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Global structure of positive solutions for second-order discrete Neumann problems involving a superlinear nonlinearity with zeros

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Abstract
This paper studies the global structure of positive solutions of a class discrete Neumann problem which includes the superlinear nonlinearity with zeros. The main results are based on the method of lower and upper solutions, *a priori* estimates and Brouwer degree theory.

MSC: 34B15; 39A12

Keywords: positive solutions; global structure; degree; difference equation

1 Introduction

The difference equations play an important role in many fields such as science, biology, engineering, and technology where discrete phenomena abound, in addition, from the advent and the rise of computers, where differential equations are solved by employing their approximative difference equations formulations; *e.g.*, see [1–12] and the references therein.

One of the difference equations that has attracted some attention is

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta y(k-1)\big]+q(k)y(k)=f\big(k,y(k)\big), \quad k\in\{1,\dots,N\}=:I,\\ &\Delta y(0)=0, \qquad \Delta y(N)=0, \end{aligned} \tag{1.1}$$

where $\Delta y(k)=y(k+1)-y(k)$ for all $k\in\mathbb{Z}$, $p:\{0,1,\dots,N\}\to\mathbb{R}$, $q:I\to\mathbb{R}$ are functions.

In 2003, Cabada and Otero-Espinar [3] studied the existence of solutions of (1.1) by the method of upper and lower solutions whenever $p(\cdot)\equiv 1$, $q(\cdot)=0$. Anderson *et al.* [2] obtained the existence of solutions of (1.1) by a fixed point theorem whenever $p(\cdot)\equiv 1$ and Lu and Gao [10] obtained the existence of positive solutions of (1.1) by the fixed point theorem in cones. Jun Ji, Bo Yang, Candito, G. D'Aguì and Gao also studied the discrete Neumann problem by different methods, see [1, 4, 6–9] and references therein.

However, very little work has been done for the existence of positive solutions of second-order difference equation involving superlinear nonlinearity with zeros. It is worth pointing out that Ma obtained the existence of multiple solutions for some discrete Sturm-

Liouville problems; see [11]. Inspired by the above work, we study the global structure of positive solutions of the following discrete Neumann boundary value problem:

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta y(k-1)\big] + q(k)y(k) = \lambda a(k)f\big(k,y(k)\big), \quad k \in I, \\ &\Delta y(0) = 0, \qquad \Delta y(N) = 0, \end{aligned} \tag{1.2}$$

where the functions $p : \{0, 1, \dots, N\} \to (0, \infty)$, $q, a : I \to [0, \infty)$ with $a(k) > 0$ on $k \in I$ and the nonlinearity f satisfies:

(C1) $f \in C(I \times [0, \infty), [0, \infty))$ and there exists a function $m : I \to (0, \infty)$, satisfying $m(k_1) + m(k_2) \le m(\frac{k_1+k_2}{2})$ with $k_1, k_2 \in I$, such that $f(k, 0) = f(k, m(k)) = 0$ and $f(k, y) > 0$ if $0 < y < m(k)$.

(C2) There exists a function $h : I \to (0, \infty)$ such that

$$\lim_{y\to 0^+} \frac{f(k,y)}{y} = h(k) \quad \text{uniformly in } k \in I. \tag{1.3}$$

(C3) There exists a subset $I_1 \subset I$ such that

$$\lim_{y\to\infty} \frac{f(k,y)}{y} = \infty \quad \text{uniformly in } k \in I_1.$$

(C4) The function $f_y := \frac{\partial f}{\partial y}$ exists and is continuous in the set $\{(k, y) : k \in I, y \in [0, m(k)]\}$; further,

$$f_y(k,y) < y^{-1}f(k,y), \quad (k,y) \in \big\{(k,y) : k \in I, y \in \big(0, m(k)\big)\big\}. \tag{1.4}$$

Through careful analysis we have found that the nonlinearity has a zero at a variable positive value and has linear growth at zero and locally superlinear growth at infinity. The effect of the variable zero and the condition of superlinear growth in a small interval are the main differences when comparing to the results in [10, 11]. For the results concerning the existence of positive solutions for nonlinear differential equation boundary value problems involving nonlinearity with zero points, see, *e.g.* [13–15] and references therein.

Let $\lambda_{1,ah}$ be the first eigenvalue of the eigenvalue problem

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta y(k-1)\big] + q(k)y(k) = \lambda a(k)h(k)y(k), \quad k \in I, \\ &\Delta y(0) = 0, \qquad \Delta y(N) = 0. \end{aligned} \tag{1.5}$$

It is well known (*cf.* Kelly and Peterson [8], Gao [6]) that $\lambda_{1,ah} > 0$ is positive and simple, and that it is a unique eigenvalue with positive eigenfunctions $\varphi_{1,ah}$.

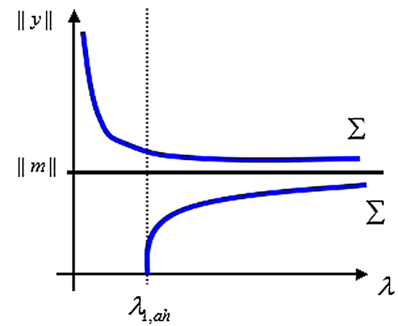
Theorem 1.1 *Let* (C1)-(C3) *hold. Then for every* $0 < \lambda < \lambda_{1,ah}$, *the problem* (1.2) *has at least one positive solution.*

Theorem 1.2 *Let* (C1)-(C4) *hold. Then for every* $\lambda > \lambda_{1,ah}$, *the problem* (1.2) *has at least two ordered positive solutions.*

Theorem 1.3 *Let* (C1)-(C4) *hold. Suppose* $\{y_\lambda\}$ *is a family of positive solutions of* (1.2). *Then*

(i) *one has* $\|y_\lambda\| \to \infty$ *as* $\lambda \to 0^+$;

Figure 1 The branch Σ of positive solutions of (1.2).



