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Abstract: The relationship between asset demand and information quality in rational expectations economies is analyzed. First we derive a number of new summary descriptive statistics that measure four basic characteristics of investment style: asset selection, market timing, aggressiveness, and specialization. Then we relate these statistics to the divergence between a given investor's information structure and the market average information structure. Finally, we demonstrate that informational differentials can be identified, and consistently estimated, using OLS from the time series of observed asset demand.

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Information Quality, Performance Measurement, and Security Demand in Rational Expectations Economies

Performance Measurement is the first step in allocating investment funds and determining compensation schemes for portfolio managers. As Admati, Bhattacharya, Pfleiderer and Ross (1986) point out, the *fundamental* issue in performance measurement is to assess, using observable variables, the *quality* of private information possessed by portfolio managers. Portfolio returns relative to a benchmark return appear to be a natural choice for the observable variable to be used in this assessment. However, as shown by Dybvig and Ross (1985a, 1985b) and Admati and Ross (1985), except in very special cases, inferring the quality of private information from measured portfolio returns relative to a benchmark may generally be difficult.

Our goal is to attack the problem of performance measurement in a rational expectations setting from a different perspective. Instead of examining the relationship between excess returns and information quality, we examine the relationship between the *structure of asset demand* and information quality. Our analysis focuses on two issues: (i) *Demand analysis*–how does the structure of an investor's information regarding asset returns affect his demand for risky assets? (ii) *Structural estimation*–how can the quality of an investor's information be estimated from the observed time series of realized demands?

Demand analysis in a rational expectations economy, in essence, involves relating information to asset demand. Each investor receives a private signal regarding the payoffs from a subset (not necessarily proper) of the assets traded in the economy. The signals received about the payoffs from different assets may be correlated with each other. An investor's information structure is captured by the joint distribution of private signals. When information structures are heterogeneous, the signals received by different investors will be not be identically distributed. These differences in signal structures can lead to differences in the trading patterns, determined not so much by the absolute informational position of the investor but rather by the *informational divergence* between the precision of the signals the investor receives and the average precision of the signals received by all the investors in the economy. Absent informational divergence, investors hold portfolios that are, on average, strictly proportional to asset supplies (that is, to the market portfolio).

While informational divergence is necessary for investment styles that deviate, on average, from holding the market portfolio, the exact relationship between investment style and informational divergence is subtle. To explore this relationship, we first derive a set of summary statistics that measure the characteristics of an investor's "investment style," such as the extent of asset selection and market timing activity and the degree of portfolio specialization. Then we derive a number of conditions, some necessary and others sufficient, for increases in informational divergence to result in increased magnitudes for these characteristics.

Next, we turn to the problem of structural estimation. That is, we reverse the process discussed above and ask the following question: how can informational divergences, which are not directly observable, be estimated from observed portfolio behavior? We show such estimation is possible. The estimation technique rests on the exploitation of two salient features of our model–the linear structure of the rational-expectations demand vector and implicit restrictions that the equilibrium demand relationships place on the variance-covariance matrix of error terms from the demand estimation equation. By identifying both the matrix of demand equation coefficients and the variance-covariance matrix for the error terms, we are able to identify the informational divergence of a given investor.[1]

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The above results should be useful in addressing the problem of performance measurement in a number of ways. First, the structural decomposition of demand may be valuable as a summary statistical descriptor of the trading patterns followed by portfolio managers. Second, the comparative statics derived in the paper should shed light on how private information impacts the character of portfolio demand. Third, the estimates derived in our analysis of the identification problem should prove useful in assessing the quality of private information possessed by investors.[2]

The paper is organized as follows. In Section I, we derive the rational-expectations demand vector. In Section II, we analyze the relationship between asset demand and risk tolerance and informational differentials. In Section III, we attack the demand identification problem. Section IV concludes our analysis. Some of the more involved proofs are collected in the Appendix.

I. Rational Expectations Equilibrium in a Large Multi-Asset Economy

Our first task is to determine equilibrium-investor demand for risky assets in a large, noisy, rational-expectations economy. In this economy, investors receive private information signals about the payoff realizations of possibly different subsets of assets. These investors, indexed by j, have negative exponential utility functions over their end-of-period wealth levels, with risk aversion coefficients ρ_j. Each investor seeks to maximize the expected utility of his end-of-period wealth, which comes from the payoffs to his optimal portfolio of assets created at the beginning of the period.

All the random vectors in this economy, namely the risky asset payoff vector, the per capita supply vector of the risky assets, and the private signal error vectors, are joint-normally distributed. The rational expectations equilibrium price vector is also a normally distributed random vector. This distributional setup of the economy gives us investors' equilibrium conditional demand vectors as linear functions of the conditioning vectors.

Presented below are the structures of the equilibrium price and the demand vectors for investors. Our framework is the large-economy rational expectations model of Admati (1985) slightly generalized, in a fashion explained below, to incorporate a greater degree of informational heterogeneity between investors. For more detail see Admati (1985).

Rational Expectations Price Conjecture: Let $\tilde{P}$ represent the per capita price vector of the risky assets, $\tilde{F}$ represent the per capita payoff vector of the risky assets, and $\tilde{Z}$ represent the per capita supply vector of risky assets. Let A_1 and A_2 be n x n matrices of coefficients and A_0 an n x 1 vector of coefficients. As in Admati (1985), it is assumed that investors have a rational expectations price conjecture of the form shown below:

$$\tilde{P} = A_0 + A_1\tilde{F} - A_2\tilde{Z}.$$

Asset Structure: There are n risky assets trading on the financial market. Asset portfolios are created at the start of the period and all payoffs are realized at the end of the period. The distribution of per capita payoffs and supplies is as follows:

$$\begin{bmatrix}\tilde{F}\\ \tilde{Z}\end{bmatrix} \sim N\left[\begin{bmatrix}\bar{F}\\ \bar{Z}\end{bmatrix},\begin{bmatrix}V & O\\ O & U\end{bmatrix}\right],$$ V and U are positive definite matrices.

The assumption of positive definiteness of the variance-covariance matrix, V, of the per capita payoff vector implies that, *ex ante*, there are no redundant assets. U represents the variance-covariance matrix of the per capita asset supply vector. There is also a risk-free asset in zero net supply, which has a current price set at one and a non - stochastic, per capita payoff of R at the end of the period.

Private Signal Structure: The signal structure employed here is a slight generalization of Admati (1985). We assume that each investor receives a signal regarding the payoffs of a nonempty (but not necessarily proper) subset of the assets traded in the economy. In Admati (1985) investors receive signals about all traded assets.[3] More specifically, we assume that $\tilde{Y}_j = C_j\tilde{F} + \tilde{\varepsilon}_j$, where $\tilde{Y}_j$ is the private signal vector for investor j; C_j is the k_j

$\times$ n transformation matrix with rank equal to $k_j \leq n$; and $\tilde{\varepsilon}_j$ is investor j's private signal error vector. $\tilde{\varepsilon}_j$ is a k_j-dimensional, nondegenerate, normally distributed, mean zero random vector with a positive-definite variance-covariance matrix S_j . Please note that S_j is $k_j \times k_j$. We assume that for all $j \in J$, the error vectors $\tilde{\varepsilon}_j$ are independent across investors and that they are also independent of the per capita asset payoff ($\tilde{F}$) and supply ($\tilde{Z}$) vectors.

