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Stabilization of a laminated beam with interfacial slip by boundary controls

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Abstract

We consider two identical beams on top of each other with an adhesive in between. A considerable natural slip occurs in the structure and will not be ignored as was done in the previous investigations. In this work we take into account this slip and prove that we can stabilize the system in an exponential manner using boundary controls. The model consists of three coupled equations. The first two are related to the well-known Timoshenko system, and the third one describes the dynamic of the slip. Our result improves the few existing similar works in the literature.

MSC: 34B05; 34D05; 34H05

Keywords: exponential stabilization; vibration reduction; Timoshenko system; slip dynamic; boundary control; multiplier technique

1 Introduction

Of concern is a structure of two identical beams of uniform thickness stuck together by an adhesive. They are placed on top of each other. The structure is subject to a longitudinal displacement in addition to transversal and rotational displacements. These vibrations are undesirable, and it is our goal to stabilize the system in a fast way. These structures have gained much in popularity and are known under the name of 'laminated' beams. They are of considerable importance in engineering. The beams are attached together in such a way that a 'slip' is permitted while they are continuously in contact with each other. In certain situations the slip is purposively allowed with the objective to obtain some damping. This damping should be able to restore the system to its equilibrium state. Fastening very tightly the layers can affect anormally the performance of the structure.

A model based on the Timoshenko theory was derived in [1] (see also [2] for another one), namely

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I_\rho(3s_{tt} - \psi_{tt}) - G(\psi - w_x) - D(3s_{xx} - \psi_{xx}) = 0, \\ 3I_\rho s_{tt} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t - 3Ds_{xx} = 0, \end{cases}$$

where $x \in (0,1)$ and $t > 0$, with initial data

$$(w, \psi, s)(x,0) = (w_0, \psi_0, s_0), \qquad (w_t, \psi_t, s_t)(x,0) = (w_1, \psi_1, s_1), \quad x \in (0,1)$$

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and the cantilever boundary conditions. Here w, ψ, ρ, G, I_ρ, D, γ, β denote the transverse displacement, rotation angle, density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness and adhesive damping parameter, respectively, and s is proportional to the amount of slip along the interface. It is rather the third equation which is in the spotlight as when $s \equiv 0$ we recover the standard Timoshenko system. As is well known by now, the Timoshenko system has been studied by many authors, and many results may be found in the literature. It has been stabilized by means of different controls such as internal and/or boundary frictional and/or viscoelastic damping, dynamic boundary conditions, pointwise damping, distributed damping, heat damping, *etc.* The large number of citations cannot fit in this small paper. We refer the reader, however, to [3–12] for similar boundary controls to the ones used here and also for the interesting papers [13–15].

In [16] an exponential decay result was proved for this problem with one end fixed ($w(0,t) = \psi(0,t) = s(0,t) = 0$, $t > 0$) and $\psi(1,t) - w_x(1,t) = u_1(t)$, $s_x(1,t) = 0$, $(3s_x - \psi_x)(1,t) = u_2(t)$, $t > 0$ at the other end. They adopted the boundary control

$$u_1(t) = k_1 w_t(1,t), \qquad u_2(t) = -k_2(3s_t - \psi_t)(1,t)$$

and assumed that $r_1 := \frac{G}{\rho} \neq \frac{D}{I_\rho} =: r_2$ and $k_i \neq r_i$, $i = 1,2$. Moreover, they checked that the damping present in the third equation alone is not able to stabilize the structure in an exponential manner.

Under different boundary controls, namely

$$\psi(0,t) - w_x(0,t) = -k_1 w_t(0,t) - w(0,t), \qquad 3s_x(1,t) - \psi_x(1,t) = -k_1 \xi_t(1,t) - \xi(1,t),$$

the authors in [17] proved an exponential stabilization of the system provided that the 'dominant' part of the closed loop system is itself exponentially stable.

Since then, it seems that the subject remained dormant. We would like to bring back to life this matter of treating, in a more adequate fashion, structures for which the amount of slip is considerable.

