On the non-integrability of a fifth order equation with integrable two-body dynamics

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Abstract

We consider the fifth order partial differential equation (PDE)

$$u_{4x,t} - 5u_{xxt} + 4u_t + uu_{5x} + 2u_xu_{4x} - 5uu_{3x} - 10u_xu_{xx} + 12uu_x = 0,$$

which is a generalization of the integrable Camassa-Holm equation. The fifth order PDE has exact solutions in terms of an arbitrary number of superposed pulsons, with geodesic Hamiltonian dynamics that is known to be integrable in the two-body case $N = 2$. Numerical simulations show that the pulsons are stable, dominate the initial value problem and scatter elastically. These characteristics are reminiscent of solitons in integrable systems. However, after demonstrating the non-existence of a suitable Lagrangian or bi-Hamiltonian structure, and obtaining negative results from Painlevé analysis and the Wahlquist-Estabrook method, we assert that the fifth order PDE is not integrable.

1 Introduction

This note is concerned with the fifth order partial differential equation (PDE)

$$u_{4x,t} - 5u_{xxt} + 4u_t + uu_{5x} + 2u_xu_{4x} - 5uu_{3x} - 10u_xu_{xx} + 12uu_x = 0. \quad (1.1)$$

One reason for our interest in this equation is that it admits exact solutions of the form

$$u = \sum_{j=1}^{N} p_j(t)(2e^{-|x-q_j(t)|} - e^{-2|x-q_j(t)|}), \quad (1.2)$$

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where p_j, q_j satisfy the canonical Hamiltonian dynamics generated by

$$H_N = \frac{1}{2} \sum_{j,k=1}^{N} p_j p_k (2e^{-|q_j - q_k|} - e^{-2|q_j - q_k|}). \tag{1.3}$$

Following [1], we refer to such solutions as "pulsons." The equations for the N-body pulson dynamics are equivalent to geodesic flow on an N-dimensional space with coordinates q_j and co-metric

$$g^{jk} = g(q_j - q_k), \qquad g(x) = 2e^{-|x|} - e^{-2|x|}.$$

The pulsons (1.2) are weak solutions with discontinuous second derivatives at isolated points.

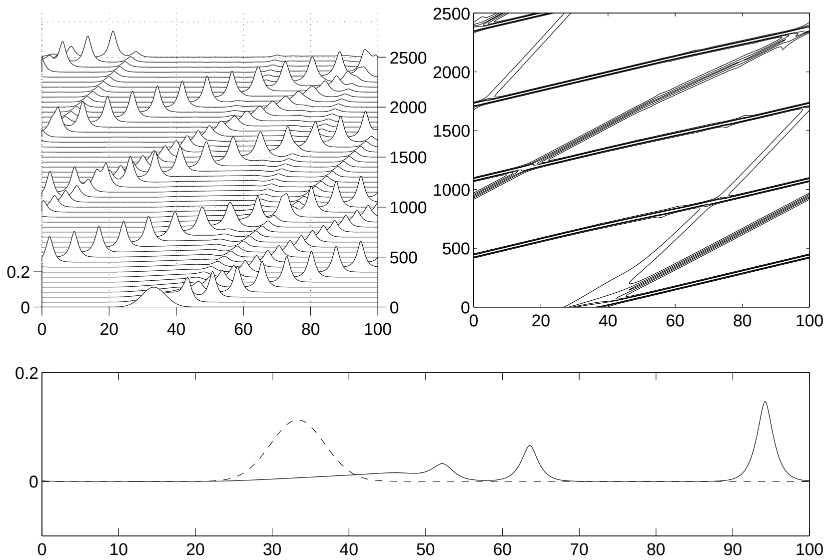


Figure 1: Pulson solutions (1.2) of equation (1.1) emerge from a Gaussian of unit area and width $\sigma = 5$ centered about $x = 33$ on a periodic domain of length $L = 100$. The fastest pulson crosses the domain four times and collides elastically with the slower ones.

