Diophantine non-integrability of a third order recurrence with the Laurent property

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February 2, 2008

Abstract

We consider a one-parameter family of third order nonlinear recurrence relations. Each member of this family satisfies the singularity confinement test, has a conserved quantity, and moreover has the Laurent property: all of the iterates are Laurent polynomials in the initial data. However, we show that these recurrences are not Diophantine integrable according to the definition proposed by Halburd (2005 *J. Phys. A: Math. Gen.* **38** L1). Explicit bounds on the asymptotic growth of the heights of iterates are obtained for a special choice of initial data. As a by-product of our analysis, infinitely many solutions are found for a certain family of Diophantine equations, studied by Mordell, that includes Markoff's equation.

For some time there has been considerable interest in maps or discrete equations which are integrable. Various different criteria have been proposed as tests for integrability in the discrete setting. One of the earliest proposals was the singularity confinement test of Grammaticos, Ramani and Papageorgiou [12], which has proved to be an extremely useful tool for isolating discrete Painlevé equations (see e.g. [28]). However, after Hietarinta and Viallet's discovery of some non-integrable equations with the singularity confinement property, they were led to introduce the zero algebraic entropy condition for integrability of rational maps [14], namely that the degree d_n of the nth iterate of a map (as a rational function of the initial data) should satisfy $\lim_{n\to\infty}(\log d_n)/n = 0$. The phenomenon of weak degree growth for integrable maps had been studied earlier by Veselov [33, 34], and the notion of algebraic entropy proposed in [14] is connected with other measures of growth such as Arnold complexity [1].

Since the work of Okamoto [22] it has been known that the Bäcklund transformations of differential Painlevé equations can be classified in terms of affine Weyl groups. Noumi and Yamada have shown that the symmetries of these ODEs are simultaneously compatible with associated discrete Painlevé equations [21], and this viewpoint has been explained geometrically by means of Cremona group actions on rational surfaces [30]. Discrete integrable systems are also related to the notions of discrete geometry [4] and discrete analytic functions [5].

In the past few years, several new techniques have been developed for isolating integrable discrete equations: analytical criteria on the one hand, and arithmetical ones on the other. The Painlevé property has been used very effectively to isolate integrable differential equations [7], and this led Ablowitz, Halburd and Herbst to extend this property to difference equations using Nevanlinna theory [2], by considering the asymptotic growth of meromorphic solutions at infinity. Roberts and Vivaldi [29] have studied the distribution of orbit

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lengths in rational maps reduced to finite fields $\mathbb{F}_p$ for different primes p, in order to identify integrable cases of such maps. Most recently [13], Halburd used Vojta's dictionary between Nevanlinna theory and Diophantine approximation to translate the concepts of [2] into a Diophantine integrability criterion for discrete equations.

There is by now a considerable literature on discrete bilinear equations, including the bilinear forms of discrete Painlevé equations [28], and bilinear partial difference equations such as the discrete Hirota equation, which can be written (with two arbitrary parameters α, β) in the form

$$\tau_{\ell+1,m,n}\tau_{\ell-1,m,n} = \alpha\,\tau_{\ell,m+1,n}\tau_{\ell,m-1,n} + \beta\,\tau_{\ell,m,n+1}\tau_{\ell,m,n-1}. \tag{1}$$

The partial difference equation (1) has continuum limits to all of the bilinear equations in the KP hierarchy of PDEs [35], and also appears as an equation for transfer matrices in quantum integrable models [18]. However, despite the huge amount of theory that has been developed for bilinear discrete equations, there is another property of such equations which appears to have been overlooked by the integrable systems community, namely the following: for initial data specified on a suitable subset of $\mathbb{Z}^3$, all of the iterates of (1) are Laurent polynomials in the initial data with coefficients in $\mathbb{Z}[\alpha,\beta]$. This *Laurent property* was originally known only to a few algebraic combinatorialists [25], and for the equation (1) it was first proved by Fomin and Zelevinsky within the framework of the theory of cluster algebras [10], while more detailed properties of the associated Laurent polynomials have been shown by Speyer [31].