(ii) *if* $f(k,y) > 0$ *for* $y \neq m(k)$, $y \neq 0$, *then*

$$y_\lambda \to m \quad \textit{pointwise in } I \quad \textit{and} \quad \|y_\lambda\| \to \|m\|, \quad \textit{as } \lambda \to \infty.$$

Remark 1.1 Theorems 1.1-1.3 give the global structure of positive solutions of (1.2) under the conditions (C1)-(C3); see Figure 1. The condition (C4) is used to obtain the second solution, by a nice homotopy argument, which is a technical hypothesis. We conjecture this condition may be not imposed here if we use another technique to deal with the problem (1.2).

Remark 1.2 Notice that (C2) means that f has asymptotically linear growth at $u = 0$. From Figure 1, we see that (1.2) has a positive solution u satisfying $\min_{k\in I} u(k) > \|m\|$ for any $\lambda \in (0, \lambda_{ah})$ and (1.2) has two positive solutions u_1 and u_2 with $\|u_1\| < \|m\|$ and $\min_{k\in I} u_2(k) > \|m\|$ for any $\lambda > \lambda_{ah}$.

If $f_0 = \infty$, then for any $\lambda > 0$ (1.2) has two positive solutions u_1 and u_2 with $\|u_1\| < \|m\|$ and $\min_{k\in I} u_2(k) > \|m\|$. Especially, under the conditions (C1)-(C3), we can show (1.2) has at least one positive solution u with $\|u\| < \|m\|$ for any $\lambda > 0$ by using the fixed point theorem and Proposition 3.3. Compared with the main results of [5], the interval of λ is sharp. However, Theorem 1.1-Theorem 1.2 and [5], Theorem 3.3, Theorem 3.7, Theorem 3.9, with $p = 2$ complement each other but cannot contain each other.

The rest of this paper is organized as follows. In Section 2, we state some notations and preliminary results. Section 3 contains the proof of existence of one solution for λ small enough and the proof of the existence of two solutions for λ large enough. Finally, we study the asymptotic behavior of the solutions compact operator equation and prove Theorem 1.3 in Section 4.

2 Preliminaries

Let $\hat{I} := \{0, 1, \ldots, N, N+1\}$, and define $E = \{y \mid y : \hat{I} \to \mathbb{R}\}$ be the space of all maps from $\hat{I}$ into $\mathbb{R}$. Then it is a Banach space with the norm $\|y\| = \max_{k\in\hat{I}} |y(k)|$.

Let $P := \{y \in E \mid y(k) \geq 0, k \in \hat{I}\}$. Then P is a cone which is normal and has a nonempty interior and $E = \overline{P - P}$.

Let $\phi(k)$, $\psi(k)$ be the solution of the initial value problem

$$-\Delta\big[p(k-1)\Delta\phi(k-1)\big] + q(k)\phi(k) = 0 \quad \text{for } k \in I,$$
$$\phi(0) = 1, \qquad \Delta\phi(0) = 0,$$

and

$$-\Delta\big[p(k-1)\Delta\psi(k-1)\big]+q(k)\psi(k)=0 \quad \text{for } k\in I,$$
$$\psi(N+1)=1, \qquad \Delta\psi(N)=0,$$

respectively. It is easy to compute and show that

(i) $\phi(k)=1+\sum_{s=1}^{k-1}(\sum_{j=s}^{k-1}\frac{1}{p(j)})q(s)\phi(s)>0$, and ϕ is increasing on $\hat{I}$;
(ii) $\psi(k)=1+\sum_{s=k+1}^{N}(\sum_{j=k}^{s-1}\frac{1}{p(j)})q(s)\psi(s)>0$, and ψ is decreasing on $\hat{I}$.

Lemma 2.1 ([10], Lemma 2.4) *Let $h:I\to\mathbb{R}$. Then the linear boundary value problem*

$$\begin{aligned}&-\Delta\big[p(k-1)\Delta y(k-1)\big]+q(k)y(k)=h(k), \quad k\in I,\\&\Delta y(0)=0, \qquad \Delta y(N)=0,\end{aligned} \tag{2.1}$$

has a unique solution

$$y(k)=\sum_{s=1}^{N}G(k,s)h(s), \quad k\in\hat{I}, \tag{2.2}$$

where

$$G(k,s)=\frac{1}{p(N)\Delta\phi(N)}\begin{cases}\phi(s)\psi(k), & 1\le s\le k\le N+1,\\ \phi(k)\psi(s), & 0\le k\le s\le N.\end{cases} \tag{2.3}$$

Moreover, if $h(k)\ge 0$ and $h\not\equiv 0$ on I, then $y(k)>0$ on $\hat{I}$.

Lemma 2.2 *$G(k,s)$ has the following properties*:

(i) $G(k,s)\ge 0$, $\forall(k,s)\in\hat{I}\times\hat{I}$; $G(k,s)>0$, $\forall(k,s)\in I\times I$.
(ii) $G(k,s)\le G(s,s)$, $\forall(k,s)\in\hat{I}\times\hat{I}$.
(iii) $G(k,s)\ge\min\{\frac{\phi(k)}{\phi(N+1)},\frac{\psi(k)}{\psi(0)}\}G(s,s)$, $\forall(k,s)\in\hat{I}\times\hat{I}$.

