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An extension for Rice's integral and applications

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Center, Huxi Campus, Chongqing,
401331, People's Republic of China**Abstract**

In this paper, we generalize Rice's integral and give some applications. Some old and new binomial identities are obtained.

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1 Introduction

For complex z and $m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define that

$$(z)_0 = 1, \quad (z)_n = z(z+1)\cdots(z+n-1) \quad (1)$$

and

$$(z+m)_{n-m} = \begin{cases} (m+z)(m+z+1)\cdots(z+n-1), & \text{when } 0 \leq m < n, \\ 1, & \text{when } m \geq n. \end{cases} \quad (2)$$

The standard Bell polynomials (see [1, p.133]). The modified Bell polynomials $L_m(x_1, x_2, \dots)$ are defined by

$$\exp\left(\sum_{k=1}^{\infty} x_k \frac{t^k}{k}\right) = 1 + \sum_{m=1}^{\infty} L_m(x_1, x_2, \dots) \frac{t^m}{m!}. \quad (3)$$

We have

$$L_m(x_1, x_2, \dots) = \sum_{m_1+2m_2+3m_3+\dots=m} \frac{m!}{m_1!m_2!m_3!\dots} \left(\frac{x_1}{1}\right)^{m_1} \left(\frac{x_2}{2}\right)^{m_2} \left(\frac{x_3}{3}\right)^{m_3} \dots \quad (4)$$

Recalling Rice's integral (see [2] and [3]), suppose that $f(z)$ is a rational function analytic on $[0, +\infty[$, and $n \in \mathbb{N}_0$, then we have

$$\sum_{k=0}^n (-1)^k \binom{n}{k} f(k) = \frac{(-1)^n}{2\pi i} \int_{\mathcal{C}} \frac{n!}{z(z-1)\cdots(z-n)} f(z) dz, \quad (5)$$

where \mathcal{C} is a positively oriented, encloses the simple poles at the integers $0, 1, 2, \dots, n$ and no others.

All technicalities and many examples of Rice’s integral are involved in [2] and [3].

Well-known is the following binomial identity: For $n \in \mathbb{N} = \{1, 2, \dots\}$ and $\theta > 0$, we have

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\theta}{\theta + k} = \prod_{k=1}^n \frac{k}{\theta + k}. \tag{6}$$

Recently, Jonathon [4] gave a simple and interesting probabilistic proof of the above binomial identity, and he further extended this binomial identity based on the probabilistic method. We state his result as follows. For all $r, n \in \mathbb{N}; \theta > 0$, we have

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{\theta}{\theta + k}\right)^r \\ &= \left(\prod_{k=1}^n \frac{k}{\theta + k}\right) \left(1 + \sum_{j=1}^{r-1} \sum_{1 \leq k_1 \leq \dots \leq k_j \leq n} \frac{\theta^j}{(\theta + k_1)(\theta + k_2) \dots (\theta + k_j)}\right). \end{aligned} \tag{7}$$

More recently, Bayad *et al.* [5, 6] established new identities for some new polynomials associated with the λ -array type polynomials. Simsek [6] gave an approach to combinatorial sums involving binomial coefficients and the Catalan number from the analysis of the Bernstein basis functions. There is much literature about the binomial identities, we refer the readers to [7–17].

In the present paper, we further extend Rice’s integral by using Cauchy’s integral formula. As some applications, some old and new binomial identities are obtained.

2 Rice’s integral

In this section, we generalize Rice’s integral using Cauchy’s integral formula.

Theorem 1 *Let $f(z)$ be a rational function analytic on $[0, +\infty[$ and $m, n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{m+k} f(k) = \frac{(-1)^n n!}{2\pi i (n-m)!} \int_C \frac{(m+z+1)_{n-m}}{z(z-1) \dots (z-n)} f(z) dz, \tag{8}$$

where C is a positively oriented, encloses the poles $0, 1, 2, \dots, n$ and no others.

Proof By using Cauchy’s integral formula, via the residues calculus, we have

$$\begin{aligned} & \frac{(-1)^n n!}{2\pi i (n-m)!} \int_C \frac{(z+m+1)_{n-m}}{z(z-1) \dots (z-n)} f(z) dz \\ &= \frac{(-1)^n n!}{(n-m)!} \sum_{k=1}^{n+1} \operatorname{Res}_{z=k-1} \frac{(z+m+1)_{n-m}}{z(z-1) \dots (z-n)} f(z) \\ &= \frac{(-1)^n n!}{(n-m)!} \sum_{k=1}^{n+1} \lim_{z \rightarrow k-1} \frac{(z+m+1)_{n-m}}{z(z-1) \dots (z-k+2)(z-k) \dots (z-n)} f(z) \\ &= \frac{(-1)^n n!}{(n-m)!} \sum_{k=1}^{n+1} \frac{(m+k)_{n-m}}{(k-1) \dots 2 \cdot 1(-1)(-2) \dots (-n+k-1)} f(k-1) \end{aligned}$$

$$\begin{aligned}
 &= \frac{n!}{(n-m)!} \sum_{k=0}^n (-1)^k \frac{(m+k+1)_{n-m}}{k!(n-k)!} f(k) \\
 &= \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{m+k} f(k).
 \end{aligned}$$

The proof is complete. □

Corollary 2 *Let $f(z)$ be a rational function analytic on $[0, +\infty[$ and $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} f(k) = \frac{(-1)^n}{2\pi i} \int_C \frac{(z+1)_n}{z(z-1)\cdots(z-n)} f(z) dz, \tag{9}$$

where C is a positively oriented, encloses the poles $0, 1, 2, \dots, n$ and no others.