One of the simplest manifestations of this Laurent phenomenon was found by Michael Somos, who considered bilinear recurrences of the form

$$\tau_{n+k}\tau_n = \sum_{j=1}^{[k/2]} \tau_{n+k-j}\tau_{n+j}, \qquad k \geq 4 \tag{2}$$

taking the initial values $\tau_0 = \tau_1 = \ldots = \tau_{k-1} = 1$. Clearly for the recurrences (2) each new iterate τ_{n+k} is a rational function of the initial data, so with this particular choice one would expect τ_n to be rational numbers, but it was observed numerically that for the Somos-4 recurrence

$$\tau_{n+4}\tau_n = \alpha\,\tau_{n+3}\tau_{n+1} + \beta\,(\tau_{n+2})^2 \tag{3}$$

with parameters $\alpha = \beta = 1$ and starting with four ones, an integer sequence

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, \ldots \tag{4}$$

results. Similar empirical observations showed that Somos-5, -6 and -7 also yield integer sequences, while the Somos-k recurrences for $k \geq 8$ do not. The first inductive proof of the integrality of the sequence (4) appeared in [11], but it was realized that the deeper reason behind this lay in the fact that the recurrence (3) has the Laurent property, that is $\tau_n \in \mathbb{Z}[\alpha,\beta,\tau_0^{\pm1},\tau_1^{\pm1},\tau_2^{\pm1},\tau_3^{\pm1}]$ for all $n \in \mathbb{Z}$. In other words, the iterates of (3) are Laurent polynomials in the four initial data whose coefficients are in $\mathbb{Z}[\alpha,\beta]$, so if $\alpha = \beta = \tau_0 = \tau_1 = \tau_2 = \tau_3 = 1$ then the sequence must consist entirely of integers.

Fomin and Zelevinsky found a suitable setting within which to prove the Laurent property for a variety of discrete equations [10], including certain recurrences of the form

$$\tau_{n+k}\tau_n = F(\tau_{n+1},\ldots,\tau_{n+k-1}) \tag{5}$$

for particular polynomials F (mostly, but not all, quadratic forms in their arguments), as well as integrable two- and three-dimensional bilinear recurrences like (1). More recently, the connection between the iterates of Somos-4 recurrences and sequences of points on elliptic curves was explained in the PhD thesis of Swart [32], while independently the author found the explicit solution of the initial value problem for both Somos-4 [15] and Somos-5 [16] in terms of elliptic sigma functions; see also [24] for a connection with continued fractions. An essential observation in [15] and [16] was that each of these bilinear recurrences (fourth and fifth order respectively) could be understood in terms of a suitable integrable mapping of the plane, with each one corresponding to a particular degenerate case of the family of maps studied by Quispel, Roberts and Thompson [27]. Somos sequences are of considerable interest to number theorists due to the way that new prime factors appear therein [8, 9].

A natural question raised by all of the above is the following: if discrete equations with the Laurent property like (1) and (3) are integrable, then are all discrete equations with the Laurent property integrable? The converse is clearly not possible: most integrable rational maps do not have iterates that are Laurent polynomials in the initial data. Thus it is sensible to restrict ourselves to a suitable sub-class of maps, namely those defined by recurrences of the form (5). The purpose of this note is to show that the answer to this question is a negative one, by taking an example not considered in [10], namely the one-parameter family of third order recurrences given by

$$\tau_{n+3}\tau_n = \tau_{n+2}^2 + \tau_{n+1}^2 + J. \tag{6}$$

In the case $J = 0$, it was noticed by Dana Scott [11] that with $\tau_0 = \tau_1 = \tau_2 = 1$ an integer sequence results:

$$1, 1, 1, 2, 5, 29, 433, 37666, 48928105, 5528778008357, \ldots \tag{7}$$

In fact, (6) is related to some old problems in number theory: the numbers (7) are a subsequence of the Markoff numbers that arise in the theory of indefinite quadratic forms [19], while the case of arbitrary J is related to a Diophantine equation considered by Mordell [20]. Below we give two different proofs that the recurrence (6) has the Laurent property; the first proof also shows that it passes the singularity confinement test, while the second one gives a stronger result based on the existence of a conserved quantity. However, when Halburd's Diophantine integrability test is applied to a particular class of sequences generated by (6), it is seen that the growth of the heights of iterates is double exponential (which can already be guessed from the rapid growth of the terms in (7)). For the the particular sequences being considered, quite sharp bounds on this double exponential growth are obtained. Hence we have an interesting example of a non-integrable recurrence with the Laurent property.

The recurrence (6) defines a rational map of three-dimensional affine space, i.e.

$$(x_n, y_n, z_n) \mapsto (y_n, z_n, (y_n^2 + z_n^2 + J)/x_n) = (x_{n+1}, y_{n+1}, z_{n+1}), \tag{8}$$

upon setting $(\tau_n, \tau_{n+1}, \tau_{n+2}) = (x_n, y_n, z_n)$. Before considering the properties of this map in more detail, we present the first proof that it has the Laurent property, since this will simultaneously show that it satisfies singularity confinement as well. Letting $\tau_0 = a$, $\tau_1 = b$, $\tau_2 = c$ and putting the recurrence into MAPLE, we see that the next two iterates are

$$\tau_3 = \frac{b^2 + c^2 + J}{a}, \quad \tau_4 = \frac{b^4 + a^2c^2 + 2b^2c^2 + c^4 + (a^2 + 2b^2 + 2c^2)J + J^2}{a^2 b},$$

and τ_5 has a^4b^2c in the denominator and its numerator is a polynomial in $\mathbb{Z}[a, b, c, J]$ with 29 terms. The first miracle occurs when $n = 3$, because upon dividing the right hand side of (6) by τ_3 the numerator $b^2 + c^2 + J$ cancels, so that $\tau_6 \in \mathcal{R} = \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, J]$ with $a^7b^4c^2$ in the denominator and a polynomial with 104 terms in the numerator.

Now to prove the Laurent property, we can use the fact that modulo monomial factors $a^j b^k c^\ell$ the ring $\mathcal{R}$ of Laurent polynomials is a unique factorization domain (see [10] - the case $J=0$ is covered by Theorem 1.10). Thus we can consider divisibility of the τ_n modulo such monomials. As the inductive hypothesis, assume that $\tau_j \in \mathcal{R}$ for $0 \leq j \leq n+5$, that any three adjacent terms $\tau_j, \tau_{j+1}, \tau_{j+2}$ are pairwise coprime (modulo monomials) in this range, and also that $\tau_{j\pm 1}^2 + J$ is coprime to τ_j. To prove that the next iterate τ_{n+6} is in $\mathcal{R}$, consider the three adjacent Laurent polynomials $x = \tau_{n+1}$, $y = \tau_{n+2}$, $z = \tau_{n+3}$. From successive applications of (6) we have

$$x^2 + y^2 + J = \tau_{n+3}\tau_n \equiv 0 \mod z, \tag{9}$$

as well as $\tau_{n+4} \equiv (y^2+J)/x$ and $\tau_{n+4} \equiv (y^4 + J(x^2+2y^2)+J^2)/(x^2y)$ (both mod$z$), which together imply that

$$\tau_{n+6}\tau_{n+3} \equiv \frac{(y^2+J)(x^2+y^2+J)(y^4+J(x^2+2y^2)+J^2)}{x^4y^2} \equiv 0 \mod z \tag{10}$$