We discuss here the same model and boundary conditions as in [16]. We prove an exponential stabilization result without assuming the conditions $r_1 := \frac{G}{\rho} \neq \frac{D}{I_\rho} =: r_2$ and $k_i \neq r_i$, $i = 1,2$. These conditions are dropped, and instead we will assume that $\rho G < I_\rho$. Our approach is different from the one used in [16]. Namely, we will consider the system

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I_\rho(3s_{tt} - \psi_{tt}) - G(\psi - w_x) - (3s - \psi)_{xx} = 0, \\ I_\rho s_{tt} + G(\psi - w_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t - s_{xx} = 0 \end{cases} \tag{1}$$

for $x \in (0,1)$, $t > 0$, with the boundary conditions

$$\begin{cases} w(0,t) = \psi(0,t) = s(0,t) = 0, \\ (\psi - w_x)(1,t) = k_1 w_t(1,t), \qquad s_x(1,t) = 0, \\ (3s - \psi)_x(1,t) = -k_2(3s - \psi)_t(1,t) \end{cases} \tag{2}$$

for $t \geq 0$.

The well-posedness follows easily from a slight modification of the arguments in [16, 17] (see also references in [1]). We have weak solutions in $(V_*^1 \times L^2)^3$ and strong solutions in $(V_*^2 \times H^1)^3$, where

$$V_*^k = \{v : v \in H^k(0,1) : v(0) = 0\}, \quad k = 1, 2.$$

We shall focus here on the asymptotic behavior of solutions and in particular on the exponential stabilization of the system.

In the next section we prove that the energy is decreasing, define the different functionals we will utilize later and prepare some useful lemmas. Section 2 is devoted to the statement and proof of our main result. The last section is a short one containing our conclusion.

2 Some useful preliminary results

The energy of system (1)-(2) is given by

$$\begin{aligned} E(t) = \frac{1}{2}\big[&3I_\rho\|s_t\|^2 + 4\gamma\|s\|^2 + 3\|s_x\|^2 + I_\rho\|3s_t - \psi_t\|^2 + \|(3s-\psi)_x\|^2 \\ &+ \rho\|w_t\|^2 + G\|\psi - w_x\|^2\big], \quad t \geq 0, \end{aligned} \tag{3}$$

where $\|\cdot\|$ denotes the norm in $L^2(0,1)$.

Proposition 1 *The energy $E(t)$ is decreasing and in fact we have*

$$E'(t) = -Gk_1 w_t^2(1) - 4\beta\|s_t\|^2 - k_2(3s-\psi)_t^2(1), \quad t \geq 0.$$

Proof This follows directly by multiplying the first equation in (1) by w_t (the second by $(3s-\psi)_t$ and the third by s_t) and integrating over $(0,1)$. Integration by parts and the boundary conditions will also be used. Indeed, we obtain

$$\frac{\rho}{2}\frac{d}{dt}\|w_t\|^2 + G\big((\psi - w_x)_x, w_t\big) = 0, \quad t \geq 0$$

or

$$\frac{\rho}{2}\frac{d}{dt}\|w_t\|^2 - G(\psi - w_x, w_{xt}) + \big[G(\psi - w_x)w_t\big]_0^1 = 0, \quad t \geq 0$$

and by our boundary conditions

$$\frac{\rho}{2}\frac{d}{dt}\|w_t\|^2 - G(\psi - w_x, w_{xt}) + Gk_1 w_t^2(1) = 0, \quad t \geq 0.$$

The observation

$$G(\psi - w_x, w_{xt}) = -G\big(\psi - w_x, (\psi - w_x - \psi)_t\big) = -\frac{G}{2}\frac{d}{dt}\|\psi - w_x\|^2 + G(\psi - w_x, \psi_t)$$

leads to

$$\frac{1}{2}\frac{d}{dt}\big[\rho\|w_t\|^2 + G\|\psi - w_x\|^2\big] - G(\psi - w_x, \psi_t) + Gk_1 w_t^2(1) = 0, \quad t \geq 0. \tag{4}$$

Working with the third equation of (1) we arrive at

$$I_\rho(s_{tt},s_t) + G(\psi - w_x, s_t) + \frac{4\gamma}{3}(s,s_t) + \frac{4\beta}{3}\|s_t\|^2 - (s_{xx},s_t) = 0$$

or

$$\frac{1}{2}\frac{d}{dt}\left[I_\rho\|s_t\|^2 + \frac{4\gamma}{3}\|s\|^2 + \|s_x\|^2\right] + G(\psi - w_x, s_t) + \frac{4\beta}{3}\|s_t\|^2 = 0, \quad t \geq 0. \tag{5}$$