Let $b:I\to\mathbb{R}$ be a function and consider the following eigenvalue problem:

$$\begin{aligned}&-\Delta\big[p(k-1)\Delta y(k-1)\big]+q(k)y(k)=\lambda b(k)y(k), \quad k\in I,\\&\Delta y(0)=0, \qquad \Delta y(N)=0.\end{aligned} \tag{2.4}$$

It is well known (*cf.* Ma *et al.* [12], Theorem 1; Gao [6], Theorem 1.1; Ji and Yang [7], Theorem 2.6) that the problem (2.4) has finite simple eigenvalues and there exist a sequence of positive eigenvalues

$$\lambda_{1,b}<\lambda_{2,b}<\cdots<\lambda_{M-1,b}<\lambda_{M,b},$$

provided $b>0$ in a subset of I and $M\le N$; there exist a sequence of negative eigenvalues

$$\lambda_{-(N-M),b}<\lambda_{-(N-M-1),b}<\cdots<\lambda_{-2,b}<\lambda_{-1,b},$$

provided $b<0$ in a subset of I.

In particular, $\lambda_{1,b} > 0$ is the positive principal eigenvalue of (2.4) and the associated eigenfunction $\varphi_{1,b}$ satisfies $\varphi_{1,b}(k) > 0$, $k \in I$, while $\varphi_{i,b}$ changes its sign $i - 1$ for $|i| > 1$. In addition, it yields, for any $y \in E$,

$$\sum_{k=1}^{N} p(k)\left|\Delta y(k)\right|^2 - \sum_{k=1}^{N} q(k)\left|y(k)\right|^2 \geq \lambda_{1,b} \sum_{k=1}^{N} b(k)\left|y(k)\right|^2,$$

where equality holds if and only if y is a multiple of $\varphi_{1,b}$ (see [6, 8]).

First, we introduce the following strict monotonicity property with respect to the weight function b for the problem (2.4) (see Gao [6], Lemma 2.5 and Ji and Yang [7], Theorem 3.1).

Lemma 2.3 *Let b and $\tilde{b}$ be two bounded weights with $b \leq \tilde{b}$, and let $j \in \{-N, \dots, -1, 1, \dots, N\}$. Then the eigenvalue $\lambda_{j,b} > \lambda_{j,\tilde{b}}$.*

Let $A : E \to E$ be defined by

$$Ay(k) = \lambda \sum_{s=1}^{N} G(k,s)a(s)f\big(s, y(s)\big), \tag{2.5}$$

and $P_0 = \{y \in P \mid \min_{k \in I} y(k) \geq c_0 \|y\|\}$, here $c_0 = \min_{k \in I}\{\frac{\phi(k)}{\phi(N+1)}, \frac{\psi(k)}{\psi(0)}\}$. It is easy to see that A is a completely continuous operator. By Lemma 2.1 and Lemma 2.2, it follows that $A(P_0) \subset P_0$ and its nontrivial fixed points in P_0 correspond to the positive solutions of (1.2).

3 The existence of one solution and two solutions for (1.2)

In this section, we will show the existence of a solution for $\lambda \in (0, \lambda_{1,ah})$ and the existence of two solutions for $\lambda \in (\lambda_{1,ah}, \infty)$.

Lemma 3.1 *Suppose* (C1)-(C3) *hold. Then for all $\Lambda, B > 0$, there exists $R > 0$ such that for any fixed $\lambda > \Lambda$ and $y \in \{y \in P_0 \mid \|y\| \geq R\}$, we have*

$$\|Ay\| > B\|y\|.$$

Proof By the condition (C3), for any fixed $M_0 > 0$, there exists $X > 0$ such that $y > X$ implies $f(k,y) \geq M_0 y$, $\forall k \in I_1$. By Lemma 2.1 and Lemma 2.2, it is easy to see that

$$y(k) \geq \min_{k \in I_1}\left\{\frac{\phi(k)}{\phi(N+1)}, \frac{\psi(k)}{\psi(0)}\right\}\|y\| \geq c_0\|y\|, \quad \text{for } k \in I_1.$$

Then, if we choose $\|y\| \geq R > \frac{X}{c_0}$, by using $f \geq 0$, it implies that

$$\begin{aligned}
\|Ay\| \geq Ay(2) &= \lambda \sum_{s=1}^{N} G(2,s)a(s)f\big(s, y(s)\big) \\
&\geq \lambda \sum_{s \in I_1} G(2,s)a(s)f\big(s, y(s)\big) \\
&\geq \lambda \sum_{s \in I_1} G(2,s)a(s)M_0 c_0 \|y\| \\
&= \left(\lambda M_0 c_0 \sum_{s \in I_1} G(2,s)a(s)\right)\|y\|.
\end{aligned} \tag{3.1}$$

Set $M_0 > B(\Lambda c_0 \sum_{s\in I_1} G(2,s)a(s))^{-1}$. Then we obtain

$$\|Ay\| > \lambda B\Lambda^{-1}\|y\| \geq B\|y\|. \qquad \square$$

Lemma 3.2 *Suppose* (C1) *and* (C2) *hold. Then for any fixed* $\lambda \in (0, \lambda_{1,ah})$, *there exist a norm* $\|\cdot\|_*$ *equivalent to* $\|\cdot\|$ *and* $r > 0$ *such that for all* $y \in \{y \in P_0 \mid \|y\|_* = r\}$, *we have*

$$\|Ay\|_* < \|y\|_*.$$

Proof For some $\varepsilon > 0$ small enough, it follows from $\lambda \in (0, \lambda_{1,ah})$ that

$$0 < \lambda < (1+\varepsilon)\lambda < \lambda_{1,ah}.$$

Next, take $M_1 > 0$ such that

$$M_1\lambda_{1,ah} \sup_{k\in\hat{I}} \frac{\omega(k)}{\varphi_{1,ah}(k)} < \frac{\lambda_{1,ah}}{\lambda(1+\varepsilon)} - 1,$$

where $\omega(k) = \sum_{s=1}^{N} G(k,s)a(s)h(s)$.

