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A modified kernel method for a time-fractional inverse diffusion problem

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Abstract

In this paper, we consider a time-fractional inverse diffusion problem, where the data is given at $x = 1$ and the solution is sought in the interval $0 \leq x < 1$. Such a problem is obtained from the classical diffusion equation by replacing the first-order time derivative by the Caputo fractional derivative of order $\alpha \in (0, 1)$. We show that a time-fractional inverse diffusion problem is severely ill-posed and we further apply a modified kernel method to solve it based on the solution in the frequency domain. The corresponding convergence estimates are provided. Finally, an example is constructed to show the feasibility and efficiency of the proposed method.

MSC: 35R25; 35R30; 47A52

Keywords: time-fractional inverse diffusion problem; ill-posed problem; regularization method; error estimate

1 Introduction

In the past decades, studies on the problems of the partial differential equation mainly focused on direct problems and inverse problems of integer order differential equation, and some numerical techniques have been proposed to solve integer order differential equation [1–8]. However, fractional derivatives calculus and fractional differential equations have been used recently to solve a range of problems in mechanical engineering [9], viscoelasticity [10], electron transport [11], dissipation [12], heat conduction [13, 14], and high-frequency financial data [15].

The time-fractional diffusion equation arises by replacing the standard time partial derivative in the diffusion equation with a time-fractional partial derivative. It is usually used to describe the anomalous diffusion (superdiffusion, non-Gaussian diffusion, subdiffusion) which is not consistent with the classical Fick (or Fourier) law [16]. The direct problem, *i.e.*, initial value problem and initial boundary for time-fractional diffusion equation have been studied extensively in the past few years [17–22]. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. We only know the noisy data on a part of the boundary or at some interior points of the concerned domain. This leads to an inverse and ill-posed problem of the fractional diffusion equation, which means the solution does not depend continuously on the given known conditions. In this paper, we investigate an inverse problem of the time-fractional diffusion equation. This kind of ill-posed problem is important in many branches of engineering sciences [23, 24].

Due to the difficulty of the fractional derivative and the ill-posedness, to the authors' knowledge, the results on inverse problem of the time-fractional diffusion equation are very few. The uniqueness of an inverse problem for a one-dimensional fractional diffusion equation was given in [25]. Cheng and Fu [26] gave an iteration regularization method for a time-fractional inverse diffusion problem. Zheng and Wei [27, 28] investigated a time-fractional inverse diffusion problem by using a spectral regularization method and a modified equation method.

In this article, we consider the following time-fractional inverse diffusion problem (TFIDP):

$$\begin{cases} -u_x(x,t) = {}_0D_t^\alpha u(x,t), & x > 0, t > 0, 0 < \alpha < 1, \\ u(x,0) = 0, & x \geq 0, \\ u(1,t) = f(t), & t \geq 0, \\ \lim_{x\to\infty} u(x,t) = 0, & t \geq 0, \end{cases} \tag{1}$$

where the time-fractional derivative ${}_0D_t^\alpha u(x,t)$ is the Caputo fractional derivative of order $\alpha \in (0,1)$ defined by [29]

$${}_0D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1, \tag{2}$$

$${}_0D_t^\alpha u(x,t) = \frac{\partial u(x,t)}{\partial t}, \quad \alpha = 1, \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function.

The TFIDP is an inverse problem and is severely ill-posed. That means the solution does not depend continuously on the given data, and any small perturbation in the given data may cause large changes to the solution. In this paper, we will present a modified kernel method to construct a stable approximation solution of the TFIDP.

The rest of this paper is organized as follows. In Section 2, we demonstrate ill-posedness of the time-fractional inverse diffusion problem. In Section 3, a modified kernel method is used to solve problem (1), and we also obtain the convergence estimates between the regularization solution and the exact solution based on the *a priori* assumptions for the exact solution. In Section 4, an example is illustrated to show the main results. Finally, we conclude this paper in Section 5.

2 Ill-posedness of a time-fractional inverse diffusion problem

In order to apply the Fourier analysis of the time-fractional inverse diffusion problem, we extend all the functions to the whole line $-\infty < t < \infty$ by defining them to be zero for $t < 0$ whenever it is necessary. We also assume all the functions involving t variable are in $L^2(\mathbb{R})$. Here, and in the following sections, we use the corresponding L^2 norm, as defined below:

$$\|g\| = \left(\int_{-\infty}^{\infty} \left|g(t)\right|^2 dt \right)^{\frac{1}{2}}.$$

Let

$$\hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) e^{-i\xi t} dt$$

be the Fourier transform of the function $g(t) \in L^2(\mathbb{R})$ and $\|\cdot\|_p$ denote the H_p norm, *i.e.*,

$$\|g\|_p = \left(\int_{-\infty}^{\infty} (1+\xi^2)^p \left|\hat{g}(\xi)\right|^2 d\xi\right)^{\frac{1}{2}}.$$