Summing up (4) and (5) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[3I_\rho\|s_t\|^2 + 4\gamma\|s\|^2 + 3\|s_x\|^2 + \rho\|w_t\|^2 + G\|\psi - w_x\|^2\right]\\ &\quad + G(\psi - w_x, 3s_t - \psi_t) + 4\beta\|s_t\|^2 + Gk_1 w_t^2(1) = 0, \quad t \geq 0.\end{aligned} \tag{6}$$

The second equation in (1) allows us to write, for $t \geq 0$,

$$\begin{aligned}&\frac{I_\rho}{2}\frac{d}{dt}\|3s_t - \psi_t\|^2\\ &\quad = \bigl(G(\psi - w_x) + (3s - \psi)_{xx}, (3s - \psi)_t\bigr)\\ &\quad = G\bigl(\psi - w_x, (3s - \psi)_t\bigr) - \bigl((3s - \psi)_x, (3s - \psi)_{xt}\bigr) + \bigl[(3s - \psi)_x(3s - \psi)_t\bigr]_0^1\\ &\quad = G\bigl(\psi - w_x, (3s - \psi)_t\bigr) - \bigl((3s - \psi)_x, (3s - \psi)_{xt}\bigr) - k_2(3s - \psi)_t^2(1)\end{aligned}$$

or

$$\frac{1}{2}\frac{d}{dt}\left[I_\rho\|3s_t - \psi_t\|^2 + \bigl\|(3s - \psi)_x\bigr\|^2\right] = G\bigl(\psi - w_x, (3s - \psi)_t\bigr) - k_2(3s - \psi)_t^2(1). \tag{7}$$

Our assertion follows at once from (6) and (7). □

Although from this proposition we see that the energy is uniformly bounded and decreasing, it is not clear how to prove exponential decay from this functional at this stage. We need to establish a new functional $F(t)$ which is suitable enough to derive an exponential decay. The strategy (which is by now standard) consists in starting with the energy $E(t)$ and modifying it by adding new adequate terms (functionals) which may be estimated below and above by similar terms already existing in the expression of the energy (leading to the equivalence of both functionals) and whose derivatives provide us with the missing terms in the energy after differentiation. That is, the goal is to obtain an inequality of the form $F'(t) \leq -\kappa F(t)$ for some positive constant κ.

We claim that the functional

$$F(t) := E(t) + \sum_{i=1}^{5}\delta_i L_i(t), \quad t \geq 0$$

with

$$L_1(t) = I_\rho(s_t,s) - \rho(w_t,S), \qquad L_2(t) = I_\rho(3s_t - \psi_t, 3s - \psi),$$

$$L_3(t) = -\rho(xw_t, \psi - w_x), \qquad L_4(t) = I_\rho\big(x(3s_x - \psi_x), 3s_t - \psi_t\big),$$
$$L_5(t) = -\rho(w_t, w),$$

where

$$S(x,t) = -\int_0^x s(r,t)\,dr, \tag{8}$$

for $\delta_i > 0$, $i = 1,\ldots,5$, to be determined, is an appropriate one.

It is easy to see that $F(t)$ and $E(t)$ are equivalent. Simple use of the Cauchy-Schwarz inequality and the Poincaré inequality will do.

Lemma 1 *The derivative of $L_1(t)$ along solutions of* (1)-(2) *is estimated by*

$$L_1'(t) \le \left(I_\rho + \frac{\rho^2}{4\varepsilon_0} + \frac{4\beta^2}{9\varepsilon_0}\right)\|s_t\|^2 + (\varepsilon_0 - 1)\|s_x\|^2 + \left(\varepsilon_0 - \frac{4\gamma}{3}\right)\|s\|^2$$
$$+ \varepsilon_0\|w_t\|^2 + \frac{k_1^2G^2}{4\varepsilon_0}w_t^2(1), \quad t \ge 0, \varepsilon_0 > 0.$$