The PDE (1.1) is one of a family of integral partial differential equations considered in [1], given by

$$m_t + um_x + 2u_x m = 0, \qquad u = g * m, \tag{1.4}$$

where $u(x,t)$ is defined in terms of $m(x,t)$ by the convolution integral

$$g * m := \int_{-\infty}^{\infty} g(x-y) m(y,t)\, dy.$$

The integral kernel $g(x)$ is taken to be an even function, and for any g the equation (1.4) has the Lie-Poisson Hamiltonian form

$$m_t = -(m\partial_x + \partial_x m)\frac{\delta H}{\delta m} \tag{1.5}$$

where

$$H = \frac{1}{2}\int m\, g * m\, dx = \frac{1}{2}\int mu\, dx. \tag{1.6}$$

Any equation in this family admits pulson solutions

$$u(x,t) = \sum_{j=1}^{N} p_j(t) g(x - q_j(t))$$

for arbitrary N, with p_j, q_j satisfying the canonical Hamilton's equations

$$\frac{dp_j}{dt} = -\frac{\partial H_N}{\partial q_j} = -p_j \sum_{k=1}^{N} p_k g'(q_j - q_k), \qquad \frac{dq_j}{dt} = \frac{\partial H_N}{\partial p_j} = \sum_{k=1}^{N} p_k g(q_j - q_k). \tag{1.7}$$

generated by the Hamiltonian

$$H_N = \frac{1}{2}\sum_{j,k} p_j p_k\, g(q_j - q_k).$$

The equations (1.7) correspond to geodesic motion on a manifold with co-metric $g^{jk} = g(q_j - q_k)$. A significant result of [1] is that the two-body dynamics ($N = 2$) is integrable for any choice of kernel g, and numerical calculations show that this elastic two-pulson scattering dominates the initial value problem.

Three special cases are isolated in [1], namely (up to suitable scaling)

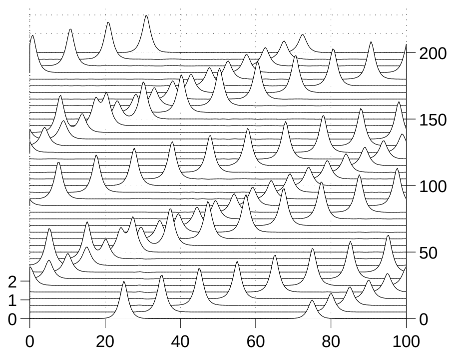
- $g(x) = \delta(x)$ - Riemann shocks,
- $g(x) = 1 - |x|$, $|x| < 1$ - compactons,
- $g(x) = e^{-|x|}$ - peakons.

For each of these cases both the integral PDE (1.4) and the corresponding finite-dimensional system (1.7) (for any N) are integrable. Of most relevance here is the third case, where $g(x) = e^{-|x|}$, which is the (scaled) Green's function for the Helmholtz operator, satisfying the identity

$$(1 - \partial_x^2) g(x) = 2\delta(x).$$

In that case after rescaling we may take

$$m = u - u_{xx}, \tag{1.8}$$



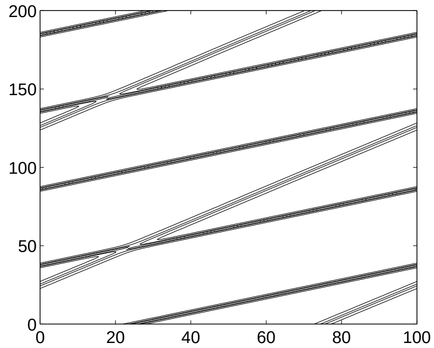


Figure 2: Two rear-end collisions of pulson solutions (1.2) of equation (1.1). The initial positions are $x = 25$ and $x = 75$. The faster pulson moves at twice the speed of the slower one. For this ratio of speeds, both collisions result in a phase shift to the right for the faster space-time trajectory, but no phase shift for the slower one.

and the equation (1.4) is just a PDE for $u(x,t)$, namely

$$u_t - u_{xxt} - uu_{3x} - 2u_x u_{xx} + 3uu_x = 0, \tag{1.9}$$

which is the dispersionless form of the integrable Camassa-Holm equation for shallow water waves [2, 3]. For Camassa-Holm the pulson solutions take the form of peakons or peaked solitons, i.e.

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}. \tag{1.10}$$

The fifth order equation arises from a different choice of Green's function. Using the identity

$$(4 - \partial_x^2)(1 - \partial_x^2) g(x) = 12\delta(x), \qquad g(x) = 2e^{-|x|} - e^{-2|x|} \tag{1.11}$$

we find that (after suitable scaling) this choice of g yields

$$m = u_{4x} - 5u_{xx} + 4u, \tag{1.12}$$

and then the equation (1.4) becomes the fifth order equation (1.1). Thus the PDE (1.1) should be regarded as a natural higher order generalization of the Camassa-Holm equation.