Proof Taking $m = 0$ in Theorem 1, Corollary 2 follows. □

Remark 3 Taking $m = n$ in Theorem 1 and noting that $(z+m+1)_0 = 1$, we get Rice’s integral (5) immediately. Obviously, Theorem 1 is an extension of Rice’s integral (5).

3 Applications

In this section, we give some applications of Theorem 1 and Corollary 2. We can obtain many old and new binomial identities.

Below we give an extension of binomial identity (7) and deduce some new binomial identities.

Theorem 4 *Let $r, l, n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \leq m \leq n$; $\theta > 0$, $\beta \neq 0$.*

When $\theta \notin \{m+1, m+2, \dots, n\}$ and $r > 0$, we have

$$\begin{aligned}
 &\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{m+k} \left(\frac{\beta}{\theta+k}\right)^r \\
 &= \left(\prod_{k=1}^n \frac{k}{\theta+k}\right) \left(\prod_{k=m+1}^n \frac{k-\theta}{k-m}\right) \\
 &\quad \times \left(\left(\frac{\beta}{\theta}\right)^r + \sum_{j=1}^{r-1} \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\beta^r}{\theta(\theta+s_1)(\theta+s_2)\cdots(\theta+s_j)} \right. \\
 &\quad \left. + \sum_{i+l=r-1} \sum_{i=1}^{n-m} \sum_{m+1 \leq k_1 < \dots < k_i \leq n} \sum_{0 \leq s_1 \leq \dots \leq s_l \leq n} \frac{\beta^r}{\theta(k_1-\theta)\cdots(k_i-\theta)(\theta+s_1)\cdots(\theta+s_l)} \right). \tag{10}
 \end{aligned}$$

When $\theta \in \{m+1, m+2, \dots, n\}$ and $r > 1$, we have

$$\begin{aligned}
 &\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{m+k} \left(\frac{\beta}{\theta+k}\right)^r \\
 &= \frac{1}{(n-m)!} \left(\prod_{k=1}^n \frac{k}{\theta+k}\right)
 \end{aligned}$$

$$\begin{aligned} & \times \left(\prod_{k=m+1, k \neq \theta}^n (k - \theta) \right) \left(\frac{\beta^r}{\theta^{r-1}} + \sum_{j=1}^{r-2} \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\beta^r}{\theta(\theta + s_1)(\theta + s_2) \dots (\theta + s_j)} \right. \\ & \left. + \sum_{i+l=r-2} \sum_{i=1}^{n-m-1} \sum_{\substack{k_1, \dots, k_i \neq \theta \\ m+1 \leq k_1 < \dots < k_i \leq n}} \sum_{0 \leq s_1 \leq \dots \leq s_l \leq n} \frac{\beta^r}{\theta(k_1 - \theta) \dots (k_i - \theta)(\theta + s_1) \dots (\theta + s_l)} \right). \end{aligned} \tag{11}$$

Proof Taking $f(z) = \left(\frac{\beta}{\theta+z}\right)^r$ in (8), and C is a positively oriented, encloses the poles $0, 1, 2, \dots, n$ and no others, by means of the residues calculus, we obtain the following.