Aggregation: For brevity, and comparability with the extant literature, we model a continuum economy. We assume that $1/\rho_j$ and S_j^{-1} are uniformly bounded, and the investor set is the interval J = [0, 1].[4] We use $1/\bar{\rho}$ to represent the average risk tolerance of investors, that is $1/\bar{\rho} \equiv \int_0^1 (1/\rho_j)\, dj$. Similarly, Q represents the average precision of the private signals, weighted by the investors' risk tolerance coefficients, that is

$$Q = \int_0^1 \left(\frac{1}{\rho_j}\right)[C_j^T S_j^{-1} C_j]\, dj.$$

Equilibrium Pricing Functional: A rational expectations equilibrium price functional is a self-fulfilling price conjecture of the form $\tilde{P} = A_0 + A_1\tilde{F} - A_2\tilde{Z}$, A_2 *nonsingular*. The nonsingularity of A_2 is part of Admati's *definition* of a large-economy, rational - expectations-equilibrium price conjecture. She shows that, in any finite investor economy, the equilibrium price conjecture must have this property and argues that the limiting large economy should also have it. Admati demonstrates that, within the class of price conjectures defined above, a unique rational expectations equilibrium price conjecture exists. As shown in Admati (1985), the relevant coefficients of the equilibrium price vector are given by

$$A_0 = (1/R)\, [\bar{\rho}Q + Q^T U^{-1} Q + V^{-1}]^{-1}\ [V^{-1}\bar{F} + Q^T U^{-1}\bar{Z}],$$

$$A_1 = (1/R)[\bar{\rho}Q + Q^T U^{-1} Q + V^{-1}]^{-1}\ [\bar{\rho}\, Q + Q^T U^{-1} Q], \text{ and}$$

$A_2 = (1/R)\,[\bar{\rho} Q + Q^T U^{-1} Q + V^{-1}]^{-1}\ [\bar{\rho}\, I + Q^T U^{-1}]$, A_2 nonsingular.

II. Analysis of Investor Demand

We decompose an investor's informational demand into different components representing different aspects of investment style and investigate the effect of information quality on each of these components of demand. To initiate the formal analysis, we require an explicit representation of an investor's risky asset demand. This is provided in Proposition 1.

Proposition 1. The asset demand vector of the j^{th} investor, $\tilde{D}_j$, is given by

$$\tilde{D}_j[\circ \mid \tilde{Y}_j, \tilde{P}] = \alpha_{0j} + \beta_j A_0 + [\alpha_{1j} C_j + \beta_j A_1]\,\tilde{F} + \alpha_{1j}\,\varepsilon_j - \beta_j A_2\,\tilde{Z}, \text{ where}$$

$$\alpha_{0j} + \beta_j A_0 = -(\Theta_j/\rho_j)\,[V^{-1} + Q^T U^{-1} Q + \bar{\rho}\ Q]^{-1}\,[[V^{-1}]\bar{F} + Q^T U^{-1}\ \bar{Z}],$$

$$\alpha_{1j} C_j + \beta_j A_1 = (\Theta_j/\rho_j)\,[V^{-1} + Q^T U^{-1} Q + \bar{\rho}\ Q]^{-1}\,V^{-1}, \tag{1}$$

$$\alpha_{1j} = 1/\rho_j\, C_j^T S_j^{-1},$$

$$\beta_j A_2 = -1/\rho_j\,[\bar{\rho}\, I + \Theta_j\,[V^{-1} + Q^T U^{-1} Q + \bar{\rho}\ Q]^{-1}\,[\bar{\rho}\, I + Q^T U^{-1}]],$$

where Θ_j, which we henceforth refer to as the investor's *informational difference* matrix, is defined as $\Theta_j = C_j^T S_j^{-1} C_j - \bar{\rho}\, Q$.

Proof. This result follows from direct manipulation of the demand and price equations.[5] ■

To determine the effect of private information on security demand, we first define the demand vector in the absence of any private information. When there is no private information, then $S_j^{-1} = 0$ for all investors and, hence, the demand equation for any investor j, reduces to

$$\tilde{D}_j^{o} = (\bar{\rho}/\rho_j)\,\tilde{Z}. \tag{2}$$

Equation (2) shows that in the absence of private information, every agent's asset demand is a linear function of the per capita asset supply (i.e., the standard CAPM result holds). Thus, we can represent the jth agent's asset demand, $\tilde{D}_j$, as the sum of two components, $\tilde{D}_j^o$ and $\tilde{D}_j^I$, where $\tilde{D}_j^I$ is the demand induced by private information and $\tilde{D}_j^o$ is the demand in an otherwise identical economy in which no agent receives any private information, i.e., $\tilde{D}_j^I = \tilde{D}_j - \tilde{D}_j^o$. Using the basic properties of conditional expectations, it is possible to decompose the private information induced demand vector, $\tilde{D}_j^I$, into three components as follows:

$$\tilde{D}_j^I \overset{dist.}{=} E[\tilde{D}_j^I] + \sqrt{VAR(E[\tilde{D}_j^I|\tilde{Z}])}\ \ \zeta + \sqrt{VAR[\tilde{D}_j^I|\tilde{Z}]}\ \ \chi\,, \tag{3}$$

where ζ and χ are independent, standard, normally distributed n-dimensional random vectors.[6] The first component, $E[\tilde{D}_j^I]$, is the mean demand: it measures the *average*, or expected, demand. The second component represents the degree of variation in demand that is proportional to the per capita asset supply vector, $\tilde{Z}$. Thus, $VAR(E[\tilde{D}_j^I \mid \tilde{Z}])$ measures *market timing*. On the other hand, $VAR[\tilde{D}_j^I \mid \tilde{Z}]$ measures the variation that is uncorrelated with asset supply and, thus, measures *asset selection* activity.

The expected information-driven demand, $E[\tilde{D}_j^I]$, is given by

$$\overline{D}_j^I = E[\tilde{D}_j^I] = \frac{\bar{\rho}}{\rho_j}\Theta_j\Delta^{-1}\ \overline{Z}, \tag{4}$$

where $\Delta = [V^{-1} + Q^T\ U^{-1}\ Q + \bar{\rho}\,Q]$. Equation (4) shows that the two key investor-specific parameters of the demand function are the investor's risk aversion, ρ, and his informational difference matrix, Θ ($= C^T S^{-1}\ C - \bar{\rho}\,Q$).

The informational difference matrix, Θ, measures the difference between the precision of the investor's information and the precision of the information of the average market participant. If the investor's information quality is equal to that of the market, that is, $\Theta_j = 0$, then the investor's expected informational demand is zero. In other words, although a particular realized signal may lead such an investor to take an "unbalanced" portfolio position, these positions average out in the long-run and, thus, the average asset demand of the investor closely tracks the asset supply vector which in the standard CAPM nomenclature is the benchmark market portfolio. The two key economy-wide parameters of the model are Δ and $\bar{Z}$. Of course, $\bar{Z}$ is the vector of expected asset supplies. To better understand Δ, note that when $\Theta_j = 0$, then $VAR[\tilde{F} \mid \tilde{Y}_j, \tilde{P}] = \Delta$. Thus, Δ can be thought of as the variance-covariance matrix of asset payoffs as conjectured by the "typical" or average investor. For this reason, we will call Δ the *average payoff variance matrix*.

Using the orthogonal projection theorem we can decompose the expected information-driven demand, $\bar{D}^I \equiv E[\tilde{D}_j{}^I]$, into two components, one of which lies on the market line and another which is perpendicular to the market line. This yields the complete decomposition of the demand vector that we will utilize in the remainder of the paper.