2 Hamiltonian and Lagrangian considerations

In a forthcoming article [4] we discuss a more general family of integral PDEs of the form

$$m_t + um_x + bu_x m = 0, \qquad u = g * m, \tag{2.1}$$

where b is an arbitrary parameter; (1.4) corresponds to the particular case $b = 2$, and numerical results have recently been established for different b values in [5]. In the case of the peakon kernel $g = e^{-|x|}$, with m given by (1.8), the equations in this class were tested by the method of asymptotic integrability [6], and only the cases $b = 2, 3$ were isolated as potentially integrable. For $b = 2$ the integrability of the Camassa-Holm equation by inverse scattering was already known [2, 3], but for the new equation

$$u_t - u_{xxt} - uu_{3x} - 3u_xu_{xx} + 4uu_x = 0, \tag{2.2}$$

with $b = 3$ the integrability was proved in [7] by the construction of the Lax pair. The two integrable cases $b = 2, 3$ were also found recently via the perturbative symmetry approach [8]. For any $b \neq -1$, the peakon family (2.1) with $g = e^{-|x|}$ arises as the dispersionless limit at the quadratic order in the asymptotic expansion for shallow water waves [9].

Another motivation for our interest in the fifth order equation (1.1) is that it is expressed naturally in terms of the quantity m (1.12), given by the product of two Helmholtz operators acting on u. Such a product appears in the fifth order operator

$$B_0 = \partial_x(4 - \partial_x^2)(1 - \partial_x^2),$$

which we found [7] to provide the first Hamiltonian structure for the the new integrable equation (2.2). This led us to the conjecture that the operator B_0 should appear naturally in the theory of higher order integrable equations such as (1.1).

All the equations of the form (1.4) have the Lie-Poisson Hamiltonian structure given by (1.5), but for integrability we expect a bi-Hamiltonian structure. In the case of the Camassa-Holm equation (1.9) there are two ways to derive a second Hamiltonian structure. The first is by inspection using a conservation law, noting that (1.9) may be written as

$$m_t = \left(uu_{xx} + \frac{1}{2}u_x^2 - \frac{3}{2}u^2\right)_x = \partial_x \frac{\delta\tilde{H}}{\delta u} = \partial_x(1 - \partial_x^2)\frac{\delta\tilde{H}}{\delta m} \tag{2.3}$$

for

$$\tilde{H} = -\frac{1}{2}\int (uu_x^2 + u^3)\,dx. \tag{2.4}$$

The identity (2.3) gives the second Hamiltonian structure for Camassa-Holm, and $m\partial_x + \partial_x m$, $\partial_x(1 - \partial_x^2)$ constitute a compatible bi-Hamiltonian pair.

Similarly $\int m\,dx$ is conserved for (1.1), with m given by (1.12). The conservation law is explicitly

$$(u_{4x} - 5u_{xx} + 4u)_t = -\left(uu_{4x} + u_xu_{3x} - \frac{1}{2}u_{xx}^2 - 5uu_{xx} - \frac{5}{2}u_x^2 + 6u^2\right)_x =: \mathcal{F}_x. \tag{2.5}$$

By analogy with the Camassa-Holm equation, this would suggest that a suitable constant coefficient Hamiltonian operator might be $\partial_x(4-\partial_x^2)(1-\partial_x^2)$ (which we know to be a Hamiltonian operator for the new equation (2.2)). This would require the right hand side of (2.5) to take the form

$$\partial_x \frac{\delta K}{\delta u} = \partial_x(4-\partial_x^2)(1-\partial_x^2)\frac{\delta K}{\delta m}.$$

However, for the flux of (2.5) we find

$$\mathcal{F} \neq \frac{\delta K}{\delta u}$$

for any local density functional K of u, and we suppose that the operators $\partial_x(4-\partial_x^2)(1-\partial_x^2)$ and $m\partial_x + \partial_x m$ must be incompatible.

