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# Composition of piecewise pseudo almost periodic functions and applications to abstract impulsive differential equations

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## Abstract

In this paper, we introduce the concept of piecewise pseudo almost periodic functions on a Banach space and establish some composition theorems of piecewise pseudo almost periodic functions. We apply these composition theorems to investigate the existence of piecewise pseudo almost periodic (mild) solutions to abstract impulsive differential equations. In addition, the stability of piecewise pseudo almost periodic solutions is considered.

**Keywords:** piecewise pseudo almost periodic functions; composition theorem; mild solutions; impulsive differential equations

## 1 Introduction

The notion of pseudo almost periodic functions was introduced by Zhang as a natural generalization of the classical concept of almost periodic functions in [1, 2]. Since then, such a notion has attracted many researchers' interest. The topics of these functions and their generations have been widely investigated in many publications such as [3–18] and the references therein. In particular, in [11, 12], Diagana introduced the concept of Stepanov-like pseudo almost periodic functions and gave some properties including the composition theorem; the authors in [17] proposed the concept of pseudo almost periodic functions on time scales and established some results about the existence of pseudo almost periodic solutions to dynamic equations on time scales; in [10], a new concept which is called weighted pseudo-almost periodicity implements in a natural fashion the notion of pseudo-almost periodicity.

On the other hand, the study of impulsive differential equations is important [19–21] because many evolution processes, optimal control models in economics, stimulated neural networks, population models, artificial intelligence, and robotics are characterized by the fact that at certain moments of time they undergo abrupt changes of state. The existence of solutions is among the most attractive topics in the qualitative theory of impulsive differential equations [19, 21–25]. Likewise, the existence of almost periodic solutions of abstract impulsive differential equations has been considered by many authors; see, e.g., [26–28]. However, there are few papers concerned with pseudo almost periodic functions on impulsive systems.

Motivated by the above, our main propose of this paper is to introduce the concept of piecewise pseudo almost periodic functions on a Banach space and establish some com-

position theorems of piecewise pseudo almost periodic functions. Finally, we give some results about the existence and stability of piecewise pseudo almost periodic solutions to the following abstract impulsive differential equation:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & t \in R, t \neq t_i, i \in Z, \\ \Delta u(t_i) \triangleq u(t_i^+) - u(t_i^-) = I_i(u(t_i)), \end{cases} \quad (1.1)$$

where  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $\{T(t) : t \geq 0\}$  on a Banach space  $X$ ,  $f$ ,  $I_i$ , and  $t_i$  satisfy suitable conditions that will be established later. In addition, the notations  $u(t_i^+)$  and  $u(t_i^-)$  represent the right-hand side and the left-hand side limits of  $u(\cdot)$  at  $t_i$ , respectively.

## 2 Preliminaries

Throughout this paper, we denote by  $X$  a Banach space; let  $\mathbb{T}$  be the set consisting of all real sequences  $\{t_i\}_{i \in \mathbb{Z}}$  such that  $\gamma = \inf_{i \in \mathbb{Z}}(t_{i+1} - t_i) > 0$ . For  $\{t_i\}_{i \in \mathbb{Z}} \in \mathbb{T}$ , let  $PC(R, X)$  be the space formed by all bounded piecewise continuous functions  $\phi : R \rightarrow X$  such that  $\phi(\cdot)$  is continuous at  $t$  for any  $t \notin \{t_i\}_{i \in \mathbb{Z}}$  and  $\phi(t_i) = \phi(t_i^-)$  for all  $i \in \mathbb{Z}$ ; let  $\Omega$  be a subset of  $X$  and  $PC(R \times \Omega, X)$  be the space formed by all piecewise continuous functions  $\phi : R \times \Omega \rightarrow X$  such that for any  $x \in \Omega$ ,  $\phi(\cdot, x) \in PC(R, X)$  and for any  $t \in R$ ,  $\phi(t, \cdot)$  is continuous at  $x \in \Omega$ .

**Definition 2.1** [26] A function  $\phi \in PC(R, X)$  is said to be piecewise almost periodic if the following conditions are fulfilled:

- (1)  $\{t_i^j = t_{i+j} - t_i\}, i \in \mathbb{Z}, j = 0, \pm 1, \pm 2, \dots$  are equipotentially almost periodic; that is, for any  $\epsilon > 0$ , there exists a relatively dense set  $Q_\epsilon$  of  $R$  such that for each  $\tau \in Q_\epsilon$ , there is an integer  $q \in \mathbb{Z}$  such that  $|t_{i+q} - t_i - \tau| < \epsilon$  for all  $i \in \mathbb{Z}$ .
- (2) For any  $\epsilon > 0$ , there exists a positive number  $\delta = \delta(\epsilon)$  such that if the points  $t'$  and  $t''$  belong to the same interval of continuity of  $\phi$  and  $|t' - t''| < \delta$ , then  $\|\phi(t') - \phi(t'')\| < \epsilon$ .
- (3) For every  $\epsilon > 0$ , there exists a relatively dense set  $\Omega(\epsilon)$  in  $R$  such that if  $\tau \in \Omega(\epsilon)$ , then

$$\|\phi(t + \tau) - \phi(t)\| < \epsilon$$

for all  $t \in R$  satisfying the condition  $|t - t_i| > \epsilon, i \in \mathbb{Z}$ . The number  $\tau$  is called an  $\epsilon$ -almost period of  $\phi$ .

We denote by  $AP_T(R, X)$  the space of all piecewise almost periodic functions. Obviously, the space  $AP_T(R, X)$  endowed with the sup norm defined by  $\|\phi\|_\infty = \sup_{t \in R} \|\phi(t)\|$  for any  $\phi \in AP_T(R, X)$  is a Banach space.

Throughout the rest of this paper, let  $UPC(R, X)$  be the space of all functions  $\phi \in PC(R, X)$  such that  $\phi$  satisfies the condition (2) in Definition 2.1.

**Lemma 2.2** [26] Let  $\phi \in AP_T(R, X)$ , then the range of  $\phi, R(\phi)$ , is a relatively compact subset of  $X$ .

**Definition 2.3**  $f \in PC(R \times \Omega, X)$  is said to be piecewise almost periodic in  $t$  uniformly in  $x \in \Omega$  if for each compact set  $K \subseteq \Omega, \{f(\cdot, x) : x \in K\}$  is uniformly bounded, and given

$\epsilon > 0$ , there exists a relatively dense set  $\Omega(\epsilon)$  such that

$$\|f(t + \tau, x) - f(t, x)\| < \epsilon$$

for all  $x \in K$ ,  $\tau \in \Omega(\epsilon)$ , and  $t \in R$ ,  $|t - t_i| > \epsilon$ . Denote by  $AP_T(R \times \Omega, X)$  the set of all such functions.

Set

$$PAP_T^0(R, X) := \left\{ \phi \in PC(R, X) : \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \|\phi(t)\| dt = 0 \right\},$$

$$PAP_T^0(R \times \Omega, X) := \left\{ \Phi \in PC(R \times \Omega, X) : \lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \|\Phi(t, s)\| dt = 0 \text{ uniformly with respect to } s \in K, \right. \\ \left. \text{where } K \text{ is an arbitrary compact subset of } \Omega \right\}.$$

**Lemma 2.4** Suppose  $\phi \in PC(R, X)$ .  $\phi \in PAP_T^0(R, X)$  if and only if for any  $\epsilon > 0$ ,

$$\lim_{r \rightarrow \infty} \frac{m(M_{r,\epsilon}(\phi))}{2r} = 0,$$

where  $M_{r,\epsilon}(\phi) = \{t \in [-r, r] : \|\phi(t)\| \geq \epsilon\}$  and  $m$  is the Lebesgue measure on  $R$ .

*Proof* Sufficiency. Since  $M_\phi = \sup_{t \in R} \|\phi(t)\| < \infty$ , by the hypothesis, for any  $\epsilon > 0$ , there exists  $r_0 > 0$  such that for  $r > r_0$ ,

$$\frac{m(M_{r,\epsilon}(\phi))}{2r} < \frac{\epsilon}{M_\phi}.$$

Then

$$\frac{1}{2r} \int_{-r}^r \|\phi(t)\| dt = \frac{1}{2r} \left[ \int_{M_{r,\epsilon}(\phi)} \|\phi(t)\| dt + \int_{[-r,r] \setminus M_{r,\epsilon}(\phi)} \|\phi(t)\| dt \right] \\ \leq \frac{m(M_{r,\epsilon}(\phi))}{2r} M_\phi + \epsilon < 2\epsilon$$

for  $r > r_0$ . This implies that  $\phi \in PAP_T^0(R, X)$ .