Proposition 2. Let S_n (P_n) represent the space of $n \times n$ symmetric (positive semidefinite) matrices. Let $j \in J$ be an investor. Then there exists a scalar, κ, an n-dimensional demand vector, $\tilde{D}_j{}^I$, two $n \times n$ positive semidefinite matrices, $\mathbf{T}(\Theta_j, \rho_j)$ and $\mathbf{A}(\Theta_j, \rho_j)$, and two n-dimensional vectors, $\bar{D}^{\parallel}(j)$ and $\bar{D}^{\perp}(j)$, such that

$$\tilde{D}_j{}^I \stackrel{\text{dist.}}{=} \sqrt{\mathbf{T}(\Theta_j, \rho_j)}\,\zeta + \sqrt{\mathbf{A}(\Theta_j, \rho_j)}\,\chi + \bar{D}^{\parallel}(j) + \bar{D}^{\perp}(j), \tag{5}$$

where

$\mathbf{T}: S_n \times \mathbb{R} \rightarrow P_n$ is the function defined by

$$T(\Theta, \rho) = \frac{\Theta \Delta^{-1} [\bar{\rho} I + Q^T U^{-1}] U [\bar{\rho} I + U^{-1} Q] \Delta^{-1} \Theta}{\rho^2}, \tag{6}$$

and $\mathbf{A}: S_n \times \mathbb{R} \rightarrow P_n$ is the function defined by

$$A(\Theta, \rho) = \frac{\Theta \Delta^{-1} V^{-1} \Delta^{-1} \Theta + \bar{\rho} Q + \Theta}{\rho^2}, \tag{7}$$

and $\kappa: S_n \rightarrow \mathbb{R}$ is the function defined by

$$\kappa(\Theta) = \frac{\bar{Z}^T \Theta \Delta^{-1} \bar{Z}}{\bar{Z}^T \bar{Z}} . \tag{8}$$

Finally, $D^{\|}(j)$ ($\bar{D}^{\perp}(j)$) is the component of *ex ante* demand that is collinear (orthogonal) to $\bar{Z}$, defined by

$$\bar{D}^{\|}(j) = (\bar{\rho}/\rho_j)\, \kappa(\Theta_j)\, \bar{Z}, \tag{9}$$

$$\bar{D}^{\perp}(j) = \frac{\bar{\rho}}{\rho_j} (\Theta(j)\, \Delta^{-1} - \kappa(\Theta_j)\, I)\bar{Z}, \tag{10}$$

and ζ and χ are independent, n-dimensional, standard, normally distributed random vectors.

Proof: The first two terms follow by explicit computations of $VAR(E[\tilde{D}_j^I|Z])$ and $E(VAR[\tilde{D}_j^I|Z])$ (using expression (1) and equation (3)).[7] The decomposition of $E[\tilde{D}_j^I]$ follows by applying the Projection Theorem (see Luenberger 1969, Theorem 2, page 51) to the subspace of $\mathbb{R}^n$ spanned by the mean per capita supply vector $\bar{Z}$. ■

The matrix algebra required to state the decomposition provided by Proposition 2 makes it appear somewhat complex. Unfortunately, this matrix algebra *is* required if we are going to deal with the very high dimensional demand vectors that characterize real-world asset demands. Though the simple intuition underlying the decomposition is obscured by the algebraic treatment, it is transparent in the following diagramatic presentation of the two-asset case in Figure 1. The investor's risk aversion coefficient ρ is given by 4; his informational differential Θ is given by $\begin{bmatrix} 12 & -16 \\ -16 & 28 \end{bmatrix}$; $\bar{\rho}$, the average risk aversion is given by 2; $\bar{Z}$, the expected per capita supply of risky assets is given by (2,

$2)^T$; and the average payoff variance matrix, Δ, is given by $\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}$. The scatter of the small hollow circles represents possible realized asset demands. These demands are a function of the realized value of the investor's private signal. Some of the variations in the investor's demand represent increases or decreases in holdings of the market portfolio; these variations are parallel to the market line, and some of the variations are orthogonal to the market line. The expected magnitude of parallel variations we call market timing and represent by $\mathbf{T}(\Theta, \rho)$. The expected magnitude of orthogonal variations we term asset selection activity and represent by $\mathbf{A}(\Theta, \rho)$. In the example, $\mathbf{T}(\Theta, \rho) = \begin{bmatrix} 2.8125 & -5.62501 \\ -5.62501 & 14.0625 \end{bmatrix}$ and $\mathbf{A}(\Theta, \rho) = \begin{bmatrix} 3.000 & -4.625 \\ -4.625 & 8.000 \end{bmatrix}$. The large filled circle at the center of the scatter of realized demands represents the expected demand for risky assets, $\bar{D}$. In the example, $\bar{D} = (1, 3)^T$. In the figure, we see that the difference between demand in the absence of private information, $\bar{D}^o$, and the expected demand given private information, $\bar{D}$, i.e., $\bar{D}^I$, can be decomposed into two components: agressiveness, the increase in the investor's expected holdings in the market portfolio, measured by $\bar{D}^{\parallel}$, and specialization, the emergence of a systematic deviation between the investor's demand and the market line, measured by $\bar{D}^{\perp}$.

In Figure 1, $D^{\parallel} = (1, 1)^T$ and $\bar{D}^{\perp} = (-1, 1)^T$. Thus, the receipt of private information leads the investor to hold a portfolio that exhibits higher systematic risk than would have been the case in the absence of private information. Also, the receipt of information leads the investor to hold a portfolio that is much more heavily weighted toward asset 2 than is the market portfolio. Further, variability has been added to the investor's demand vector by the market timing and asset selection activity in response to the realized private signals. How can one account for these effects of information on investment style? We address this question in the rest of this section.

To initiate the analysis, we define a measure of portfolio specialization based on the distance between the *ex ante* demand vector and the market line. By the Projection theorem (see Luenberger 1969, Theorem 2, page 51), this distance is given by $\| \bar{D}^{\perp} \|$, the norm of the component of demand that is orthogonal to the mean asset supply vector $\bar{Z}$. The Euclidean distance of investor j's *ex ante* demand from the market line in turn is given by

$$\| \bar{D}^{\perp}(j) \| = \sqrt{\bar{D}^{\perp}(j) \cdot \bar{D}^{\perp}(j)} = \frac{\bar{\rho}}{\rho_j} \| (\Theta(j)\, \Delta^{-1} - \kappa(\Theta_j)\, I\,)\bar{Z} \|. \tag{11}$$

Simple computations yield

$$\| \bar{D}^{\perp}(j) \| = \sqrt{\mathbf{N}(\rho_j, \Theta_j)}, \tag{12}$$

where

$$\mathbf{N}(\rho, \Theta) = \left(\frac{\bar{\rho}}{\rho} \right)^2 \bar{Z}^T \Delta^{-1}\ \Theta\, \Xi\, \Theta\, \Delta^{-1}\ \bar{Z}, \tag{13}$$

and $\Xi = I - (\bar{Z}\,\bar{Z}^T / \bar{Z}^T\,\bar{Z})$ is a projection matrix.

The above calculations clarify the relationship between investors' information structures and diversification. Note that (11) and (13) imply that $\| D^{\perp}(j) \| = 0 \Leftrightarrow \Theta(j)\,\Delta^{-1}\,\bar{Z} = \kappa(\Theta_j)\,\bar{Z}$. This equation holds if and only if there exists a scalar λ such that $\Theta(j)^{-1}\,\bar{Z} = \lambda\,\Delta^{-1}\,\bar{Z}$, that is, $\bar{Z}$ is an eigenvector of $\Theta(j)\,\Delta^{-1}$; this result is recorded in Lemma 1.

