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A family of linearisable recurrences with the Laurent property

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ABSTRACT

We consider a family of nonlinear recurrences with the Laurent property. Although these recurrences are not generated by mutations in a cluster algebra, they fit within the broader framework of Laurent phenomenon algebras, as introduced recently by Lam and Pylyavskyy. Furthermore, each member of this family is shown to be linearisable in two different ways, in the sense that its iterates satisfy both a linear relation with constant coefficients and a linear relation with periodic coefficients. Associated monodromy matrices and first integrals are constructed, and the connection with the dressing chain for Schrödinger operators is also explained.

1. Introduction

A nonlinear recurrence relation is said to possess the Laurent property if all of its iterates are Laurent polynomials in the initial data with integer coefficients. Particular recurrences of the form

$$x_{n+N}x_n = F(x_{n+1}, \dots, x_{n+N-1}), \quad (1.1)$$

with F being a polynomial, first gained wider notice through the article by Gale [9], which highlighted the fact that such nonlinear relations can unexpectedly generate integer sequences. The second order recurrence

$$x_{n+2}x_n = x_{n+1}^2 + 1 \quad (1.2)$$

is one of the simplest examples of this type. With the initial values $x_0 = 1, x_1 = 1$, this generates a sequence beginning $1, 1, 2, 5, 13, 34, 89, 233, \dots$, and it turns out that x_n is an integer for all n , but it is not obvious why this should be so. The fact that (1.2) has the Laurent property provides an explanation: if x_0 and x_1 are considered as variables, then each x_n is an element of the Laurent polynomial ring $\mathbb{Z}[x_0^{\pm 1}, x_1^{\pm 1}]$, and so generates an integer when evaluated at $x_0 = x_1 = 1$.

One of the motivations behind Fomin and Zelevinsky's cluster algebras, introduced in [4], was to provide an axiomatic framework for the Laurent property, which arises in many different areas of mathematics. A cluster is an N -tuple $\mathbf{x} = (x_1, x_2, \dots, x_N)$ which can be mutated in direction j for each choice of $j \in \{1, \dots, N\}$, to produce a new cluster \mathbf{x}' with components $x'_k = x_k$ for $k \neq j$ and x'_j determined by the exchange relation

$$x'_j x_j = \prod_{b_{ij} > 0} x_i^{b_{ij}} + \prod_{b_{ij} < 0} x_i^{-b_{ij}} \quad (1.3)$$

for a coefficient-free (geometric type) cluster algebra, where $B = (b_{ij})$ is an associated skew-symmetrizable $N \times N$ integer matrix. There is also an associated operation of matrix mutation, $B \rightarrow B'$, the precise details of which are not needed here.

In general, sequences of mutations in a cluster algebra do not generate orbits of a fixed map, since the exponents in (1.3) both depend on j and vary under the mutation of the matrix B .

However, Fordy and Marsh [6] found criteria for a skew-symmetric matrix B (or the equivalent quiver) to vary periodically under a cyclic sequence of mutations, and in particular showed how this can correspond to the iteration of a single recurrence relation of the form (1.1), with F being a sum of two monomials obtained from an exchange relation (1.3). For the classification of mutation-periodic quivers in [6], the building-blocks were provided by the affine A -type quivers (referred to there as “primitives”), and it was also shown that sequences of cluster variables generated from these affine quivers satisfy linear relations with constant coefficients. The same result was found independently in [1], where it was conjectured (and also proved for type D) that there should be linear relations for cluster variables associated with all the affine Dynkin types; the full conjecture was proved in [15].

In [7] it was shown that the $\tilde{A}_{1,N-1}$ exchange relations produce integrable maps for N even; in the non-commutative setting, the exchange relations for N odd were considered in [2]. The simplest case ($N = 2$) is just the recurrence (1.2), which preserves the Poisson bracket $\{x_n, x_{n+1}\} = x_n x_{n+1}$, and has a conserved quantity C (independent of n) that appears as a non-trivial coefficient in the linear relation for the sequence of cluster variables (x_n) , that is

$$x_{n+2} - Cx_{n+1} + x_n = 0, \quad \text{with} \quad C = \frac{x_n}{x_{n+1}} + \frac{x_{n+1}}{x_n} + \frac{1}{x_n x_{n+1}}.$$

Whenever a nonlinear recurrence is such that its iterates satisfy a linear relation, as is the case here, we say that it is *linearisable*.

Q-systems, which arise from the Bethe ansatz for quantum integrable models, provide further examples of nonlinear recurrences which are obtained from sequences of cluster mutations [14], and are also linearisable in the above sense [2]. They correspond to characters of representations of Yangian algebras, as well as being reductions of discrete Hirota equations [17]. The simplest case is the A_1 Q-system, which coincides with (1.2). In addition to the aforementioned examples coming from affine A -type quivers, yet another family of linearisable recurrences obtained from cluster mutations is considered in [8].

In this paper we will consider the family of nonlinear recurrences given by

$$x_{n+N}x_n = x_{n+N-1}x_{n+1} + a \sum_{i=1}^{N-1} x_{n+i}, \quad (1.4)$$

where $N \geq 2$ is an integer and a is a constant parameter. We will show that all of these recurrences have the Laurent property, and they are linearisable. However, observe that the right hand side of (1.4) has N terms, which means that for $N \geq 3$ it cannot be obtained from an exchange relation in a cluster algebra, since for that to be so the polynomial F should have the same binomial form as (1.3); the case $N = 2$ is not an exchange relation either (but see Example 4.11 in [8]).

The inspiration for (1.4) comes from results of Heideman and Hogan [11], who considered odd order recurrences of the form

$$x_{n+2k+1}x_n = x_{n+2k}x_{n+1} + x_{n+k} + x_{n+k+1}, \quad (1.5)$$

and showed that an integer sequence is generated for initial data $x_0 = x_1 = \dots = x_{2k} = 1$. Their argument is based on the fact that each of these integer sequences also satisfies a linear recurrence relation, which can be restated thus:

PROPOSITION 1.1 ([11, Lemma 2]). *For each integer $k \geq 1$, the iterates of recurrence (1.5) with the initial data $x_i = 1$, $i = 0, \dots, 2k$, satisfy the homogeneous linear relation*

$$x_{n+6k} - (2k^2 + 8k + 4)(x_{n+4k} - x_{n+2k}) - x_n = 0 \quad \forall n \in \mathbb{Z}. \quad (1.6)$$

When $N = 3$ the recurrence (1.4) coincides with (1.5) for $k = 1$, and is a special case of a more general third order recurrence that is shown to be linearisable in [13].