Proof As simple differentiation of $L_1(t)$ and use of the first and third equations in system (1) yield

$$\begin{aligned}
L_1'(t) &= I_\rho(s_t,s_t) + I_\rho(s_{tt},s) - \rho(w_{tt},S) - \rho(w_t,S_t) \\
&= I_\rho\|s_t\|^2 - G(\psi - w_x, s) - \frac{4\gamma}{3}\|s\|^2 - \frac{4\beta}{3}(s_t,s) - \|s_x\|^2 \\
&\quad + G\big((\psi - w_x)_x, S\big) - \rho(w_t,S_t), \quad t \ge 0.
\end{aligned}$$

Observe that

$$\begin{aligned}
G\big((\psi - w_x)_x, S\big) &= G\big[(\psi - w_x)S\big]_0^1 - G(\psi - w_x, S_x) \\
&= G(\psi - w_x)(1)S(1) + G(\psi - w_x, s) \\
&\le \varepsilon_0\|s_x\|^2 + \frac{k_1^2G^2}{4\varepsilon_0}w_t^2(1) + G(\psi - w_x, s), \quad t \ge 0.
\end{aligned}$$

Therefore

$$L_1'(t) \le \left(I_\rho + \frac{\rho^2}{4\varepsilon_0} + \frac{4\beta^2}{9\varepsilon_0}\right)\|s_t\|^2 + (\varepsilon_0 - 1)\|s_x\|^2 + \left(\varepsilon_0 - \frac{4\gamma}{3}\right)\|s\|^2$$
$$+ \varepsilon_0\|w_t\|^2 + \frac{k_1^2G^2}{4\varepsilon_0}w_t^2(1), \quad t \ge 0. \qquad \square$$

Lemma 2 *The derivative of $L_2(t)$ along solutions of* (1)-(2) *satisfies*

$$L_2'(t) \le I_\rho\|3s_t - \psi_t\|^2 + \left(\varepsilon_0 - \frac{1}{2}\right)\big\|(3s - \psi)_x\big\|^2 + \frac{G^2}{2}\|\psi - w_x\|^2$$
$$+ \frac{k_2^2}{4\varepsilon_0}(3s - \psi)_t^2(1), \quad t \ge 0, \varepsilon_0 > 0.$$

Proof Using the second equation of (1), we find

$$\begin{aligned} I_\rho \frac{d}{dt}(3s_t-\psi_t,3s-\psi) &= I_\rho\|3s_t-\psi_t\|^2+G(\psi-w_x,3s-\psi)+\big[(3s_x-\psi_x)(3s-\psi)\big]_0^1-\|3s_x-\psi_x\|^2 \\ &\le I_\rho\|3s_t-\psi_t\|^2+G(\psi-w_x,3s-\psi)+(\varepsilon_0-1)\big\|(3s-\psi)_x\big\|^2 \\ &\quad+\frac{k_2^2}{4\varepsilon_0}(3s-\psi)_t^2(1),\quad t\ge 0. \end{aligned}$$

The estimate

$$G(\psi-w_x,3s-\psi)\le\frac{1}{2}\big\|(3s-\psi)_x\big\|^2+\frac{G^2}{2}\|\psi-w_x\|^2$$

implies that

$$\begin{aligned} L_2'(t) &\le I_\rho\|3s_t-\psi_t\|^2+\left(\varepsilon_0-\frac{1}{2}\right)\big\|(3s-\psi)_x\big\|^2+\frac{G^2}{2}\|\psi-w_x\|^2 \\ &\quad+\frac{k_2^2}{4\varepsilon_0}(3s-\psi)_t^2(1),\quad t\ge 0. \end{aligned}$$

□

Lemma 3 *The derivative of the functional* $L_3(t)$ *along solutions of* (1)-(2) *may be estimated as follows*:

$$\begin{aligned} L_3'(t) &\le \frac{\rho}{2}\|\psi_t-3s_t\|^2+\varepsilon_0\|w_t\|^2+\frac{9\rho^2}{4\varepsilon_0}\|s_t\|^2+\frac{\rho+Gk_1^2}{2}w_t^2(1) \\ &\quad-\frac{G}{2}\|\psi-w_x\|^2,\quad t\ge 0,\varepsilon_0>0. \end{aligned}$$