- When $\theta \notin \{m + 1, m + 2, \dots, n\}$, we have

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{m+k} \left(\frac{\beta}{\theta+k}\right)^r \\ &= \frac{(-1)^n n!}{2\pi i (n-m)!} \int_C \frac{(m+z+1)_{n-m}}{z(z-1) \dots (z-n)} \left(\frac{\beta}{\theta+z}\right)^r dz \\ &= \frac{(-1)^{n+1} n!}{(n-m)!} \operatorname{Res}_{z=\theta} \frac{(m+z+1)_{n-m}}{z(z-1) \dots (z-n)} \left(\frac{\beta}{\theta+z}\right)^r \\ &= \frac{(-1)^{n+1} n!}{(n-m)!} [(\theta+z)^{r-1}] \frac{(z+1+m) \dots (z+n) \beta^r}{z(z-1) \dots (z-n)} \\ &= \frac{n!}{(n-m)!} [z^{r-1}] \left(\prod_{k=1}^n \frac{1}{\theta+k} \right) \left(\prod_{k=m+1}^n (k-\theta) \right) \frac{(1 + \frac{z}{m+1-\theta}) \dots (1 + \frac{z}{n-\theta}) \beta^r}{\theta(1 - \frac{z}{\theta})(1 - \frac{z}{\theta+1}) \dots (1 - \frac{z}{\theta+n})} \\ &= \left(\prod_{k=1}^n \frac{k}{\theta+k} \right) \left(\prod_{k=m+1}^n \frac{k-\theta}{k-m} \right) [z^{r-l-1}] \left(1 + \sum_{i=1}^{n-m} \sum_{m+1 \leq k_1 < \dots < k_i \leq n} \frac{z^i}{(k_1 - \theta) \dots (k_i - \theta)} \right) \\ & \quad \times \sum_{0 \leq s_1 \leq \dots \leq s_l \leq n} \frac{\beta^r}{\theta(\theta + s_1)(\theta + s_2) \dots (\theta + s_l)} \\ &= \left(\prod_{k=1}^n \frac{k}{\theta+k} \right) \left(\prod_{k=m+1}^n \frac{k-\theta}{k-m} \right) [z^{r-l-1}] \left(1 + \sum_{i=1}^{n-m} \sum_{m+1 \leq k_1 < \dots < k_i \leq n} \frac{z^i}{(k_1 - \theta) \dots (k_i - \theta)} \right) \\ & \quad \times \left(\frac{\beta^r}{\theta^{l+1}} + \sum_{j=1}^l \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\beta^r}{\theta(\theta + s_1)(\theta + s_2) \dots (\theta + s_j)} \right) \\ &= \left(\prod_{k=1}^n \frac{k}{\theta+k} \right) \left(\prod_{k=m+1}^n \frac{k-\theta}{k-m} \right) \left(\left(\frac{\beta}{\theta}\right)^r + \sum_{j=1}^{r-1} \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\beta^r}{\theta(\theta + s_1)(\theta + s_2) \dots (\theta + s_j)} \right. \\ & \quad \left. + \sum_{i+l=r-1} \sum_{i=1}^{n-m} \sum_{m+1 \leq k_1 < \dots < k_i \leq n} \sum_{0 \leq s_1 \leq \dots \leq s_l \leq n} \frac{\beta^r}{\theta(k_1 - \theta) \dots (k_i - \theta)(\theta + s_1) \dots (\theta + s_l)} \right). \end{aligned}$$

- When $\theta \in \{m + 1, m + 2, \dots, n\}$, we have

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{m+k} \left(\frac{\beta}{\theta+k}\right)^r \\ &= \frac{(-1)^n n!}{2\pi i (n-m)!} \int_C \frac{(m+z+1)_{n-m}}{z(z-1) \dots (z-n)} \left(\frac{\beta}{\theta+z}\right)^r dz \end{aligned}$$

$$\begin{aligned}
 &= \frac{(-1)^{n+1}n!}{(n-m)!} \operatorname{Res}_{z=-\theta} \frac{(m+z+1)_{n-m}}{z(z-1)\cdots(z-n)} \left(\frac{\beta}{\theta+z}\right)^r \\
 &= \frac{(-1)^{n+1}n!}{(n-m)!} \operatorname{Res}_{z=-\theta} \frac{(z+1+m)\cdots(z+\theta-1)(z+\theta+1)\cdots(z+n-\theta)}{z(z-1)\cdots(z-n)} \frac{\beta^r}{(\theta+z)^{r-1}} \\
 &= \frac{(-1)^{n+1}n!}{(n-m)!} [(\theta+z)^{r-2}] \frac{(z+1+m)\cdots(z+\theta-1)(z+\theta+1)\cdots(z+n-\theta)\beta^r}{z(z-1)\cdots(z-n)} \\
 &= \frac{(-1)^{n+1}n!}{(n-m)!} [z^{r-2}] \frac{(z+1+m-\theta)\cdots(z-1)(z+1)\cdots(z+n-\theta)\beta^r}{(z-\theta)(z-\theta-1)\cdots(z-n-\theta)} \\
 &= \frac{n!}{(n-m)!} [z^{r-2}] \left(\prod_{k=1}^n \frac{1}{\theta+k}\right) \left(\prod_{k=m+1, k\neq\theta}^n (k-\theta)\right) \\
 &\quad \times \frac{(1+\frac{z}{m+1-\theta})\cdots(1-z)(1+z)\cdots(1+\frac{z}{n-\theta})\beta^r}{\theta(1-\frac{z}{\theta})(1-\frac{z}{\theta+1})\cdots(1-\frac{z}{\theta+n})} \\
 &= \frac{1}{(n-m)!} \left(\prod_{k=1}^n \frac{k}{\theta+k}\right) \left(\prod_{k=m+1, k\neq\theta}^n (k-\theta)\right) [z^{r-l-2}] \\
 &\quad \times \left(1 + \sum_{i=1}^{n-m-1} \sum_{\substack{k_1, \dots, k_i \neq \theta \\ m+1 \leq k_1 < \dots < k_i \leq n}} \frac{z^i}{(k_1-\theta)\cdots(k_i-\theta)}\right) \\
 &\quad \times \sum_{0 \leq s_1 \leq \dots \leq s_l \leq n} \frac{\beta^r}{\theta(\theta+s_1)(\theta+s_2)\cdots(\theta+s_l)} \\
 &= \frac{1}{(n-m)!} \left(\prod_{k=1}^n \frac{k}{\theta+k}\right) \left(\prod_{k=m+1, k\neq\theta}^n (k-\theta)\right) [z^{r-l-2}] \\
 &\quad \times \left(1 + \sum_{i=1}^{n-m-1} \sum_{\substack{k_1, \dots, k_i \neq \theta \\ m+1 \leq k_1 < \dots < k_i \leq n}} \frac{z^i}{(k_1-\theta)\cdots(k_i-\theta)}\right) \\
 &\quad \times \left(\frac{\beta^r}{\theta^{l+1}} + \sum_{j=1}^l \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\beta^r}{\theta(\theta+s_1)(\theta+s_2)\cdots(\theta+s_j)}\right) \\
 &= \frac{1}{(n-m)!} \left(\prod_{k=1}^n \frac{k}{\theta+k}\right) \left(\prod_{k=m+1, k\neq\theta}^n (k-\theta)\right) \\
 &\quad \times \left(\frac{\beta^r}{\theta^{r-1}} + \sum_{j=1}^{r-2} \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\beta^r}{\theta(\theta+s_1)(\theta+s_2)\cdots(\theta+s_j)}\right) \\
 &\quad + \sum_{i+l=r-2} \sum_{i=1}^{n-m-1} \sum_{\substack{k_1, \dots, k_i \neq \theta \\ m+1 \leq k_1 < \dots < k_i \leq n}} \sum_{0 \leq s_1 \leq \dots \leq s_l \leq n} \frac{\beta^r}{\theta(k_1-\theta)\cdots(k_i-\theta)(\theta+s_1)\cdots(\theta+s_l)} \Big).
 \end{aligned}$$

