Algebraic curves, integer sequences and a discrete Painlevé transcendent

A.N.W. Hone
Institute of Mathematics, Statistics & Actuarial Science,
University of Kent,
*Canterbury CT2 7NF, UK.**

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Abstract

We consider some bilinear recurrences that have applications in number theory. The explicit solution of a general three-term bilinear recurrence relation of fourth order is given in terms of the Weierstrass sigma function for an associated elliptic curve. The recurrences can generate integer sequences, including the Somos 4 sequence and elliptic divisibility sequences. An interpretation via the theory of integrable systems suggests the relation between certain higher order recurrences and hyperelliptic curves of higher genus. Analogous sequences associated with a q-discrete Painlevé I equation are briefly considered.

1 Introduction

Question: What's the next number in this sequence:

$$1, 1, 2, 3, \ldots ?$$

If you thought the answer is 5, then you guessed wrong! Actually the required answer is 7, and it continues

$$1, 1, 2, 3, 7, 23, 59, 314, \ldots, \qquad (1.1)$$

*E-mail: anwh@kent.ac.uk
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but it's a terrible question, as any four numbers can serve as initial data for a recurrence of fourth order or higher. Unfortunately this is the sort of question that is routinely given to schoolchildren - I noticed it in a recent GCSE exam set to 15-year-olds in Britain.

So if it's not Fibonacci, which comes from a second order linear recurrence relation, then how is the sequence (1.1) generated? It comes from the *bilinear* fourth order recurrence

$$\tau_{n+2} = \frac{\tau_{n+1}\tau_{n-1} + (\tau_n)^2}{\tau_{n-2}}, \qquad \tau_0 = \tau_1 = \tau_2 = \tau_3 = 1, \quad (1.2)$$

and goes by the name of the Somos 4 sequence in Sloane's catalogue [14]. It turns out that with the special choice of initial conditions as in (1.2), all numerators cancel with their denominators in such a way that all the terms in the sequence are integers.

The remarkable fact is that the sequence (1.1) is related to the elliptic curve

$$y^2 = 4x^3 - 4x + 1. \quad (1.3)$$

Morever, there is a whole family of sequences defined by fourth order bilinear recurrences, of the form

$$\tau_{n+2}\tau_{n-2} = \alpha\, \tau_{n+1}\tau_{n-1} + \beta\, (\tau_n)^2 \quad (1.4)$$

with constant coefficients α, β. These include a type of integer sequences, known as elliptic divisibility sequences, [16, 13], and their generalizations [12, 15]. The connection with elliptic curves is a general property of the family of fourth order bilinear recurrences. The following theorem is proved in [7].

Theorem. *The general solution of the quadratic recurrence relation (1.4) for $\alpha \neq 0$ takes the form*

$$\tau_n = A\, B^n \frac{\sigma(z_0 + n\kappa)}{\sigma(\kappa)^{n^2}}, \quad (1.5)$$

where κ and z_0 are non-zero complex numbers, the constants A and B are given by

$$A = \frac{\tau_0}{\sigma(z_0)}, \qquad B = \frac{\sigma(\kappa)\sigma(z_0)\,\tau_1}{\sigma(z_0+\kappa)\,\tau_0}, \tag{1.6}$$

and σ denotes the Weierstrass sigma function associated to the elliptic curve $y^2 = 4x^3 - g_2x - g_3$.

The complex constants z_0, κ should be considered as points on the Jacobian of the curve, hence only defined modulo the period lattice, and together with the invariants g_2, g_3 they are determined uniquely in terms of the constants α, β and the initial data $\tau_0 \neq 0, \tau_1, \tau_2, \tau_3$ specified for the fourth order recurrence (1.4). The precise construction of the curve from α, β and the initial data is presented in [7]. Note that in the case $\alpha = 0$ excluded by the Theorem, the recurrence (1.4) decouples into two trivial recurrences for even and odd terms, so that $\tau_{2k} = \tau_0(\tau_2/\tau_0)^k\beta^{k(k-1)/2}$, and $\tau_{2k+1} = \tau_1(\tau_3/\tau_1)^k\beta^{k(k-1)/2}$.