The second way to derive the Hamiltonian structure (2.3) for Camassa-Holm is via the action (integral of Lagrangian density)

$$S = \int\int \mathcal{L}[\phi]\, dx\, dt := \int\int \frac{1}{2}\left(\phi_x\phi_t - \phi_{3x}\phi_t + \phi_x\phi_{xx}^2 + \phi_x^3\right)\, dx\, dt,$$

for $u = \phi_x$. A Legendre transformation yields the conjugate momentum

$$\frac{\partial \mathcal{L}}{\partial \phi_t} = \frac{1}{2}(\phi_x - \phi_{3x}) = \frac{m}{2},$$

and the same Hamiltonian as (2.4) above, i.e.

$$\tilde{H} = \int \left(\frac{1}{2}m\phi_t - \mathcal{L}\right)\, dx.$$

Trying the same approach for (1.1), we set $u = \phi_x$ and rewrite it as

$$\phi_{5x,t} - 5\phi_{3x,t} + 4\phi_{xt} + \phi_x\phi_{6x} + 2\phi_{xx}\phi_{5x} - 5\phi_x\phi_{4x} - 10\phi_{xx}\phi_{3x} + 12\phi_x\phi_{xx} = 0. \tag{2.6}$$

However, the equation (2.6) cannot be derived from a local Lagrangian density $\mathcal{L}[\phi]$ due to the presence of the terms $\phi_x\phi_{6x} + 2\phi_{xx}\phi_{5x}$.

The first nonlocal Hamiltonian structure for the Camassa-Holm equation is obtained by applying the recursion operator to $m\partial_x + \partial_x m$. This means that (1.9) can be written in the Hamiltonian form

$$m_t = (m\partial_x + \partial_x m)(\partial_x^3 - \partial_x)^{-1}(m\partial_x + \partial_x m)\frac{\delta \hat{H}}{\delta m}, \qquad \hat{H} = \int m\, dx.$$

With the same $\hat{H}$, the analogous identity for (1.1) is

$$m_t = B\frac{\delta \hat{H}}{\delta m} \equiv (m\partial_x + \partial_x m)(\partial_x^5 - 5\partial_x^3 + 4\partial_x)^{-1}(m\partial_x + \partial_x m)\frac{\delta \hat{H}}{\delta m}, \tag{2.7}$$

but from the above considerations we would expect that the formal nonlocal operator B on the right hand side of (2.7) is not Hamiltonian, and indeed using the functional equations derived in [4] it is possible to show that it fails to satisfy the Jacobi identity.

3 Reciprocal transformation and Painlevé analysis

Having failed to find the sort of Lagrangian or bi-Hamiltonian structure for (1.1) that we would reasonably expect, we proceed to see what Painlevé analysis can tell us about this fifth order equation. However, we note that both the Camassa-Holm equation (1.9) and the new equation (2.2) provide examples of the weak Painlevé property [10], with algebraic branching in the solutions. For these equations we have found it convenient to use reciprocal transformations (see [11] for definitions), which transform to equations with pole singularities, and indeed in [7] this was the key to our discovery of the Lax pair for (2.2). Hodograph transformations of this kind have been used before to remove branching from classes of evolution equations [12, 13], but here we are dealing with non-evolutionary PDEs.

To make the results of our analysis more general, we will consider the whole class of equations

$$m_t + um_x + bu_x m = 0, \qquad m = u_{4x} - 5u_{xx} + 4u, \qquad b \neq 0, \tag{3.1}$$

for arbitrary nonzero b, which is the particular family of equations (2.1) corresponding to the integral kernel (1.11), and includes (1.1) in the special case $b = 2$. Each equation in the class (3.1) has the conservation law

$$(m^{1/b})_t = -(m^{1/b}u)_x,$$

and so introducing a new dependent variable p according to

$$p^b = -m \tag{3.2}$$

means that we may consistently define a reciprocal transformation to new independent variables X, T given by

$$dX = p\,dx - pu\,dt, \qquad dT = dt. \tag{3.3}$$

Transforming the derivatives we have the new conservation law

$$(p^{-1})_T = u_X. \tag{3.4}$$

Rewriting the relation (1.12) in terms of ∂_X and using (3.4) to eliminate derivatives of u we obtain the identity

$$u = \frac{1}{4}\left(5 - (p\partial_X)^2\right) \cdot (p\partial_X) \cdot p(p^{-1})_T - \frac{p^b}{4}, \tag{3.5}$$

which means that (3.4) can be written as an equation for p alone, i.e.