Lemma 1. The j^{th} investor's expected demand is proportional to expected asset supplies if and only if there exists a scalar λ such that $\Theta(j)Z = \lambda\,\Delta\,\bar{Z}$.

When the conditions of Lemma 1 are satisfied, we will say that $\Theta(j)$ is *comparable with* the average payoff variance matrix. To understand the intuition behind this condition, note that for any two portfolios of assets $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots y_n)^T$, $y^T\,\Delta\,x$ is simply the covariance between the payoffs on portfolios x and y conditioned on the information of the average investor. Also, $y^T\,\Theta\,x$ can be thought of as the differential between the investor's information and the average market information regarding the "co-movement" between the payoffs on portfolio x and portfolio y. Note also that, $\Theta(j)\bar{\bar{Z}} = \lambda \Delta \bar{Z}$ if and only if $x^T \Theta(j)\bar{\bar{Z}} = \lambda x^T \Delta \bar{Z}$ for every portfolio x. *Thus, the comparability condition, $\Theta(j)\bar{\bar{Z}} = \lambda \Delta \bar{Z}$, holds if and only if, for all portfolios x, the investor's informational differential regarding the co-movement in payoffs between x and the market portfolio, $\bar{Z}$, is proportional to the covariance (conditioned on the average investor's information set) between the payoffs on x and the market portfolio.*

As the above lemma shows, comparability with the average payoff variance matrix has important consequences. Diagrammatically, it corresponds to $\bar{D}^I$, the expected information-driven demand, lying on the ex ante market line. When the comparability condition is not satisfied, differences between an investor's information structure and the average structure lead to the holding of specialized portfolios that do not, even on average, lie on the market line. A natural question to ask is, how are these deviations related to the quality of the investor's information? To analyze this question, we need to measure the quality of the investor's information. Information in a multiple-asset economy is not represented by a simple scalar, measuring signal precision, but rather by a precision matrix determined both by the variances of private information signals about different assets and their covariances with each other. Thus, there is no single, natural way of ordering information structures in a multi-asset economy. However, we require some method of comparing the informativeness of different information structures. Two types of orderings are considered. The first of these is the well-known positive definite ordering of symmetric matrices. (See Dhrymes 1978, page 485.)

Definition 1. (Superior Information): Let Θ_1 and Θ_2 be two symmetric matrices. Θ_1 is superior to Θ_2 ($\Theta_2 \prec \Theta_1$) if $\Theta_1 - \Theta_2$ is positive definite. If $0 \prec \Theta_1$ then we simply say that Θ_1 is superior.

We also consider the impact of imposing another ordering on information structures, which we term *uniform superiority* of information.

Definition 2. (Uniformly Superior Information): For two symmetric matrices Θ_1 and Θ_2, Θ_1 is uniformly superior to Θ_2, ($\Theta_2 \prec\prec \Theta_1$) if and only if Max $\sigma(\Theta_2) <$ Min $\sigma(\Theta_1)$, where $\sigma(\Theta_i)$ represents the set of eigenvalues of the matrix Θ_i.

Uniform superiority thus requires that even the smallest of the eigenvalues of the matrix Θ_1 be larger than the largest eigenvalue of the matrix Θ_2. Admittedly, this ordering is not very "intuitive." However, some insight can be gained by thinking of the magnitudes of the eigenvalues of a matrix as measures of the size of the matrix. For a variance-covariance matrix, this is apparent: the largest eigenvalue measures the maximum variance attainable by a weighted combination of the random variables in which the length of the weight vector is restricted to unity. The next largest eigenvalue measures the maximum variance attainable by a weighted combination of the random variables in which the length of the weight vector is restricted to unity and is perpendicular to the maximizing weight vector, and so forth. Thus, uniform ordering requires the size of the larger matrix to exceed the smaller one regardless of which pair of eigenvalues is used to measure the size of each of the matrices. When the informational divergence matrices Θ are diagonal, the contrast between uniform superiority and superiority is straightforward: uniform superiority requires that the smallest element along the diagonal of the dominating matrix exceed the largest diagonal element of the dominated matrix while superiority requires only componentwise dominance along the diagonal.

Given these definitions of information quality, what predictions can we make, independent of the particular asset payoff structure, about how the quality of an investor's private information affects the extent of the "specialization" of his portfolio? (Specialization is defined as the degree to which the portfolio differs on average from the market portfolio.) The answer to this question is fairly subtle. For example, superior information, i.e., dominance in the positive-definite ordering, need not lead in general to a greater degree of portfolio specialization. Consider, for example, the case of an investor whose relative information matrix (Θ) is very large in the positive-definite ordering but is comparable with the average variance matrix Δ, versus an investor whose relative information matrix is very small in the positive-definite ordering but which is not comparable with Δ. According to the above result, Lemma 1, the first investor holds a completely nonspecialized portfolio, while the second investor, whose information is of lower quality, holds a specialized portfolio. Thus, better information does not, in general, translate into greater specialization and, thus, a general result linking information quality and specialization is not possible. An additional condition is required: *comparability* between the two information structures. This is a generalization of the idea of comparability with Δ, the average payoff variance matrix, and is defined below.

Definition 3. Two invertible matrices, M_1 and M_2, are comparable, which we write as $M_1 \sim_Z M_2$ if there exists a scalar λ such that $M_1^{-1}\bar{Z} = \lambda M_2^{-1}\bar{Z}$.

In fact, if we impose this comparability condition, we can identify the nature of the relationship between a measure of the absolute divergence of an investor's information structure, namely, $\Theta\,\Theta$, and the degree of portfolio specialization. Part (a) of the next result shows that, for information structures with comparable relations to the market portfolio, a larger informational divergence implies a greater degree of portfolio specialization. Part (b) shows that the comparability condition is, in fact, necessary if we want to be able to draw predictions about specialization that are invariant across different parameterizations of the joint distribution of asset payoffs.

Proposition 3. **(a)** Let 1 and 2 be otherwise identical investors endowed with relative information structures, Θ_1 and Θ_2, respectively, such that $\Theta_1 \sim_Z \Theta_2$. If $\Theta_1\,\Theta_1 \succ \Theta_2\,\Theta_2$, investor 1's portfolio will exhibit more specialization than 2's, i.e., $\mathbf{N}(\Theta_1, \rho) \geq \mathbf{N}(\Theta_2, \rho)$. **(b)** Conversely, if the comparability condition fails, that is, $\Theta_1 \not\sim_Z \Theta_2$, then even if investor 1 has superior information and $\Theta_1 \succ 0$ and $\Theta_1\,\Theta_1 \succ \Theta_2\,\Theta_2$ holds, there exist asset payoff variance-covariance matrices, Δ, under which $\mathbf{N}(\Theta_1, \rho) < \mathbf{N}(\Theta_2, \rho)$, that is, investor 2 holds a more specialized portfolio.

Proof. See Appendix.