Necessity. If it is not, there exists  $\epsilon_0 > 0$  such that  $\frac{m(M_{r,\epsilon_0}(\phi))}{2r} \not\rightarrow 0$  as  $r \rightarrow \infty$ . That is, there exists  $\delta > 0$  such that for any  $n$ ,

$$\frac{m(M_{r_n,\epsilon_0}(\phi))}{2r_n} \geq \delta$$

for some  $r_n > n$ . Then

$$\frac{1}{2r_n} \int_{-r_n}^{r_n} \|\phi(t)\| dt = \frac{1}{2r_n} \left[ \int_{M_{r_n,\epsilon_0}(\phi)} \|\phi(t)\| dt + \int_{[-r_n,r_n] \setminus M_{r_n,\epsilon_0}(\phi)} \|\phi(t)\| dt \right] \\ \geq \frac{1}{2r_n} \int_{M_{r_n,\epsilon_0}(\phi)} \|\phi(t)\| dt$$

$$\begin{aligned} &\geq \frac{m(M_{r_n, \epsilon_0}(\phi))}{2r_n} \epsilon_0 \\ &\geq \delta \epsilon_0, \end{aligned}$$

which contradicts the fact that  $\phi \in PAP_T^0(R, X)$ . This completes the proof.  $\square$

**Remark 2.5** The proof of Lemma 2.4 is essentially contained in Liang *et al.*'s result (see Lemma 2.1 in [29]) or a more general case (see Proposition 3.1 and Corollary 3.2 in [30]). We have included it for the reader's convenience.

**Lemma 2.6**  $PAP_T^0(R, X)$  is a translation invariant set of  $PC(R, X)$ .

*Proof* For any  $s \in R$ ,  $\phi \in PAP_T^0(R, X)$ ,  $\epsilon > 0$ ,  $r > 0$ , we have

$$\begin{aligned} M_{r, \epsilon}(R_s \phi) &= \{t \in [-r, r] : \|R_s \phi(t)\| \geq \epsilon\} \\ &= \{t \in [-r, r] : \|\phi(t + s)\| \geq \epsilon\} \\ &= \{t \in [-r + s, r + s] : \|\phi(t)\| \geq \epsilon\} \subseteq \{t \in [-r - |s|, r + |s|] : \|\phi(t)\| \geq \epsilon\}. \end{aligned}$$

So,

$$\frac{1}{2r} m(M_{r, \epsilon}(R_s \phi)) \leq \frac{1}{2r} m(M_{r+|s|, \epsilon}(\phi)) = \frac{2(r + |s|)}{2r} \frac{1}{2(r + |s|)} m(M_{r+|s|, \epsilon}(\phi)).$$

Since  $\phi \in PAP_T^0(R, X)$ , then by Lemma 2.4, we have

$$\frac{1}{2(r + |s|)} m(M_{r+|s|, \epsilon}(\phi)) \rightarrow 0, \quad r \rightarrow \infty.$$

Furthermore,  $\lim_{r \rightarrow \infty} \frac{2(r+|s|)}{2r} = 1$ , so

$$\frac{1}{2r} m(M_{r, \epsilon}(R_s \phi)) \rightarrow 0, \quad r \rightarrow \infty.$$

Again, using Lemma 2.4, we know  $R_s \phi \in PAP_T^0(R, X)$ . The proof is complete.  $\square$

**Definition 2.7** A function  $f \in PC(R, X)$  is said to be piecewise pseudo almost periodic if it can be decomposed as  $f = g + h$ , where  $g \in AP_T(R, X)$  and  $h \in PAP_T^0(R, X)$ . Denote by  $PAP_T(R, X)$  the set of all such functions.

$PAP_T(R, X)$  is a Banach space with the sup norm  $\|\cdot\|_\infty$ .

**Lemma 2.8** The decomposition of piecewise pseudo almost periodic functions is unique.

*Proof* By Definition 2.7, we only need to prove that  $f \equiv 0$  when  $f \in AP_T(R, X)$  as well as being in  $PAP_T^0(R, X)$ . Suppose the contrary, then there exists at least one number  $x_0 \in R$  such that  $f(x_0) \neq 0$ . Without loss of generality, we may assume that  $x_0 \neq t_i$  ( $i \in Z$ ) since we can replace  $x_0$  by  $x_1$  in a small neighborhood of  $x_0$ . By Lemma 76 in [20], we can choose two numbers  $l > 0$  and  $0 < \delta < \min\{\frac{\gamma}{2}, \min_{i \in Z} |x_0 - t_i|\}$  such that any interval of length  $l$  contains a subinterval of length  $2\delta$  whose points must all be  $\frac{\|f(x_0)\|}{3}$ -almost period and  $t', t'' \in (t_i, t_{i+1})$ ,

$i \in \mathbb{Z}$ ,  $|t' - t''| < \delta$  imply that  $\|f(t') - f(t'')\| < \frac{\|f(x_0)\|}{3}$ . Consider now any interval of length  $l$ ,  $(r - \delta - x_0, r - \delta - x_0 + l)$ ,  $r$  is a real number, there exists a  $\frac{\|f(x_0)\|}{3}$ -almost period  $\tau$  of  $f$  which belongs to this interval, thus  $x_0 + \tau \in (r - \delta, r - \delta + l)$ . Assume  $|x - x_0| < \delta$ , then  $x + \tau$  will range over an interval of length  $2\delta$ , and

$$\begin{aligned} \|f(x + \tau)\| &\geq \|f(x_0)\| - \|f(x) - f(x_0)\| - \|f(x + \tau) - f(x)\| \\ &> \|f(x_0)\| - \frac{\|f(x_0)\|}{3} - \frac{\|f(x_0)\|}{3} = \frac{\|f(x_0)\|}{3}, \end{aligned}$$

which shows that any interval of length  $l$  contains a subinterval of length  $2\delta$  at all points satisfying  $\|f(x)\| > \frac{\|f(x_0)\|}{3}$ , so

$$\frac{1}{2nl} \int_{-nl}^{nl} \|f(x)\| \, dx \geq \frac{1}{2nl} \sum_{k=-(n-1)}^n \int_{(k-1)l}^{kl} \|f(x)\| \, dx > \frac{1}{2nl} \cdot 2n \cdot 2\delta \cdot \frac{\|f(x_0)\|}{3} = \frac{2\delta \|f(x_0)\|}{3l},$$

which contradicts the fact that  $f \in PAP_T^0(R, X)$ . The proof is complete.  $\square$

**Definition 2.9** Let  $PAP_T(R \times \Omega, X)$  consist of all functions  $F \in PC(R \times \Omega, X)$  such that  $F = G + H$ , where  $G \in AP_T(R \times \Omega, X)$  and  $H \in PAP_T^0(R \times \Omega, X)$ .

### 3 Composition theorems

**Theorem 3.1** Suppose  $f \in PAP_T(R \times \Omega, X)$ . Assume that the following conditions hold:

- (i)  $\{f(t, x) : t \in R, x \in K\}$  is bounded for every bounded subset  $K \subseteq \Omega$ .
- (ii)  $f(t, \cdot)$  is uniformly continuous in each bounded subset of  $\Omega$  uniformly in  $t \in R$ .

If  $\phi \in PAP_T(R, X)$  such that  $R(\phi) \subset \Omega$ , then  $f(\cdot, \phi(\cdot)) \in PAP_T(R, X)$ .

*Proof* Since  $f \in PAP_T(R \times \Omega, X)$  and  $\phi \in PAP_T(R, X)$ , by Definitions 2.7 and 2.9, we have  $f = g + h$  and  $\phi = \phi_1 + \phi_2$  with  $g \in AP_T(R \times \Omega, X)$ ,  $h \in PAP_T^0(R \times \Omega, X)$ ,  $\phi_1 \in AP_T(R, X)$ ,  $\phi_2 \in PAP_T^0(R, X)$ . So, the function  $f(\cdot, \phi(\cdot))$  can be decomposed as

$$\begin{aligned} f(\cdot, \phi(\cdot)) &= g(\cdot, \phi_1(\cdot)) + f(\cdot, \phi(\cdot)) - g(\cdot, \phi_1(\cdot)) \\ &= g(\cdot, \phi_1(\cdot)) + f(\cdot, \phi(\cdot)) - f(\cdot, \phi_1(\cdot)) + h(\cdot, \phi_1(\cdot)). \end{aligned}$$

By Lemma 2.2,  $R(\phi_1)$  is relatively compact in  $X$ ,  $g(t, \cdot)$  is uniformly continuous in  $R(\phi_1)$  uniformly in  $t \in R$ . By a proof similar to Theorem 3.1 in [27],  $g(\cdot, \phi_1(\cdot)) \in AP_T(R, X)$ . To show that  $f(\cdot, \phi(\cdot)) \in PAP_T(R, X)$ , we need to show that  $f(\cdot, \phi(\cdot)) - f(\cdot, \phi_1(\cdot)) + h(\cdot, \phi_1(\cdot)) \in PAP_T^0(R, X)$ .