In the case of (1.4) it turns out that the integer sequence generated by choosing all N data to take the value 1 is related to the Fibonacci numbers, which we denote by f_n (with the convention that $f_0 = 0, f_1 = 1$). The analogue of Proposition 1.1 is as follows.

PROPOSITION 1.2. *For each integer $N \geq 2$, the iterates of recurrence (1.4) with the initial data $x_i = 1, i = 0, \dots, p = N - 1$, satisfy the homogeneous linear relation*

$$x_{n+3p} - (f_{2p+2} + pf_{2p} - f_{2p-2} + 1)(x_{n+2p} - x_{n+p}) - x_n = 0 \quad \forall n \in \mathbb{Z}. \quad (1.7)$$

1.1. Outline of the paper

Although (1.4) cannot be obtained from a cluster algebra, in the next section we briefly explain how it fits into the broader framework of Laurent phenomenon algebras (LP algebras), introduced in [16], which immediately shows that the Laurent property holds. In the third section we present Theorem 3.3, our main result on the linearisability of (1.4); in addition to a linear relation with coefficients that are first integrals (i.e. conserved quantities like C above), the iterates also satisfy a linear relation with periodic coefficients. Although the result essentially follows by adapting methods from [8], the complete proof involves the detailed structure of monodromy matrices associated with an integrable Hamiltonian system, namely the periodic dressing chain of [19]; this connection is explained in section 4. Section 5 contains our conclusions, while the properties of the particular family of integer sequences in Proposition 1.2 are reserved for an appendix.

2. Laurent phenomenon algebras

A seed in a cluster algebra is a pair (\mathbf{x}, B) , where \mathbf{x} is a cluster and B is a skew-symmetrizable integer matrix; for each index j , an associated exchange polynomial is given by the right hand side of (1.3). In the setting of LP algebras, introduced recently by Lam and Pylyavskyy [16], the exchange polynomials are allowed to be irreducible polynomials with an arbitrary number of terms, rather than just binomials. The Laurent phenomenon for certain recurrences of the type (1.1) with non-binomial F was already observed in Gale’s article [9], with Fomin and Zelevinsky’s Caterpillar Lemma eventually providing a means to prove the Laurent property in this more general setting [5]. The axioms of an LP algebra, which we briefly describe below, guarantee that mutations of an initial seed always produce Laurent polynomials in the initial cluster variables.

A seed in an LP algebra of rank N is a pair of N -tuples $t = (\mathbf{x}, \mathbf{F})$, where \mathbf{x} is a cluster, and the entries of \mathbf{F} are exchange polynomials $F_j \in \mathcal{P} := S[x_1, \dots, x_N]$, where S is a coefficient ring. The exchange polynomials must satisfy the conditions that (1) F_j is irreducible in \mathcal{P} and not divisible by any variable x_i , and (2) F_j does not involve the variable x_j , for $j = 1, \dots, N$. A set of exchange Laurent polynomials $\{\hat{F}_1, \dots, \hat{F}_N\}$ is also defined such that, for all j , (i) there are $a_i \in \mathbb{Z}_{\leq 0}$ for each $i \neq j$ with $\hat{F}_j = F_j \prod_{i \neq j} x_i^{a_i}$, and (ii) $\hat{F}_i|_{x_j \leftarrow F_j/x} \in S[x_1^{\pm 1}, \dots, x_{j-1}^{\pm 1}, x^{\pm 1}, x_{j+1}^{\pm 1}, \dots, x_N^{\pm 1}]$ and is not divisible by F_j . Mutation in direction j gives $(\mathbf{x}, \mathbf{F}) \rightarrow (\mathbf{x}', \mathbf{F}')$, where the cluster variables of the new seed are given by $x'_k = x_k$ for $k \neq j$ and $x'_j = \hat{F}_j/x_j$. For the new exchange polynomials, set

$$G_k = F_k \Big|_{x_j \leftarrow \frac{\hat{F}_j|_{x_k \leftarrow 0}}{x'_j}}$$

and define H_k by removing all common factors with $\hat{F}_j|_{x_k \leftarrow 0}$ from G_k ; then $F'_k = MH_k$ where M is a Laurent monomial in the new cluster variables. Lam and Pylyavskyy proved in [16, Theorem 5.1] that the Laurent phenomenon holds for LP algebras, which means that cluster variables obtained by arbitrary sequences of mutations belong to $S[x_1^{\pm 1}, \dots, x_N^{\pm 1}]$.

To see how each of the recurrences (1.4) fits into the LP algebra framework, we take the coefficient ring $S = \mathbb{Z}[a]$. The initial seed is then given by $t_1 = \{(x_1, F_1), \dots, (x_N, F_N)\}$, with the irreducible exchange polynomials

$$F_1 = x_N x_2 + a \sum_{i=2}^N x_i, \quad F_N = x_{N-1} x_1 + a \sum_{i=1}^{N-1} x_i, \quad F_k = a + x_{k-1} + x_{k+1} \quad \text{otherwise.}$$

Mutating t_1 in direction 1 gives a new seed $t_2 = \{(x_{N+1}, F'_1), (x_2, F'_2), \dots, (x_N, F'_N)\}$, containing one new cluster variable $x'_1 = x_{N+1}$, where

$$F'_1 = F_1, \quad F'_2 = x_{N+1} x_3 + a \sum_{i=3}^{N+1} x_i, \quad F'_N = a + x_{N-1} + x_{N+1}, \quad F'_k = F_k \quad \text{otherwise.}$$

The seeds t_1 and t_2 are similar to each other, in the sense of 3.4 in [16] (see also section 7 therein): they are transformed one to the other by $(x_1, x_2, \dots, x_N) \rightarrow (x_2, x_3, \dots, x_{N+1})$, corresponding to a single iteration of (1.4), and successive iterations are generated by mutating t_2 in direction 2, and so forth. For the rest of the paper, it will be more convenient to take x_0, x_1, \dots, x_{N-1} as initial data, and introduce the ring of Laurent polynomials

$$\mathcal{R}_N := \mathbb{Z}[a, x_0^{\pm 1}, \dots, x_{N-1}^{\pm 1}],$$

in order to make the following statement.