To illustrate these concepts, consider the following numerical example. Fix the economy-wide parameters $\bar{Z}$ and Δ, as well as the average risk aversion level $\bar{\rho}$ as in the example developed in Figure 1. Let $\Theta_1 = \begin{bmatrix} 8 & -2 \\ -2 & 8 \end{bmatrix}$, $\Theta_2 = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$, $\Theta_3 = \begin{bmatrix} 24 & -12 \\ -12 & 36 \end{bmatrix}$, and $\Theta_4 = \begin{bmatrix} 8 & -4 \\ -4 & 12 \end{bmatrix}$; $\rho_j = 4$, for all $j = 1, 2, 3, 4$. In this example, Θ_1 and Θ_2 are comparable with each other but not with the average payoff variance matrix Δ, while Θ_3 and Θ_4 are comparable with the average payoff variance matrix. Further, it is easy to check that $\Theta_1\Theta_1 - \Theta_2\Theta_2$ and $\Theta_3\Theta_3 - \Theta_4\Theta_4$ are positive definite and, thus, $\Theta_1\Theta_1 \succ \Theta_2\Theta_2$ and $\Theta_3\Theta_3 \succ \Theta_4\Theta_4$. As Proposition **3** asserts, investor 1 holds a more specialized portfolio than investor 2, while as Lemma **1** asserts, the two investors, 3 and 4, with informational differentials comparable with the average payoff variance matrix hold completely nonspecialized portfolios.[8]

The quality of an investor's information structure affects the average amount of aggressiveness exhibited by the investor, as well as affecting the specialization component of his ex ante demand. This is measured naturally by $\kappa(\Theta)$. For, as can be seen by inspecting Proposition 2, when $\kappa(\Theta) > (<)\ 0$, the proportion of the market portfolio purchased by the investor is larger (smaller) than the proportion implied by his level of *average information* demand. The proof of the next result shows that superior information leads to a more aggressive investment policy when an investor's information structure is comparable with the average payoff variance matrix, Δ.

Proposition 4. If an investor possesses superior (inferior) information and his information structure is comparable with the average payoff variance matrix, then he will hold a portfolio with higher (lower) systematic risk than that held by an otherwise identical investor with average quality information, i.e., if $\Theta \sim_Z \Delta$ and $\Theta \succ (\prec) 0$, then, $\kappa(\Theta) > (<) 0$.

Proof. See Appendix.

The quality of an investor's information also affects the extent to which his portfolio allocations exhibit *market timing* and *asset selectivity.* The degree of market timing is a function of the investor's *ex ante* informational differential alone. Some characteristics of the timing activity are immediately apparent from the definitions of **T** and **A**, provided in Proposition 2. Notably, market timing depends only on Θ, the degree to which an investor's information structure *diverges* from the market average. On the other hand, asset selectivity depends on both the magnitude of the divergence of an investor's informational structure and the direction of that divergence. Thus, a divergence between an investor's information structure and the average information structure always leads to increased market timing activity but does not necessarily lead to increased asset-selection activity. To formalize these insights, we require some measure of the magnitude of each of the matrices $\mathbf{T}(\Theta, \rho)$ and $\mathbf{A}(\Theta, \rho)$. We will use the spectral norm, $|||\cdot|||$, over matrices, defined by $||| M ||| \equiv \text{Max}\{\|Mx\|: \|x\| = 1\}$ to obtain a simple representation of this magnitude. This measure associates with each matrix a number that measures the "size" of the matrix by the extent to which the lengths of unit vectors transformed by the matrix are expanded. Using this definition, the next Proposition formalizes the discussion provided above.

Proposition 5. Consider two otherwise identical investors, 1 and 2, endowed with information structures, Θ_1 and Θ_2, respectively, such that $0 \prec \Theta_1$, $0 \prec \Theta_2$ and $\Theta_2 \prec\prec \Theta_1$.Then, investor 1 will engage in strictly more market timing and asset selection than investor 2, i.e., $|||\mathbf{T}(\Theta_2,\rho)||| \leq |||\mathbf{T}(\Theta_1, \rho)|||$, and $||| \mathbf{A}(\Theta_2, \rho) ||| \leq |||\mathbf{A}(\Theta_1, \rho)|||$. Similarly, if both investors possess inferior information, i.e., $\Theta_1 \prec 0$, $\Theta_2 \prec 0$ and $\Theta_2 \prec\prec \Theta_1$, then investor 2 will engage in strictly more market timing than investor 1, i.e., $|||\mathbf{T}(\Theta_1, \rho)||| \leq |||\mathbf{T}(\Theta_2, \rho)|||$.

Proof. See Appendix.

III. Identification of Investor Characteristics From Data

In the previous section, we determined the effect of the structure of an investor's informational differential, Θ, on the pattern of his portfolio holdings. We show that fairly strong comparative static relations can be established between asset demand and the relative informational structure. This fairly strong relationship between asset demand and information leads to an interesting question: is it possible to "back out" an investor's informational structure from the pattern of his trades? An affirmative answer to this question would show that the demand analysis can be used to answer the fundamental question of performance measurement: the identification of traders with superior information.

In this section, we show that, in the rational expectations setting, it is possible to identify relative information structures from the time series of investor asset demand. The required data for such an estimation are given as follows. First, asset demand data for the given investor are collected. That is, the number of units of different assets held in the portfolio at the end of each observation interval is recorded. Next, the price vector for the economy for each observation period is also recorded. Finally, the payoff vector, that is, the end-of-the-period price vector plus the vector of cash flows from the assets (e.g., dividends) that may have occurred during the observation interval is obtained. The estimation procedure is based on the observation that the pricing equations imply the existence of a vector γ_1, and matrices γ_2, γ_3 such that

$$\tilde{D}_j = \gamma_{j1} + \gamma_{j2}\,\tilde{F} + \gamma_{j3}\,\tilde{P} + \tilde{\eta}_j, \qquad \text{(RC)}$$

where $\tilde{\eta}_j$ is a zero-mean, normally distributed random N-vector, independent of both $\tilde{F}$ and $\tilde{P}$. In fact, the pricing equations derived above imply the following structural restrictions on the regression coefficients:

$$\gamma_{j1} = \alpha_{0j},\ \gamma_{j2} = \alpha_{1j}\, C_j\ ,\ \gamma_{3j} = \beta_j\ , \text{ and } \tilde{\eta}_j = \alpha_{1j}\, \tilde{\varepsilon}_j. \qquad \text{(RE)}$$

Note that $VAR[\tilde{\varepsilon}_j] = S_j$ and that $\gamma_{j2} = (1/\rho_j)C_j^T S_j^{-1} C_j$ and $\alpha_{1j} = (1/\rho_j)C_j^T S_j^{-1}$. These results, combined with the definition of $\tilde{\eta}_j$, given above, imply that $VAR[\tilde{\eta}_j] = \alpha_{1j} VAR[\tilde{\varepsilon}_j]\ \alpha_{1j}^T = (1/\rho_j^2)\ C_j^T\ S_j^{-1}\ C_j$. Taken together, these results imply the crucial identity

$$\gamma_{j2} = \rho_j\ VAR[\tilde{\eta}_j].^9 \qquad (14)$$

When the investor's signal is not full dimensional, $VAR[\tilde{\eta}_j]$ is singular. Thus, in general, we cannot invert $VAR[\tilde{\eta}_j]$ to obtain an estimate of ρ_j. However, the desired result follows from applying any linear functional to both sides of (14), and a convenient choice for such a functional is the Trace. Taking the Trace of both sides yields

$$\rho_j \equiv \frac{\text{Trace}(\gamma_{j2})}{\text{Trace}(VAR[\tilde{\eta}_j])}. \qquad (15)$$