First, we show that  $f(\cdot, \phi(\cdot)) - f(\cdot, \phi_1(\cdot)) \in PAP_T^0(R, X)$ . Let  $K$  be a bounded subset of  $\Omega$  such that  $R(\phi) \subseteq K$ ,  $R(\phi_1) \subseteq K$ . By (ii),  $f(t, \cdot)$  is uniformly continuous in  $R(\phi_1)$  uniformly in  $t \in R$ , given  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $y_1, y_2 \in K$  and  $\|y_1 - y_2\| < \delta$  implies that

$$\|f(t, y_1) - f(t, y_2)\| < \epsilon, \quad t \in R.$$

Thus, for each  $t \in R$ ,  $\|\phi_2(t)\| < \delta$  implies that for all  $t \in R$ ,

$$\|f(t, \phi(t)) - f(t, \phi_1(t))\| < \epsilon,$$

where  $\phi_2(t) = \phi(t) - \phi_1(t)$ . For  $r > 0$ , let  $M_{r,\delta}(\phi_2) = \{t \in [-r, r] : \|\phi_2(t)\| \geq \delta\}$ , so we get

$$M_{r,\epsilon}(f, \phi, \phi_1) \triangleq M_{r,\epsilon}(f(\cdot, \phi(\cdot)) - f(\cdot, \phi_1(\cdot))) \subseteq M_{r,\delta}(\phi_2).$$

Since  $\phi_2 \in PAP_T^0(R, X)$ , by Lemma 2.4, we have

$$\frac{m(M_{r,\delta}(\phi_2))}{2r} \rightarrow 0 \quad \text{as } r \rightarrow \infty,$$

hence

$$\frac{m(M_{r,\epsilon}(f, \phi, \phi_1))}{2r} \rightarrow 0 \quad \text{as } r \rightarrow \infty,$$

this implies  $f(\cdot, \phi(\cdot)) - f(\cdot, \phi_1(\cdot)) \in PAP_T^0(R, X)$  by Lemma 2.4.

It remains to show  $h(\cdot, \phi_1(\cdot)) \in PAP_T^0(R, X)$ . Note that  $f = g + h$  and  $g(t, \cdot)$  is uniformly continuous in  $R(\phi_1)$  uniformly in  $t \in R$ . By the hypothesis (ii),  $f(t, \cdot)$  is uniformly continuous in  $R(\phi_1)$  uniformly in  $t \in R$ , so is  $h$ . Since  $R(\phi_1)$  is relatively compact in  $X$ , for  $\epsilon > 0$ , one can find a finite number  $n$  of open balls  $O_k$  with center  $z_k \in R(x_1)$ ,  $k = 1, 2, \dots, n$  and radius  $\delta(z_k, \frac{\epsilon}{2})$  such that  $R(\phi_1) \subseteq \bigcup_{k=1}^n O_k$  and

$$\|h(t, z) - h(t, z_k)\| < \frac{\epsilon}{2}, \quad z \in O_k, t \in R.$$

The set

$$B_k = \{t \in R : \phi_1(t) \in O_k\}$$

is open and  $R = \bigcup_{k=1}^n B_k$ , let  $E_k = B_k \setminus \bigcup_{j=1}^{k-1} B_j$ , then  $E_i \cap E_j = \emptyset$  when  $i \neq j$ ,  $1 \leq i, j \leq n$ .

Since each  $h(\cdot, z_k) \in PAP_T^0(R, X)$ , there is a number  $r_0$  such that

$$\sum_{k=1}^n \frac{1}{2r} \int_{-r}^r \|h(t, z_k)\| dt < \frac{\epsilon}{2}, \quad r > r_0.$$

Then

$$\begin{aligned} \frac{1}{2r} \int_{-r}^r \|h(t, \phi_1(t))\| dt &= \frac{1}{2r} \sum_{k=1}^n \int_{E_k \cap [-r, r]} \|h(t, \phi_1(t))\| dt \\ &\leq \frac{1}{2r} \sum_{k=1}^n \left[ \int_{E_k \cap [-r, r]} \|h(t, \phi_1(t)) - h(t, z_k)\| dt \right. \\ &\quad \left. + \int_{E_k \cap [-r, r]} \|h(t, z_k)\| dt \right] \\ &\leq \frac{\epsilon}{2} + \sum_{k=1}^n \frac{1}{2r} \int_{-r}^r \|h(t, z_k)\| dt \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This implies that  $h(\cdot, \phi_1(\cdot)) \in PAP_T^0(R, X)$ . Thus,  $f(\cdot, \phi(\cdot)) \in PAP_T(R, X)$ . This completes the proof.  $\square$

**Remark 3.2** A result similar to Theorem 3.1 was obtained by Agarwal *et al.* for weighted pseudo-almost periodic functions (see Theorem 3.2 in [30]).

Since the uniform continuity is weaker than the Lipschitz continuity, the next corollary is a straightforward consequence of the previous theorem.

**Corollary 3.3** *Let  $f \in PAP_T(R \times \Omega, X)$ ,  $\phi \in PAP_T(R, X)$  and  $R(\phi) \subset \Omega$ . Assume further that there exists a number  $L > 0$  satisfying*

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \quad \forall t \in R, x, y \in \Omega,$$

*then the function  $t \mapsto f(t, \phi(t))$  belongs to  $PAP_T(R, X)$ .*

**Theorem 3.4** *Assume the sequence of vector-valued functions  $\{I_i\}_{i \in Z}$  is pseudo almost periodic, i.e., for any  $x \in \Omega$ ,  $\{I_i(x), i \in Z\}$  is a pseudo almost periodic sequence. Suppose  $\{I_i(x) : i \in Z, x \in K\}$  is bounded for every bounded subset  $K \subseteq \Omega$ ,  $I_i(x)$  is uniformly continuous in  $x \in \Omega$  uniformly in  $i \in Z$ . If  $\phi \in PAP_T(R, X) \cap UPC(R, X)$  such that  $R(\phi) \subset \Omega$ , then  $I_i(\phi(t_i))$  is pseudo almost periodic.*

*Proof* Fix  $\phi \in PAP_T(R, X) \cap UPC(R, X)$ , first we show  $\phi(t_i)$  is pseudo almost periodic. By Definition 2.7, we have  $\phi = \phi_1 + \phi_2$ , where  $\phi_1 \in AP_T(R, X)$ ,  $\phi_2 \in PAP_T^0(R, X)$ . It follows from Lemma 37 in [20] that the sequence  $\phi_1(t_i)$  is almost periodic. To show  $\phi(t_i)$  is pseudo almost periodic, we need to show that  $\phi_2(t_i) \in PAP_0(Z, X)$ . By the hypothesis,  $\phi, \phi_1 \in UPC(R, X)$ , so is  $\phi_2$ . Let  $0 < \epsilon < 1$ , there exists  $0 < \delta < \min\{1, \gamma\}$  such that for  $t \in (t_i - \delta, t_i)$ ,  $i \in Z$ , we have

$$\|\phi_2(t)\| \geq (1 - \epsilon)\|\phi_2(t_i)\|, \quad i \in Z.$$

Since  $\{t_i^j\}$ ,  $i \in Z, j = 0, \pm 1, \pm 2, \dots$  are equipotentially almost periodic,  $\{t_i^1\}$  is an almost periodic sequence. Here we assume a bound of  $\{t_i^1\}$  is  $M_t$  and  $|t_i| \geq |t_{-i}|$ ; therefore,

$$\begin{aligned} \frac{1}{2t_i} \int_{-t_i}^{t_i} \|\phi_2(t)\| dt &\geq \frac{1}{2t_i} \sum_{j=-i+1}^i \int_{t_{j-1}}^{t_j} \|\phi_2(t)\| dt \\ &\geq \frac{1}{2t_i} \sum_{j=-i+1}^i \int_{t_j-\delta}^{t_j} \|\phi_2(t)\| dt \\ &\geq \frac{1}{2t_i} \sum_{j=-i+1}^i \delta(1 - \epsilon)\|\phi_2(t_j)\| \\ &\geq \frac{\delta(1 - \epsilon)}{M_t} \frac{1}{2i} \sum_{j=-i+1}^i \|\phi_2(t_j)\|. \end{aligned}$$

Since  $\phi_2 \in PAP_T^0(R, X)$ , it follows from the inequality above that  $\phi_2(t_i) \in PAP_0(Z, X)$ . Hence,  $\phi(t_i)$  is pseudo almost periodic.