THEOREM 2.1. *For each N the Laurent property holds for (1.4), i.e. $x_n \in \mathcal{R}_N \quad \forall n \in \mathbb{Z}$.*

Working in the ambient field of fractions $\mathbb{Q}(a, x_0, \dots, x_{N-1})$, it is also worth noting that, from the form of (1.4), the iterates x_n can be written as subtraction-free rational expressions in a and the initial data, and hence are all non-vanishing.

3. Linearisability

In order to show that the iterates of (1.4) satisfy linear relations, we begin by noting that the nonlinear recurrence can be rewritten using a 2×2 determinant as

$$D_n = a \sum_{i=1}^{N-1} x_{n+i}, \quad \text{where} \quad D_n := \begin{vmatrix} x_n & x_{n+N-1} \\ x_{n+1} & x_{n+N} \end{vmatrix}, \quad (3.1)$$

and proceed to consider the 3×3 matrix

$$\Psi_n = \begin{pmatrix} x_n & x_{n+N-1} & x_{n+2N-2} \\ x_{n+1} & x_{n+N} & x_{n+2N-1} \\ x_{n+2} & x_{n+N+1} & x_{n+2N} \end{pmatrix}. \quad (3.2)$$

Iterating (1.4) with initial data $(x_0, x_1, \dots, x_{N-1})$ is equivalent to iterating the birational map

$$\varphi : \quad (x_0, x_1, \dots, x_{N-1}) \mapsto \left(x_1, x_2, \dots, \left(x_{N-1} x_1 + a \sum_{i=1}^{N-1} x_i \right) / x_0 \right). \quad (3.3)$$

The key to obtaining linear relations for the variables x_n is the observation that the determinant of the matrix Ψ_n is invariant under $n \rightarrow n + 1$, meaning that it provides a conserved quantity for the map φ .

LEMMA 3.1. *The determinant of the matrix Ψ_n is invariant under the map φ defined by the recurrence (1.4). In other words, if it is written in terms of the initial data as*

$$\det \Psi_n = \mu(x_0, x_1, \dots, x_{N-1}), \quad (3.4)$$

then the function μ is a first integral, i.e. it satisfies $\varphi^* \mu = \mu$.

Proof. To begin with, we note that the identity

$$D_{n+1} - D_n = a(x_{n+N} - x_{n+1}) \quad (3.5)$$

follows from (3.1). We wish to show that $\det \Psi_{n+1} = \det \Psi_n$ for all n . However, since all formulae remain valid under shifting each of the indices by an arbitrary amount, it is sufficient to perform the calculation with $n = 0$ for ease of notation. Using the method of Dodgson condensation [3] to expand the 3×3 determinant in terms of 2×2 minors, we calculate

$$\begin{aligned} \det \Psi_0 &= \frac{1}{x_N} \begin{vmatrix} D_0 & D_{N-1} \\ D_1 & D_N \end{vmatrix} = \frac{1}{x_N} \begin{vmatrix} D_1 + a(x_1 - x_N) & D_N + a(x_N - x_{2N-1}) \\ D_1 & D_N \end{vmatrix} \\ &= \frac{a}{x_N} (D_N(x_1 - x_N) - D_1(x_N - x_{2N-1})), \end{aligned}$$

where we have used (3.5). Similarly, by using (3.5) again, we have

$$\begin{aligned} \det \Psi_1 &= \frac{1}{x_{N+1}} \begin{vmatrix} D_1 & D_N \\ D_2 & D_{N+1} \end{vmatrix} = \frac{1}{x_N} \begin{vmatrix} D_1 & D_N \\ D_1 + a(x_{N+1} - x_2) & D_N + a(x_{2N} - x_{N+1}) \end{vmatrix} \\ &= \frac{a}{x_N} (D_1(x_{2N} - x_{N+1}) - D_N(x_{N+1} - x_2)). \end{aligned}$$

Taking the difference and multiplying by a common denominator gives

$$\begin{aligned} \frac{x_N x_{N+1}}{a} (\det \Psi_1 - \det \Psi_0) &= D_N (-x_N(x_{N+1} - x_2) - x_{N+1}(x_1 - x_N)) \\ &\quad + D_1 (x_N(x_{2N} - x_{N+1}) + x_{N+1}(x_N - x_{2N-1})) \\ &= -D_N D_1 + D_1 D_N = 0, \end{aligned}$$

as required. Hence the determinant of the 3×3 matrix Ψ_n is a conserved quantity (independent of n). Starting from the matrix (3.2) with $n = 0$, $\det \Psi_0$ can be rewritten as a rational function of the initial values x_0, \dots, x_{N-1} , denoted μ as in (3.4), by repeatedly using (1.4) to express x_{2N} as a rational function of terms of lower index, then x_{2N-1} , etc. By construction, the pullback of this function satisfies $\varphi^* \mu = \mu \cdot \varphi = \mu$, so μ is a first integral for the map (3.3). \square

REMARK 1. The first integral μ is a rational function, or more precisely a Laurent polynomial, in terms of the variables x_0, \dots, x_{N-1} . To verify that it is not identically zero, it is enough to check that one specific choice of initial values gives a non-vanishing value of μ . In particular, for the sequence beginning with all N initial values equal to 1, the value of μ can be calculated by using the formulae (6.1) found in the Appendix, which gives $\mu(1, 1, \dots, 1) = \mathfrak{f}_{2p} p^2 \neq 0$ (where $p = N - 1$).