These results allow the consistent estimation of managerial risk aversion, assuming that the single-period rational expectations equilibrium demand equations hold at each point in time and that the signal realizations are uncorrelated across time. Given that these assumptions hold, and, for notional simplicity, dropping the investor index (which is in any case held constant throughout the analysis), we obtain, for each asset i = 1, 2, ... N, the following regression equation:

$$\tilde{D}_i(t) = \gamma_{i1} + \sum_{k=1}^{N} \gamma_{ik2}\, \tilde{F}_k(t) + \sum_{k=1}^{N} \gamma_{ik3} \tilde{P}_k(t) + \tilde{\eta}_i(t)\ , \quad t = 1,2, \dots T. \qquad (16)$$

In this equation, the demand for the i^{th} asset ($\tilde{D}_i$) is regressed on the end-of-period payoffs to each asset at time t, $\tilde{F}_k(t)$, as well as start-of-period prices, $\tilde{P}_k(t)$. Given that the data matrix in each of the N-regressions is the same, it is possible to efficiently estimate each of the regressions *en seriatim* (see Zellner 1962).[10] These estimates then yield consistent estimates of the matrices γ_1 and γ_2, which we denote by $\hat{\gamma}_1$ and $\hat{\gamma}_2$. The variance-covariance matrix, VAR[$\tilde{\eta}$], can be consistently estimated using the estimated residuals from the N-regression equations. The estimated variance-covariance matrix is denoted by $\hat{\Sigma}$. The continuity of the Trace operator and scalar division, combined with (15), then implies that

$$\hat{\rho} \equiv \text{Trace}(\hat{\gamma}_2)/\text{Trace}(\hat{\Sigma}) \tag{17}$$

is a consistent estimator of ρ.[11]

It is equally easy, in principle, to consistently estimate the investor's relative informational differential Θ. Note that equations (1), (RE), and (RC) imply that

$$\Theta = \rho\,(\gamma_2 + \gamma_3\,A_1)\,(I - R\,A_1)^{-1} \tag{18}$$

where $A_1 = \text{COV}[\tilde{P},\ \tilde{F}]\,(\text{VAR}[\tilde{F}])^{-1}$. A_1 can be consistently estimated from the data matrix of asset payoffs and prices.[12] Let $\hat{A}_1$ represent any consistent estimator of A_1. γ_1 and γ_2 are consistently estimatable by $\hat{\gamma}_1$ and $\hat{\gamma}_2$. Finally, ρ can be consistently estimated using (17) and R, the per-period payoff on the risk-free asset, is observable in principle. Thus, the joint continuity of the right-hand side expression in (18) in all of its arguments implies that Θ can be consistently estimated. This result is formalized in the next proposition.

Proposition 6. Define the statistic $\hat{\Theta}$ by

$$\hat{\Theta} \equiv \hat{\rho}\,(\hat{\gamma}_2 + \hat{\gamma}_3\,\hat{A}_1)\,(I - R\ \hat{A}_1)^{-1} \tag{19}$$

where $\hat{\rho}$, $\hat{\gamma}_2$, $\hat{\gamma}_3$, $\hat{A}_1$ are as defined above. Then $\hat{\Theta}$ is a consistent estimator of Θ.

IV. Conclusion

In this paper we analyze the relationship between portfolio composition and information quality. Summary statistics are derived representing basic aspects of investment style, such as market timing, asset-selection activity, and specialization. The effect on investor demand of the structure of his relative informational advantage over the market is considered. Finally, we show that it is possible to consistently estimate an investor's informational advantage from the time series of his asset demands.

Appendix

The proofs in this appendix require some standard definitions: Let $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ represent the standard inner product on $\mathbb{R}^n$. If x and y $\in \mathbb{R}^n$ then we write $x \perp y$ if and only if $x \cdot y = 0$. Also, let $\| x \|$ represent the Euclidean norm of x. Let $\mathcal{M}_n$ represent the space of $n \times n$ matrices. Let $\mathcal{S}_n$ ($\mathcal{P}_n$) represent the space of $n \times n$ symmetric (positive semidefinite) matrices. For all $M \in \mathcal{M}_n$, let $\mathcal{N}(M) = \{x \in \mathbb{R}^n: Mx = 0\}$, $\mathcal{N}(M)$ is thus the null space of the matrix M. For any vector $x \in \mathbb{R}^n$, let $[x]$ represent the linear subspace generated by the vector x

Proof of Proposition 3. Note that $\mathbf{N}(\Theta_1, \rho)$ - $\mathbf{N}(\Theta_2, \rho)$ has the same sign as

$$\bar{Z}^T\Delta^{-1} \{ \Theta_1 \Xi \Theta_1 - \Theta_2 \Xi \Theta_2\}\Delta^{-1} \bar{Z}. \tag{A1}$$

We first prove that the conditions of the Proposition imply that for all $x \in \mathbb{R}^n$, $x^T(\Theta_1 \Xi \Theta_1 - \Theta_2 \Xi \Theta_2)x \geq 0$. This implies *a fortiori* that (A1) holds and, thus, establishes part (a) of the proposition. To see this, first note that, because $\Theta_1 \sim_z \Theta_2$, $\Theta_1^{-1}([\bar{Z}]) = \Theta_2^{-1}([\bar{Z}])$. Let $\mathcal{V}$ represent this subspace; i.e., $\mathcal{V} = \Theta_1^{-1}([\bar{Z}]) = \Theta_2^{-1}([\bar{Z}])$. Next, note that any vector $x \in \mathbb{R}^n$ can be written as $x = \alpha v + w$, where α is a scalar, $v \in \mathcal{V}$, and $w \in \mathcal{V}^\perp$. Hence $\Theta_i x = \alpha \Theta_i v + \Theta_i w$, i = 1, 2. $\Theta_i(v) \in [\bar{Z}]$ and thus, because Ξ is the projection matrix of $\mathbb{R}^n$ onto the orthogonal complement of $[\bar{Z}]$, $\Xi \Theta_i v = 0$. On the other hand, because Θ_i is a symmetric invertible matrix, $\mathcal{V}^\perp = (\theta_i^{-1}([\bar{Z}]))^\perp = \theta_i^{-1}([\bar{Z}]^\perp)$, i = 1, 2 (Luenberger 1969, Theorem 1, page 158). Thus, $\Theta_i w \in [\bar{Z}]^\perp$, which implies that $\Xi \Theta_i w = \Theta_i w$. Together these facts imply that

$$\Xi \Theta_i x = \Theta_i w, i = 1, 2, \tag{A2}$$

where w is the projection of x onto $\mathcal{V}^\perp$. But because Ξ represents the projection of $\mathbb{R}^n$ onto $[\bar{Z}]^\perp$, Ξ is idempotent .We thus have $\Theta_1 \Xi \Theta_1 = \Theta_1 \Xi \Xi \Theta_1$. This fact combined with (A2) implies that $x^T\Theta_i\ \Xi\ \Theta_i x = w^T\Theta_i\ \Xi\ \Theta_i w$, i= 1, 2, where w is the projection of x onto $\mathcal{V}^\perp$. This shows that, for all $x \in \mathbb{R}^n$,

$$x^T(\Theta_1\ \Xi\ \Theta_1 - \Theta_2\ \Xi\ \Theta_2)x = w^T(\Theta_1\ \Theta_1 - \Theta_2\ \Theta_2)w, \tag{A3}$$

where, again, w is the projection of x onto $\mathcal{V}^\perp$. The result then follows by the assumption that $\Theta_1\ \Theta_1 - \Theta_2\ \Theta_2$ is positive definite.