$$(p^{-1})_T = \left(\frac{1}{4}\left((p\partial_X)^2 - 5\right) p(\log p)_{XT} - \frac{p^b}{4}\right)_X. \tag{3.6}$$

The fifth order equation (3.6) is the reciprocal transform of (3.1). Rather than carrying out the full Painlevé test for the transformed equation, it is sufficient for our purposes to follow [14] and apply the Painlevé test for ODEs to the travelling wave reduction of (3.6). Hence we set $p = p(z)$, $z = X - cT$ and the resulting fifth order ODE may be integrated twice to get the third order ODE

$$\frac{5}{8}\left(\frac{p'}{p}\right)^2 - \frac{1}{4}\left(p'p''' - \frac{1}{2}(p'')^2 - \frac{(p')^2p''}{p} + \frac{1}{2}\frac{(p')^4}{p^2}\right) - \frac{1}{2p^2} = \frac{c^{-1}p^{b-1}}{4(b-1)} + \frac{d}{p} + e, \tag{3.7}$$

$b \neq 1$, where d, e are arbitrary constants and c is the wave speed, with prime denoting d/dz. For $b = 1$ there is a $\log p$ term on the right hand side, and so this case has logarithmic branching and is immediately excluded by the Painlevé test. Similarly, because of the p^{b-1} term all non-integer values of b have branching and are discarded.

We proceed to apply Painlevé analysis to (3.7) for integer $b \neq 0, 1$, seeking leading order behaviour at a movable point z_0 of the form $p \sim a(z - z_0)^\mu$ for integer exponent μ. For all integers $b \leq -2$ the only possible balance is $\mu = 4/(3 - b)$, which is non-integer and hence gives algebraic branching. In the special case $b = -1$ there are four possible balances with $\mu = 1$, with a and the resonances depending on the value of c; we have checked that no value of c gives all integer resonances, so the Painlevé test is failed. For the remaining cases of integer $b \geq 2$ we find $\mu = 1$ with $a^2 = 1$ or $a^2 = 4$. For integer $b \geq 4$ there is also the balance $\mu = 4/(3-b)$ which is in general non-integer, except for the special cases $\mu = -4, -2, -1$ for $b = 4, 5, 7$ respectively. Thus all integer values of b apart from $b = 2, 3, 4, 5, 7$ are ruled out by the (strong) Painlevé test due to algebraic branching; but they could still be analysed by the weak Painlevé test if we allow such branching.

Let us consider in more detail the first two types of balance for integer $b \geq 2$. When $p \sim \pm(z - z_0)$ we have a non-principal balance with resonances $r = -1, -1, 3$. Interestingly, the resonance condition at $r = 3$ is failed when $b = 2$, the obstruction being the c^{-1} term (so for these balances the test is only passed in the limit $c \to \infty$), but satisfied for all integer $b \geq 3$. However, for the principal balances $p \sim \pm 2(z - z_0)$ the resonances are $r = -1, 1/2, 3/2$ which means there is algebraic branching and so the (strong) Painlevé test is failed for any b. We have further checked whether the weak Painlevé test of [10] could be satisfied by allowing an expansion in powers of $(z - z_0)^{1/2}$ in the principal balance. However, the resonance condition is satisfied at $r = 1/2$ but failed at $r = 3/2$, meaning that this expansion with square root branching cannot represent the general solution as it doesn't contain enough arbitrary constants. The arbitrariness can only be restored by adding infinitely many terms in powers of $\log(z - z_0)$, and so no form of Painlevé property can be recovered. The existence of logarithmic branching in both the principal and non-principal balances is a strong indication of non-integrability.

It is interesting to observe that when the first term on the right hand side of (3.7) is absent (the limit $c \to \infty$), it admits exact solutions in terms of trigonometric/hyperbolic functions, corresponding to the first order reductions

$$(p')^2 = 1 + 2dp + \frac{1}{3}(8e - d^2)p^2, \qquad (p')^2 = 4 + 8dp + \frac{8}{3}(2d^2 - e)p^2.$$

In fact we can also see that the original equation (1.1) fails the weak Painlevé test directly. For the Camassa-Holm equation (1.9) the test is satisfied by a principal balance

$$u \sim -\phi_t/\phi_x + a\phi^{\frac{2}{3}} + \dots$$

with resonances $-1, 0, 2/3$, with the singular manifold $\phi(x,t)$ and $a(x,t)$ being arbitrary. For (1.1) there is an analogous balance

$$u \sim -\phi_t/\phi_x + a\phi^{\frac{4}{3}} + \dots,$$

with resonances $-1, 0, 4/3, (1 \pm \sqrt{41})/6$; the presence of irrational resonances implies logarithmic branching.