COROLLARY 3.2. *If the sequence (x_n) satisfies (1.4), then $\det \hat{\Psi}_n = 0$, where*

$$\hat{\Psi}_n = \begin{pmatrix} x_n & x_{n+N-1} & x_{n+2N-2} & x_{n+3N-3} \\ x_{n+1} & x_{n+N} & x_{n+2N-1} & x_{n+3N-2} \\ x_{n+2} & x_{n+N+1} & x_{n+2N} & x_{n+3N-1} \\ x_{n+3} & x_{n+N+2} & x_{n+2N+1} & x_{n+3N} \end{pmatrix}. \quad (3.6)$$

Proof. Note that from (3.1) it follows that D_n is a subtraction-free rational expression in the initial data, hence is non-vanishing for all n . Applying Dodgson condensation again yields

$$\det \hat{\Psi}_n = \frac{\det \Psi_n \det \Psi_{n+N} - \det \Psi_{n+1} \det \Psi_{n+N-1}}{D_{n+N}} = 0,$$

since, by Lemma 3.1, $\det \Psi_n$ is independent of n . \square

The fact that the 4×4 matrix in (3.6) has a non-trivial kernel is enough to show that sequences (x_n) generated by (1.4) also satisfy linear recurrence relations. There are two types of relation, corresponding to the kernel of $\hat{\Psi}_n$, and to the kernel of its transpose, respectively. In order to describe these relations more explicitly, it is helpful to introduce the functions

$$J_n = \frac{x_{n+2} + x_n + a}{x_{n+1}}, \quad (3.7)$$

which turn out to be periodic with respect to n .

THEOREM 3.3. *The iterates of the recurrence (1.4) satisfy the pair of homogeneous linear relations*

$$x_{n+3N-3} - K(x_{n+2N-2} - x_{n+N-1}) - x_n = 0, \quad (3.8)$$

$$x_{n+3} - (1 + J_{n+1})x_{n+2} + (1 + J_n)x_{n+1} - x_n = 0, \quad (3.9)$$

where K is a first integral (independent of n), and the coefficients of (3.9) are periodic, such that $J_{n+N-1} = J_n$ for all n .

Proof. From Corollary 3.2, the existence of the two linear relations follows by arguments which are almost identical to those used for the proof of Lemma 5.1 in [8]. We first sketch the essential argument for (3.8), before discussing further details of the coefficients in each relation.

Without loss of generality, a non-zero vector \mathbf{w}_n in the kernel of $\hat{\Psi}_n$ can be written in normalised form as $\mathbf{w}_n = (A_n, B_n, C_n, -1)^T$. Indeed, the last entry cannot vanish, for otherwise there would be a non-trivial kernel for the 3×3 matrix Ψ_n given by (3.2), contradicting the remark that μ is not identically zero. The first three rows of the equation $\hat{\Psi}_n \mathbf{w}_n = 0$ produce the linear system

$$\Psi_n \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix} = \begin{pmatrix} x_{n+3N-3} \\ x_{n+3N-2} \\ x_{n+3N-1} \end{pmatrix}, \quad (3.10)$$

while the last three rows of the same equation give

$$\Psi_{n+1} \begin{pmatrix} A_n \\ B_n \\ C_n \end{pmatrix} = \begin{pmatrix} x_{n+3N-2} \\ x_{n+3N-1} \\ x_{n+3N} \end{pmatrix}. \quad (3.11)$$

Each of the systems (3.10) and (3.11) can be solved for the vector $(A_n, B_n, C_n)^T$, and the two different expressions that result differ by a shift $n \rightarrow n + 1$, which implies that this vector must be independent of n . Upon applying Cramer's rule to the system (3.10), the first component can be simplified as $A_n = \det \Psi_{n+N-1} / \det \Psi_n = 1$, by Lemma 3.1. Supposing that they are not trivially constant, the other two components are first integrals, so we may set $K^{(1)} = C_n$ and $K^{(2)} = -B_n$, and then the equation for \mathbf{w}_n gives the relation

$$x_{n+3N-3} - K^{(1)}x_{n+2N-2} + K^{(2)}x_{n+N-1} - x_n = 0, \quad (3.12)$$

where the coefficients are all independent of n .

By considering the kernel of $\hat{\Psi}_n^T$, an almost identical argument shows that (x_n) also satisfies a four-term linear relation with coefficients that are periodic with period $N - 1$. However, this four-term relation can be obtained more directly in another way. Upon setting $E_n := x_{n+N}x_n - x_{n+N-1}x_{n+1} - a(x_{n+1} + \dots + x_{n+N-1})$, which vanishes for all n whenever (1.4) holds, taking the forward difference $E_{n+1} - E_n = 0$ and rearranging yields

$$\frac{x_{n+N+1} + x_{n+N-1} + a}{x_{n+N}} = \frac{x_{n+2} + x_n + a}{x_{n+1}}. \tag{3.13}$$

Comparison of the above identity with (3.7) reveals that the sequence of quantities J_n is periodic with period $p = N - 1$: the left hand side of (3.13) is J_{n+p} , and the right hand side is J_n . Now rearranging (3.7) clearly gives an inhomogeneous linear equation, namely

$$x_{n+2} - J_n x_{n+1} + x_n + a = 0. \tag{3.14}$$

The forward difference operator applied to the latter equation produces (3.9). □

As we shall see, the coefficients K and J_n are themselves Laurent polynomials in the initial data, meaning that the each of the linear relations (3.12) and (3.9) offers an alternative proof of Theorem 2.1. The periodicity of the quantities (3.7) immediately implies the following.

COROLLARY 3.4. *For all $n \in \mathbb{Z}$ the iterates of (1.4) satisfy $x_n \in \mathbb{Z}[a, x_0, x_1, J_0, J_1, \dots, J_{N-2}]$.*

Note that the latter result gives a stronger version of the Laurent property for the nonlinear recurrence (1.4). Indeed, it is clear from the formula (3.7) that $J_n \in \mathcal{R}_N$ for $n = 0, \dots, N - 3$, while (using (1.4) to substitute for x_N) the same formula for $n = N - 2$ gives

$$J_{N-2} = \frac{x_{N-2}}{x_{N-1}} + \frac{x_1}{x_0} + \frac{a}{x_{N-1}x_0} \sum_{i=0}^{N-1} x_i \in \mathcal{R}_N.$$

Observe that the foregoing proof of Theorem 3.3 is not yet complete: so far we have not shown that the coefficients $K^{(1)}$ and $K^{(2)}$ in (3.12) coincide, which is required for the relation (3.8) to hold. Cramer’s rule applied to the linear system (3.10) produces formulae for these coefficients as ratios of 3×3 determinants, but it is not apparent from these formulae why $K^{(1)} = K^{(2)} = K$. An explanation for this coincidence is postponed until the next section, but first we present an example.