Then, applying the Fourier transforming with respect to t to problem (1), we have [29]

$$\hat{u}_x(x,\xi) = -(i\xi)^\alpha \hat{u}(x,\xi), \tag{4}$$

$$\hat{u}(1,\xi) = \hat{g}(\xi). \tag{5}$$

The solution of the above problem can be given by

$$\hat{u}(x,\xi) = e^{l(\xi)(1-x)}\hat{g}(\xi), \tag{6}$$

where

$$l(\xi) = (i\xi)^\alpha = \begin{cases} |\xi|^\alpha(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}), & \xi \geq 0, \\ |\xi|^\alpha(\cos\frac{\alpha\pi}{2} - i\sin\frac{\alpha\pi}{2}), & \xi < 0. \end{cases}$$

For the above problem, note that $l(\xi)$ has a positive real part $|\xi|^\alpha \cos\frac{\alpha\pi}{2}$, the small error in the high-frequency components will be amplified by the factor $e^{|\xi|^\alpha \cos\frac{\alpha\pi}{2}(1-x)}$. Therefore the TFIDP for recovering the temperature $u(x,t)$ from the measured data $g^\delta(t)$ is severely ill-posed. Here, suppose that the measured data $g^\delta(t) \in L^2(\mathbb{R})$ satisfy

$$\left\|g^\delta(t) - g(t)\right\| \leq \delta, \tag{7}$$

where the constant $\delta > 0$ is the noise level.

To solve the problem (1), a natural way to stabilize the problem is to eliminate the high frequencies or to replace the 'kernel' $e^{l(\xi)(1-x)}$ by a bounded approximation.

We now list two kernels of regularization methods for solving the time-fractional inverse diffusion problem.

The first is

$$e^{(i\xi)^\alpha(1-x)}\chi_{\max},$$

where $\chi_{\max}$ denotes the characteristic function of interval $[-\xi_{\max}, \xi_{\max}]$, and $\xi_{\max}$ is a regularization parameter. It corresponds to a spectral regularization method; see [27].

The second is

$$e^{\frac{(i\xi)^\alpha}{1+\mu\xi^2}(1-x)},$$

where μ is a regularization parameter. It corresponds to a modified equation method; see [28].

In this article, we propose a regularization method by modifying the 'kernel' to deal with the difficulty of problem (1) as follows:

$$\frac{e^{l(\xi)(1-x)}}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}},$$

where β is a regularization parameter.

Now, we will apply this regularization strategy to solve a time-fractional inverse diffusion problem and prove that this regularization strategy is feasible.

3 A modified kernel method and error estimates

In this section, we will construct regularization solution by modifying the 'kernel' and obtain convergence estimates.

Here, we give an approximate solution of problem (1) by perturbing the kernel of (6) as follows:

$$\hat{u}^{\beta,\delta}(x,\xi) = \frac{e^{l(\xi)(1-x)}}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}}\hat{g}^\delta(\xi). \tag{8}$$

In order to obtain our main results, we first give two important lemmas. Here, we set $\eta = |\xi|^\alpha \cos\frac{\alpha\pi}{2}$.

Lemma 1 *If* $0 < x < 1$, $0 < \alpha < 1$, *then we have*

$$\sup_{\eta\geq 0}\frac{e^{\eta(1-x)}}{1+\beta^2 e^{2\eta}} \leq \beta^{x-1}. \tag{9}$$

Proof Let

$$F(\eta) = \frac{e^{\eta(1-x)}}{1+\beta^2 e^{2\eta}},$$

by elementary calculation, we see that $\eta = \frac{1}{2}\ln(\frac{1-x}{1+x}\beta^{-2})$ is the maximum point of $F(\eta)$. So, we know

$$\left(1+\beta^2 e^{2\eta}\right)^{-1} = \left(1+\frac{1-x}{1+x}\right)^{-1} = \left(\frac{2}{1+x}\right)^{-1} = \frac{1+x}{2} \leq 1 \tag{10}$$

and

$$e^{\eta(1-x)} = \left(\frac{1-x}{1+x}\beta^{-2}\right)^{\frac{1-x}{2}} = \left(\frac{1-x}{1+x}\right)^{\frac{1-x}{2}}\beta^{x-1} \leq \beta^{x-1}. \tag{11}$$

Combining (10) and (11), we have

$$\sup_{\eta\geq 0}\frac{e^{\eta(1-x)}}{1+\beta^2 e^{2\eta}} \leq \beta^{x-1}.$$

□

Lemma 2 *If* $0 < x < 1$, $0 < \alpha < 1$, *then we have*

$$\sup_{\eta\geq 0}\frac{e^{\eta(2-x)}}{1+\beta^2 e^{2\eta}} \leq \beta^{x-2}. \tag{12}$$

Proof The proof of this lemma is similar to that of Lemma 1, we get the inequality (12).