Proof In view of the first and third equations in (1) and the definition of $L_3(t)$, we see that

$$\begin{aligned} L_3'(t) &= -\rho\big(xw_t,(\psi-w_x)_t\big)-\rho(xw_{tt},\psi-w_x) \\ &= -\rho(xw_t,\psi_t)+\rho(xw_t,w_{xt})+G\big(x(\psi-w_x)_x,\psi-w_x\big) \\ &= -\rho(xw_t,\psi_t-3s_t)-3\rho(xw_t,s_t)+\frac{\rho}{2}\left(x,\frac{dw_t^2}{dx}\right)+\frac{G}{2}\left(x,\frac{d(\psi-w_x)^2}{dx}\right) \\ &= -\rho(xw_t,\psi_t-3s_t)-3\rho(xw_t,s_t)+\frac{\rho}{2}\big[xw_t^2\big]_0^1+\frac{G}{2}\big[x(\psi-w_x)^2\big]_0^1 \\ &\quad-\frac{\rho}{2}\|w_t\|^2-\frac{G}{2}\|\psi-w_x\|^2,\quad t\ge 0. \end{aligned}$$

Therefore, from our boundary conditions

$$\begin{aligned} L_3'(t) &= -\rho(xw_t,\psi_t-3s_t)-3\rho(xw_t,s_t)+\frac{\rho}{2}w_t^2(1) \\ &\quad+\frac{G}{2}(\psi-w_x)^2(1)-\frac{\rho}{2}\|w_t\|^2-\frac{G}{2}\|\psi-w_x\|^2,\quad t\ge 0. \end{aligned}$$

Next, the estimations

$$-\rho(xw_t,\psi_t-3s_t)\le\frac{\rho}{2}\|w_t\|^2+\frac{\rho}{2}\|\psi_t-3s_t\|^2$$

and

$$-3\rho(xw_t, s_t) \le \varepsilon_0\|w_t\|^2 + \frac{9\rho^2}{4\varepsilon_0}\|s_t\|^2$$

imply that

$$\begin{aligned} L_3'(t) &\le \frac{\rho}{2}\|w_t\|^2 + \frac{\rho}{2}\|\psi_t - 3s_t\|^2 + \varepsilon_0\|w_t\|^2 + \frac{9\rho^2}{4\varepsilon_0}\|s_t\|^2 \\ &\quad + \frac{\rho}{2}w_t^2(1) + \frac{G}{2}(\psi - w_x)^2(1) - \frac{\rho}{2}\|w_t\|^2 - \frac{G}{2}\|\psi - w_x\|^2 \\ &\le \frac{\rho}{2}\|\psi_t - 3s_t\|^2 + \varepsilon_0\|w_t\|^2 + \frac{9\rho^2}{4\varepsilon_0}\|s_t\|^2 \\ &\quad + \frac{\rho + Gk_1^2}{2}w_t^2(1) - \frac{G}{2}\|\psi - w_x\|^2, \quad t \ge 0. \end{aligned}$$

□

Lemma 4 *Along solutions of* (1)-(2), *it holds that*

$$L_4'(t) \le \frac{I_\rho + k_2^2}{2}(3s_t - \psi_t)^2(1) - \frac{I_\rho}{2}\|3s_t - \psi_t\|^2 + \frac{G^2}{2}\|\psi - w_x\|^2, \quad t \ge 0.$$

Proof Clearly

$$\begin{aligned} L_4'(t) &= I_\rho\big(x(3s_x - \psi_x)_t, 3s_t - \psi_t\big) + I_\rho\big(x(3s_x - \psi_x), 3s_{tt} - \psi_{tt}\big) \\ &= \frac{I_\rho}{2}\left(x, \frac{d(3s_t - \psi_t)^2}{dx}\right) + \big(x(3s_x - \psi_x), G(\psi - w_x) + (3s - \psi)_{xx}\big) \\ &= \frac{I_\rho}{2}(3s_t - \psi_t)^2(1) - \frac{I_\rho}{2}\|3s_t - \psi_t\|^2 + G\big(x(3s_x - \psi_x), \psi - w_x\big) \\ &\quad + \frac{1}{2}(3s_x - \psi_x)^2(1) - \frac{1}{2}\|3s_x - \psi_x\|^2, \quad t \ge 0, \end{aligned}$$