EXAMPLE 3.5. *For $N = 3$ the nonlinear recurrence (1.4) is*

$$x_{n+3}x_n = x_{n+2}x_{n+1} + a(x_{n+2} + x_{n+1}), \tag{3.15}$$

which is the same as Heideman and Hogan’s recurrence (1.5) for $k = 1$ and $a = 1$, and is also a special case of the third order recurrence considered in section 5 of [13]. In accordance with (3.12), the iterates of (3.15) satisfy $x_{n+6} - K(x_{n+4} - x_{n+2}) - x_n = 0$, where the first integral K can be expressed as a Laurent polynomial in the initial data x_0, x_1, x_2 as follows:

$$K = 1 + \frac{x_0}{x_2} + \frac{x_2}{x_0} + a \left(\frac{x_0}{x_1x_2} + \frac{x_2}{x_0x_1} + 2 \sum_{i=0}^2 \frac{1}{x_i} \right) + a^2 \left(\frac{1}{x_0x_1} + \frac{1}{x_1x_2} + \frac{1}{x_2x_0} \right). \tag{3.16}$$

This agrees with the result of Theorem 5.1 in [13], and with the particular integer sequence considered in [11]: putting the initial data $x_0 = x_1 = x_2 = 1$ with $a = 1$ into (3.16) gives the constant value $K = 14$, in accordance with Proposition 1.1 for $k = 1$.

The iterates of (3.15) also satisfy the homogeneous relation (3.9) with period 2 coefficients

$$J_0 = \frac{x_2 + x_0 + a}{x_1}, \quad J_1 = \frac{x_1 x_0 + x_2 x_1 + a x_0 + a x_1 + a x_2}{x_2 x_0}. \quad (3.17)$$

4. Monodromy and the dressing chain

Any function of the periodic quantities J_0, J_1, \dots, J_{N-2} that is invariant under cyclic permutations provides a first integral for the N -dimensional map (3.3) corresponding to (1.4). The $N - 1$ quantities J_i are independent functions of x_0, x_1, \dots, x_{N-1} , so a choice of $N - 1$ independent cyclically symmetric functions of these J_i gives the maximum number of independent first integrals for such a map in N dimensions, as long as it is not purely periodic. Hence the coefficients in (3.12) should be functions of the J_i , and writing them explicitly as such will allow us to show that $K^{(1)} = K^{(2)}$, by applying monodromy arguments.

4.1. 3×3 monodromy matrices

From the linear relation (3.9), the matrix Ψ_n satisfies

$$\Psi_{n+1} = \mathbf{L}_n \Psi_n, \quad \text{where} \quad \mathbf{L}_n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 - J_n & 1 + J_{n+1} \end{pmatrix}. \quad (4.1)$$

Shifting the indices $N - 2$ times in the above linear equation for Ψ_n gives the monodromy over the period of the coefficients, such that

$$\Psi_{n+p} = \mathbf{M}_n \Psi_n \quad \text{with} \quad \mathbf{M}_n = \mathbf{L}_{n+N-2} \mathbf{L}_{n+N-3} \cdots \mathbf{L}_n,$$

where $p = N - 1$ is the period. From the second linear relation (3.12), the matrix Ψ_n also satisfies

$$\Psi_{n+p} = \Psi_n \hat{\mathbf{L}}, \quad \text{where} \quad \hat{\mathbf{L}} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -K^{(2)} \\ 0 & 1 & K^{(1)} \end{pmatrix}. \quad (4.2)$$

By Remark 1, Ψ_n is invertible, so we can rewrite these two monodromy equations as $\mathbf{M}_n = \Psi_{n+p} \Psi_n^{-1}$ and $\hat{\mathbf{L}} = \Psi_n^{-1} \Psi_{n+p}$, and then taking the trace of each yields

$$K^{(1)} = \text{tr} \hat{\mathbf{L}} = \text{tr} \mathbf{M}_n \quad \text{for all } n. \quad (4.3)$$

Thus we can write the invariant $K^{(1)}$ in terms of the periodic functions J_i . Similarly, from $\mathbf{M}_n^{-1} = \Psi_n \Psi_{n+p}^{-1}$ and $\hat{\mathbf{L}}^{-1} = \Psi_{n+p}^{-1} \Psi_n$, we also have

$$K^{(2)} = \text{tr} \hat{\mathbf{L}}^{-1} = \text{tr} \mathbf{M}_n^{-1} \quad \text{for all } n. \quad (4.4)$$

However, it is still not obvious from the form of the monodromy matrix \mathbf{M}_n that $\text{tr} \mathbf{M}_n = \text{tr} \mathbf{M}_n^{-1}$, which is required for the two non-trivial coefficients in (3.12) to be the same. Before considering the general case, we give a couple of examples.

EXAMPLE 4.1. For the third order case (3.15), as in Example 3.5, we have the monodromy matrix $\mathbf{M}_0 = \mathbf{L}_1 \mathbf{L}_0$, where the matrix in (4.1) has period 2: $\mathbf{L}_{n+2} = \mathbf{L}_n$ for all n . By taking traces it can be verified directly that $K^{(1)} = K^{(2)} = K$, where

$$K = \text{tr} \mathbf{M}_0 = \text{tr} \mathbf{M}_0^{-1} = J_0 J_1 - 1. \quad (4.5)$$

The above identity, expressing K in terms of J_0 and J_1 , can also be checked by comparing the Laurent polynomials in (3.16) and (3.17).

EXAMPLE 4.2. For $N = 4$ the nonlinear recurrence (1.4) becomes

$$x_{n+4}x_n = x_{n+3}x_{n+1} + a(x_{n+1} + x_{n+2} + x_{n+3}).$$

The functions defined by (3.7), which appear as entries in (4.1), have period 3. They are

$$J_0 = \frac{x_0 + x_2 + a}{x_1}, \quad J_1 = \frac{x_1 + x_3 + a}{x_2}, \quad J_2 = \frac{x_0x_2 + x_1x_3 + a(x_0 + x_1 + x_2 + x_3)}{x_0x_3},$$

with $J_{n+3} = J_n$ for all n . Upon taking the trace of monodromy matrix $\mathbf{M}_0 = \mathbf{L}_2\mathbf{L}_1\mathbf{L}_0$, and that of its inverse, it follows from (4.3) and (4.4) that $K^{(1)} = K^{(2)} = K$, where

$$K = J_0J_1J_2 - (J_0 + J_1 + J_2) + 1 \tag{4.6}$$

is the explicit formula for K as a symmetric function of J_0, J_1, J_2 .