4 Prolongation algebra method

While Painlevé analysis is a good heuristic tool for isolating potentially integrable equations, it can never be said to provide definite proof of non-integrability. If one gives a precise definition of integrability in terms of existence of infinitely many commuting symmetries, then the symmetry approach of Shabat et al [15] gives necessary conditions for integrability (but does not provide a constructive way to find a Lax pair or linearization when such conditions are satisfied). The symmetry approach has only very recently been extended [8] so that it can be applied to nonlocal or non-evolution equations such as (1.1), (1.9). As an alternative, we apply the prolongation algebra method of Wahlquist and Estabrook [16], and directly seek a Lax pair for (1.1) in the form of a compatible linear system

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi \tag{4.1}$$

for suitable matrices U, V (usually taking values in the fundamental representation of a semi-simple Lie algebra) which should depend on u and its derivatives, and on a spectral parameter. We have found the clear presentation of the method in [17] very useful.

The compatibility of the system (4.1) yields the zero curvature equation

$$U_t - V_x + [U, V] = 0, \tag{4.2}$$

and the essence of the Wahlquist-Estabrook method is that given the original PDE (in this case (1.1)) one may use (4.2) to derive the functional dependence of U, V on u, u_x, etc. A negative result means that no Lax pair of a suitable form exists, suggesting that the equation is not integrable, but of course this is sensitive to the initial assumptions that are made on the functional form of U, V.

For ease of notation we will denote the nth derivative $u_{nx} = u_n$. Given that (1.1) can be written as a conservation law for m as in (2.5), a reasonable ansatz is to assume that

$$U = U(m), \qquad V = V(u, u_1, u_2, u_3, u_4)$$

(with dependence on the spectral parameter suppressed). Given the known form of the zero curvature representations for the equations (1.9), (2.2) we further assume that U is linear in m, so that

$$U = Am + B \equiv (u_4 - 5u_2 + 4u)A + B,$$

where A, B are constant matrices (independent of x, t, but potentially dependent on the spectral parameter). Substituting this ansatz into (4.2), and using (2.5) to eliminate the t derivative m_t, we find

$$\begin{aligned}(-uu_5 - 2u_1u_4 + 5uu_3 + 10u_1u_2 - 12uu_1)A - u_5V_{u_4} - u_4V_{u_3} - u_3V_{u_2} - u_2V_{u_1} \\ - u_1V_u + (u_4 - 5u_2 + 4u)[A, V] + [B, V] = 0 \qquad (4.3)\end{aligned}$$

(with subscripts on V denoting partial derivatives). None of the matrices depend on u_5, so (4.3) is linear in u_5. In particular, the coefficient of u_5 must vanish, giving the equation $V_{u_4} = -uA$ which integrates immediately to yield

$$V = -uu_4A + \Gamma(u, u_1, u_2, u_3), \qquad (4.4)$$

where Γ is so far arbitrary and must be determined from the remaining terms in (4.3).

At the next step we substitute for V in the rest of (4.3) to obtain

$$\begin{aligned}(-u_1u_4 + 5uu_3 + 10u_1u_2 - 12uu_1)A - u_4\Gamma_{u_3} - u_3\Gamma_{u_2} - u_2\Gamma_{u_1} - u_1\Gamma_u \\ + (u_4 - 5u_2 + 4u)[A, \Gamma] + uu_4[A, B] + [B, \Gamma] = 0. \qquad (4.5)\end{aligned}$$

The coefficient of u_4 gives the equation

$$\Gamma_{u_3} = [A, \Gamma + uB] - u_1A$$

which can be integrated exactly as

$$\Gamma = e^{u_3A}\Delta(u, u_1, u_2)e^{-u_3A} - u_1u_3A - uB, \qquad (4.6)$$

where Δ is the arbitrary function of integration. From (4.6) we see the presence of $Ad \exp u_3 A = \exp(ad\, u_3 A)$ acting on Δ, which would imply exponential-type dependence on u_3 in the Lax pair unless $(ad\, u_3 A)^n \Delta = 0$ for some positive integer n. Such exponential dependence would seem unlikely given that the original equation (1.1) is polynomial in u and its derivatives, and we will seek assumptions that prohibit infinitely many non-zero commutators occurring in (4.6).

