from (2.14h) or differentiate (3.11) and eliminate k by using (3.11) and both (2.14g) and (2.14h). The discrete Miura transformations (2.16) and (2.17) follow from the definitions (2.4) and (2.13) exploiting the various relations such as (3.8) and (3.10) above. The consistency of all these connections and relations follow from the structures given in [18] from which also isomonodromic deformation systems (i.e. Lax pairs) for all the equations can be derived as well as explicit gauge transformations between them. The construction of those is in preparation.

4 Some Special Limits

In this section, we consider limits of the equations presented in §2. In particular, we present new limits exhibiting discrete versions of Schwarzian P_{II} (1.2). We want to stress again, that the discrete and continuous equations of the previous sections are not related via continuum limit (which amount, in fact, to coalescence limits on the parameters of the equations), but that their relation is exact in the sense that the discrete and continuous flows are compatible, meaning that one can be considered to be a symmetry of the other and vice versa. However, in this section we will perform limits on the parameters to show that the discrete equations contain many of the previously known discrete Painlevé equations under coalescence, in particular the discrete analogues of P_{II}. In fact, taking the limit $r \to \infty$, $m/r \to \xi$, it is easy to see that (2.5c) reduces to

$$\frac{n+1}{x_n x_{n+1}+1} + \frac{n}{x_n x_{n-1}+1} = n + \bar{\mu} + \frac{1}{2}\xi \left(x_n - \frac{1}{x_n} \right) , \tag{4.1}$$

which is the the alternate dP_{II} equation of [24, 25].

The limit to the standard discrete P_{II} (dP_{II}) is a little more subtle, since we need to work in an oblique direction. In fact, to obtain dP_{II} from (2.5c) we write $r = 1 + \delta$ and take the limit $\delta \to 0$, whilst taking $n = n' - m$, where $n, m \to \infty$ such that n' is fixed (in the limit) and $\delta m \to \eta$ finite. In that limit we have also $(1 - x_n)/(1 + x_n) \to a_{n+1}$ and (2.5c) reduces to the following equation:

$$(n'+1)(1 - x_n^2) - (\mu + \nu)(1 + x_n)^2 = 2\eta x_n \left(\frac{1 - x_{n+1}}{1 + x_{n+1}} + \frac{1 - x_{n-1}}{1 + x_{n-1}} \right) , \tag{4.2}$$

where we consider x_n to be a function of n' rather than n. In terms of a_n, (omitting from now on primes on the n' variable), this equation reads

$$\frac{1}{2}\eta(a_{n+1} + a_{n-1}) = \frac{-\mu + \nu + n a_n}{1 - a_n^2} , \tag{4.3}$$

which is dP_{II} [16, 26].

In the above coalescence limits, the discrete Schwarzian equation (2.1d) reduces to

$$4\eta \frac{(z_{n+2} - z_{n+1})(z_n - z_{n-1})}{(z_{n+2} - z_n)(z_{n+1} - z_{n-1})} = \frac{1}{2} + \nu + n - \bar{\mu} \frac{z_{n+1} + z_n}{z_{n+1} - z_n} , \tag{4.4}$$

which is the discrete Schwarzian P_{II}, i.e. the discrete analogue of (1.2), cf. [10], whereas in the alternate limit the equation becomes

$$\left[\bar{\mu} \frac{z_{n+1}}{z_{n+1} - z_n} - (n+1) \frac{z_{n+2} - z_{n+1}}{z_{n+2} - z_n} \right] \left[\bar{\mu} \frac{z_n}{z_{n+1} - z_n} - n \frac{z_n - z_{n-1}}{z_{n+1} - z_{n-1}} \right] = \xi^2 , \tag{4.5}$$

cf. [15]. In both equations the "discrete Schwarzian derivative", which is the canonical cross-ratio of the four entries z_{n-1}, z_n, z_{n+1}, z_{n+2} is prominent. Finally, in these limits the modified equations reduce to

$$(h_{n+1} + h_n)(h_n + h_{n-1}) = 4\eta \frac{h_n^2 - 1}{\bar{\mu} h_n - (\frac{1}{2} + \nu + n)} , \tag{4.6}$$

which is the discrete analogue of the thirty-fourth equation in the classification list of Painlevé-Gambier (see [21]). Its discrete analogue dP_{XXXIV}, i.e. (4.6), was first given in [27] (see also [28]). The alternate version, following from the other coalescence limit, reads

$$\left(\bar{\mu} - \frac{2(n+1)}{h_{n+1} + h_n} \right) \left(\bar{\mu} - \frac{2n}{h_{n-1} + h_n} \right) = \frac{\xi^2}{h_n^2 - 1} , \tag{4.7}$$

and was first given in [15].

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