and therefore

$$\begin{aligned} L_4'(t) &\le \frac{I_\rho}{2}(3s_t - \psi_t)^2(1) - \frac{I_\rho}{2}\|3s_t - \psi_t\|^2 + \frac{1}{2}\|3s_x - \psi_x\|^2 \\ &\quad + \frac{G^2}{2}\|\psi - w_x\|^2 + \frac{1}{2}(3s_x - \psi_x)^2(1) - \frac{1}{2}\|3s_x - \psi_x\|^2, \quad t \ge 0, \end{aligned}$$

or simply

$$L_4'(t) \le \frac{I_\rho + k_2^2}{2}(3s_t - \psi_t)^2(1) - \frac{I_\rho}{2}\|3s_t - \psi_t\|^2 + \frac{G^2}{2}\|\psi - w_x\|^2, \quad t \ge 0.$$

□

Lemma 5 *An estimation of the derivative of* $L_5(t)$ *along solutions of* (1)-(2) *is given by*

$$\begin{aligned} L_5'(t) &\le -\rho\|w_t\|^2 + \left(2\varepsilon_0 + \frac{3G}{2}\right)\|\psi - w_x\|^2 + 4\varepsilon_0\big[\big\|(\psi - 3s)_x\big\|^2 + 9\|s\|^2\big] \\ &\quad + G\|\psi_x\|^2 + \frac{G^2k_1^2}{4\varepsilon_0}w_t^2(1), \quad t \ge 0, \varepsilon_0 > 0. \end{aligned}$$

Proof It is easy to see, from the first equation in (1) and the boundary conditions, that

$$\begin{aligned} L_5'(t) &= -\rho\|w_t\|^2 - \rho(w_{tt}, w) = -\rho\|w_t\|^2 + G\big((\psi - w_x)_x, w\big) \\ &= -\rho\|w_t\|^2 + G\big[(\psi - w_x)w\big]_0^1 - G(\psi - w_x, w_x) \\ &= -\rho\|w_t\|^2 + G(\psi - w_x)(1)w(1) - G(\psi - w_x, w_x), \quad t \ge 0. \end{aligned} \tag{9}$$

The last two terms in (9) may be estimated as follows:

$$G(\psi - w_x)(1)w(1) \le \varepsilon_0\|w_x\|^2 + \frac{G^2k_1^2}{4\varepsilon_0}w_t^2(1) \le 2\varepsilon_0\|\psi - w_x\|^2 + 2\varepsilon_0\|\psi\|^2 + \frac{G^2k_1^2}{4\varepsilon_0}w_t^2(1)$$

so

$$G(\psi - w_x)(1)w(1) \le 2\varepsilon_0\|\psi - w_x\|^2 + 4\varepsilon_0\big[\big\|(\psi - 3s)_x\big\|^2 + 9\|s\|^2\big] + \frac{G^2k_1^2}{4\varepsilon_0}w_t^2(1),$$

and

$$\begin{aligned} G(\psi - w_x, w_x) &\le \frac{1}{2}G\|\psi - w_x\|^2 + \frac{G}{2}\|w_x\|^2 \le \frac{G}{2}\|\psi - w_x\|^2 + \frac{G}{2}\big[2\|\psi - w_x\|^2 + 2\|\psi_x\|^2\big] \\ &\le \frac{3G}{2}\|\psi - w_x\|^2 + G\|\psi_x\|^2, \quad t \ge 0. \end{aligned}$$

Taking into account the previous relations, we find

$$\begin{aligned} L_5'(t) &\le -\rho\|w_t\|^2 + 2\varepsilon_0\|\psi - w_x\|^2 + 4\varepsilon_0\big[\big\|(\psi - 3s)_x\big\|^2 + 9\|s\|^2\big] + \frac{G^2k_1^2}{4\varepsilon_0}w_t^2(1) \\ &\quad + \frac{3G}{2}\|\psi - w_x\|^2 + G\|\psi_x\|^2 \\ &\le -\rho\|w_t\|^2 + \left(2\varepsilon_0 + \frac{3G}{2}\right)\|\psi - w_x\|^2 + 4\varepsilon_0\big[\big\|(\psi - 3s)_x\big\|^2 + 9\|s\|^2\big] \\ &\quad + G\|\psi_x\|^2 + \frac{G^2k_1^2}{4\varepsilon_0}w_t^2(1), \quad t \ge 0. \end{aligned}$$

□

3 Main result

Using the previous lemmas we obtain the following result.