Consider the norm $\|y\|_* = \inf\{\eta : \eta(\varphi_{1,ah} + M_1) \geq y\} = \|\frac{y}{\varphi_{1,ah}+M_1}\|$, which is equivalent to $\|\cdot\|$.

From (C1)-(C2), there exists a $\rho = \rho(\varepsilon) > 0$ such that $0 < y < \rho$ implies that $f(k,y) < (1+\varepsilon)h(k)y(k)$ for all $k \in I$. Let $r > 0$ be such that $r(\|\varphi_{1,ah}\| + M_1) < \rho$, so that $\|y\|_* = r$ implies $\|y\| < \rho$.

If $y \in P_0$ with $\|y\|_* = r$, we imply

$$\begin{aligned}
Ay(k) &= \lambda \sum_{s=1}^{N} G(k,s)a(s)f\bigl(s,y(s)\bigr) \\
&\leq \lambda \sum_{s=1}^{N} G(k,s)a(s)(1+\varepsilon)h(s)\frac{\varphi_{1,ah}(s)+M_1}{\varphi_{1,ah}(s)+M_1}y(s) \\
&\leq \lambda \sum_{s=1}^{N} G(k,s)a(s)(1+\varepsilon)h(s)\bigl(\varphi_{1,ah}(s)+M_1\bigr)\|y\|_* \\
&= \lambda(1+\varepsilon)\left\{\sum_{s=1}^{N} G(k,s)a(s)h(s)\varphi_{1,ah}(s) + M_1\sum_{s=1}^{N} G(k,s)a(s)h(s)\right\}\|y\|_* \\
&= (1+\varepsilon)\left\{\frac{\lambda}{\lambda_{1,ah}}\varphi_{1,ah}(k) + \lambda M_1\omega(k)\right\}\|y\|_* \\
&= (1+\varepsilon)\varphi_{1,ah}(k)\frac{\lambda}{\lambda_{1,ah}}\left\{1 + \lambda_{1,ah}M_1\frac{\omega(k)}{\varphi_{1,ah}(k)}\right\}\|y\|_* \\
&\leq \varphi_{1,ah}(k)\|y\|_* < \bigl(\varphi_{1,ah}(k)+M_1\bigr)\|y\|_*.
\end{aligned}$$

Therefore $\|Ay\|_* < \|y\|_*$. $\square$

Lemma 3.1 and Lemma 3.2 may imply the existence of a positive solution of (1.2).

Proof of Theorem 1.1 The existence of a positive solution is a consequence of the fixed point theorem (see Deimling [16], Theorem 20.1) by virtue of Lemma 3.1 and Lemma 3.2. □

Now we shall show (1.2) has at least two positive solutions for $\lambda > \lambda_{1,ah}$. First of all, we start with the existence of a first solution y which satisfies $y(k) \leq m(k)$.

Proposition 3.3 *Suppose* (C1)-(C3) *hold. Then for every* $\lambda > \lambda_{1,ah}$, *the problem* (1.2) *has a positive solution* y *which satisfies* $y(k) \leq m(k)$.

Proof By (C1)-(C3), we may use the method of lower and upper solutions (see [3]).

From (C1), m is positive and $f(k, m(k)) = 0$, we see that $m(k)$ is always an upper solution of (1.2). Clearly, it is a strict upper solution, because $m(k) > 0$ and the concave of m implies that it cannot satisfy $\Delta y(0) = \Delta y(N) = 0$. We recall that a strict lower solution of (1.2) is a lower solution which is not a solution of (1.2).

Let $\lambda > \lambda_{1,ah}$. By (C2), fix a $\tau \in (0,1)$, there exists sufficiently small $\varrho = \varrho(\tau)$ such that

$$(1-\tau)h(k)y(k) < f\bigl(k, y(k)\bigr), \quad y \in (0, \varrho]. \tag{3.2}$$

Hence, if τ is chosen such that $\lambda_{1,ah} < (1-\tau)\lambda$, then

$$\lambda_{1,ah}a(k)h(k)y(k) < \lambda a(k)f\bigl(k, y(k)\bigr), \quad \text{for } y \in (0, \varrho], \tag{3.3}$$

and if $\epsilon > 0$ is such that $\epsilon\|\varphi_{1,ah}\| < \varrho$, then

$$\begin{aligned} -\Delta\bigl[p(k-1)\Delta\bigl(\epsilon\varphi_{1,ah}(k-1)\bigr)\bigr] + q(k)\epsilon\varphi_{1,ah}(k) &= \lambda_{1,ah}a(k)h(k)\bigl(\epsilon\varphi_{1,ah}(k)\bigr) \\ &< \lambda a(k)f\bigl(k, \epsilon\varphi_{1,ah}(k)\bigr), \end{aligned}$$

that is, $\epsilon\varphi_{1,ah}$ is a (strict) lower solution for (1.2). Finally, ϵ can always be chosen such that $\epsilon\|\varphi_{1,ah}\| \leq m(k)$. Thus, the method of lower and upper solutions implies that there exists a solution y satisfying $0 < \epsilon\varphi_{1,ah} \leq y \leq m$. □

Remark 3.1 Obviously, the choice of τ, the values of ϱ and ϵ in the preceding proof depends on λ. However, once chosen τ for a given value of λ, the same choice works for any larger value of λ. Thus, for any given $\hat{\lambda} > \lambda_{1,ah}$, it is possible to find a unique function $\epsilon\varphi_{1,ah}$ which is a lower solution for any $\lambda > \hat{\lambda}$.