Now, we show  $I_i(\phi(t_i))$  is pseudo almost periodic. Let

$$I(t, x) = I_n(x) + (t - n)[I_{n+1}(x) - I_n(x)], \quad n \leq t < n + 1, n \in Z,$$

$$\Phi(t) = \phi(t_n) + (t - n)[\phi(t_{n+1}) - \phi(t_n)], \quad n \leq t < n + 1, n \in Z.$$

Since  $I_n, \phi(t_n)$  are two pseudo almost periodic sequences, by Lemma 1.7.12 in [31], we know that  $I \in PAP(R \times \Omega, X), \Phi \in PAP(R, X)$ . For every  $t \in R$ , there exists a number  $n \in Z$  such that  $|t - n| \leq 1$ ,

$$\begin{aligned} \|I(t, x)\| &\leq \|I_n(x)\| + |t - n|[\|I_{n+1}(x)\| + \|I_n(x)\|] \\ &\leq 2\|I_n(x)\| + \|I_{n+1}(x)\|. \end{aligned}$$

Since  $\{I_n(x) : n \in Z, x \in K\}$  is bounded for every bounded set  $K \subseteq \Omega, \{I(t, x) : t \in R, x \in K\}$  is bounded for every bounded set  $K \subseteq \Omega$ . For every  $x_1, x_2 \in \Omega$ , we have

$$\begin{aligned} \|I(t, x_1) - I(t, x_2)\| &\leq \|I_n(x_1) - I_n(x_2)\| + |t - n|[\|I_{n+1}(x_1) - I_{n+1}(x_2)\| + \|I_n(x_1) - I_n(x_2)\|] \\ &\leq 2\|I_n(x_1) - I_n(x_2)\| + \|I_{n+1}(x_1) - I_{n+1}(x_2)\|. \end{aligned}$$

Noting that  $I_i(x)$  is uniformly continuous in  $x \in \Omega$  uniformly in  $i \in Z$ , we then get that  $I(t, x)$  is uniformly continuous in  $x \in \Omega$  uniformly in  $t \in R$ . Then by Theorem 2.1 in [15],  $I(\cdot, \Phi(\cdot)) \in PAP(R, X)$ . Again, using Lemma 1.7.12 in [31], we have that  $I(i, \Phi(i))$  is a pseudo almost periodic sequence, that is,  $I_i(\phi(t_i))$  is pseudo almost periodic. This completes the proof.  $\square$

**Corollary 3.5** *Assume the sequence of vector-valued functions  $\{I_i\}_{i \in Z}$  is pseudo almost periodic, if there is a number  $L > 0$  such that*

$$\|I_i(x) - I_i(y)\| \leq L\|x - y\|$$

*for all  $x, y \in \Omega, i \in Z$ , if  $\phi \in PAP_T(R, X) \cap UPC(R, X)$  such that  $R(\phi) \subset \Omega$ , then  $I_i(\phi(t_i))$  is pseudo almost periodic.*

#### 4 Piecewise pseudo almost periodic solutions

In this section, we investigate the existence and stability of a piecewise pseudo almost periodic solution to Eq. (1.1). Before starting our main results in this section, we recall the definition of a mild solution to Eq. (1.1).

**Definition 4.1** A function  $u : R \rightarrow X$  is called a mild solution of Eq. (1.1) if for any  $t \in R, t > \sigma, \sigma \neq t_i, i \in Z$ ,

$$u(t) = T(t - \sigma)u(\sigma) + \int_{\sigma}^t T(t - s)f(s, u(s)) ds + \sum_{\sigma < t_i < t} T(t - t_i)I_i(u(t_i)).$$



In fact, using the semigroup theory, we know

$$u(t) = T(t - \sigma)u(\sigma) + \int_{\sigma}^t T(t - s)f(s, u(s)) ds, \quad t > \sigma,$$

is a mild solution to

$$u'(t) = Au(t) + f(t, u(t)).$$

For any  $\sigma \in R$ , we can find  $i \in Z$ ,  $t_{i-1} < \sigma \leq t_i$ , for  $t \in (\sigma, t_i]$ ,

$$u(t) = T(t - \sigma)u(\sigma) + \int_{\sigma}^t T(t - s)f(s, u(s)) ds,$$

by using  $u(t_i^+) - u(t_i^-) = I_i(u(t_i))$ , we have

$$u(t_i^+) = T(t_i - \sigma)u(\sigma) + \int_{\sigma}^{t_i} T(t_i - s)f(s, u(s)) ds + I_i(u(t_i)),$$

then we have

$$\begin{aligned} u(t) &= T(t - t_i)u(t_i^+) + \int_{t_i}^t T(t - s)f(s, u(s)) ds \\ &= T(t - t_i) \left[ T(t_i - \sigma)u(\sigma) + \int_{\sigma}^{t_i} T(t_i - s)f(s, u(s)) ds + I_i(u(t_i)) \right] \\ &\quad + \int_{t_i}^t T(t - s)f(s, u(s)) ds \\ &= T(t - \sigma)u(\sigma) + \int_{\sigma}^{t_i} T(t - s)f(s, u(s)) ds + T(t - t_i)I_i(u(t_i)) \\ &\quad + \int_{t_i}^t T(t - s)f(s, u(s)) ds \\ &= T(t - \sigma)u(\sigma) + \int_{\sigma}^t T(t - s)f(s, u(s)) ds + T(t - t_i)I_i(u(t_i)). \end{aligned}$$

Reiterating this procedure, we get

$$u(t) = T(t - \sigma)u(\sigma) + \int_{\sigma}^t T(t - s)f(s, u(s)) ds + \sum_{\sigma < t_i < t} T(t - t_i)I_i(u(t_i)).$$

First, we study the existence of a piecewise pseudo almost periodic mild solution of Eq. (1.1) when the perturbations  $f, I_i$  ( $i \in Z$ ) are not Lipschitz continuous. We need a criterion of the relatively compact set in  $PC(R, X)$ . We list the following result about the relatively compact set; one may refer to [32–34] for more details.

Let  $h : R \rightarrow R$  be a continuous function such that  $h(t) \geq 1$  for all  $t \in R$  and  $h(t) \rightarrow \infty$  as  $|t| \rightarrow \infty$ . We consider the space

$$(PC)_h^0(R, X) = \left\{ \phi \in PC(R, X) : \lim_{|t| \rightarrow \infty} \frac{\|\phi(t)\|}{h(t)} = 0 \right\}.$$

Endowed with the norm  $\|\phi\|_h = \sup_{t \in R} \frac{\|\phi(t)\|}{h(t)}$ , it is a Banach space.

**Lemma 4.2** *A set  $B \subseteq (PC)_h^0(R, X)$  is a relatively compact set if and only if*

- (1)  $\lim_{|t| \rightarrow \infty} \frac{\|\phi(t)\|}{h(t)} = 0$  uniformly for  $\phi \in B$ .
- (2)  $B(t) = \{\phi(t) : \phi \in B\}$  is relatively compact in  $X$  for every  $t \in R$ .
- (3) The set  $B$  is equicontinuous on each interval  $(t_i, t_{i+1})$  ( $i \in Z$ ).

*Proof* Let  $(PC)^0(R, X) = \{\phi \in PC(R, X) : \lim_{|t| \rightarrow \infty} \|\phi(t)\| = 0\}$ . By an analogous argument in [33, 34],  $(PC)^0(R, X)$  is isometrically isomorphic with the space  $(PC)_h^0(R, X)$ . In order to prove Lemma 4.2, we only need to show that  $B' \subseteq (PC)^0(R, X)$  is a relatively compact set if and only if

- (11)  $\lim_{|t| \rightarrow \infty} \|f(t)\| = 0$  uniformly for  $f \in B'$ .
- (22)  $B'(t) = \{f(t) : f \in B'\}$  is relatively compact in  $X$  for every  $t \in R$ .
- (33) The set  $B'$  is equicontinuous on each interval  $(t_i, t_{i+1})$  ( $i \in Z$ ).

Sufficiency. By (11), for any  $\epsilon > 0$ , there exists  $\delta_1 > 0$  such that

$$\|f(t)\| < \epsilon, \quad |t| > \delta_1, f \in B'. \tag{4.1}$$

By (33), for the above  $\epsilon$ , there exists  $\delta : 0 < \delta < \min\{\delta_1, \gamma\}$  such that  $t', t'' \in (t_i, t_{i+1})$ ,  $i \in Z$ ,  $|t' - t''| < \delta$ ,

$$\|f(t') - f(t'')\| < \epsilon, \quad f \in B'.$$

For the interval  $[-\delta_1, \delta_1]$ , there exists a set  $S = \{s_1, s_2, \dots, s_q\} \subset [-\delta_1, \delta_1]$ ,  $s_j \neq t_i$ ,  $j = 1, 2, \dots, q$  such that  $|t - s_j| < \delta$  and

$$\|f(t) - f(s_j)\| < \epsilon, \quad j = 1, 2, \dots, q, f \in B'. \tag{4.2}$$

For any sequence  $\{f_k : k \geq 1\} \subseteq B'$ , by (22), we can extract a subsequence that converges at each point  $t \in R$ . Since  $S$  is finite, then for the above  $\epsilon > 0$ , there exists  $n_0 \in N$ ,

$$\|f_m(t) - f_n(t)\| < \epsilon, \quad m, n \geq n_0, t \in S. \tag{4.3}$$

So, for  $t \in [-\delta_1, \delta_1]$ , by (4.2) and (4.3),

$$\|f_m(t) - f_n(t)\| \leq \|f_m(t) - f_m(s_j)\| + \|f_m(s_j) - f_n(s_j)\| + \|f_n(s_j) - f_n(t)\| < 3\epsilon.$$

For  $|t| > \delta_1$ , by (4.1),

$$\|f_m(t) - f_n(t)\| < 2\epsilon.$$

Thus,  $\{f_k : k \geq 1\}$  is uniformly convergent on  $R$ ,  $B' \subseteq (PC)^0(R, X)$  is a relatively compact set.