Theorem 1 *For the energy E defined above, there exist two positive constants K and κ_0 such that*

$$E(t) \le Ke^{-\kappa_0 t}, \quad t \ge 0$$

provided that $\rho < I_\rho/G$.

Proof In view of Lemmas 1-5, we see that

$$\begin{aligned} F'(t) &\le -Gk_1w_t^2(1) - 4\beta\|s_t\|^2 - k_2(3s - \psi)_t^2(1) + \delta_1\left(I_\rho + \frac{\rho^2}{4\varepsilon_0} + \frac{4\beta^2}{9\varepsilon_0}\right)\|s_t\|^2 \\ &\quad + \delta_1(\varepsilon_0 - 1)\|s_x\|^2 + \delta_1\left(\varepsilon_0 - \frac{4\gamma}{3}\right)\|s\|^2 + \varepsilon_0\delta_1\|w_t\|^2 + \frac{\delta_1k_1^2G^2}{4\varepsilon_0}w_t^2(1) \end{aligned}$$

$$+ \delta_2 I_\rho \|3s_t - \psi_t\|^2 + \delta_2\left(\varepsilon_0 - \frac{1}{2}\right)\left\|(3s-\psi)_x\right\|^2 + \frac{\delta_2 G^2}{2}\|\psi - w_x\|^2$$
$$+ \frac{k_2^2\delta_2}{4\varepsilon_0}(3s-\psi)_t^2(1) + \frac{\delta_3\rho}{2}\|\psi_t - 3s_t\|^2 + \varepsilon_0\delta_3\|w_t\|^2 + \frac{9\rho^2\delta_3}{4\varepsilon_0}\|s_t\|^2$$
$$+ \frac{(\rho + Gk_1^2)\delta_3}{2}w_t^2(1) - \frac{G\delta_3}{2}\|\psi - w_x\|^2 + \frac{(I_\rho + k_2^2)\delta_4}{2}(3s_t - \psi_t)^2(1)$$
$$- \frac{I_\rho\delta_4}{2}\|3s_t - \psi_t\|^2 + \frac{G^2\delta_4}{2}\|\psi - w_x\|^2 - \delta_5\rho\|w_t\|^2 + G\delta_5\|\psi_x\|^2$$
$$+ \frac{G^2k_1^2}{4\varepsilon_0}\delta_5 w_t^2(1) + \delta_5\left(2\varepsilon_0 + \frac{3G}{2}\right)\|\psi - w_x\|^2 + 4\varepsilon_0\delta_5\left[\left\|(\psi - 3s)_x\right\|^2 + 9\|s\|^2\right]$$