In the following we will show the existence of a second solution for all $\lambda > \lambda_{1,ah}$. From (C1)-(C3), we fix λ and denote by $y_1 = y_1(\lambda)$ the solution of (1.2) found above.

Consider the auxiliary problem

$$\begin{aligned} &-\Delta\bigl[p(k-1)\Delta\bigl(y_1(k-1) + w(k-1)\bigr)\bigr] + q(k)\bigl(y_1(k) + w(k)\bigr) = \lambda a(k)f\bigl(k, y_1 + w^+\bigr), \\ &\Delta w(0) = 0, \qquad \Delta w(N) = 0, \end{aligned} \tag{3.4}$$

where $w^+ = \max\{w, 0\}$, $w^- = \max\{-w, 0\}$, and $w = w^+ - w^-$.

It is easy to see that $w \geq 0$ is a nontrivial solution of (3.4), then $y_1 + w$ is a second positive solution of (1.2), which satisfies $y_1 + w \geq y_1$.

For $\vartheta, \tau \in [0,1]$ and $\rho \geq 0$, we consider the following parameterized family operators:

$$A_{\vartheta,\tau,\rho} w(k) = \lambda\vartheta \sum_{k=1}^{N} G(k,s)a(s)\frac{f(s,y_1(s) + \tau w^+(s))}{\tau} + \rho,$$

$$A_{\vartheta,0,\rho} w(k) = \lambda\vartheta \sum_{k=1}^{N} G(k,s)a(s)f_y\big(s,y_1(s)\big)w^+(s) + \rho.$$

Clearly, it follows from (C4) that A is a continuous operator with respect to the parameters ϑ, τ, ρ.

From this definition, a solution of (3.4) is a fixed point of $A_{1,1,0}$, since $\lambda a(k)f(k,v_1) = -\Delta[p(k-1)\Delta y(k-1)] + q(k)y(k)$. On the other hand, a fixed point of $A_{1,1,\rho}$ is a solution of the more general problem,

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta\big(y_1(k-1) + w(k-1)\big)\big] + q(k)\big(y_1(k) + w(k)\big) \\ &\quad = \lambda a(k)\big[f\big(k,y_1 + w^+\big) + \rho\big], \\ &\Delta w(0) = 0, \qquad \Delta w(N) = 0, \end{aligned} \tag{3.5}$$

while a fixed point of $A_{\vartheta,0,0}$ is a solution of

$$\begin{aligned} &-\Delta\big[p(k-1)\Delta w(k-1)\big] + q(k)w(k) = \lambda\vartheta a(k)f_y\big(k,y_1(k)\big)w^+(k), \quad k \in I, \\ &\Delta w(0) = 0, \qquad \Delta w(N) = 0. \end{aligned} \tag{3.6}$$

Our purpose is to find a nontrivial fixed point of $A_{1,1,0}$. We need the following lemmas.

Lemma 3.4 *Suppose* (C1), (C2), *and* (C4) *hold*. *Then if w is a solution of* (3.4) (*or of* (3.6)), *then* $w \geq 0$.

Proof Let w be a solution of (3.4) and w^- be a test function. Then

$$\begin{aligned} &\lambda \sum_{k=1}^{N} a(k)f\big(k,y_1(k) + w^+(k)\big)w^-(k) \\ &\quad = \sum_{k=1}^{N} -\Delta\big[p(k-1)\Delta\big(y_1(k-1) + w(k-1)\big)\big]w^-(k) + \sum_{k=1}^{N} q(k)y_1(k)w^-(k) \\ &\qquad + \sum_{k=1}^{N} q(k)w(k)w^-(k) \\ &\quad = \sum_{k=1}^{N} p(k)\Delta y_1(k)\Delta w^-(k) + \sum_{k=1}^{N} p(k)\Delta w(k)\Delta w^-(k) + \sum_{k=1}^{N} q(k)y_1(k)w^-(k) \\ &\qquad + \sum_{k=1}^{N} q(k)w(k)w^-(k) \end{aligned}$$

$$= \sum_{k=1}^{N} p(k)\Delta y_1(k)\Delta w^-(k) + \sum_{k=1}^{N} p(k)\Delta w^+(k)\Delta w^-(k) - \sum_{k=1}^{N} p(k)\big(\Delta w^-(k)\big)^2$$
$$+ \sum_{k=1}^{N} q(k)y_1(k)w^-(k) + \sum_{k=1}^{N} q(k)w^+(k)w^-(k) - \sum_{k=1}^{N} q(k)\big(w^-(k)\big)^2.$$

Consequently, this yields

$$\lambda \sum_{k=1}^{N} a(k)f\big(k, y_1(k)\big)w^-(k)$$
$$= \sum_{k=1}^{N} p(k)\Delta y_1(k)\Delta w^-(k) + \sum_{k=1}^{N} q(k)y_1(k)w^-(k) - \sum_{k=1}^{N} p(k)\big(\Delta w^-(k)\big)^2$$
$$- \sum_{k=1}^{N} q(k)\big(w^-(k)\big)^2.$$