This proof is complete. □

Setting $r = 1$ and $\theta = \beta$ in (10) of Theorem 4 and $r = 2$ and $\theta = \beta$ in (11) of Theorem 4, we have the following new binomial identity.

Corollary 5 *Let $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, $0 \leq m \leq n$; $\theta > 0$.*

When $\theta \notin \{m + 1, m + 2, \dots, n\}$, we have

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{m+k} \frac{\theta}{\theta+k} = \left(\prod_{k=1}^n \frac{k}{\theta+k} \right) \left(\prod_{k=m+1}^n \frac{k-\theta}{k-m} \right). \tag{12}$$

When $\theta \in \{m + 1, m + 2, \dots, n\}$, we have

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{m+k} \left(\frac{\theta}{\theta+k} \right)^2 \\ &= \frac{\theta}{(n-m)!} \left(\prod_{k=1}^n \frac{k}{\theta+k} \right) \left(\prod_{k=m+1, k \neq \theta}^n (k-\theta) \right). \end{aligned} \tag{13}$$

Corollary 6 (see [4]) For $r, n \in \mathbb{N}; \theta > 0$, we have

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{\theta}{\theta+k} \right)^r \\ &= \left(\prod_{k=1}^n \frac{k}{\theta+k} \right) \left(1 + \sum_{j=1}^{r-1} \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\theta^j}{(\theta+s_1)(\theta+s_2)\dots(\theta+s_j)} \right). \end{aligned} \tag{14}$$

Proof Take $m = n, \beta = \theta$ in (10) of Theorem 4, it is a must that $\theta \notin \{m + 1, m + 2, \dots, n\}$, and define that $\prod_{k=n+1}^n = 1, \sum_{i=1}^0 = 0$, then Corollary 6 follows. \square

Setting $r = 0$ in (14), defining the empty sum $\sum_{j=1}^{-1} = -1$, we have the following familiar formula.

Corollary 7 For all $n \in \mathbb{N}$ and $\theta > 0$,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0. \tag{15}$$

Setting $r = 1$ in (14), defining the empty sum $\sum_{j=1}^0 = 0$, we have the following.

Corollary 8 (see [4]) For all $n \in \mathbb{N}$ and $\theta > 0$,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{\theta}{\theta+k} = \prod_{k=1}^n \frac{k}{\theta+k}. \tag{16}$$

Setting $m = 2$ in (14), we have the following.

Corollary 9 (see [4]) For all $n \in \mathbb{N}$ and $\theta > 0$,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{\theta}{\theta+k} \right)^2 = \left(\prod_{k=1}^n \frac{k}{\theta+k} \right) \left(1 + \sum_{k=1}^n \frac{\theta}{\theta+k} \right). \tag{17}$$

Corollary 10 Let $r, l, n \in \mathbb{N}$ and $\theta > 0$.

When $\theta \notin \{1, 2, \dots, n\}$,

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \left(\frac{\theta}{\theta+k}\right)^r \\ &= \left(\prod_{k=1}^n \frac{k-\theta}{\theta+k} \right) \left(1 + \sum_{j=1}^{r-1} \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\theta^j}{(\theta+s_1)(\theta+s_2)\dots(\theta+s_j)} \right. \\ & \quad \left. + \sum_{i+l=r-1} \sum_{i=1}^n \sum_{1 \leq k_1 < \dots < k_i \leq n} \sum_{0 \leq s_1 \leq \dots \leq s_l \leq n} \frac{\theta^{r-1}}{(k_1-\theta)\dots(k_i-\theta)(\theta+s_1)\dots(\theta+s_l)} \right). \end{aligned} \tag{18}$$

When $\theta \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \left(\frac{\theta}{\theta+k}\right)^r \\ &= \left(\prod_{k=1}^n \frac{1}{\theta+k} \right) \left(\prod_{k=1, k \neq \theta}^n (k-\theta) \right) \left(\theta + \sum_{j=1}^{r-2} \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\theta^j}{(\theta+s_1)(\theta+s_2)\dots(\theta+s_j)} \right. \\ & \quad \left. + \sum_{i+l=r-2} \sum_{i=1}^{n-1} \sum_{\substack{k_1, \dots, k_i \neq \theta \\ 1 \leq k_1 < \dots < k_i \leq n}} \sum_{0 \leq s_1 \leq \dots \leq s_l \leq n} \frac{\theta^{r-1}}{(k_1-\theta)\dots(k_i-\theta)(\theta+s_1)\dots(\theta+s_l)} \right). \end{aligned} \tag{19}$$

Proof Take $m = 0, \beta = \theta$ in Theorem 4, then Corollary 10 follows. □

Setting $r = 0$ in (18) and defining the suitable empty sum, we have the following familiar formula.