The terms in the Somos 4 sequence (1.1) are given by formula (1.5) with

$$\kappa = 2\omega_1 - \int_1^\infty (4t^3 - 4t + 1)^{-\frac{1}{2}}\,dt = 1.859185431$$

$$z_0 = 2\omega_3 + \int_{-1}^\infty (4t^3 - 4t + 1)^{-\frac{1}{2}}\,dt = 0.204680500 + 1.225694691i,$$

where the integrals are taken along suitable paths in the t plane avoiding the three real roots of the cubic, and have been evaluated to 9 decimal places using version 8 of the MAPLE computer algebra package. The real and imaginary half-periods are $\omega_1 = 1.496729323$ and $\omega_3 = 1.225694691i$ respectively. The other constants appearing in (1.5) are just

$$\sigma(\kappa) = 1.555836426, \qquad A = 0.112724016 - 0.824911686i,$$

$$B = 0.215971963 + 0.616028193i.$$

However, in this case the sequence of arguments of the sigma function can be written more succinctly as

$$z_0 + n\kappa \equiv (2n-3)\hat{z}_0, \qquad \hat{z}_0 = 0.929592715 + \omega_3,$$

so that the iterates of the recurrence correspond to the sequence of points $(2n-3)P$ on the curve (1.3), where $P = (0,1)$; a referee was helpful in pointing this out.

There is currently much number theoretical interest in elliptic divisibility sequences (integer sequences satisfying $\tau_n|\tau_m$ whenever $n|m$), and in Somos sequences, since apparently they are a source of large prime numbers [4, 5].

2 Symplectic structure and integrability

By the change of variables

$$f_n = \frac{\tau_{n+1}\tau_{n-1}}{(\tau_n)^2}, \tag{2.1}$$

the bilinear recurrence is related to a second order difference equation, and so τ_n is a tau-function.

Proposition. *For $\alpha \neq 0$ the second order nonlinear map*

$$f_{n+1} = \frac{1}{f_{n-1}f_n}\left(\alpha + \frac{\beta}{f_n}\right) \tag{2.2}$$

has the general solution

$$f_n = \wp(\kappa) - \wp(z_0 + n\kappa) \tag{2.3}$$

with complex parameters z_0 and $\kappa \neq 0$, where the $\wp$ is the Weierstrass elliptic function associated to the curve

$$y^2 = 4x^3 - g_2x - g_3, \tag{2.4}$$

and the points $(\wp(\kappa), \wp'(\kappa))$ and $(\wp(z_0), \wp'(z_0))$ on the curve, together with the values of the invariants g_2, g_3, are determined explicitly by the parameters α, β and the non-zero initial data f_0, f_1 of the map (2.2).

The second order discrete equation (2.2) can be thought of as a mapping of the plane, $(f_{n-1}, f_n) \to (f_n, f_{n+1})$, with the first integral

$$\lambda = \frac{1}{3}\left[f_n + f_{n+1} + \frac{1}{4\alpha}\left(f_n f_{n+1} - \frac{\beta}{f_n f_{n+1}} - \frac{\alpha}{f_n} - \frac{\alpha}{f_{n+1}}\right)^2\right].$$

satisfying $\lambda(f_n, f_{n+1}) = \lambda(f_{n-1}, f_n)$. This two-dimensional mapping is *not* symplectic, because $(f_{n+1})^{-1}\, df_n \wedge df_{n+1} = (f_{n-1})^{-1}\, df_{n-1} \wedge df_n$.

To get a symplectic map, we make the transformation to symplectic coordinates (q_n, p_n) on the curve (with two-form $dp_n \wedge dq_n$), by setting $\mu = \sqrt{\alpha}$ and

$$q_n = \lambda - f_n, \qquad p_n = \frac{1}{\mu} f_n^2 (f_{n+1} - f_{n-1}). \tag{2.5}$$

With these variables the mapping can be rewritten as

$$\begin{aligned} q_{n+1} &= \Phi_n^2 - q_n - \lambda, \\ p_{n+1} &= -2\Phi_n^3 + (4q_n + 2\lambda)\Phi_n - p_n, \end{aligned} \tag{2.6}$$

where

$$\Phi_n = \frac{p_n - \mu}{2(q_n - \lambda)}, \; \mu(q_n, p_n) = \sqrt{p_n^2 + 4(\lambda^3 - q_n^3) - g_2(\lambda - q_n)}.$$

Note that λ and g_2 should be interpreted as constant parameters for the map (2.6), while μ is interpreted as a function on phase space.