Necessity. Since  $B' \subseteq (PC)^0(R, X)$  is relatively compact, for any  $\epsilon > 0$ , there exist a finite number of functions  $f_1, f_2, \dots, f_m$  of  $B'$  such that

$$\|f - f_j\| < \epsilon, \quad j = 1, 2, \dots, m, f \in B'. \tag{4.4}$$

This finite set of functions  $f_1, f_2, \dots, f_m$  is equicontinuous; that is, for the above  $\epsilon$ , there exists a number  $\delta_2 > 0$  such that  $t', t'' \in (t_i, t_{i+1})$ ,  $i \in Z$ ,  $|t' - t''| < \delta_2$ , we have  $\|f(t') - f(t'')\| < \epsilon$ , using (4.4), for any  $f \in B'$ ,

$$\|f(t') - f(t'')\| \leq \|f(t') - f_j(t')\| + \|f_j(t') - f_j(t'')\| + \|f_j(t'') - f(t'')\| < 3\epsilon,$$

which shows (33). Since  $f_j \in B'$ , then for the above  $\epsilon$ , there exist numbers  $\zeta_j > 0$  such that

$$\|f_j(t)\| < \epsilon, \quad |t| > \zeta_j, j = 1, 2, \dots, m. \tag{4.5}$$

Let  $\delta_3 = \max\{\zeta_1, \zeta_2, \dots, \zeta_m\}$ , by (4.4) and (4.5), for any  $f \in B'$ ,

$$\|f(t)\| \leq \|f(t) - f_j(t)\| + \|f_j(t)\| < 2\epsilon, \quad |t| > \delta_3,$$

which shows (11). Since  $B'$  is relatively compact, then for any sequence  $\{f_k : k \geq 1\} \subseteq B'$ , there exists a subsequence that converges uniformly on  $R$ . Fix  $t \in R$ , from the sequence  $\{f_k(t) : k \geq 1\} \subseteq X$ , there exists a convergent subsequence. Therefore, for fixed  $t \in R$ , the set  $\{f(t) : f \in B'\}$  is relatively compact, which shows (22). The proof is complete.  $\square$

**Remark 4.3** In the  $C(R, X)$ -setting, a result similar to Lemma 4.2 was proved in Henriquez and Lizama [33] (see also [34]).

The first existence result is based upon the Schauder fixed point theorem.

**Theorem 4.4** *Suppose Eq. (1.1) satisfies the following conditions:*

- (A1) *A is the infinitesimal generator of an exponentially stable  $C_0$ -semigroup  $\{T(t) : t \geq 0\}$ ; that is, there exist numbers  $\delta, M > 0$  such that  $\|T(t)\| \leq Me^{-\delta t}$ ,  $t \geq 0$ . Moreover,  $T(t)$  is compact for  $t > 0$ .*
- (A2)  *$f \in PAP_T(R \times \Omega, X)$ , and  $f(t, \cdot)$  is uniformly continuous in each bounded subset of  $\Omega$  uniformly in  $t \in R$ ;  $I_i$  is a pseudo almost periodic sequence,  $I_i(x)$  is uniformly continuous in  $x \in \Omega$  uniformly in  $i \in Z$ .*
- (A3) *For any  $L > 0$ ,  $C_{1L} = \sup_{t \in R, \|x\| \leq L} \|f(t, x)\| < \infty$ ,  $C_{2L} = \sup_{i \in Z, \|x\| \leq L} \|I_i(x)\| < \infty$ . Moreover, there exists a number  $L_0 > 0$  such that  $\frac{M}{\delta} C_{1L_0} + \frac{M}{1-e^{-\delta\gamma}} C_{2L_0} \leq L_0$ .*

*Then Eq. (1.1) has a piecewise pseudo almost periodic solution.*

*Proof* Let  $D = \{\phi \in PAP_T(R, X) \cap UPC(R, X) : \|\phi\| \leq L_0\}$ . Define an operator  $\Gamma$  on  $D$  by

$$\Gamma \phi(t) = \int_{-\infty}^t T(t-s)f(s, \phi(s)) ds + \sum_{t_i < t} T(t-t_i)I_i(\phi(t_i)).$$

We next show that  $\Gamma$  has a fixed point in  $D$ . We divide the proof into several steps.

**Step 1.** For every  $\phi \in D$ ,  $\Gamma \phi \in PAP_T(R, X)$ .

Fix  $\phi \in D$ , by (A2) and Theorem 3.1, we have  $f(\cdot, \phi(\cdot)) \in PAP_T(R, X)$ , then we have by Definition 2.7 that  $f(\cdot, \phi(\cdot)) = \phi_1(\cdot) + \phi_2(\cdot)$ , where  $\phi_1 \in AP_T(R, X)$ ,  $\phi_2 \in PAP_T^0(R, X)$ , so

$$\int_{-\infty}^t T(t-s)f(s, \phi(s)) ds = \int_{-\infty}^t T(t-s)\phi_1(s) ds + \int_{-\infty}^t T(t-s)\phi_2(s) ds \triangleq I_1(t) + I_2(t).$$

Meanwhile, given  $\epsilon > 0$ , there exists a relatively dense set  $\Omega_1$  such that for  $\tau \in \Omega_1, t \in R, |t - t_i| > \epsilon$ ,

$$\|\phi_1(t + \tau) - \phi_1(t)\| < \epsilon.$$

Thus, by (A1),

$$\begin{aligned} \|I_1(t + \tau) - I_1(t)\| &= \left\| \int_{-\infty}^{t+\tau} T(t + \tau - s)\phi_1(s) ds - \int_{-\infty}^t T(t - s)\phi_1(s) ds \right\| \\ &= \left\| \int_{-\infty}^t T(t - s)[\phi_1(s + \tau) - \phi_1(s)] ds \right\| \\ &\leq \int_{-\infty}^t Me^{-\delta(t-s)} \|\phi_1(s + \tau) - \phi_1(s)\| ds \\ &\leq \frac{M}{\delta}\epsilon, \end{aligned}$$

this implies that  $I_1 \in AP_T(R, X)$ . Since  $PAP_T^0(R, X)$  is translation invariant, then for  $\phi_2 \in PAP_T^0(R, X)$ , one can find  $r_0 > 0$  such that

$$\frac{1}{2r} \int_{-r}^r \|\phi_2(t - s)\| dt < \epsilon, \quad r > r_0, s \in R. \tag{4.6}$$

Then by (A1), one obtains that

$$\begin{aligned} \frac{1}{2r} \int_{-r}^r \|I_2(t)\| dt &= \frac{1}{2r} \int_{-r}^r \left\| \int_{-\infty}^t T(t - s)\phi_2(s) ds \right\| dt \\ &\leq \frac{1}{2r} \int_{-r}^r dt \int_{-\infty}^t Me^{-\delta(t-s)} \|\phi_2(s)\| ds \\ &= \frac{1}{2r} \int_{-r}^r dt \int_0^\infty Me^{-\delta s} \|\phi_2(t - s)\| ds \\ &= \int_0^\infty Me^{-\delta s} ds \frac{1}{2r} \int_{-r}^r \|\phi_2(t - s)\| dt. \end{aligned}$$

By using the Lebesgue dominated convergence theorem and (4.6), we have  $I_2 \in PAP_T^0(R, X)$ . Thus,  $\int_{-\infty}^\cdot T(\cdot - s)f(s, \phi(s)) ds \in PAP_T(R, X)$ .