4.2. Connection with the dressing chain

A one-dimensional Schrödinger operator can be factorised as $L = -(\partial + f)(\partial - f)$, where ∂ denotes the operator of differentiation with respect to the independent variable, z say, and $f = f(z)$. The dressing chain, as described in [19], is the set of ordinary differential equations

$$(f_i + f_{i+1})' = f_i^2 - f_{i+1}^2 + \alpha_i, \tag{4.7}$$

where α_i are constant parameters, and the prime denotes the z derivative. These equations arise from successive Darboux transformations $L_i \rightarrow L_{i+1}$ that are used to generate a sequence of Schrödinger operators (L_i); the nature of the Darboux transformation is such that each operator is obtained from the previous one by a reordering of factors and a constant shift:

$$L_i = -(\partial + f_i)(\partial - f_i) \longrightarrow L_{i+1} = -(\partial - f_i)(\partial + f_i) + \alpha_i = -(\partial + f_{i+1})(\partial - f_{i+1}).$$

Of particular interest is the periodic case, where $L_{i+p} = L_i$ for all i . In that case, all indices in (4.7) should be read mod p , and this becomes a finite-dimensional system for the f_i . As noted in [19], the properties of this system depend sensitively on the parameters α_i : when $\sum_i \alpha_i \neq 0$ the general solutions correspond to Painlevé transcendents or their higher order analogues (see [18] and references for further details), while for $\sum_i \alpha_i = 0$ the solutions are of an algebro-geometric nature, corresponding to finite gap solutions of the KdV equation; the latter connection goes back to results of Weiss [20].

Here we are concerned with the case $\sum_i \alpha_i = 0$ only, so following [19] we use parameters β_i such that $\alpha_i = \beta_i - \beta_{i+1} \pmod p$. For p odd there is an invertible transformation f_i to new coordinates, that is

$$J_i = f_i + f_{i+1}, \quad i = 1, \dots, p, \tag{4.8}$$

and in [19] (where the coordinates J_i are denoted g_i) it was shown that in this case the periodic dressing chain is a bi-Hamiltonian integrable system, meaning that it has a pencil of compatible Poisson brackets together with the appropriate number of Poisson-commuting first integrals to satisfy the requirements of Liouville’s theorem. (For p odd the transformation (4.8) is not invertible, but there is a degenerate Poisson bracket for the dressing chain such that it is an integrable system on a generic symplectic leaf.) A recent development was the observation in [8] that (with all $\beta_i = 0$) a combination of these compatible brackets for the dressing chain coordinates J_i arises by reduction from the log-canonical Poisson structure for the cluster variables in cluster algebras associated with affine A -type Dynkin quivers. In this context, a further observation was that the linear relations between cluster variables found in [6] (see also [1, 7, 15]), which are of the form

$$x_{n+2p} - \kappa x_{n+p} + x_n = 0, \tag{4.9}$$

have a non-trivial coefficient κ which is the generating function for the first integrals of the dressing chain (expressed in terms of suitable variables J_i). The key to understanding these observations is the following result, which is omitted from the published version of [8] but appeared in the original preprint.

LEMMA 4.3. *If a 2×2 monodromy matrix is defined by*

$$\mathbf{M}^* = \begin{pmatrix} 0 & \zeta_1 \\ 1 & J_1 \end{pmatrix} \begin{pmatrix} 0 & \zeta_2 \\ 1 & J_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & \zeta_p \\ 1 & J_p \end{pmatrix}, \quad (4.10)$$

then (with indices read mod p) the trace is given by the explicit formula

$$\mathrm{tr} \mathbf{M}^* = \prod_{i=1}^p \left(1 + \zeta_i \frac{\partial^2}{\partial J_i \partial J_{i+1}} \right) \prod_{n=1}^p J_n.$$

Proof. This is very similar to the proof of Theorem 2 in [19], where a Lax equation for the dressing chain is given for a different monodromy matrix, written in terms of the variables f_i . The above identity for the trace of \mathbf{M}^* follows by calculating the coefficient of each monomial in the ζ_j on either side of the equation, using the relations

$$\begin{pmatrix} 0 & 0 \\ 1 & J_1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & J_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 0 \\ 1 & J_\ell \end{pmatrix} = J_1 J_2 \cdots J_{\ell-1} \begin{pmatrix} 0 & 0 \\ 1 & J_\ell \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 1 & J_j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & J_k \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & J_k \end{pmatrix},$$

and noting that in the latter relation the central matrix on the left is nilpotent. \square

REMARK 2. For the period p dressing chain (4.7) with $\alpha_i = \beta_i - \beta_{i+1}$, the Poisson-commuting first integrals are found in terms of the variables J_i by setting $\zeta_i = \beta_i - \lambda$ in (4.10) and expanding $\mathrm{tr} \mathbf{M}^*$ in powers of the spectral parameter λ .

The 2×2 monodromy matrix (4.10) also provides the key to understanding the detailed structure of the linear relation (3.8) associated with (1.4).

THEOREM 4.4. *In terms of the quantities J_n defined in (3.7), the coefficient K in (3.8) is given by the formula*

$$K = 1 + \prod_{i=1}^p \left(1 - \frac{\partial^2}{\partial J_i \partial J_{i+1}} \right) \prod_{n=1}^p J_n. \quad (4.11)$$

Proof. Upon introducing the 2×2 matrices

$$\Phi_n = \begin{pmatrix} x_n & x_{n+1} \\ x_{n+p} & x_{n+p+1} \end{pmatrix}, \quad \mathbf{L}_n^* = \begin{pmatrix} 0 & -1 \\ 1 & J_1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

the inhomogeneous linear equation (3.14) implies that

$$\Phi_{n+1} = \Phi_n \mathbf{L}_n^* - a \mathbf{C}. \quad (4.12)$$

Seeking an appropriate inhomogeneous counterpart of (3.8), we define the matrix

$$\mathbf{M}_n^* = \mathbf{L}_n^* \mathbf{L}_{n+1}^* \cdots \mathbf{L}_{n+p-1}^*,$$

which (up to cyclic permutations) is a special case of the monodromy matrix \mathbf{M}^* above, obtained from (4.10) by setting $\zeta_i = -1$ for all i . Now, by applying a shift to (4.12), we have

$$\Phi_{n+2} = (\Phi_n \mathbf{L}_n^* - a\mathbf{C})\mathbf{L}_{n+1}^* - a\mathbf{C} = \Phi_n \mathbf{L}_n^* \mathbf{L}_{n+1}^* - a\mathbf{C}(\mathbf{L}_{n+1}^* + \mathbf{I})$$