The symplectic map (2.6) is integrable, with first integral

$$g_3(q_n, p_n) = 4q_n^3 - g_2 q_n - p_n^2, \tag{2.7}$$

and has a 2×2 discrete Lax pair [7]. With a *different* choice of symplectic structure on the elliptic curve, it is equivalent to the one-particle discrete Garnier system in [8].

It is natural to wonder how all of the above might generalize to the case of the bilinear recurrences

$$\tau_{n+g+1}\tau_{n-g-1} = \sum_{j=0}^{g} \alpha_j\, \tau_{n+j}\tau_{n-j}, \tag{2.8}$$

which for $g \geq 2$ is the natural higher order analogue of (1.4), with $g+1$ parameters $\alpha_0, \alpha_1, \ldots, \alpha_g$. The arithmetic of sequences defined by such recurrences is discussed in [6]. We are able to state the following

Conjecture. *The general solution of the bilinear recurrence (2.8) for $g \geq 2$ is given in terms of the Riemann theta-function associated to an algebraic curve Γ of genus g. Using the change of variables (2.1), the bilinear recurrence yields a nonlinear discrete equation*[1] *of order $2g$, which is equivalent to an integrable symplectic map that linearizes on* $\mathrm{Jac}(\Gamma)$*, the Jacobian of the algebraic curve.*

The complete proof of this conjecture is work in progress with Harry Braden, but here we briefly outline its main ingredients and explain why the result is plausible. The result can be seen as a direct consequence of certain vector addition theorems for theta-functions obtained by Buchstaber and Krichever [1]. However, it turns out that the recurrence (2.8) is naturally related to a hyperelliptic curve Γ of the form

$$y^2 = 4x^{2g+1} + \sum_{j=1}^{2g} c_j x^{2g-j}, \tag{2.9}$$

(the KdV spectral curves [8]) rather than to an arbitrary curve of genus g. Furthermore, it is most convenient to ex-

[1] The recurrence is $f_n^{g+1}\prod_{k=1}^{g+1}(f_{n+k}f_{n-k})^{g+1-k} = \alpha_0 + \sum_{j=1}^{g}\alpha_j f_n^j \prod_{k=1}^{j}(f_{n+k}f_{n-k})^{j-k}$.

press the general solution using the Kleinian sigma function of the curve (this is equivalent to an expression in theta-functions - see [2]). In that case the formula for τ_n is almost identical to (1.5) except that $z_0 \in \mathrm{Jac}(\Gamma)$ is a vector in the g-dimensional Jacobian corresponding to the reduced divisor $D_0 = (P_1 - \infty) + (P_2 - \infty) + \ldots + (P_g - \infty)$ of g points on the curve, and $\kappa \in \mathrm{Jac}(\Gamma)$ corresponds to the reduced divisor $D = P - \infty$ of another point P (with a derivative of σ taken on the theta divisor).

The iterates of the recurrence correspond to the sequence of divisors $D_0 + nD$, which gives a linear flow $z_0 + n\kappa$ on $\mathrm{Jac}(\Gamma)$. The division polynomials for hyperelliptic curves, corresponding to the special case of multiples of a single point, $nD = n(P - \infty)$, have been considered by Cantor [3], who gave examples of these recurrences. Matsutani has also done some detailed calculations of addition formulae in genus two [10].

3 An integrable q-difference analogue

The second order nonlinear difference equation (2.2) has a non-autonomous, q-difference version given by

$$f_{n+1}f_{n-1} = \frac{\alpha q^n f_n + \beta}{f_n^2}. \tag{3.1}$$

This map satisfies the singularity confinement test and has a 4×4 Lax pair [11], as well as a continuum limit to the first Painlevé equation (PI). Therefore (3.1) may be referred to as qdPI. The bilinearization of qdPI is achieved by the same formula (2.1), to obtain the the q-difference analogue of (1.4), that is

$$\tau_{n+2}\tau_{n-2} = \alpha q^n \,\tau_{n+1}\tau_{n-1} + \beta\,(\tau_n)^2. \tag{3.2}$$

It is a remarkable fact that the bilinear equation (3.2) shares some of the divisibility properties of the autonomous recurrences (1.4), in the sense that given constants τ_{-2}, τ_{-1}, τ_0, τ_1 as initial data, all of the τ_n for $n \geq 2$ are polynomials in q (with coefficients being rational in α, β and the four initial data). So far we have only found an inductive argument for this, but ideally τ_n should be given by a determinant, as for *linear* q-special functions [9]. When the formula (1.5) for the solution to the autonomous recurrence (1.4 is specialized to rational curves, then there are sequences where τ_n satisfies a linear recurrence. However, for (3.2) it is possible to prove that this can only occur for the special values $q = \pm 1, \pm i$.