Moreover, by (A3) and Theorem 3.4,  $I_i(\phi(t_i))$  is a pseudo almost periodic sequence, then  $I_i(\phi(t_i)) = \beta_i + \gamma_i$  with  $\beta_i$  is an almost periodic sequence and  $\gamma_i \in PAP_0(Z, X)$ , so

$$\sum_{t_i < t} T(t - t_i)I_i(\phi(t_i)) = \sum_{t_i < t} T(t - t_i)\beta_i + \sum_{t_i < t} T(t - t_i)\gamma_i \triangleq II_1(t) + II_2(t).$$

Since  $\{t_i^j\}, i \in Z, j = 0, \pm 1, \pm 2, \dots$ , are equipotentially almost periodic, then by Lemma 35 in [20], for any  $\epsilon > 0$ , there exist relatively dense sets of real numbers  $\Omega_2$  and integers  $\Omega_3$  such that the following relations hold:

- (1) For every  $\tau \in \Omega_2$ , there exists at least one number  $q \in \Omega_3$  such that

$$|t_i^q - \tau| < \epsilon, \quad i \in Z.$$

$$(2) \|\beta_{i+q} - \beta_i\| < \epsilon, q \in \Omega_3, i \in Z.$$

So,

$$\begin{aligned} \|II_1(t + \tau) - II_1(t)\| &= \left\| \sum_{t_i < t + \tau} T(t + \tau - t_i)\beta_i - \sum_{t_i < t} T(t - t_i)\beta_i \right\| \\ &\leq \sum_{t_i < t} \|T(t - t_i)\| \|\beta_{i+q} - \beta_i\| \\ &\leq \sum_{t_i < t} M e^{-\delta(t-t_i)} \epsilon < \frac{M}{1 - e^{-\delta\gamma}} \epsilon. \end{aligned}$$

This shows  $II_1 \in AP_T(R, X)$ . It remains to show  $II_2 \in PAP_T^0(R, X)$ . For any  $r > 0$ , there exist  $i(r), j(r)$  such that

$$t_{i(r)-1} < -r \leq t_{i(r)} < \dots < t_{j(r)} \leq r < t_{j(r)+1}.$$

Since  $\gamma_i \in PAP_0(Z, X)$ ,  $M_{\gamma_i} = \sup_{i \in Z} \|\gamma_i\| < \infty$ , so

$$\begin{aligned} \frac{1}{2r} \int_{-r}^r \|II_2(t)\| dt &= \frac{1}{2r} \int_{-r}^r \left\| \sum_{t_i < t} T(t - t_i)\gamma_i \right\| dt \\ &\leq \frac{1}{2r} \int_{-r}^r \sum_{t_i < t} M e^{-\delta(t-t_i)} \|\gamma_i\| dt \\ &\leq \frac{1}{2r} \sum_{t_i < -r} M e^{-\delta(-r-t_i)} \|\gamma_i\| \int_{-r}^r e^{-\delta(t+r)} dt \\ &\quad + \frac{1}{2r} \sum_{-r < t_i < r} \|\gamma_i\| \int_{-r}^r M e^{-\delta(t-t_i)} dt \\ &\leq \frac{1}{2r} \sum_{t_i < -r} \frac{M}{\delta} M_{\gamma_i} e^{-\delta(-r-t_i)} + \frac{1}{2r} \sum_{-r < t_i < r} \frac{M}{\delta} \|\gamma_i\| \\ &\leq \frac{1}{2r} \frac{M M_{\gamma_i}}{\delta} \frac{1}{1 - e^{-\delta\gamma}} + \frac{M}{\delta} \frac{1}{t_{j(r)} - t_{i(r)}} \sum_{k=i(r)}^{j(r)} \|\gamma_k\| \\ &\leq \frac{1}{2r} \frac{M M_{\gamma_i}}{\delta} \frac{1}{1 - e^{-\delta\gamma}} + \frac{M}{\delta\gamma} \frac{1}{j(r) - i(r)} \sum_{k=i(r)}^{j(r)} \|\gamma_k\|. \end{aligned}$$

Since  $\gamma_i \in PAP_0(Z, X)$ , for  $r \rightarrow \infty$ ,  $j(r) - i(r) \rightarrow \infty$ ,

$$\frac{1}{j(r) - i(r)} \sum_{k=i(r)}^{j(r)} \|\gamma_k\| \rightarrow 0, \quad \text{as } r \rightarrow \infty.$$

Clearly,  $\lim_{r \rightarrow \infty} \frac{1}{2r} \frac{M M_{\gamma_i}}{\delta(1 - e^{-\delta\gamma})} = 0$ , so

$$\frac{1}{2r} \int_{-r}^r \|II_2(t)\| dt \rightarrow 0, \quad \text{as } r \rightarrow \infty.$$

Thus,  $\sum_{t_i < \cdot} T(\cdot - t_i)I_i(\phi(t_i)) \in PAP_T(R, X)$ , then for every  $\phi \in D$ ,  $\Gamma\phi \in PAP_T(R, X)$ .

Step 2. For every  $\phi \in D$ ,  $\|\Gamma\phi\| \leq L_0$ .

For every  $\phi \in D$ , by (A1) and (A3), we have

$$\begin{aligned} \|\Gamma\phi(t)\| &= \left\| \int_{-\infty}^t T(t-s)f(s, \phi(s)) ds + \sum_{t_i < t} T(t-t_i)I_i(\phi(t_i)) \right\| \\ &\leq \int_{-\infty}^t Me^{-\delta(t-s)} \|f(s, \phi(s))\| ds + \sum_{t_i < t} Me^{-\delta(t-t_i)} \|I_i(\phi(t_i))\| \\ &\leq C_{1L_0} \int_{-\infty}^t Me^{-\delta(t-s)} ds + C_{2L_0} \sum_{t_i < t} Me^{-\delta(t-t_i)} \\ &\leq \frac{M}{\delta} C_{1L_0} + \frac{M}{1-e^{-\delta\gamma}} C_{2L_0} \leq L_0, \end{aligned}$$

then  $\|\Gamma\phi\| \leq L_0$ .

Step 3. For every  $\phi \in D$ ,  $\Gamma\phi \in UPC(R, X)$ .

Suppose  $\phi \in D$ ,  $t', t'' \in (t_i, t_{i+1})$ ,  $i \in \mathbb{Z}$ ,  $\epsilon > 0$ , since  $\{T(t) : t \geq 0\}$  is a  $C_0$ -semigroup (see [35, 36]) and  $\|T(t)\| \leq Me^{-\delta t}$ ,  $t \geq 0$ , there exists  $0 < \mu < \frac{\epsilon}{3MC_{1L_0}}$  such that  $0 < t' - t'' < \mu$  implies

$$\|T(t' - t'') - I\| < \min \left\{ \frac{\delta\epsilon}{3MC_{1L_0}}, \frac{(1-e^{-\delta\gamma})\epsilon}{3MC_{2L_0}} \right\}.$$

Then

$$\begin{aligned} &\|\Gamma\phi(t') - \Gamma\phi(t'')\| \\ &= \left\| \int_{-\infty}^{t'} T(t'-s)f(s, \phi(s)) ds - \int_{-\infty}^{t''} T(t''-s)f(s, \phi(s)) ds \right. \\ &\quad \left. + \sum_{t_i < t'} T(t'-t_i)I_i(\phi(t_i)) - \sum_{t_i < t''} T(t''-t_i)I_i(\phi(t_i)) \right\| \\ &\leq \left\| \int_{-\infty}^{t''} [T(t'-s) - T(t''-s)]f(s, \phi(s)) ds \right\| + \left\| \int_{t''}^{t'} T(t'-s)f(s, \phi(s)) ds \right\| \\ &\quad + \left\| \sum_{t_i < t''} [T(t'-t_i) - T(t''-t_i)]I_i(\phi(t_i)) \right\| \\ &\leq \int_{-\infty}^{t''} \|T(t''-s)\| \|T(t'-t'') - I\| \|f(s, \phi(s))\| ds + \int_{t''}^{t'} \|T(t'-s)\| \|f(s, \phi(s))\| ds \\ &\quad + \sum_{t_i < t''} \|T(t''-t_i)\| \|T(t'-t'') - I\| \|I_i(\phi(t_i))\| \\ &\leq C_{1L_0} \frac{\delta\epsilon}{3MC_{1L_0}} \int_{-\infty}^{t''} Me^{-\delta(t''-s)} ds + C_{1L_0} M \frac{\epsilon}{3MC_{1L_0}} + C_{2L_0} \frac{(1-e^{-\delta\gamma})\epsilon}{3MC_{2L_0}} \sum_{t_i < t''} Me^{-\delta(t''-t_i)} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

which shows  $\Gamma\phi \in UPC(R, X)$ .

Combing Step 1, Step 2, and Step 3, it follows that  $\Gamma D \subseteq D$ .

Step 4.  $\Gamma$  is continuous.