by (9), and hence $\tau_{n+6} \in \mathcal{R}$. Next suppose that an irreducible factor $p \in \mathcal{R}$ is such that $p|\tau_{n+6}$ and $p|\tau_{n+5}$; then $p|(\tau_{n+6}\tau_{n+3} - \tau_{n+5}^6) = \tau_{n+4}^2 + J$, which contradicts the coprimality of τ_{n+5} and $\tau_{n+4}^2 + J$, so τ_{n+6} and τ_{n+5} must be coprime. The same argument shows that τ_{n+6} and τ_{n+4} must also be coprime, so that the three adjacent terms τ_{n+4}, τ_{n+5} and τ_{n+6} are pairwise coprime. Similarly it follows that $\tau_{n+5}^2 + J$ is coprime to τ_{n+6}, and also $\tau_{n+6}^2 + J$ is coprime to τ_{n+5}, which completes the inductive step. This proves that $\tau_n \in \mathcal{R}$ for all positive values of the index n, and the same conclusion holds for negative indices by the reversibility of the recurrence.

To see why singularity confinement also follows from the preceding argument, observe that a singularity can only occur when one of the iterates vanishes. Thus, without loss of generality suppose that $z = \tau_{n+3}$ vanishes, and we take $\tau_n = w$ and $y^2 = -x^2 - J + \epsilon w$ so that $z = \epsilon \to 0$. The above expressions can quickly be evaluated in this limit by noting that all terms congruent to z are $O(\epsilon)$, and also $y = y_0 + O(\epsilon)$ with $y_0 = \sqrt{-x^2 - J}$. Thus $\tau_{n+4} = -x + O(\epsilon)$, $\tau_{n+5} = (x^2+J)/y_0 + O(\epsilon)$, and clearly from (10) we have $\tau_{n+6}\,\epsilon = O(\epsilon)$ so $\tau_{n+6} = O(1)$ and the singularity is confined. This argument for confinement will work for many other recurrences with the Laurent property, such as those in [10]. For the recurrence (6), a stronger version of the Laurent property actually holds, as a consequence of the following

Proposition 1 *The quantity $N_n = (\tau_{n+2}^2 + \tau_{n+1}^2 + \tau_n^2 + J)/(\tau_{n+2}\tau_{n+1}\tau_n)$ is an invariant for the recurrence (6), so that $N_n = N_0 = N$ with*

$$N = \frac{a^2+b^2+c^2+J}{abc}. \tag{11}$$

Moreover, for the same initial data $\tau_0 = a, \tau_1 = b, \tau_2 = c$, the iterates of (6) are the same as those of the recurrence

$$\tau_{n+3} = N\tau_{n+2}\tau_{n+1} - \tau_n, \tag{12}$$

and all τ_n lie in the ring $\mathbb{Z}[a,b,c,N] \subset \mathcal{R} = \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, J]$.

The proof is quite straightforward. The fact that N_n is a conserved quantity for (6) follows from the direct calculation that $N_{n+1} - N_n = 0$, so clearly the iterates of the map (8) lie on the cubic surface

$$x^2 + y^2 + z^2 - Nxyz + J = 0, \tag{13}$$

which is non-singular for $J \neq 0, -4/N^2$. This surface admits some obvious symmetries: it is unchanged by permutation of (x, y, z), and also for fixed y and z it is a quadratic equation in x and hence is preserved by the involution $\mathcal{I} : (x, y, z) \mapsto (x^\dagger, y, z)$ where

$$x^\dagger = (y^2 + z^2 + J)/x = Nyz - x \tag{14}$$

is the other root of the quadratic. Taking the cyclic permutation $\mathcal{P} : (x, y, z) \mapsto (y, z, x)$, then performing $\mathcal{I}$ followed by $\mathcal{P}$ yields the transformation $(x, y, z) \mapsto (y, z, x^\dagger)$. If the first equality for $x^\dagger$ in (14) is used in this transformation, then the rational map (8) corresponding to the recurrence (6) arises. Similarly, the alternative recurrence (12) comes from the second equality in (14), and one could start with this and observe that it has the conserved quantity $J_n = N\tau_{n+2}\tau_{n+1}\tau_n - \tau_{n+2}^2 - \tau_{n+1}^2 - \tau_n^2$. It is also obvious that the iterates of (12) lie in $\mathbb{Z}[a, b, c, N]$, and with N given by the formula (11) this is a subring of $\mathcal{R}$.