Substituting for Γ from (4.6) in the u_4-independent terms of (4.5) and applying $Ad \exp(-u_3 A)$ we get

$$(u_2 u_3 + 5uu_3 + 10u_1 u_2 - 12uu_1)A - u_3 \Delta_{u_2} - u_2 \Delta_{u_1} - u_1 \Delta_u + (-5u_2 + 4u)[A, \Delta] + [e^{-u_3 A} B e^{u_3 A}, \Delta] + (u_1 u_3 + 5uu_2 - 4u^2) e^{-u_3 A} C e^{u_3 A} = 0, \tag{4.7}$$

where we have set

$$C = [A, B].$$

Potentially (4.7) is an infinite power series in u_3, each coefficient of which must vanish. The simplest assumption we can make to terminate the series is to take

$$[A, C] = 0 \tag{4.8}$$

which implies

$$Ad\, e^{-u_3 A}(B) = e^{ad(-u_3 A)}(B) = B - u_3 C, \quad Ad\, e^{-u_3 A}(C) = e^{ad(-u_3 A)}(C) = C,$$

and hence (4.7) becomes linear in u_3. A fortunate consequence of (4.8) is that the coefficient of u_3 gives

$$\Delta_{u_2} = (u_2 + 5u)A - [C, \Delta] + u_1 C,$$

which integrates exactly without further assumption to yield

$$\Delta = e^{-u_2 C} E(u, u_1) e^{u_2 C} + \left(\frac{1}{2}u_2^2 + 5uu_2\right) A + u_1 u_2 C. \tag{4.9}$$

The remaining terms in (4.7), after acting with $Ad \exp u_2 C$, now become

$$(5u_1 u_2 - 12uu_1)A - \left(\frac{3}{2}u_2^2 + 4u^2\right) C - u_2 E_{u_1} - u_1 E_u + (-5u_2 + 4u)[A, E] + [e^{u_2 C} B e^{-u_2 C}, E] + u_1 u_2 [e^{u_2 C} B e^{-u_2 C}, C] = 0. \tag{4.10}$$

Once again we are faced with an infinite power series, this time in u_2. Before looking for further simplifying assumptions, we note that the coefficient of the term linear in u_2 is just

$$-E_{u_1} + u_1(5A + [B, C]) - [5A + [B, C], E] = 0,$$

which integrates immediately to

$$E = e^{-u_1 D} Z(u) e^{u_1 D} + \frac{1}{2} u_1^2 D, \qquad D = 5A + [B, C]. \tag{4.11}$$

To analyse the other terms in (4.10) we find it convenient to introduce the quantities

$$F = [C, [C, B]], \qquad G = [D, B],$$

and note that the identities

$$[A, D] = 0 = [A, F], \qquad [A, G] = -[C, D] = F, \qquad [[B, C], D] = 0 \tag{4.12}$$

all hold.

The coefficient of u_2^2 in (4.10) is then

$$-\frac{3}{2} C + \frac{1}{2} [F, E] - u_1 F = 0, \tag{4.13}$$

and the coefficient of u_2^0 gives

$$\begin{aligned} &-12 u u_1 A - 4u^2 e^{u_1 D} C e^{-u_1 D} - u_1 Z_u \\ &+ 4u [A, Z] + \left[e^{u_1 D} B e^{-u_1 D}, Z + \frac{1}{2} u_1^2 D \right] = 0 \end{aligned} \tag{4.14}$$

(after substituting for E from (4.11) and acting with $Ad\, e^{u_1 D}$). We shall not need to consider the equations $[(ad\, C)^n B, E] - n u_1 (ad\, C)^n B = 0$ occurring at u_2^n, $n \geq 3$. Instead we look at the coefficient of u_1 in (4.14), which is

$$-Z_u + [G, Z] - 4u^2 F - 12 u A = 0. \tag{4.15}$$

We are unable to integrate this directly without making a further assumption, the simplest possible being

$$[F, G] = 0 \tag{4.16}$$

which implies

$$Z = e^{Gu} \Theta e^{-Gu} + \frac{2}{3} u^3 F - 6u^2 A. \tag{4.17}$$

We must now use the remaining equations to determine the commutation relations for the constant Lie algebra elements A, B, C, D, F, G, Θ.