Let  $\{\phi_n\} \subset D \subset PC(R, X)$ ,  $\phi_n \rightarrow \phi$  in  $PC(R, X)$  as  $n \rightarrow \infty$ . Then we can find a bounded subset  $K \subseteq \Omega$  such that  $R(\phi) \subseteq K$ ,  $R(\phi_n) \subseteq K$ ,  $n \in N$ . By (A2), given  $\epsilon > 0$ , there exists  $\delta' > 0$  such that  $x, y \in K$  and  $\|x - y\| < \delta'$  implies that

$$\|f(t, x) - f(t, y)\| < \epsilon, \quad \text{for all } t \in R,$$

and

$$\|I_i(x) - I_i(y)\| < \epsilon, \quad \text{for all } i \in Z.$$

For the above  $\delta'$ , there exists  $n_0$  such that  $\|\phi_n(t) - \phi(t)\| < \delta'$  for  $n > n_0$  and  $t \in R$ , then for  $n > n_0$ , we have

$$\|f(t, \phi_n(t)) - f(t, \phi(t))\| < \epsilon, \quad \text{for all } t \in R,$$

and

$$\|I_i(\phi_n(t_i)) - I_i(\phi(t_i))\| < \epsilon, \quad \text{for all } i \in Z.$$

Hence,

$$\begin{aligned} & \|\Gamma\phi_n(t) - \Gamma\phi(t)\| \\ &= \left\| \int_{-\infty}^t T(t-s)[f(s, \phi_n(s)) - f(s, \phi(s))] ds + \sum_{t_i < t} T(t-t_i)[I_i(\phi_n(t_i)) - I_i(\phi(t_i))] \right\| \\ &\leq \int_{-\infty}^t Me^{-\delta(t-s)} \|f(s, \phi_n(s)) - f(s, \phi(s))\| ds + \sum_{t_i < t} Me^{-\delta(t-t_i)} \|I_i(\phi_n(t_i)) - I_i(\phi(t_i))\| \\ &\leq \int_{-\infty}^t Me^{-\delta(t-s)} \epsilon ds + \sum_{t_i < t} Me^{-\delta(t-t_i)} \epsilon \\ &\leq \left[ \frac{M}{\delta} + \frac{M}{1 - e^{-\delta\gamma}} \right] \epsilon, \end{aligned}$$

from which it follows that  $\Gamma$  is continuous.

Step 5.  $D(t) = \{\Gamma\phi(t) : \phi \in D\}$  is a relatively compact subset of  $X$  for each  $t \in R$ .

For each  $t \in R$ ,  $0 < \epsilon < 1$ ,  $\phi \in D$ , define

$$\begin{aligned} \Gamma^\epsilon \phi(t) &= \int_{-\infty}^{t-\epsilon} T(t-s)f(s, \phi(s)) ds + \sum_{t_i < t-\epsilon} T(t-t_i)I_i(\phi(t_i)) \\ &= T(\epsilon) \left[ \int_{-\infty}^{t-\epsilon} T(t-\epsilon-s)f(s, \phi(s)) ds + \sum_{t_i < t-\epsilon} T(t-\epsilon-t_i)I_i(\phi(t_i)) \right] \\ &= T(\epsilon)\Gamma\phi(t-\epsilon). \end{aligned}$$

Since  $\{\Gamma\phi(t-\epsilon) : \phi \in D\}$  is bounded and  $T(\epsilon)$  is compact,  $\{\Gamma^\epsilon \phi(t) : \phi \in D\}$  is a relatively compact subset of  $X$ . Moreover, for  $\epsilon$  is small enough, the point  $t$  and  $t - \epsilon$  belong to the

same interval of continuity of  $\phi$ , then

$$\Gamma\phi(t) - \Gamma^\epsilon\phi(t) = \int_{t-\epsilon}^t T(t-s)f(s, \phi(s)) ds.$$

So,

$$\|\Gamma\phi(t) - \Gamma^\epsilon\phi(t)\| \leq \int_{t-\epsilon}^t Me^{-\delta(t-s)} \|f(s, \phi(s))\| ds \leq MC_{1L_0}\epsilon.$$

Thus,  $D(t) = \{\Gamma\phi(t) : \phi \in D\}$  is a relatively compact subset of  $X$  for each  $t \in R$ .

By Step 3, we know that  $\{\Gamma\phi : \phi \in D\}$  is equipotentially continuous at each interval  $(t_i, t_{i+1})$  ( $i \in Z$ ). Since  $\{\Gamma\phi : \phi \in D\} \subset (PC)_h^0(R, X)$  and  $\{\Gamma\phi : \phi \in D\}$  satisfies the conditions of Lemma 4.2,  $\{\Gamma\phi : \phi \in D\}$  is a relatively compact set, then  $\Gamma$  is a compact operator. Since  $D$  is a closed convex set, it follows from the Schauder fixed point theorem that  $\Gamma$  has a fixed point  $\phi$  in  $D$ . So, the fixed point  $\phi$  satisfies the integral equation

$$\phi(t) = \int_{-\infty}^t T(t-s)f(s, \phi(s)) ds + \sum_{t_i < t} T(t-t_i)I_i(\phi(t_i))$$

for all  $t \in R$ . Fix  $a, a \neq t_i, i \in Z$ , we have

$$\phi(a) = \int_{-\infty}^a T(a-s)f(s, \phi(s)) ds + \sum_{t_i < a} T(a-t_i)I_i(\phi(t_i)).$$

Since  $\{T(t) : t \geq 0\}$  is a  $C_0$ -semigroup, so

$$\begin{aligned} \phi(t) &= \int_{-\infty}^t T(t-s)f(s, \phi(s)) ds + \sum_{t_i < t} T(t-t_i)I_i(\phi(t_i)) \\ &= \int_{-\infty}^a T(t-s)f(s, \phi(s)) ds + \sum_{t_i < a} T(t-t_i)I_i(\phi(t_i)) \\ &\quad + \int_a^t T(t-s)f(s, \phi(s)) ds + \sum_{a < t_i < t} T(t-t_i)I_i(\phi(t_i)) \\ &= T(t-a)\phi(a) + \int_a^t T(t-s)f(s, \phi(s)) ds + \sum_{a < t_i < t} T(t-t_i)I_i(\phi(t_i)). \end{aligned}$$

Hence,  $\phi$  is a piecewise pseudo almost periodic mild solution to Eq. (1.1). □

The following existence result is based on the contraction principle.

**Theorem 4.5** *Assume the following conditions hold:*

(A1') *A is the infinitesimal generator of an exponentially stable  $C_0$ -semigroup  $\{T(t) : t \geq 0\}$ ; that is, there exist numbers  $\delta, M > 0$  such that  $\|T(t)\| \leq Me^{-\delta t}, t \geq 0$ .*



(A2')  $f \in PAP_T(R \times \Omega, X)$ , and  $f$  satisfies the Lipschitz condition in the following sense:

$$\|f(t, x) - f(t, y)\| \leq L_1 \|x - y\|, \quad t \in R, x, y \in \Omega,$$

where  $L_1 > 0$ .

(A3')  $I_i$  is a pseudo almost periodic sequence, and there exists a number  $L_2 > 0$  such that

$$\|I_i(x) - I_i(y)\| \leq L_2 \|x - y\|,$$

for all  $x, y \in \Omega, i \in Z$ .

If  $\frac{M}{\delta}L_1 + \frac{M}{1-e^{-\delta\gamma}}L_2 < 1$ , then Eq. (1.1) has a piecewise pseudo almost periodic solution.

*Proof* Define the operator  $\Gamma$  on  $PAP_T(R, X) \cap UPC(R, X)$  as in the proof of Theorem 4.4.

Fix  $\phi \in PAP_T(R, X) \cap UPC(R, X)$ , by (A2') and (A3'), we have

$$\|f(t, \phi(t))\| \leq \|f(t, \phi(t)) - f(t, 0)\| + \|f(t, 0)\| \leq L_1 \|\phi(t)\| + \|f(t, 0)\|,$$

and

$$\|I_i(\phi(t_i))\| \leq \|I_i(\phi(t_i)) - I_i(0)\| + \|I_i(0)\| \leq L_2 \|\phi(t_i)\| + \|I_i(0)\|.$$

Then  $f(\cdot, \phi(\cdot)) \in PC(R, X)$  and  $I_i(\phi(t_i))$  is a bounded sequence, it follows from the proof of Theorem 4.4 that  $\Gamma\phi \in PAP_T(R, X) \cap UPC(R, X)$ . So,  $\Gamma(PAP_T(R, X) \cap UPC(R, X)) \subset PAP_T(R, X) \cap UPC(R, X)$ . It suffices now to show that the operator  $\Gamma$  has a fixed point in  $PAP_T(R, X) \cap UPC(R, X)$ . For  $\phi, \varphi \in PAP_T(R, X) \cap UPC(R, X)$ ,

$$\begin{aligned} & \|\Gamma\phi(t) - \Gamma\varphi(t)\| \\ &= \left\| \int_{-\infty}^t T(t-s)[f(s, \phi(s)) - f(s, \varphi(s))] ds + \sum_{t_i < t} T(t-t_i)[I_i(\phi(t_i)) - I_i(\varphi(t_i))] \right\| \\ &\leq \int_{-\infty}^t Me^{-\delta(t-s)} \|f(s, \phi(s)) - f(s, \varphi(s))\| ds + \sum_{t_i < t} Me^{-\delta(t-t_i)} \|I_i(\phi(t_i)) - I_i(\varphi(t_i))\| \\ &\leq \int_{-\infty}^t Me^{-\delta(t-s)} L_1 \|\phi(s) - \varphi(s)\| ds + \sum_{t_i < t} Me^{-\delta(t-t_i)} L_2 \|\phi(t_i) - \varphi(t_i)\| \\ &\leq \left[ \int_{-\infty}^t L_1 Me^{-\delta(t-s)} ds + \sum_{t_i < t} L_2 Me^{-\delta(t-t_i)} \right] \|\phi - \varphi\| \\ &\leq \left[ \frac{M}{\delta} L_1 + \frac{M}{1-e^{-\delta\gamma}} L_2 \right] \|\phi - \varphi\|. \end{aligned}$$

Since  $\frac{M}{\delta}L_1 + \frac{M}{1-e^{-\delta\gamma}}L_2 < 1$ ,  $\Gamma$  is a contraction,  $\Gamma$  has a fixed point in  $PAP_T(R, X) \cap UPC(R, X)$ , then Eq. (1.1) has a piecewise pseudo almost periodic solution.  $\square$

Finally, we investigate the stability of a piecewise pseudo almost periodic solution to Eq. (1.1). By using the generalized Gronwall-Bellman inequality (see Lemma 1 in [20]) and Lipschitz conditions, it can be formulated as follows.