Here we should point out that the aforementioned symmetries were used extensively by Mordell [20] when he considered the Diophantine problem of finding integer triples (x, y, z) satisfying (13) for fixed integers N and J. The case $N = -2$, $J = -n \in \mathbb{Z}$ was treated in great detail, since this is related to the problem of finding which integers can be written as the sum of four cubes. However, he remarked that "*I know of no test except trial for the existence of solutions for given n.*" The particular case $N = 3$, $J = 0$ is another distinguished instance of (13) known as Markoff's equation, which is related to the spectrum of indefinite binary quadratic forms [19]. Although a great deal more is known about this case, there are still difficult open problems associated with it [3, 6, 23]. The case $J = 0$ has also acquired a combinatorial interpretation very recently [17, 26]. The above Proposition implies that, given any initial solution triple of integers (a, b, c), the recurrence (6) (or (12), which is equivalent) will generate infinitely many such triples provided that it leads to neither a zero nor a periodic orbit. In fact, by the above discussion of singularity confinement, a single zero (e.g. a triple $(x, y, 0)$) is not a problem in the sense that one can introduce a small parameter ϵ and analytically continue through the apparent singularity. However, a solution like $(x, 0, 0)$ can also occur if $-J$ is a perfect square: see the Lemma and the Theorem on p.506 of [20]; this is a more serious obstacle. On the other hand, given an initial solution (x_0, y_0, z_0) of (13), by that Lemma one can still generate infinitely many solutions (x_n, x_{n+1}, z_0) with the same value of z by setting $x_1 = y_0$ and iterating the *integrable* recurrence

$$x_{n+1}x_{n-1} = x_n^2 + z_0^2 + J \tag{15}$$

(with $N^2 z_0^2 \geq 4$ for an infinity of integer solutions) or, equivalently, the linear recurrence

$$x_{n+1} - Nz_0 x_n + x_{n-1} = 0. \tag{16}$$

Halburd's Diophantine integrability test [13] applies to rational maps with iterates and parameters taking values in $\mathbb{Q}$ or a number field. For $x \in \mathbb{Q}$ with $x = p/q$ as a fraction in lowest terms, the height is $H(x) = \max\{|p|, |q|\}$, while the logarithmic height is $h(x) = \log H(x)$. A rational map of this kind is defined to be *Diophantine integrable* if its iterates x_n are such that $h(x_n)$ grows no faster than a polynomial in n. To apply the test to the recurrence (6) we shall restrict ourselves to the case when all of the iterates τ_n are integers, so that $h(\tau_n) = \log|\tau_n|$. More generally we can consider real or complex values of τ_n, and set $\Lambda_n = \log|\tau_n|$. Now suppose that $\Lambda_n \sim C\lambda^n \to \infty$ as $n \to \infty$ for real $\lambda > 1$ and some $C > 0$. Then clearly $\tau_n^{-1} \to 0$ and also $\Lambda_{n+1} - \Lambda_{n+2} \sim C\lambda^{n+1}(1 - \lambda) \to -\infty$, whence $\tau_{n+1}/\tau_{n+2} \to 0$ as $n \to \infty$. So taking the logarithm of (6) and substituting in these asymptotics gives

$$\Lambda_{n+3} - 2\Lambda_{n+2} + \Lambda_n = \log\left|1 + \frac{\tau_{n+1}^2}{\tau_{n+2}^2} + \frac{J}{\tau_{n+2}^2}\right| \to 0. \tag{17}$$

The characteristic polynomial for the linear recurrence on the left hand side of (17) gives $\lambda^3 - 2\lambda^2 + 1 = 0$, with the largest root being the golden mean, $\lambda = (1+\sqrt{5})/2 > 1$, which is consistent with the original assumptions on the asymptotics of Λ_n. Thus if integer sequences of this kind exist, then their logarithmic heights grow exponentially with n with $\lim_{n\to\infty}(\log h(\tau_n))/n = \log((1+\sqrt{5})/2) \approx 0.4812$. We proceed to show existence.