Corollary 11 ([7]) *For all $n \in \mathbb{N}$ and $\theta > 0$,*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} = (-1)^n. \tag{20}$$

Setting $r = 1$ and $\theta = \beta$ in (18) and defining the empty sum $\sum_{j=1}^0 = 0$, we have the following new binomial identity.

Corollary 12 *For all $n \in \mathbb{N}$ and $\theta > 0$,*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \frac{\theta}{\theta+k} = \left(\prod_{k=1}^n \frac{k-\theta}{\theta+k} \right). \tag{21}$$

Setting $r = 2$ and $\theta = \beta$ in (18) and defining the empty sum $\sum_{j=1}^0 = 0$, we have the following new binomial identity.

Corollary 13 *For all $n \in \mathbb{N}$ and $\theta > 0$,*

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \left(\frac{\theta}{\theta+k}\right)^2 = \left(\prod_{k=1}^n \frac{k-\theta}{\theta+k} \right) \left(1 + \sum_{k=1}^n \frac{\theta}{\theta+k} \right). \tag{22}$$

Theorem 14 For $r, l, n \in \mathbb{N}$, $0 \leq m \leq n$, $\beta \neq 0$.

When $\theta \notin \{0, 1, 2, \dots, n\}$, we have

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{m+k} \left(\frac{\beta}{k-\theta}\right)^r \\ &= (-1)^{n+1} \binom{\theta-1}{n}^{-1} \binom{n+\theta}{m+\theta} \left(-\left(-\frac{\beta}{\theta}\right)^r + \sum_{j=1}^{r-1} \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\beta^r}{\theta(s_1-\theta) \dots (s_j-\theta)}\right. \\ & \quad \left. + \sum_{i+l=r-1} \sum_{i=1}^{n-m} \sum_{m+1 \leq k_1 < \dots < k_i \leq n} \sum_{0 \leq s_1 \leq \dots \leq s_l \leq n} \frac{\beta^r}{\theta(k_1+\theta) \dots (k_i+\theta)(s_1-\theta) \dots (s_l-\theta)}\right). \end{aligned} \tag{23}$$

When $\theta \in \{0, 1, 2, \dots, n\}$, we have

$$\begin{aligned} & \sum_{k=0, k \neq \theta}^n (-1)^k \binom{n}{k} \binom{n+k}{m+k} \left(\frac{\beta}{k-\theta}\right)^r \\ &= (-1)^{\theta+1} \binom{n}{\theta} \binom{n+\theta}{m+\theta} \left(\left(-\frac{\beta}{\theta}\right)^r + \sum_{j=1}^r \sum_{\substack{s_1, \dots, s_j \neq \theta \\ 1 \leq s_1 \leq \dots \leq s_j \leq n}} \frac{\beta^r}{(s_1-\theta) \dots (s_j-\theta)}\right. \\ & \quad \left. + \sum_{i+l=r} \sum_{i=1}^{n-m} \sum_{m+1 \leq k_1 < \dots < k_i \leq n} \sum_{\substack{s_1, \dots, s_l \neq \theta \\ 0 \leq s_1 \leq \dots \leq s_l \leq n}} \frac{\beta^r}{(k_1+\theta) \dots (k_i+\theta)(s_1-\theta) \dots (s_l-\theta)}\right). \end{aligned} \tag{24}$$

Proof By means of Theorem 1 and the residues calculus, we obtain the following.

- When $\theta \notin \{0, 1, 2, \dots, n\}$, taking $f(z) = (\frac{\beta}{z-\theta})^r$ in (8), and \mathcal{C} is a positively oriented, encloses the poles $0, 1, 2, \dots, n$ and no others.