Looking at the coefficient of u_1^0 in (4.13) we see that

$$-\frac{3}{2} C + \frac{1}{2} [F, Z] = 0,$$

and using (4.12) and (4.16) with (4.17) yields

$$[F, \Theta] = 3 e^{-Gu} C e^{Gu},$$

which immediately implies

$$[F, \Theta] = 3C, \qquad [C, G] = 0. \tag{4.18}$$

Using the Jacobi identity we also have

$$0 = [C, G] = [C, [D, B]] = -[B, [C, D]] - [D, [B, C]] = [B, F],$$

which by further applications of the Jacobi identity gives

$$[C, F] = 0 = [D, F]. \tag{4.19}$$

Now we return to the equation (4.13) and use (4.11) to evaluate the coefficient of u_1 as

$$-\frac{1}{2}[F, [D, Z]] - F = 0. \tag{4.20}$$

Substituting for Z as in (4.17) and taking the constant coefficient u^0 in (4.20) we use (4.12), (4.18), (4.19) to find

$$0 = -\frac{1}{2}[F, [D, \Theta]] - F = \frac{1}{2}\Big([\Theta, [F, D]] + [D, [\Theta, F]]\Big) - F$$

$$= -\frac{3}{2}[D, C] - F = -\frac{5}{2}F.$$

Then $F = 0$ implies $C = 0$ from the first equation in (4.18), and it is straightforward to show that the Lax pair (4.1) collapses down to the trivial case $[U, V] = 0$, with (4.2) reducing to the scalar equation $m_t = \mathcal{F}_x$ as in (2.5).

5 Conclusions

Since the fifth order PDE (1.1) is in the class of pulson equations studied by Fringer and Holm [1] it admits exact solutions in the form of a direct superposition of an arbitrary number of pulsons (as in Figure 1). These particular N-pulson solutions take the precise form (1.2), and by the general results of [1] we know that for $N = 2$ the Hamiltonian equations (1.7) describing the two-body dynamics are integrable. However, several different considerations provide strong evidence that the fifth order PDE (1.1) is not integrable in the sense of admitting a Lax pair and being solvable by the inverse scattering transform.

It is well known that integrable PDEs such as the Korteweg-deVries equation or Camassa-Holm [2, 3] admit a compatible pair of Hamiltonian structures which together define a recursion operator generating infinitely many higher symmetries. We have tried unsuccessfully to find an analogous bi-Hamiltonian or Lagrangian formulation for the fifth order equation (1.1), but as far as we are aware it admits only the single Hamiltonian structure (1.5).

Both the Camassa-Holm equation (1.9) and the new integrable equation (2.2) isolated by Degasperis and Procesi [6] exhibit the weak Painlevé property of [10], with algebraic branching in local expansions representing a general solution. There are many examples of Liouville integrable systems in finite dimensions [18] and Lax integrable PDEs [13] with this property. For evolution equations transformations of hodograph type can restore the strong Painlevé property [12, 13], and similarly in [7, 4] we have used reciprocal transformations for the non-evolutionary equations (1.9), (2.2). In order to apply Painlevé analysis to (1.1) we have found it convenient to employ a reciprocal transformation which removes the branching at leading order, but further analysis of the travelling wave reduction shows that there is still algebraic branching in the principal balances due to half-integer resonances. Furthermore, in both principal and non-principal balances a resonance condition is failed, and so after the transformation even the weak Painlevé test cannot be satisfied.

We have also applied an integrability test which is perhaps less fashionable nowadays, namely the prolongation algebra method of Wahlquist and Estabrook [16]. By making certain simple assumptions we find that no polynomial Lax pair of a suitable form exists for (1.1).

We cannot expect the N-body pulson dynamical system to be integrable for arbitrary $N > 2$, since this would imply the existence of an infinite integrable subsector within a non-integrable PDE. However, it would be good to find an analytical explanation for the apparent soliton-like behaviour of the pulson solutions, and their numerical stability as evidenced by Figure 2. Further analytical and numerical studies will be required to understand the stability properties of the pulson solutions.

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