(with \mathbf{I} being the 2×2 identity matrix), and so by induction, after a total of $p - 1$ shifts, we find that

$$\Phi_{n+p} = \Phi_n \mathbf{M}_n^* - a\mathbf{C}\mathbf{K}_n, \quad (4.13)$$

where

$$\mathbf{K}_n = \mathbf{L}_{n+1}^* \cdots \mathbf{L}_{n+p-1}^* + \mathbf{L}_{n+2}^* \cdots \mathbf{L}_{n+p-1}^* + \cdots + \mathbf{L}_{n+p-1}^* + \mathbf{I}.$$

Note that the entries of \mathbf{L}_n^* , and hence those of \mathbf{M}_n^* and \mathbf{K}_n , are all periodic with period p , so from (4.13) it follows that

$$\Phi_{n+2p} = \Phi_{n+p} \mathbf{M}_n^* - a\mathbf{C}\mathbf{K}_n = \Phi_n (\mathbf{M}_n^*)^2 - a\mathbf{C}\mathbf{K}_n (\mathbf{M}_n^* + \mathbf{I}).$$

The latter equation can be further simplified with the use of the Cayley-Hamilton theorem, which gives

$$(\mathbf{M}_n^*)^2 - \kappa \mathbf{M}_n^* + \mathbf{I} = 0, \quad \text{with } \kappa = \text{tr } \mathbf{M}_n^*,$$

and hence (after using (4.13) once again) we find

$$\Phi_{n+2p} = \kappa \Phi_{n+p} - \Phi_n - a\mathbf{C}\mathbf{K}_n (\mathbf{M}_n^* + (1 - \kappa)\mathbf{I}).$$

The (1, 1) entry of the above matrix equation yields an inhomogeneous version of (4.9), namely

$$x_{n+2p} - \kappa x_{n+p} + x_n + a\tilde{J}_n = 0, \quad (4.14)$$

where the quantity \tilde{J}_n is periodic with period p . We claim that this is the desired inhomogeneous counterpart of (3.8). Indeed, applying the shift $n \rightarrow n + p$ to (4.14) and then subtracting the original equation gives

$$x_{n+3p} - (\kappa + 1)(x_{n+2p} - x_{n+p}) - x_n = 0,$$

which is (3.8) with $K = \kappa + 1$, and the formula (4.11) then follows from Lemma 4.3. This also verifies that $K^{(1)} = K^{(2)}$ in (3.12), completing the missing step in the proof of Theorem 3.3. \square

REMARK 3. Observe that 1 is a root of the characteristic polynomial of the linear relation (3.8), which means that this homogeneous equation is a total difference. Hence, using \mathcal{S} to denote the shift operator, we obtain the relation

$$(\mathcal{S}^{3p-1} + \cdots + \mathcal{S}^{2p} - \kappa(\mathcal{S}^{2p-1} + \cdots + \mathcal{S}^p) + \mathcal{S}^{p-1} + \cdots + \mathcal{S} + 1) x_n + a\tilde{K} = 0,$$

where \tilde{K} is a first integral. In the case $p = N - 1 = 2$, a formula for \tilde{K} as a Laurent polynomial in the initial data can be found in [13] (see Theorem 5.1 therein). Although we do not have a general formula for \tilde{K} , analogous to (4.11), direct calculation shows that $\tilde{K} = J_0 + J_1 + 4$ for $p = 2$, and $\tilde{K} = J_0 J_1 + J_1 J_2 + J_2 J_0 + 2(J_0 + J_1 + J_2) + 3$ for $p = 3$. Moreover, in the cases $p = 2, 3, 4, 5$ we have verified with computer algebra that \tilde{K} is given in terms of the 3×3 determinant (3.4) by

$$\tilde{K} = \frac{\mu}{a^3},$$

and we conjecture that this is always so.

REMARK 4. The form of the inhomogeneous relation (4.14) means that the solution of the initial value problem for (1.4) can be written explicitly using Chebyshev polynomials of the

first and second kind with argument $\kappa/2$, that is

$$T_j(\cos \theta) = \cos n\theta, \quad U_j(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}, \quad \text{where } \kappa = 2 \cos \theta.$$

For details of the solution in the case $N = 3$, see Proposition 5.2 in [13].

5. Conclusions

We have shown that each of the nonlinear recurrences (1.4) fits into the framework of Lam and Pylyavskyy's LP algebras, implying that the Laurent property holds. Furthermore, each recurrence is linearisable in two different ways: with a constant-coefficient homogeneous linear relation, and with another such relation with periodic coefficients. The connection between the first integral K in (3.8) and the Poisson-commuting first integrals of the dressing chain is intriguing: it raises the question of whether LP algebras might admit natural Poisson and/or presymplectic structures, analogous to those for cluster algebras [10].

The recurrences (1.5) considered by Heideman and Hogan also possess the Laurent property. (See [12] for a more detailed discussion.) It can also be shown that, beyond the linear relation (1.6) for a particular integer sequence, the result of Proposition 1.1 extends to arbitrary initial data. In this case there is also an analogue of (3.9), with periodic coefficients, but with a more complicated structure. These results will be presented elsewhere.

6. Appendix

Here we consider the particular family of integer sequences that is generated by iterating each recurrence (1.4) with all N initial data set to the value 1, and prove the following result.

THEOREM 6.1. *For each $N \geq 2$, the recurrence (1.4) with initial values $x_n = 1$ for $n = 0, \dots, p = N - 1$ generates a sequence of integers (x_n) which is symmetric in the sense that $x_n = x_{p-n}$ for all $n \in \mathbb{Z}$. The first few terms are given by*

$$x_{p+j} = f_{2j} p + 1, \quad x_{2p+j} = f_{2j} f_{2p} p^2 + \left(f_{2j} (f_{2p+2} + 1) - f_{2j-2} f_{2p} \right) p + 1, \quad j = 0, \dots, p. \quad (6.1)$$

In general, for each $k \geq 0$ the terms x_{kp+j} for $1 \leq j \leq p$ can be written as polynomials in p whose coefficients are themselves polynomials in Fibonacci numbers with even index.