$$\begin{aligned} & \sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{m+k} \left(\frac{\beta}{k-\theta}\right)^r \\ &= \frac{(-1)^n n!}{2\pi i (n-m)!} \int_{\mathcal{C}} \frac{(m+z+1)_{n-m}}{z(z-1) \dots (z-n)} \left(\frac{\beta}{z-\theta}\right)^r dz \\ &= \frac{(-1)^{n+1} n!}{(n-m)!} \operatorname{Res}_{z=\theta} \frac{(m+z+1)_{n-m}}{z(z-1) \dots (z-n)} \left(\frac{\beta}{z-\theta}\right)^r \\ &= \frac{(-1)^{n+1} n!}{(n-m)!} \operatorname{Res}_{z=\theta} \frac{(m+z+1)_{n-m}}{z(z-1) \dots (z-n)} \frac{\beta^r}{(z-\theta)^r} \\ &= \frac{(-1)^{n+1} n!}{(n-m)!} [(z-\theta)^{r-1}] \frac{(z+1+m)(z+2+m) \dots (z+n)\beta^r}{z(z-1) \dots (z-n)} \\ &= \frac{(-1)^{n+1} n!}{(n-m)!} [z^{r-1}] \frac{(z+1+m+\theta)(z+2+m+\theta) \dots (z+n+\theta)\beta^r}{(z+\theta)(z+\theta-1) \dots (z+\theta-n)} \\ &= (-1)^{n+1} \frac{n!(\theta-n-1)!(m+1+\theta)_{n-m}}{(n-m)! \theta!} [z^{r-1}] \frac{(1+\frac{z}{m+1+\theta}) \dots (1+\frac{z}{n+\theta})\beta^r}{(1+\frac{z}{\theta})(1+\frac{z}{\theta-1}) \dots (1+\frac{z}{\theta-n})} \\ &= (-1)^{n+1} \binom{\theta-1}{n}^{-1} \binom{n+\theta}{m+\theta} [z^{r-l-1}] \left(1 + \sum_{i=1}^{n-m} \sum_{m+1 \leq k_1 < \dots < k_i \leq n} \frac{z^i}{(k_1+\theta) \dots (k_i+\theta)}\right) \end{aligned}$$

$$\begin{aligned}
 & \times \sum_{0 \leq s_1 \leq \dots \leq s_l \leq n} \frac{\beta^r}{\theta(s_1 - \theta) \dots (s_l - \theta)} \\
 &= (-1)^{n+1} \binom{\theta - 1}{n}^{-1} \binom{n + \theta}{m + \theta} [z^{r-l-1}] \left(1 + \sum_{i=1}^{n-m} \sum_{m+1 \leq k_1 < \dots < k_i \leq n} \frac{z^i}{(k_1 + \theta) \dots (k_i + \theta)} \right) \\
 & \times \left((-1)^l \frac{\beta^r}{\theta^{l+1}} + \sum_{j=1}^l \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\beta^r}{\theta(s_1 - \theta) \dots (s_j - \theta)} \right) \\
 &= (-1)^{n+1} \binom{\theta - 1}{n}^{-1} \binom{n + \theta}{m + \theta} \left(-\left(-\frac{\beta}{\theta}\right)^r + \sum_{j=1}^{r-1} \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\beta^r}{\theta(s_1 - \theta) \dots (s_j - \theta)} \right. \\
 & \left. + \sum_{i+l=r-1} \sum_{i=1}^{n-m} \sum_{m+1 \leq k_1 < \dots < k_i \leq n} \sum_{0 \leq s_1 \leq \dots \leq s_l \leq n} \frac{\beta^r}{\theta(k_1 + \theta) \dots (k_i + \theta)(s_1 - \theta) \dots (s_l - \theta)} \right).
 \end{aligned}$$

- When $\theta \in \{0, 1, 2, \dots, n\}$, taking $f(z) = (\frac{\beta}{z-\theta})^r$ in (8), and \mathcal{C} is a positively oriented, encloses the poles $0, 1, \dots, \theta - 1, \theta + 1, \dots, n$ and no others.