The recurrence relation (3.2) is invariant under gauge transformations $\tau_n \longrightarrow \tilde{A}\tilde{B}^n \tau_n$, while the transformation $\tau_n \longrightarrow \tilde{C}^{n^2}\tau_n$ means that α can be rescaled to 1. Let us consider the particular sequence for $\alpha = \beta = 1$ defined by

$$\tau_{n+2} = \frac{q^n\, \tau_{n+1}\tau_{n-1} + (\tau_n)^2}{\tau_{n-2}}, \qquad \tau_{-2} = \tau_{-1} = \tau_0 = \tau_1 = 1,$$

which is a non-autonomous version of the Somos 4 sequence (1.2). We would like to make the following observations:

- For $n \geq -2$, $\tau_n = \tau_n(q)$ is a polynomial in q of degree $d_n = n^3/18 - n/6 + \frac{2}{9\sqrt{3}} \sin(2\pi n/3)$.
- The coefficients of $\tau_n(q)$ are all positive integers; the coefficient of q^{d_n} is $2^{[\frac{n+1}{3}]}$.
- The roots of $\tau_n(q)$ appear to lie in a circle of maximal radius $\sqrt{2}$ in the complex q plane, and coalesce to certain special values, one of which is -1.3877957 to 7 decimal places.

The rigorous analysis of the properties of the qdPI tau-functions will be the subject of future work[2].

[2]During the SIDE 6 meeting, Sasha Bobenko and Vassilios Papageorgiou informed me that this should be related to the Laurent phenomenon for certain recurrences recently studied by Fomin & Zelevinsky, `arXiv.math.RT/0311493`

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References

[1] V.M. Buchstaber and I.M. Krichever, Teoret. Mat. Fiz. 94 (1993) 200-212.

[2] V.M. Buchstaber, V.Z. Enolskii and D.V. Leykin, *Hyperelliptic Kleinian functions and applications*, in 'Solitons, Geometry and Topology: On the Crossroad,' (eds. V.M. Buchstaber and S.P. Novikov), AMS Translations Series 2, Vol. 179, American Mathematical Society (1997) 1-34.

[3] D. Cantor, J. reine angew. Math. 447 (1994) 91-145.

[4] M. Einsiedler, G. Everest and T. Ward, LMS Journal of Computation and Mathematics 4 (2001) 1-13.

[5] G. Everest and T. Ward, New York J. Math. 6 (2000) 331-342.

[6] G. Everest, A. van der Poorten, I. Shparlinski and T. Ward, *Recurrence Sequences*, AMS Mathematical Surveys and Monographs, vol. 104, American Mathematical Society (2003).

[7] A.N.W. Hone, *Elliptic curves and quadratic recurrence sequences*, Bull. Lond. Math. Soc. 37 (2005) 161-171; Corrigendum, *ibid.* 38 (2006) 741-742.

[8] A.N.W. Hone, V.B. Kuznetsov and O. Ragnisco, J. Phys. A: Math. Gen. 32 (1999) L299-L306.

[9] M.E.H. Ismail, H. Prodinger and D. Stanton, Sem. Loth. Comb. 44 B44a (2000).

[10] S. Matsutani, Phys. Lett. A 300 (2002) 233-242.

[11] Y. Ohta, K.M. Tamizhmani, B. Grammaticos and A. Ramani, `solv-int/9910007`

[12] R. Robinson, Proc. Amer. Math. Soc. 116 (1992) 613-619.

[13] R. Shipsey, *Elliptic divisibility sequences*, PhD thesis, University of London (2000).

[14] N.J.A. Sloane, *On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences`, sequence A006720.

[15] C.S. Swart, *Elliptic curves and related sequences*, PhD thesis, University of London (2003).

[16] M. Ward, Amer. J. Math. 70 (1948) 31-74; Duke Math. J. 15 (1948) 941-946.