□

In the following theorems, the convergence estimates between the regularization solution and the exact solution will be given based on an *a priori* choice of the regularization parameter.

Theorem 1 *Suppose that $u^{\beta,\delta}(x,t)$ is the regularization solution for problem* (1) *with noisy data $g^\delta(t)$ and that $u(x,t)$ is the exact solution for problem* (1) *with the exact data $g(t)$. Let the assumption $\|g^\delta(t)-g(t)\| \le \delta$ be satisfied and let $\|u(0,\cdot)\| \le E$ hold. If we choose*

$$\beta = \frac{\delta}{E}, \tag{13}$$

then for every $x \in (0,1)$, we obtain the following error estimate:

$$\left\| u^{\beta,\delta}(x,\cdot) - u(x,\cdot) \right\| \le 2\delta^x E^{1-x}. \tag{14}$$

Proof By Parseval's identity and the triangle inequality, we have

$$\begin{aligned}
\left\| u^{\beta,\delta}(x,\cdot) - u(x,\cdot) \right\| &= \left\| \hat{u}^{\beta,\delta}(x,\cdot) - \hat{u}(x,\cdot) \right\| \\
&= \left\| \frac{e^{l(\xi)(1-x)}}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}} \hat{g}^\delta(\xi) - e^{l(\xi)(1-x)}\hat{g}(\xi) \right\| \\
&\le \left(\int_{-\infty}^{\infty} \left| \frac{e^{l(\xi)(1-x)}}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}} \hat{g}^\delta(\xi) - \frac{e^{l(\xi)(1-x)}}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}} \hat{g}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{-\infty}^{\infty} \left| \frac{e^{l(\xi)(1-x)}}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}} \hat{g}(\xi) - e^{l(\xi)(1-x)}\hat{g}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\
&= \left(\int_{-\infty}^{\infty} \left| \frac{e^{l(\xi)(1-x)}}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}} \hat{g}^\delta(\xi) - \frac{e^{l(\xi)(1-x)}}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}} \hat{g}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\
&\quad + \left(\int_{-\infty}^{\infty} \left| \left(\frac{e^{-l(\xi)x}}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}} - e^{-l(\xi)x} \right) e^{l(\xi)}\hat{g}(\xi) \right|^2 d\xi \right)^{\frac{1}{2}} \\
&\le \delta \sup_{\xi\in\mathbb{R}} \frac{|e^{l(\xi)(1-x)}|}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}} \\
&\quad + \sup_{\xi\in\mathbb{R}} \left| \frac{\beta^2 e^{(2-x)|\xi|^\alpha \cos\frac{\alpha\pi}{2} - ix\,\mathrm{sign}(\xi)|\xi|^\alpha \sin\frac{\alpha\pi}{2}}}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}} \right| \left\| \hat{u}(0,\cdot) \right\| \\
&\le \delta \sup_{\xi\in\mathbb{R}} K_1(\xi) + E \sup_{\xi\in\mathbb{R}} K_2(\xi),
\end{aligned}$$

where

$$\begin{aligned}
K_1(\xi) &= \frac{e^{|\xi|^\alpha \cos\frac{\alpha\pi}{2}(1-x)}}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}}, \\
K_2(\xi) &= \frac{\beta^2 e^{(2-x)|\xi|^\alpha \cos\frac{\alpha\pi}{2}}}{1+\beta^2 e^{2|\xi|^\alpha \cos\frac{\alpha\pi}{2}}}.
\end{aligned}$$

Combining (9) and (12), and setting $\eta = |\xi|^\alpha \cos\frac{\alpha\pi}{2}$, we get

$$\begin{aligned}
K_1(\xi) &= \frac{e^{\eta(1-x)}}{1+\beta^2 e^{2\eta}} \le \beta^{x-1}, \\
K_2(\xi) &= \frac{\beta^2 e^{(2-x)\eta}}{1+\beta^2 e^{2\eta}} \le \beta^2 \beta^{x-2} = \beta^x.
\end{aligned}$$

Therefore,

$$\left\|u^{\beta,\delta}(x,\cdot)-u(x,\cdot)\right\| \leq \delta\beta^{x-1}+E\beta^{x},$$

according to (13), we can get the error estimate

$$\left\|u^{\beta,\delta}(x,\cdot)-u(x,\cdot)\right\| \leq 2\delta^{x}E^{1-x}. \qquad \square$$

The error estimate in Theorem 1 does not give any useful information on the continuous dependence of the solution at $x = 0$. To retain the continuous dependence of the solution at $x = 0$, one has to introduce a stronger *a priori* assumption.