$$\begin{aligned}
 & \sum_{k=0, k \neq \theta}^n (-1)^k \binom{n}{k} \binom{n+k}{m+k} \left(\frac{\beta}{k-\theta}\right)^r \\
 &= \frac{(-1)^n n!}{2\pi i (n-m)!} \int_{\mathcal{C}} \frac{(m+z+1)_{n-m}}{z(z-1)\dots(z-n)} \left(\frac{\beta}{z-\theta}\right)^r dz \\
 &= \frac{(-1)^{n+1} n!}{(n-m)!} \operatorname{Res}_{z=\theta} \frac{(m+z+1)_{n-m}}{z(z-1)\dots(z-n)} \left(\frac{\beta}{z-\theta}\right)^r \\
 &= \frac{(-1)^{n+1} n!}{(n-m)!} \operatorname{Res}_{z=\theta} \frac{(m+z+1)_{n-m}}{z(z-1)\dots(z-\theta+1)(z-\theta-1)\dots(z-n)} \frac{\beta^r}{(z-\theta)^{r+1}} \\
 &= \frac{(-1)^{n+1} n!}{(n-m)!} [(z-\theta)^r] \frac{(z+1+m)(z+2+m)\dots(z+n)\beta^r}{z(z-1)\dots(z-\theta+1)(z-\theta-1)\dots(z-n)} \\
 &= \frac{(-1)^{n+1} n!}{(n-m)!} [z^r] \frac{(z+1+m+\theta)(z+2+m+\theta)\dots(z+n+\theta)\beta^r}{(z+\theta)(z+\theta-1)\dots(z+1)(z-1)\dots(z+\theta-n)} \\
 &= (-1)^{\theta+1} \frac{n!(m+1+\theta)_{n-m}}{(n-m)!\theta!(n-\theta)!} [z^r] \frac{(1+\frac{z}{m+1+\theta})\dots(1+\frac{z}{n+\theta})\beta^r}{(1+\frac{z}{\theta})(1+\frac{z}{\theta-1})\dots(1+z)(1-z)\dots(1-\frac{z}{n-\theta})} \\
 &= (-1)^{\theta+1} \binom{n}{\theta} \binom{n+\theta}{m+\theta} [z^{r-l}] \left(1 + \sum_{i=1}^{n-m} \sum_{m+1 \leq k_1 < \dots < k_i \leq n} \frac{z^i}{(k_1 + \theta) \dots (k_i + \theta)} \right) \\
 & \times \sum_{\substack{s_1, \dots, s_l \neq \theta \\ 0 \leq s_1 \leq \dots \leq s_l \leq n}} \frac{\beta^r}{(s_1 - \theta) \dots (s_l - \theta)} \\
 &= (-1)^{\theta+1} \binom{n}{\theta} \binom{n+\theta}{m+\theta} [z^{r-l}] \left(1 + \sum_{i=1}^{n-m} \sum_{m+1 \leq k_1 < \dots < k_i \leq n} \frac{z^i}{(k_1 + \theta) \dots (k_i + \theta)} \right) \\
 & \times \left(\frac{\beta^r}{(-\theta)^l} + \sum_{j=1}^l \sum_{\substack{s_1, \dots, s_j \neq \theta \\ 1 \leq s_1 \leq \dots \leq s_j \leq n}} \frac{\beta^r}{(s_1 - \theta) \dots (s_j - \theta)} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= (-1)^{\theta+1} \binom{n}{\theta} \binom{n+\theta}{m+\theta} \left(\left(-\frac{\beta}{\theta} \right)^r + \sum_{j=1}^r \sum_{\substack{s_1, \dots, s_j \neq \theta \\ 1 \leq s_1 \leq \dots \leq s_j \leq n}} \frac{\beta^r}{(s_1 - \theta) \cdots (s_j - \theta)} \right. \\
 &\quad \left. + \sum_{i+l=r} \sum_{i=1}^{n-m} \sum_{m+1 \leq k_1 < \dots < k_i \leq n} \sum_{\substack{0 \leq s_1 \leq \dots \leq s_l \leq n \\ s_1, \dots, s_l \neq \theta}} \frac{\beta^r}{(k_1 + \theta) \cdots (k_i + \theta)(s_1 - \theta) \cdots (s_l - \theta)} \right).
 \end{aligned}$$

This proof is complete. □

Corollary 15 For $r, n \in \mathbb{N}$.

When $\theta \notin \{0, 1, 2, \dots, n\}$, we have

$$\begin{aligned}
 &\sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{\theta}{k - \theta} \right)^r \\
 &= (-1)^n \binom{\theta - 1}{n}^{-1} \left(1 - \sum_{j=1}^{r-1} \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\theta^j}{(s_1 - \theta) \cdots (s_j - \theta)} \right). \tag{25}
 \end{aligned}$$

When $\theta \in \{0, 1, 2, \dots, n\}$, we have

$$\begin{aligned}
 &\sum_{k=0, k \neq \theta}^n (-1)^k \binom{n}{k} \left(\frac{\theta}{k - \theta} \right)^r \\
 &= (-1)^{\theta+1} \binom{n}{\theta} \left((-1)^r + \sum_{j=1}^r \sum_{\substack{s_1, \dots, s_j \neq \theta \\ 1 \leq s_1 \leq \dots \leq s_j \leq n}} \frac{\theta^j}{(s_1 - \theta) \cdots (s_j - \theta)} \right). \tag{26}
 \end{aligned}$$

Proof Take $m = n, \beta = \theta$ in Theorem 14 and define that $\prod_{k=n+1}^n = 1, \sum_{i=1}^0 = 0$, then Corollary 15 follows. □

Corollary 16 For $r, l, n \in \mathbb{N}$.

When $\theta \notin \{0, 1, 2, \dots, n\}$, we have

$$\begin{aligned}
 &\sum_{k=0}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \left(\frac{\theta}{k - \theta} \right)^r \\
 &= (-1)^{n+1} \binom{\theta - 1}{n}^{-1} \binom{n+\theta}{\theta} \left((-1)^{r+1} + \sum_{j=1}^{r-1} \sum_{1 \leq s_1 \leq \dots \leq s_j \leq n} \frac{\theta^j}{(s_1 - \theta) \cdots (s_j - \theta)} \right. \\
 &\quad \left. + \sum_{i+l=r-1} \sum_{i=1}^n \sum_{1 \leq k_1 < \dots < k_i \leq n} \sum_{0 \leq s_1 \leq \dots \leq s_l \leq n} \frac{\theta^{r-1}}{(k_1 + \theta) \cdots (k_i + \theta)(s_1 - \theta) \cdots (s_l - \theta)} \right). \tag{27}
 \end{aligned}$$