**Theorem 4.6** *Suppose the conditions of Theorem 4.5 hold. Assume further that  $\frac{\ln(1+ML_2)}{\gamma} + ML_1 - \delta < 0$ , then Eq. (1.1) has an exponentially stable piecewise pseudo almost periodic solution.*

*Proof* By Theorem 4.5, we know that Eq. (1.1) has a mild piecewise pseudo almost periodic solution  $u(t)$ , by using the integral form of Eq. (1.1), if  $t > \sigma$ ,  $\sigma \neq t_i$ ,  $i \in \mathbb{Z}$ ,

$$u(t) = T(t - \sigma)u(\sigma) + \int_{\sigma}^t T(t - s)f(s, u(s)) ds + \sum_{\sigma < t_i < t} T(t - t_i)I_i(u(t_i)).$$

Let  $u(t) = u(t, \sigma, \varphi)$  and  $v(t) = v(t, \sigma, \psi)$  be two solutions of Eq. (1.1), then

$$\begin{aligned} u(t) &= T(t - \sigma)\varphi + \int_{\sigma}^t T(t - s)f(s, u(s)) ds + \sum_{\sigma < t_i < t} T(t - t_i)I_i(u(t_i)), \\ v(t) &= T(t - \sigma)\psi + \int_{\sigma}^t T(t - s)f(s, v(s)) ds + \sum_{\sigma < t_i < t} T(t - t_i)I_i(v(t_i)). \end{aligned}$$

So,

$$\begin{aligned} \|u(t) - v(t)\| &\leq \|T(t - \sigma)[\varphi - \psi]\| + \left\| \int_{\sigma}^t T(t - s)[f(s, u(s)) - f(s, v(s))] ds \right\| \\ &\quad + \left\| \sum_{\sigma < t_i < t} T(t - t_i)[I_i(u(t_i)) - I_i(v(t_i))] \right\| \\ &\leq \|T(t - \sigma)\| \|\varphi - \psi\| + \int_{\sigma}^t \|T(t - s)\| \|f(s, u(s)) - f(s, v(s))\| ds \\ &\quad + \sum_{\sigma < t_i < t} \|T(t - t_i)\| \|I_i(u(t_i)) - I_i(v(t_i))\| \\ &\leq Me^{-\delta(t-\sigma)} \|\varphi - \psi\| + \int_{\sigma}^t ML_1 e^{-\delta(t-s)} \|u(s) - v(s)\| ds \\ &\quad + \sum_{\sigma < t_i < t} ML_2 e^{-\delta(t-t_i)} \|u(t_i) - v(t_i)\|. \end{aligned}$$

Then

$$\begin{aligned} e^{\delta t} \|u(t) - v(t)\| &\leq Me^{\delta\sigma} \|\varphi - \psi\| + \int_{\sigma}^t ML_1 e^{\delta s} \|u(s) - v(s)\| ds \\ &\quad + \sum_{\sigma < t_i < t} ML_2 e^{\delta t_i} \|u(t_i) - v(t_i)\|. \end{aligned} \tag{4.7}$$

Let  $y(t) = e^{\delta t} \|u(t) - v(t)\|$ , (4.7) can be rewritten in the following form:

$$y(t) \leq My(\sigma) + \int_{\sigma}^t ML_1 y(s) ds + \sum_{\sigma < t_i < t} ML_2 y(t_i).$$

By the generalized Gronwall-Bellman inequality, we have

$$y(t) \leq My(\sigma) \prod_{\sigma < t_i < t} (1 + ML_2) e^{\int_{\sigma}^t ML_1 ds} = My(\sigma) \prod_{\sigma < t_i < t} (1 + ML_2) e^{ML_1(t-\sigma)}.$$

Since  $\gamma = \inf_{i \in Z} (t_{i+1} - t_i) > 0$ , we have

$$y(t) \leq My(\sigma)(1 + ML_2)^{\frac{t-\sigma}{\gamma}} e^{ML_1(t-\sigma)} = My(\sigma)e^{\left[\frac{\ln(1+ML_2)}{\gamma} + ML_1\right](t-\sigma)}.$$

That is,

$$\|u(t) - v(t)\| \leq M\|\varphi - \psi\|e^{\left[\frac{\ln(1+ML_2)}{\gamma} + ML_1 - \delta\right](t-\sigma)}.$$

Because  $\frac{\ln(1+ML_2)}{\gamma} + ML_1 - \delta < 0$ , then Eq. (1.1) has an exponentially stable piecewise pseudo almost periodic solution. This completes the proof.  $\square$

**Example 4.7** Consider the following problem:

$$\begin{cases} \frac{\partial}{\partial t} u(t, \xi) = \frac{\partial^2}{\partial \xi^2} u(t, \xi) + \frac{1}{15} (\sin t + \sin \sqrt{2}t + g(t)) \sin u(t, \xi), \\ t \in R, t \neq t_i, i \in Z, \xi \in [0, \pi], \\ \Delta u(t_i, \xi) = \beta_i u(t_i, \xi), \quad i \in Z, \xi \in [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, \quad t \in R, \end{cases} \quad (4.8)$$

where  $g \in UPC(R, R)$  satisfies  $|g(t)| \leq 1$  ( $t \in R$ ) and  $\lim_{r \rightarrow \infty} \frac{1}{2r} \int_r^r |g(t)| dt = 0$ ,  $\beta_i = \frac{1}{15} (\sin i + \sin \sqrt{2}i + g(i))$  and  $t_i = i + \frac{1}{4} |\sin i - \sin \sqrt{2}i|$ .

Define  $X = L^2[0, \pi]$ , let  $Au = u''$ ,  $u \in D(A) = H_0^1[0, \pi] \cap H^2[0, \pi]$ . Clearly,  $A$  is the infinitesimal generator of an analytic compact semigroup  $\{T(t) : t \geq 0\}$  on  $X$  (refer to [21]) and  $\|T(t)\| \leq e^{-t}$  ( $t \geq 0$ ) with  $M = \delta = 1$ .  $\{t_i^j\}$ ,  $i \in Z, j = 0, \pm 1, \pm 2, \dots$ , are equipotentially almost periodic (refer to p.198 in [20]) and

$$\begin{aligned} t_i^1 &= t_{i+1} - t_i = 1 + \frac{1}{4} \left| \sin(i+1) - \sin \sqrt{2}(i+1) \right| - \frac{1}{4} \left| \sin i - \sin \sqrt{2}i \right| \\ &\geq 1 - \frac{1}{4} \left| \sin(i+1) - \sin i - [\sin \sqrt{2}(i+1) - \sin \sqrt{2}i] \right| \\ &\geq 1 - \frac{1}{2} \left| \sin \frac{1}{2} \cos \frac{2i+1}{2} \right| - \frac{1}{2} \left| \sin \frac{\sqrt{2}}{2} \cos \frac{\sqrt{2}(2i+1)}{2} \right| \\ &\geq 1 - \frac{1}{2} \sin \frac{1}{2} - \frac{1}{2} \sin \frac{\sqrt{2}}{2} > \frac{2}{5}. \end{aligned}$$

So,  $\gamma = \inf_{i \in Z} (t_{i+1} - t_i) > \frac{2}{5} > 0$ . Let  $f(t, u) = \frac{1}{15} (\sin t + \sin \sqrt{2}t + g(t)) \sin u$ ,  $I_i(u) = \beta_i u$ . Clearly, both  $f$  and  $I_i$  satisfy the assumptions given in Theorem 4.5 and Theorem 4.6 with  $L_1 = \frac{1}{5}$ ,  $L_2 = \frac{1}{5}$ , respectively. Moreover,  $\frac{M}{\delta} L_1 + \frac{M}{1-e^{-\delta\gamma}} L_2 < \frac{1}{5} + \frac{1}{5(1-e^{-\frac{2}{5}})} < 1$ ,  $\frac{\ln(1+ML_2)}{\gamma} + ML_1 - \delta < \frac{5}{2} \ln \frac{6}{5} + \frac{1}{5} - 1 < 0$ , all conditions in Theorem 4.6 are satisfied. Hence, Eq. (4.8) has an exponentially stable piecewise pseudo almost periodic solution.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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