To prove (b), pick Δ so that $\Theta_1 \sim_z \Delta$ (this is possible by the assumption that $\Theta_1 \succ 0$). Because $\Theta_1 \sim_z \Delta$, $\Theta_1\ \Delta^{-1}(\bar{Z})$ is in the linear span of $\bar{Z}$, $\Xi\ \Theta_1\Delta^{-1}(\bar{Z}) = 0$. Thus,

$$\bar{Z}^T\Delta^{-1}\{\ \Theta_1\ \Xi\ \Theta_1\ \}\Delta^{-1}\bar{Z} = 0.$$

Because $\Theta_2 \nsim_z \Theta_1$, $\Theta_1 \nsim_z \Delta$, $\Xi\ \Theta_2\Delta^{-1}(\bar{Z}) \neq 0$. Thus,

$$\bar{Z}^T\Delta^{-1}\{\ \Theta_2\ \Xi\ \Theta_2\}\Delta^{-1}\bar{Z} > 0.$$

The two equations above imply that $\bar{Z}^T\Delta^{-1}\{\ \Theta_1\ \Xi\ \Theta_1 - \Theta_2\ \Xi\ \Theta_2\}\Delta^{-1}\bar{Z} < 0$ which, in turn, implies that $\mathbf{N}(\Theta_1, \rho) - \mathbf{N}(\Theta_2, \rho) < 0$. ■

Proof of Proposition 4. By the assumption of superior (inferior) information, Θ is positive definite (negative definite). This implies that all of the eigenvalues of Θ are positive (negative). Further, Δ^{-1} is a positive-definite matrix. It follows from a trivial modification of a result in Horn and Johnson (1991, Theorem 7. 6. 3, page 465), that the signs of the eigenvalues of $\Theta\ \Delta^{-1}$ are the same as those of Θ. Thus, all the eigenvalues of $\Theta\Delta^{-1}$ are positive (negative). $\bar{Z}$ is an eigenvector of $\Theta\ \Delta^{-1}$, which implies that $\theta\ \Delta^{-1}\ \bar{Z} = \lambda\ \bar{Z}$ for some $\lambda > 0$ (< 0). ■

Proof of Proposition 5: We prove the result for **T** when $0 \prec \Theta$, the proof of the other parts of the theorem being entirely similar. For this proof only, define $X = (1/\rho)\ \{\Delta^{-1}[\bar{\rho}\ I + Q^TU^{-1}]\ U[\bar{\rho}\ I + U^{-1}\ Q]\Delta^{-1}\}$. Note that X is a positive semidefinite matrix and $\mathbf{T}(\Theta, \rho) =$

Θ X Θ. Expressed in this notation, the assertion of the theorem is that when $0 \prec \Theta_1$, $0 \prec \Theta_2$ and $\Theta_2 \prec \prec \Theta_1$, then $\Theta_2 X \Theta_2 \prec \Theta_1 X \Theta_1$. To see that this is indeed the case, first note that by Ostrowski's Theorem (see Horn and Johnson 1991, Theorem 4.5.9, page 224), for all k = 1, ... n,

$$\lambda_k(\Theta_2 X \Theta_2) \leq \text{Max}[\sigma(\Theta_2 \Theta_2)] \lambda_k(X). \quad \text{(A4)}$$

Where $\lambda_k(\Theta_j X \Theta_j)$, j = 1 or 2, denotes the k^{th} eigenvalue of matrix $\Theta_j X \Theta_j$ where eigenvalues are arranged in ascending order. By the assumption that $\Theta_2 \prec \prec \Theta_1$, we have that Max $\sigma(\Theta_2) <$ Min $\sigma(\Theta_1)$. This implies, because both Θ_1 and Θ_2 are positive definite, that Max $\sigma\{\Theta_2 \Theta_2\} <$ Min $\sigma\{\Theta_1 \Theta_1\}$. This fact and (A4) imply that $\lambda_k(\Theta_2 X \Theta_2) <$ Min $\sigma\{\Theta_1 \Theta_1\} \lambda_k(X)$. By Ostrowski's Theorem, Min $\sigma\{\Theta_1 \Theta_1\} \lambda_k(X) \leq \lambda_k(\Theta_1 X \Theta_1)$. Thus, $\lambda_k(\Theta_2 X \Theta_2) < \lambda_k(\Theta_1 X \Theta_1)$ for all k = 1, 2, ... n. This implies, *a fortiori*, that Max $\sigma(\Theta_2 X \Theta_2) <$ Max $\sigma(\Theta_1 X \Theta_1)$. Because $\Theta_i X \Theta_i$ is positive definite for i = 1, 2, Max $\sigma(\Theta_i X \Theta_i) = |||\Theta_i X \Theta_i|||$. This implies that $|||\Theta_2 X \Theta_2||| < |||\Theta_1 X \Theta_1|||$. ■

Endnotes

1. We thank the referee for pointing out the practical importance of identification issues and thereby motivating our analysis of the problem.

2. As a byproduct of the exercise of obtaining the estimates of information quality, the investor's risk aversion coefficient is also identified. Admati and Ross (1985) also show that risk aversion is identifiable from returns data.

3. More specifically, we generalize the model of section 6 of Admati (1985) slightly. Our generalization allows different investors to receive signals pertaining to different subsets of assets. Admati, on the other hand, assumes that the subset of assets over which investors receive private signals is common to all investors. Her assumption implies that C_j is the same for all investors. This common value of C_j is denoted in Admati (1985) by C.

4. To ensure that aggregation is possible, we also require some technical assumptions which ensure that the integrals are well defined. Notably, we assume that [0, 1] is endowed with the Borel σ- Algebra. Further, the maps $j \rightarrow S_j$ and $j \rightarrow \rho_j$ are measurable.

5. The explicit algebraic manipulations required in this derivation are available from the authors upon request.

6. $\sqrt{\cdot}$ represents the square root function which assigns a positive semidefinite square root matrix to positive semidefinite matrices. The decomposition in (3) is based on the standard variance decomposition $VAR[\tilde{D}_j^I] = VAR(E[\tilde{D}_j^I | \tilde{Z}]) + E(VAR[\tilde{D}_j^I | \tilde{Z}])$ (see Feller 1966, Problem 18, page 164), and the fact that, for normal random variables, $VAR[\tilde{D}_j^I | \tilde{Z}]$ is a constant and, thus, equal to its expectation.

7. Further details of the derivation of $VAR(E[\tilde{D}_j I| \tilde{Z}]$ and $VAR[\tilde{D}_j I| \tilde{Z}]$ from the basic demand equation (1) are available from the authors upon request.

8. All of the numerical examples provided in the text are summarized in Table 1 for the convenience of the reader.

9. The algebra required for deriving this equation is available from the authors upon request.

10. The Seemingly Unrelated Regressions (SUR) technique of Zellner (1962) could, of course, also be used to estimate the covariance matrix. However, the use of SUR will not increase efficiency in this case (see Zellner 1962). Note also that estimating Σ from the *en seriatim* regressions is straightforward: let $\hat{\varepsilon}_i$ represent the vector of estimated residuals from the i^{th} regression. The ij^{th} term of Σ, Σ_{ij}, can be consistently estimated by $(\hat{\varepsilon}_i \cdot \hat{\varepsilon}_j)/T$.

11. In other words, the limit in probability, as the number of observations increases to infinity, is ρ. See White (1984).

12. A_1 is determined by variance and covariance matrices of multivariate normal distributions. Many consistent estimators of such matrices exist. See Johnson and Kotz (1972, Chapter 35, Section 7, pages 62-70).