When $\theta \in \{0, 1, 2, \dots, n\}$, we have

$$\begin{aligned}
 &\sum_{k=0, k \neq \theta}^n (-1)^k \binom{n}{k} \binom{n+k}{k} \left(\frac{\theta}{k - \theta} \right)^r \\
 &= (-1)^{\theta+1} \binom{n}{\theta} \binom{n+\theta}{\theta}
 \end{aligned}$$

$$\begin{aligned} & \times \left((-1)^r + \sum_{j=1}^r \sum_{\substack{s_1, \dots, s_j \neq \theta \\ 1 \leq s_1 \leq \dots \leq s_j \leq n}} \frac{\theta^j}{(s_1 - \theta) \cdots (s_j - \theta)} \right. \\ & \left. + \sum_{i+l=r} \sum_{i=1}^n \sum_{1 \leq k_1 < \dots < k_i \leq n} \sum_{\substack{s_1, \dots, s_l \neq \theta \\ 0 \leq s_1 \leq \dots \leq s_l \leq n}} \frac{\theta^r}{(k_1 + \theta) \cdots (k_i + \theta)(s_1 - \theta) \cdots (s_l - \theta)} \right). \end{aligned} \tag{28}$$

Proof Take $m = 0, \beta = \theta$ in Theorem 14 and define that $\prod_{k=n+1}^n = 1, \sum_{i=1}^0 = 0$, then Corollary 16 follows. □

Remark 17 By applying Bell polynomials (3) and (4), we can deduce all the results of [18] and [19]. Our results are another representation of the corresponding results in [18] and [19]. For example, compare formula (24) with $m = 0, \theta = M, \beta = 1$ and the following formula [19]: For $0 \leq M \leq n$,

$$\begin{aligned} & \sum_{0 \leq k \leq n, k \neq M} (-1)^{k-1} \binom{n}{k} \binom{n+k}{k} \frac{1}{(k-M)^r} \\ & = (-1)^M \binom{n}{M} \binom{M+n}{M} \sum_{m_1+2m_2+3m_3+\dots=r} \frac{1}{m_1! m_2! m_3! \cdots} \left(\frac{x_1}{1}\right)^{m_1} \left(\frac{x_2}{2}\right)^{m_2} \left(\frac{x_3}{3}\right)^{m_3} \cdots, \end{aligned} \tag{29}$$

where

$$x_k = (-1)^{k-1} H_{M+n}^{(k)} - 2H_M^{(k)} + H_{n-M}^{(k)}, \quad H_n^{(r)} = \sum_{k=1}^n \frac{1}{k^r}, \quad n, r \in \mathbb{N}.$$

Obviously, formula (24) includes the above formula (29) and the main result of [18], *i.e.*, formula (24) is an extension of them.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in writing this paper, and read and approved the manuscript.

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References

1. Comtet, L: *Advanced Combinatorics: The Art of Finite and Infinite Expansions*. Reidel, Dordrecht (1974)
2. Flajolet, P, Sedgewick, R: Mellin transforms and asymptotics: finite differences and Rice's integrals. *Theor. Comput. Sci.* **144**, 101-124 (1995)
3. Nörlund, NE: *Vorlesungen über Differenzenrechnung*. Springer, Berlin (1924); Reprinted by Chelsea Publishing Company, Bronx, New York (1954)
4. Jonathon, P: A probabilistic proof of a binomial identity. *Am. Math. Mon.* **120**, 558-562 (2013)
5. Bayad, A, Simsek, Y, Srivastava, HM: Some array type polynomials associated with special numbers and polynomials. *Appl. Math. Comput.* **244**, 149-157 (2014)

6. Simsek, Y: Analysis of the Bernstein basis functions: an approach to combinatorial sums involving binomial coefficients and Catalan number. *Math. Methods Appl. Sci.* (2014). doi:10.1002/mma.327
7. Gould, HW: *Combinatorial Identities: A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*, viii+106 pp. Henry W. Gould, Morgantown (1972)
8. Liu, H, Wang, W: Harmonic number identities via hypergeometric series and Bell polynomials. *Integral Transforms Spec. Funct.* **23**, 49-68 (2012)
9. Mansour, T: Identities on harmonic and q -harmonic number sums. *Afr. Math.* **23**, 135-143 (2012)
10. Prodinger, H: Human proofs of identities by Osburn and Schneider. *Integers* **8**, #A10 (2008)
11. Prodinger, H: Mortenson's identities and partial fraction decomposition. *Util. Math.* (in press). <http://math.sun.ac.za/~hproding/paper1st.htm>
12. Tauraso, R, Prodinger, H: New multiple harmonic sum identities. *Electron. J. Comb.* **21**, #P2.43 (2014)
13. Sofo, A: Sums of derivatives of binomial coefficients. *Adv. Appl. Math.* **42**, 123-134 (2009)
14. Spieß, J: Some identities involving harmonic numbers. *Math. Comput.* **55**, 839-863 (1990)
15. Wang, W: Riordan arrays and harmonic number identities. *Comput. Math. Appl.* **60**, 1494-1509 (2010)
16. Yan, Q, Wei, C, Fan, X: q -Generalizations of Mortenson's identities and further identities. *Ramanujan J.* (2014). doi:10.1007/s11139-013-9526-7
17. Zhang, ZZ, Wang, J: A common generalization of some identities. *Integers* **5**, #A33 (2005)
18. Kirschenhofer, P: A note on alternating sums. *Electron. J. Comb.* **3**, #R7 (1996)
19. Prodinger, H: Identities involving harmonic numbers that are of interest for physicists. *Util. Math.* **83**, 291-299 (2010)

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