Since y_1 is a solution of (1.2), we conclude that

$$\sum_{k=1}^{N} p(k)\big(\Delta w^-(k)\big)^2 + \sum_{k=1}^{N} q(k)\big(w^-(k)\big)^2 = 0.$$

Furthermore, we obtain $w^- \equiv 0$. In the case that w is a solution of (3.6), by the same argument, we also have $w \geq 0$. □

Lemma 3.5 *Suppose* (C1)-(C4) *hold. Then there exists a constant* $M_3 > 0$, *which does not depend on* ρ, *such that* $A_{1,1,\rho}w = w$ *implies* $\|w\| \leq M_3$ *for any* $\rho \geq 0$.

Proof Suppose on the contrary that $\{w_n\}$ is a sequence of fixed points of $A_{1,1,\rho_n}$ with $\|w_n\| \to \infty$ and arbitrary ρ_n. Then

$$y_1(k) + w_n(k) = \lambda \sum_{s=1}^{N} G(k,s)a(s)\big[f\big(s, y_1(s) + w_n^+(s)\big) + \rho_n\big].$$

Since $y_1 + w_n$ satisfies (3.5), it is a concave and positive function. Then

$$y_1(k) + w_n^+(k) \geq y_1(k) + w_n(k) \geq c_0\|y_1 + w_n\| \quad \text{in } I_1. \tag{3.7}$$

Moreover, we get

$$y_1(k) + w_n(k) \geq \lambda \sum_{s\in I_1} G(k,s)a(s)\big[y_1(s) + w_n^+(s)\big]\frac{f(s, y_1(s) + w_n^+(s))}{y_1(s) + w_n^+(s)}$$
$$\geq \lambda \sum_{s\in I_1} G(k,s)a(s)c_0\|y_1 + w_n\|\frac{f(s, y_1(s) + w_n^+(s))}{y_1(s) + w_n^+(s)}.$$

From (C3) and (3.7), for any $M_0 > 0$, one has $\frac{f(s,y_1(s)+w_n^+(s))}{y_1(s)+w_n^+(s)} \geq M_0$ in I_1 for n large enough, then

$$y_1(k) + w_n(k) \geq (\lambda c_0 M_0) \sum_{s\in I_1} G(k,s)a(s)\|y_1 + w_n\|.$$

This leads to the contradiction

$$1 \geq \frac{y_1(k) + w_n(k)}{\|y_1 + w_n\|} \geq \lambda c_0 M_0 \sum_{s\in I_1} G(k,s)a(s), \quad k \in I_1$$

for any $M_0 > 0$. So the conclusion is true. □

Lemma 3.6 *Suppose* (C1), (C2), *and* (C4) *hold. Then, given any* $\tilde{R} > 0$, *there exists* $\tilde{\rho}(\tilde{R}) > 0$, *which does not depend on* $\lambda > \lambda_{1,ah}$, *such that* $A_{1,1,\rho}w = w$ *has no solution in* $\overline{B_{\tilde{R}}}$ *for any* $\rho > \tilde{\rho}(\tilde{R})$. *In addition*,

$$\deg(I - A_{1,1,\rho}, B_{\tilde{R}}, 0) = 0 \quad \textit{for } \rho > \tilde{\rho}(\tilde{R}). \tag{3.8}$$

Proof It is easy to see that

$$\begin{aligned} y_1(k) + w(k) &= \lambda \sum_{s=1}^{N} G(k,s)a(s)\bigl[f\bigl(s, y_1(s) + w^+(s)\bigr) + \rho\bigr] \\ &\geq \lambda\rho \sum_{s=1}^{N} G(k,s)a(s). \end{aligned}$$

Set $\|w\| \leq \tilde{R}$. Then $\|y_1\| + \tilde{R} \geq y_1(k) + w(k)$, $k \in I$, and so

$$\|y_1\| + \tilde{R} \geq \lambda\rho \sum_{s=1}^{N} G(k,s)a(s).$$

This is a contradiction for large ρ, that is, no fixed point lies in $\overline{B_{\tilde{R}}}$ and the degree in (3.8) must be zero.

Obviously, y_1 depends on λ, and $\|y_1\| \leq \|m\|$, thus $\tilde{\rho}(\tilde{R})$ can be chosen uniformly with respect to $\lambda > \lambda_{1,ah}$. □

Lemma 3.7 *Suppose* (C1)-(C4) *hold. Set* $\tilde{R} \geq M_3 + 1$, *where* M_3 *is the constant from Lemma* 3.5, *then*

$$\deg(I - A_{1,1,0}, B_{\tilde{R}}, 0) = 0. \tag{3.9}$$

Proof By the *a priori* bound estimate in Lemma 3.5, there are no fixed points on $\partial B_{\tilde{R}}$ for any $\rho > 0$. Then, it follows from (3.8) and the homotopy invariance of the degree that

$$0 = \deg(I - A_{1,1,\tilde{\rho}(\tilde{R})}, B_{\tilde{R}}, 0) = \deg(I - A_{1,1,0}, B_{\tilde{R}}, 0). \tag{3.10}$$