First of all, to see why $x_n = x_{p-n}$, observe that the nonlinear recurrence (1.4) has two obvious symmetries: translating n , and replacing n by $-n$. Combining these two symmetries, it follows that the sequence (x_{p-n}) satisfies (1.4) whenever the sequence (x_n) does; the fact that the initial data $1, 1, \dots, 1$ is symmetric under $n \rightarrow p - n$ implies that these two sequences coincide.

The rest of the proof proceeds inductively, and relies on various identities for Fibonacci numbers; note that the convention $f_0 = 0$, $f_1 = 1$ is taken for the initial Fibonacci numbers. From the Fibonacci recurrence $f_{n+2} = f_{n+1} + f_n$ it follows that the even index terms satisfy

$$f_{2n+4} = 3f_{2n+2} - f_{2n}. \quad (6.2)$$

For $n = 0, \dots, p - 1$ the relation (1.4) is satisfied by the initial data and the first set of terms in (6.1) by virtue of the identity

$$f_{2n+2} - f_{2n} = f_{2n} + f_{2n-2} + \dots + f_2 + 1, \quad (6.3)$$

which follows from (6.2) by induction. To verify the formula for the second set of terms in (6.1), it suffices to substitute these expressions into (1.4) with $n = p, \dots, 2p - 1$ and compare

coefficients of powers of p on both sides, making use of (6.3) again, together with the identity

$$\begin{vmatrix} f_{2j-2} & f_{2j} \\ f_{2j} & f_{2j+2} \end{vmatrix} = -1.$$

The fact that the latter 2×2 determinant is a constant (independent of j) follows from (6.2), while the value -1 is obtained immediately from $f_0 = 0$, $f_{\pm 2} = \pm 1$.

Proposition 1.2 is now seen to be a special case of Theorem 3.3, by making use of Theorem 6.1. Indeed, from (3.8) with $n = j - p$ for $1 \leq j \leq p$, the above choice of initial data and the terms in (6.1), as well as the symmetry of the sequence, yields $K = (x_{2p+j} - x_{-p+j})/(x_{p+j} - x_j) = (x_{2p+j} - x_{2p-j})/(x_{p+j} - x_j) = f_{2p}p + f_{2p+2} + 1 - (f_{2j-2}f_{2p} + f_{2p-2j})/f_{2j}$. In order to see why the latter expression is a constant (independent of j), and to obtain the particular form of the coefficient K that appears in (1.7), it is enough to verify the identity $f_{2j}f_{2p-2} - f_{2j-2}f_{2p} = f_{2p-2j}$, which is done by writing the even index Fibonacci numbers as $f_{2n} = \sinh n\theta/\sinh \theta$ with $\theta = \log\left(\frac{3+\sqrt{5}}{2}\right)$, and using the addition formula for the hyperbolic sine.

References

1. I. ASSEM AND C. REUTENAUER AND D. SMITH, Friezes, *Adv. Math.* 225, (2010) 3134-3165.
2. P. DI FRANCESCO, Discrete integrable systems, positivity and continued fraction rearrangements, *Lett. Math. Phys.* 96 (2011) 299-324.
3. C.L. DODGSON, Condensation of determinants, *Proc. R. Soc. Lond.*, 15, (1866) 150-155.
4. S. FOMIN AND A. ZELEVINSKY, Cluster algebras I: Foundations, *J. Amer. Math. Soc.* 15 (2002) 497-529.
5. S. FOMIN AND A. ZELEVINSKY, The Laurent Phenomenon, *Advances in Applied Mathematics* 28 (2002) 119-144.
6. A.P. FORDY AND R.J. MARSH, Cluster Mutation-Periodic Quivers and Associated Laurent Sequences, *J. Algebr. Comb.*, 34 (2009) 19-66.
7. A.P. FORDY, Mutation-Periodic Quivers, *Integrable Maps and Associated Poisson Algebras*, *Philos. Trans. R. Soc. Lond. Ser. A*, 369 (2010) 1264-1279.
8. A.P. FORDY AND A.N.W. HONE, Discrete integrable systems and Poisson algebras from cluster maps, *Commun. Math. Phys.* (2013) at press, [arXiv:1207.6072v2](https://arxiv.org/abs/1207.6072v2)
9. D. GALE, The strange and surprising saga of the Somos sequences, *Mathematical Intelligencer* 13 (1) (1991) 40-42.
10. M. GEKHTMAN, M. SHAPIRO AND A. VAINSHTEIN, Cluster algebras and Weil-Petersson forms, *Duke Math. J.* 127 (2005), 291-311.
11. P. HEIDEMAN AND E. HOGAN, A New Family of Somos-like Recurrences, *Elec. J. Comb.*, 15, (2008) #R54.
12. E. HOGAN, Experimental Mathematics Applied to the Study of Nonlinear Recurrences, PhD thesis, Rutgers University, 2011.
13. A.N.W. HONE, Nonlinear recurrence sequences and Laurent polynomials, *Number Theory and Polynomials* (ed. J. McKee and C. Smyth), *LMS Lecture Note Series*, 352, (2008) 188-210.
14. R. KEDEM, Q-Systems as cluster algebras, *J. Phys. A*. 41 (2008) 194011.
15. B. KELLER AND S. SCHEROTZKE, Linear recurrence relations for cluster variables of affine quivers, *Adv. Math.* 228, (2011) 1842-1862
16. T. LAM AND P. PYLYAVSKYY, Laurent phenomenon algebras, (2012), [arXiv:1206.2611v2](https://arxiv.org/abs/1206.2611v2)
17. W. NAHM AND S. KEEGAN, Integrable deformations of CFTs and the discrete Hirota equations, [arXiv:0905.3776v2](https://arxiv.org/abs/0905.3776v2)
18. K. TAKASAKI, Spectral Curve, Darboux Coordinates and Hamiltonian Structure of Periodic Dressing Chains, *Commun. Math. Phys.* 241 (2003) 111-142.
19. A.P. VESELOV AND A.B. SHABAT, Dressing chains and the spectral theory of the Schrödinger Operator, *Funct. Anal. Appl.*, 27, (1993) 81-96.
20. J. WEISS, Periodic fixed points of Bäcklund transformations and the KdV equation, *J. Math. Phys.*, 27 (1986) 2647-2656.

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