Proposition 2 *With initial data $(\tau_0, \tau_1, \tau_2) = (1,1,1)$ the recurrence (6) gives a sequence of monic polynomials $\tau_n = p_n(N) \in \mathbb{Z}[N]$ with $N = J+3$, and $\deg p_n = f_n - 1$ for $n > 0$ where f_n is a Fibonacci number. For real $N > 2$, $p_{n+1}(N) > p_n(N) > 1$ holds for $n \geq 3$. Furthermore, for $N > 2$ and all $n \geq 4$ these polynomials satisfy*

$$(N-1)^{f_n - 1} < p_n(N) < N^{f_n - 1}. \tag{18}$$

That these initial data give polynomials in N follows from the earlier Proposition; equation (11) gives $N = J+3$, and we have $p_3 = N-1$, $p_4 = N^2 - N - 1$, $p_5 = N^4 - 2N^3 + N - 1$, $p_6 = N^7 - 3N^6 + N^5 + 3N^4 - 2N^3 + 1$ etc. From (12), $d_n = \deg p_n$ satisfies $d_{n+3} = d_{n+2} + d_{n+1} + 1$ which implies that $d_n = f_n - 1$, with f_n being a Fibonacci number ($f_0 = 0$, $f_1 = 1$), and the p_n are clearly monic. The values $p_n(-1)$, $p_n(0)$ and $p_n(1)$ cycle with periods 4,6 and 12 respectively, while $p_n(2) = 1$ for all n, but for $N > 2$ we have $J > -1$ so by induction $p_{n+1}^2 + J > 0$ and (6) gives $p_{n+3}p_n > p_{n+2}^2$ so $p_{n+3}/p_{n+2} > (p_{n+2}/p_{n+1})(p_{n+1}/p_n) > 1$ as required. The upper bound in (18) is easy to see by induction from (12), since $p_{n+3} < N p_{n+2} p_{n+1}$. The lower bound follows from another induction, by writing $p_{n+1} = p_n \rho_n$, and noting that for $N > 2$ the ratios satisfy $\rho_n > (N-1)^{f_{n-1}}$ when $n \geq 3$, as a consequence of the inequality $\rho_{n+2} > \rho_{n+1}\rho_n$. Our main result is an immediate consequence of the preceding facts.

Theorem *For each integer $N \geq 3$, every triple $(p_n(N), p_{n+1}(N), p_{n+2}(N))$ of adjacent iterates in the sequence for $n \geq 0$ constitutes a distinct solution triple for the Diophantine equation (13) with $J = N - 3$. Moreover, for all such $N \in \mathbb{Z}$ the logarithmic height $h(p_n(N))$ of these iterates grows exponentially, and for all real $N > 2$ we have* $\lim_{n\to\infty}(\log\log p_n(N))/n = \log\lambda \approx 0.4812$ *where λ is the golden mean.*

The integrality of the terms $p_n(N)$ for $N \in \mathbb{Z}$ is obvious. The fact that these are distinct solution triples follows from the monotonicity of the sequence for $N > 2$, while the statements about asymptotic growth follow from taking logarithms of each side of (18). Similar estimates hold for $|p_n|$ when $N \leq -2$, but then $p_{3n}(N) < 0$. It is quite easy to see that $\log\lambda$ is also the value of the algebraic entropy for the recurrence.

It is hoped that the connections between Diophantine integrability, singularity confinement and the Laurent phenomenon will become clearer in the future.

Acknowledgements. The author is grateful for the support of an EPSRC Springboard Fellowship, and thanks Jack Button, Nalini Joshi, Yuri Suris and Sasha Veselov for useful discussions.

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