□

Lemma 3.8 *Suppose* (C1), (C2), *and* (C4) *hold. Then* $A_{\vartheta,0,0}w = w$ *implies* $w = 0$ *for any* $\vartheta \in [0,1]$. *Moreover,* $A_{0,0,0} = 0$ *implies that*

$$\deg(I - A_{\vartheta,0,0}, B_\rho, 0) = \deg(I, B_\rho, 0) = 1 \quad \textit{for any } \rho > 0 \textit{ and } \vartheta \in [0,1]. \tag{3.11}$$

Proof Suppose that $\vartheta > 1$, otherwise $w \equiv 0$. Define

$$\hat{b}(k) := \vartheta a(k) f_y\bigl(k, y_1(k)\bigr), \qquad \tilde{b}(k) := a(k)\frac{f(k, y_1(k))}{y_1(k)}.$$

From (C4), it follows that $\hat{b}(k) \leq\not\equiv \tilde{b}(k)$ in I and then by Lemma 2.3, we have

$$\lambda_{j,\hat{b}} > \lambda_{j,\tilde{b}} \quad \text{for any } j \in \{-N, \dots, -1, 1, \dots, N\}.$$

Since $y_1 > 0$ satisfies the equation

$$-\Delta\bigl[p(k-1)\Delta u(k-1)\bigr] + q(k)u(k) = \lambda \tilde{b}(k)u(k)$$

and $\tilde{b}(k) \geq 0$, we deduce that $\lambda = \lambda_{1,\tilde{b}}$.

On the other hand, if w is a nontrivial fixed point of $A_{\vartheta,0,0}$, then it is a solution of (3.6). Subsequently, $w \geq 0$, and it is also a solution of

$$\begin{aligned} &-\Delta\bigl[p(k-1)\Delta w(k-1)\bigr] + q(k)w(k) = \lambda \hat{b}(k)w(k), \\ &\Delta w(0) = 0, \qquad \Delta w(N) = 0. \end{aligned} \tag{3.12}$$

Hence, λ has to be an eigenvalue of (3.12), *i.e.* $\lambda = \lambda_{j,\hat{b}}$ for some $j \in \{-N, \dots, -1, 1, \dots, N\}$.

However, since $\lambda > 0$, it cannot coincide with any $\lambda_{j,\hat{b}}$ with $j < 0$, and because $\lambda = \lambda_{1,\tilde{b}} < \lambda_{1,\hat{b}}$, it cannot coincide with any $\lambda_{j,\hat{b}}$ with $j > 0$ neither. This concludes that $w \equiv 0$. □

Proposition 3.9 *Suppose* (C1)-(C4) *hold. Then the problem* (3.4) *has a nontrivial positive solution* w, *that is,* (1.2) *has a second positive solution.*

Proof Obviously, if $A_{1,\tau,0}w = w$ for some $\tau > 0$ and $w \neq 0$, then

$$\begin{aligned} &-\Delta\bigl[p(k-1)\tau\Delta w(k-1)\bigr] + q(k)\tau w(k) = \lambda a(k)\bigl[f\bigl(k, y_1 + \tau w^+(k)\bigr) - f\bigl(k, y_1(k)\bigr)\bigr], \\ &\Delta w(0) = 0, \qquad \Delta w(N) = 0. \end{aligned}$$

This implies that τw is a solution of (3.4) and so it is positive by Lemma 3.4.

To this end, we suppose on the contrary that $A_{1,\tau,0}w = w$ has no nontrivial solution for $\tau > 0$. This together with (3.11) implies that

$$1 = \deg(I - A_{1,0,0}, B_{\tilde{R}}, 0) = \deg(I - A_{1,1,0}, B_{\tilde{R}}, 0),$$

contradicting equation (3.9). □

4 Asymptotic behavior of solutions for (1.2)

Proposition 4.1 *Suppose* (C1) *and* (C2) *hold. If* $\{y_\lambda\}$ *is a family of positive solutions of* (1.2), *then* $\|y_\lambda\| \to \infty$ *as* $\lambda \to 0^+$.

Proof Suppose by way of contradiction that there exist a sequence $\lambda_n \to 0^+$ and a constant $\sigma > 0$ such that $\|y_{\lambda_n}\| \leq \sigma$. From the continuity of f and (C2), there exists a positive constant C such that $f(k,y) \leq Cy$ for any $0 \leq y \leq \sigma$. Then,

$$y_{\lambda_n}(k) = \lambda_n \sum_{s=1}^{N} G(k,s)a(s)f\bigl(s, y_{\lambda_n}(s)\bigr) \leq \lambda_n C\|y_{\lambda_n}\| \sum_{s=1}^{N} G(k,s)a(s).$$

Therefore,

$$1 \leq \lambda_n C \sum_{s=1}^{N} \phi(s)\psi(s)a(s).$$

However, since $\lambda_n \to 0^+$, this is impossible. □

Next, we study the asymptotic behavior of the positive solutions of (1.2) as $\lambda \to \infty$, we need to prove a uniform estimate for the positive solutions of (1.2), for λ large enough.

Lemma 4.2 *Suppose* (C1)-(C3) *hold. Then there exist constant* $\Lambda, \Gamma, M_3 > 0$ *such that if* y_λ *is a positive solution of* (1.2) *with* $\lambda > \Lambda$, *then*

$$\Gamma \leq \|y_\lambda\| \leq M_3.$$

Proof By Lemma 3.1, there exists $R > 0$ such that

$$\|y_\lambda\| = \|Ay_\lambda\| > \|y_\lambda\| \quad \text{for all } \lambda \leq 1, \quad \text{provided} \quad \|y_\lambda\| \geq R.$$

This is a contradiction. Hence, we obtain $\|y_\lambda\| \leq R + 1 := M_3$ for all $\lambda > 1$.