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Figure 1. Decomposition of the Demand Vector

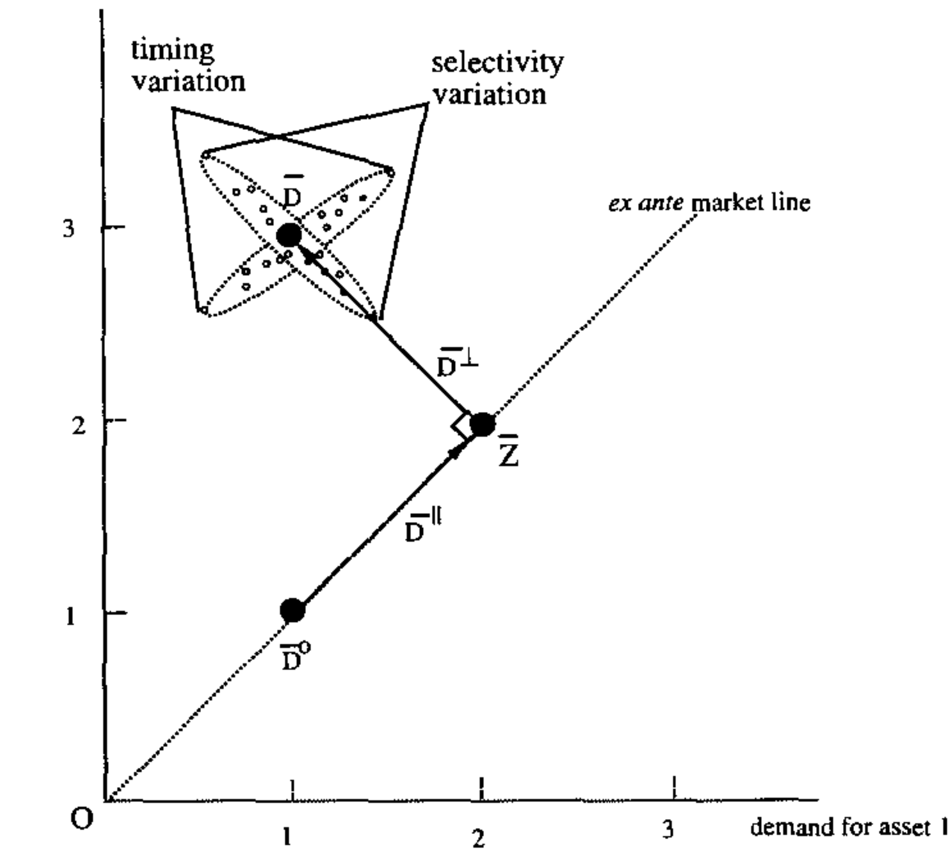

Figure 1 schematically depicts the decomposition of security demand. $\bar{D}^o$ represents the average demand for risky assets of a typical investor in an economy in which no agents have private information. In this case, the vector of asset demands is proportional to the vector of asset supplies: demand lies on the market line. The scatter of open circles represents the realized demand vectors of a typical investor with the same risk preference when endowed with private information. In this case, the investor's demand vector depends on the particular realized signal he receives. Even if one considers only the average (*ex ante*) demand across private signal realizations, $\bar{D}$, it will not, in general, equal demand in the absence of information. The effects of private information on demand are manifold. First, private information affects the investor's willingness to hold market risk (aggressiveness). The magnitude of this effect is measured by the length of the vector $\bar{D}^{\parallel}$. Second, it induces a mean deviation of the demand vector from the *ex ante* market line. We term this effect specialization and measure it by the length of the vector $\bar{D}^{\perp}$.

Table 1. Summary of Numerical Examples Presented in Manuscript

Here we summarize the examples developed in the paper. The common parameters for all the examples are: $\overline{Z} = [2, 2]^T$; $\Delta = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}$; $Q = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$; $\overline{\rho} = 2$;

$$\rho_j = 4 \text{ , all j; } U = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} ; V = \frac{1}{101}\begin{bmatrix} 72 & 20 \\ 20 & 28 \end{bmatrix} .$$

	INVESTOR 1	INVESTOR 2	INVESTOR 3	INVESTOR 4
Θ_j	$\begin{bmatrix} 8.000 & -2.000 \\ -2.000 & 8.000 \end{bmatrix}$	$\begin{bmatrix} 3.000 & -2.000 \\ -2.000 & 3.000 \end{bmatrix}$	$\begin{bmatrix} 24.000 & -12.000 \\ -12.000 & 36.000 \end{bmatrix}$	$\begin{bmatrix} 8.000 & -4.000 \\ -4.000 & 12.000 \end{bmatrix}$
$\overline{D}^I(j)$	$[2.600, 1.600]^T$	$[0.600, 0.100]^T$	$[6.000, 6.000]^T$	$[2.000, 2.000]^T$
$\overline{D}^{\parallel}(j)$	$[2.100, 2.100]^T$	$[0.350, 0.350]^T$	$[6.000, 6.000]^T$	$[2.000, 2.000]^T$
$\overline{D}^{\perp}(j)$	$[0.500, -0.500]^T$	$[0.250, -0.250]^T$	$[0.000, 0.000]^T$	$[0.000, 0.000]^T$
$\overline{D}(j)$	$[3.600, 2.600]^T$	$[1.600, 1.100]^T$	$[7.000, 7.000]^T$	$[3.000, 3.000]^T$
$\kappa(\Theta_j)$	2.100	0.350	6.000	2.000
$\lVert\overline{D}^{\perp}(j)\rVert$	0.707	0.353	0.000	0.000
$\mathbf{A}(\Theta_j, \rho_j)$	$\begin{bmatrix} 0.994 & -0.166 \\ -0.166 & 1.074 \end{bmatrix}$	$\begin{bmatrix} 0.317 & -0.183 \\ -0.183 & 0.324 \end{bmatrix}$	$\begin{bmatrix} 5.500 & -3.563 \\ -3.563 & 12.438 \end{bmatrix}$	$\begin{bmatrix} 1.000 & -0.563 \\ -0.563 & 1.938 \end{bmatrix}$
$\mathbf{T}(\Theta_j, \rho_j)$	$\begin{bmatrix} 4.866 & 3.319 \\ 3.319 & 3.178 \end{bmatrix}$	$\begin{bmatrix} 0.260 & 0.014 \\ 0.014 & 0.120 \end{bmatrix}$	$\begin{bmatrix} 25.313 & 25.313 \\ 25.313 & 50.625 \end{bmatrix}$	$\begin{bmatrix} 2.813 & 2.813 \\ 2.813 & 5.625 \end{bmatrix}$

In Table 1 above, $\overline{Z}$ represents the expected supply of the risky assets. Δ is the conditional variance-covariance matrix of asset payoffs for an investor whose signal precision equals the average signal precision. Q represents the average precision of private signals, $\overline{\rho}$ is the average coefficient of risk aversion. ρ_j is the risk aversion coefficient of the investor j = 1, 2, 3, and 4. U and V are the unconditional variance-covariance matrices for asset payoffs and asset supplies, respectively. The characteristics of investor demand are computed above. In the table presented above, Θ_j represents the divergence between investor j's information structure and the market average. $\overline{D}^I$ represents *ex ante* information-induced demand. $\overline{D}^{\parallel}$ represents information-induced demand, which is proportional to the expected supplies of risky assets (the market portfolio). It thus represents market timing activity. κ represents the proportional increase in demand for the market portfolio relative to demand in the absence of private information. $\overline{D}^{\perp}$ represents *ex ante* demand, which is orthogonal to the *ex ante* market portfolio; it thus captures portfolio specialization. **T** represents variation in asset demand that can be attributed to market timing, and **A** represents the variation that can be attributed to asset selection.