or, using

$$\|\psi_x\|^2 \le 2\left\|(\psi - 3s)_x\right\|^2 + 18\|s_x\|^2,$$

we find

$$F'(t) \le \left[-Gk_1 + \frac{\delta_1 k_1^2 G^2}{4\varepsilon_0} + \frac{\rho + Gk_1^2}{2}\delta_3 + \frac{G^2k_1^2}{4\varepsilon_0}\delta_5\right]w_t^2(1)$$
$$+ \left[-k_2 + \frac{k_2^2\delta_2}{4\varepsilon_0} + \frac{I_\rho + k_2^2}{2}\delta_4\right](3s-\psi)_t^2(1)$$
$$+ \left[-4\beta + \delta_1\left(I_\rho + \frac{\rho^2}{4\varepsilon_0} + \frac{4\beta^2}{9\varepsilon_0}\right) + \frac{9\rho^2}{4\varepsilon_0}\delta_3\right]\|s_t\|^2$$
$$+ \left[\delta_1(\varepsilon_0 - 1) + 18G\delta_5\right]\|s_x\|^2 + \left[\delta_1\left(\varepsilon_0 - \frac{4\gamma}{3}\right) + 36\varepsilon_0\delta_5\right]\|s\|^2$$
$$+ [-\delta_5\rho + \varepsilon_0\delta_1 + \varepsilon_0\delta_3]\|w_t\|^2 + \left[\frac{\rho\delta_3}{2} - \frac{I_\rho\delta_4}{2} + \delta_2 I_\rho\right]\|3s_t - \psi_t\|^2$$
$$+ \left[\delta_2\left(\varepsilon_0 - \frac{1}{2}\right) + 4\varepsilon_0\delta_5 + 2G\delta_5\right]\left\|(3s-\psi)_x\right\|^2$$
$$+ \left\{-\frac{G\delta_3}{2} + \frac{G^2\delta_2}{2} + \frac{G^2\delta_4}{2} + \delta_5\left(2\varepsilon_0 + \frac{3G}{2}\right)\right\}\|\psi - w_x\|^2, \quad t \ge 0. \tag{10}$$

We shall forget for a moment about the first three terms on the right-hand side of (10) and focus on the rest of the coefficients. We need

$$\begin{cases} \delta_1(\varepsilon_0 - 1) + 18G\delta_5 < 0, \\ \delta_1(\varepsilon_0 - \frac{4\gamma}{3}) + 36\varepsilon_0\delta_5 < 0, \\ -\delta_5\rho + \varepsilon_0\delta_1 + \varepsilon_0\delta_3 < 0, \\ \frac{\rho\delta_3}{2} - \frac{I_\rho\delta_4}{2} + \delta_2 I_\rho < 0, \\ \delta_2(\varepsilon_0 - \frac{1}{2}) + 4\varepsilon_0\delta_5 + 2G\delta_5 < 0, \\ -\frac{G\delta_3}{2} + \frac{G^2\delta_2}{2} + \frac{G^2\delta_4}{2} + \delta_5(2\varepsilon_0 + \frac{3G}{2}) < 0. \end{cases} \tag{11}$$

Next, ignoring ε_0 as we will take it small enough later, we obtain

$$\begin{cases} 18G\delta_5 < \delta_1, \\ \frac{\rho\delta_3}{2} + \delta_2 I_\rho < \frac{I_\rho\delta_4}{2}, \\ 2G\delta_5 < \frac{\delta_2}{2}, \\ \delta_2 G + G\delta_4 + 3\delta_5 < \delta_3. \end{cases} \tag{12}$$

Observe that there is only a smallness condition on δ_5, so we postpone its selection. There remains

$$\begin{cases} \frac{\rho\delta_3}{2} + \delta_2 I_\rho < \frac{I_\rho\delta_4}{2}, \\ \delta_2 G + G\delta_4 < \delta_3. \end{cases} \tag{13}$$

In turn we see that there is only a smallness condition on δ_2, therefore we need

$$\begin{cases} \rho\delta_3 < I_\rho\delta_4, \\ G\delta_4 < \delta_3. \end{cases} \tag{14}$$

These last relations (14) hold if $\rho < I_\rho/G$ by taking, for instance, $\delta_4 = (\rho + \frac{I_\rho}{G})\frac{\delta_3}{2I_\rho}$. Now we go backward and select δ_2 small enough (in terms of δ_3) so that the relations in (13) are satisfied. Next we pick δ_5 and then δ_1 so that the remaining conditions (12) are fulfilled. At this stage we may choose ε_0 (small enough to satisfy (11)). Finally, we select δ_3 so small that the three first coefficients in (10) be negative.

The inequality $F'(t) \leq -\kappa F(t)$, $t \geq 0$ implies the exponential decay of $F(t)$. This property is shared by $E(t)$ through the equivalence. □

4 Conclusion

Our main goal here is the handling of the interfacial slip and the stabilization of the system. This has been established using a different way than the one used in the literature and with much weaker conditions. Indeed, the conditions $r_1 := \frac{G}{\rho} \neq \frac{D}{I_\rho} =: r_2$ and $k_i \neq r_i$, $i = 1, 2$, are dropped and replaced by $\rho G < I_\rho$.

Competing interests

The author declares that they have no competing interests.

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