On the other hand, from (C2), choose a suitably small $\varepsilon > 0$, there exists $\Gamma > 0$ such that $f(k,y) > (h(k) - \varepsilon)y > 0$, $0 < y < \Gamma$. Let y_λ be a positive solution of (1.2) with $\|y_\lambda\| < \Gamma$. Using Lemma 2.2, it follows that

$$\begin{aligned}
\|y_\lambda\| \geq y_\lambda(2) &= \lambda \sum_{s=1}^{N} G(2,s)a(s)f\bigl(s, y_\lambda(s)\bigr) \\
&\geq \lambda \sum_{s=1}^{N} G(2,s)a(s)\bigl(h(s) - \varepsilon\bigr)y_\lambda(s) \\
&\geq \lambda \sum_{s\in I_1} G(2,s)a(s)\bigl(h(s) - \varepsilon\bigr)y_\lambda(s) \\
&\geq \lambda \sum_{s\in I_1} G(2,s)a(s)\bigl(h(s) - \varepsilon\bigr)c_0\|y_\lambda\| \\
&\geq \left(\lambda c_0 \sum_{s\in I_1} G(2,s)a(s)\bigl(h(s) - \varepsilon\bigr)\right)\|y_\lambda\|.
\end{aligned}$$

This is a contradiction if the term in brackets is greater than 1. Thus, for large enough $\lambda > 0$, it is necessary that $\|y_\lambda\| \geq \Gamma$. □

Finally, we shall give the proof of Theorem 1.3, with the following result, which reveals the asymptotic behavior of the solutions of (1.2) as $\lambda \to \infty$.

Proposition 4.3 *Suppose* (C1)-(C3) *hold and* $f(k,y) > 0$ *with* $y \neq 0$, $y \neq m(k)$. *If* $\{y_\lambda\}$ *is a family of positive solutions of* (1.2), *then*

$$y_\lambda \to m \quad \text{pointwise in } I \quad \text{and} \quad \|y_\lambda\| \to \|m\|, \quad \text{as } \lambda \to \infty.$$

Proof Set $k_0 \in I$, and let a sequence $\lambda_n \to \infty$ and $\lambda_n > \Lambda$, where Λ is given by Lemma 4.2, and assume y_n are positive solutions of (1.2) with $\lambda = \lambda_n$. Suppose on the contrary that there exists $\zeta > 0$ such that $y_n(k_0) > m(k_0) + \zeta$. Then we claim that there exists a subset $I_2 \subseteq I$ such that $k_0 \in I_2$ and

$$y_n(k) > m(k_0) + \frac{3}{4}\zeta > m(k_0) + \frac{\zeta}{4} > a(k) \quad \text{for } k \in I_2 \text{ and every } n.$$

Subsequently, $G(k_0,s)a(s)f(s,y(s))$ is a positive function for $(s,y) \in I_2 \times [m(k_0) + \zeta/4, M_3]$, where M_3 is given by Lemma 4.2, and since this set is compact, it follows that there exists $\chi > 0$ such that $G(k_0,s)a(s)f(s,y(s)) \geq \chi > 0$ in $I_2 \times [m(k_0) + \zeta/4, M_3]$, so

$$\sum_{s=1}^{N} G(k_0,s)a(s)f\bigl(s,y(s)\bigr) \geq \sum_{s\in I_2} G(k_0,s)a(s)f\bigl(s,y_\lambda(s)\bigr) \geq \sum_{s\in I_2} \chi \geq k_0\chi.$$

However,

$$M_3 \geq \|y_n\| \geq y_n(k_0) = \lambda_n \sum_{s=1}^{N} G(k_0,s)a(s)f\bigl(s,y_n(s)\bigr) \geq \lambda_n k_0 \chi > 0,$$

which is a contradiction since $\lambda_n \to \infty$.

Now suppose that $y_n(k_1) < m(k_1) - \delta$. Using again the same argument, there exists a subset $I_3 \subseteq I$ such that $k_1 \in I_3$ and

$$0 < c_0\Gamma \leq y_n(k) < m(k_1) - \frac{3}{4}\delta < m(k_1) - \frac{\delta}{4} < m(k) \quad \text{for } k \in I_3 \text{ and every } n,$$

where Γ is given by Lemma 4.2. Then there exists $\varpi > 0$ such that $G(k_1,s)a(s)f(s,y(s)) \geq \varpi > 0$ in $I_3 \times [c_0\Gamma, m(k_1) - \frac{\delta}{4}]$, so

$$\sum_{s=1}^{N} G(k_1,s)a(s)f\bigl(s,y(s)\bigr) \geq \sum_{s\in I_3} G(k_1,s)a(s)f\bigl(s,y(s)\bigr) \geq \sum_{s\in I_3} \varpi \geq k_1\varpi > 0.$$

Thus,

$$M_3 \geq \|y_n\| \geq y_n(k_1) = \lambda_n \sum_{s=1}^{N} G(k_1,s)a(s)f\bigl(s,y_n(s)\bigr) \geq \lambda_n k_1 \varpi > 0.$$

This is a contradiction, since $\lambda_n \to \infty$.

Consequently, $\|y_\lambda\| \to \|m\|$, as $\lambda \to \infty$. □

Competing interests

The author confirms that she has read SpringerOpen's guidance on competing interests and has included these in the manuscript. The author also declares that there is no conflict of interests regarding the publication of this paper.

Author's contributions

YLu completed the main study, carried out the results of this article and drafted the manuscript and checked the proofs and verified the calculation.

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