We are now in the position to formulate the convergence rate for $x = 0$.

Theorem 2 *Suppose that $u^{\beta,\delta}(x,t)$ is the regularization solution for problem* (1) *with noisy data $g^{\delta}(t)$ and that $u(x,t)$ is the exact solution for problem* (1) *with the exact data $g(t)$. Let the assumption $\|g^{\delta}(t)-g(t)\| \leq \delta$ be satisfied and let $\|u(0,\cdot)\|_p \leq E$ ($p \geq 1$) hold. If we choose*

$$\beta = \left(\frac{\delta}{E}\right)^{\gamma}, \quad \frac{1}{2} < \gamma < 1, \tag{15}$$

then, for $x = 0$, we obtain the following error estimate:

$$\begin{aligned}
&\left\|u^{\beta,\delta}(0,\cdot)-u(0,\cdot)\right\| \\
&\quad\leq \delta^{1-\gamma}E^{\gamma} + \sqrt{\pi}\,\delta^{2\gamma-\cos\frac{\alpha\pi}{2}}E^{1-2\gamma+\cos\frac{\alpha\pi}{2}}\left(\ln\frac{E}{\delta}\right)^{-p\cos\frac{\alpha\pi}{2}} \\
&\qquad + E\left[\frac{1}{2}\ln\left(\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-p}\right)\right]^{-\frac{p}{\alpha}}.
\end{aligned}$$

Proof It is similar to Theorem 1; we obtain

$$\begin{aligned}
&\left\|u^{\beta,\delta}(x,\cdot)-u(x,\cdot)\right\| \\
&\quad\leq \left(\int_{-\infty}^{\infty}\left|\frac{e^{l(\xi)(1-x)}}{1+\beta^{2}e^{2|\xi|^{\alpha}\cos\frac{\alpha\pi}{2}}}\left(\hat{g}^{\delta}(\xi)-\hat{g}(\xi)\right)\right|^{2}d\xi\right)^{\frac{1}{2}} \\
&\qquad + \left(\int_{-\infty}^{\infty}\left|\frac{\beta^{2}e^{(2-x)|\xi|^{\alpha}\cos\frac{\alpha\pi}{2}-ix\,\mathrm{sign}(\xi)|\xi|^{\alpha}\sin\frac{\alpha\pi}{2}}}{(1+\beta^{2}e^{2|\xi|^{\alpha}\cos\frac{\alpha\pi}{2}})(1+\xi^{2})^{\frac{p}{2}}}\left(1+\xi^{2}\right)^{\frac{p}{2}}\hat{u}(0,\cdot)\right|^{2}d\xi\right)^{\frac{1}{2}} \\
&\quad= K_{3}(\xi)+K_{4}(\xi).
\end{aligned}$$

Setting $\eta = |\xi|^{\alpha}\cos\frac{\alpha\pi}{2}$, and combining (9) and (12), we get

$$K_{3}(\xi) \leq \delta\sup_{\eta\in R}\frac{e^{\eta(1-x)}}{1+\beta^{2}e^{2\eta}} \leq \delta\beta^{x-1}$$

and

$$\begin{aligned}
K_4(\xi) &= \left(\int_{-\infty}^{\infty}\left|\frac{\beta^2 e^{(2-x)\eta}}{(1+\beta^2 e^{2\eta})(1+\xi^2)^{\frac{p}{2}}}\left(1+\xi^2\right)^{\frac{p}{2}}\hat{u}(0,\cdot)\right|^2 d\xi\right)^{\frac{1}{2}} \\
&\le E\left(\int_{|\xi|\le\xi_0}\left[\frac{\beta^2 e^{(2-x)\eta}}{(1+\beta^2 e^{2\eta})(1+\xi^2)^{\frac{p}{2}}}\right]^2 d\xi\right)^{\frac{1}{2}} \\
&\quad + E\left(\int_{|\xi|>\xi_0}\left[\frac{\beta^2 e^{(2-x)\eta}}{(1+\beta^2 e^{2\eta})(1+\xi^2)^{\frac{p}{2}}}\right]^2 d\xi\right)^{\frac{1}{2}} \\
&\le E\sqrt{\pi}\beta^2\left(e^{(2-x)\eta_0}|_{\eta_0=|\xi_0|^\alpha\cos\frac{\alpha\pi}{2}}\right) + E\beta^x\xi_0^{-p}.
\end{aligned}$$

Taking $\xi_0 = [\frac{1}{2}\ln(\frac{E}{\delta}(\ln\frac{E}{\delta})^{-p})]^{\frac{1}{\alpha}}$ and using (15), we get

$$\begin{aligned}
&\left\|u^{\beta,\delta}(x,\cdot) - u(x,\cdot)\right\| \\
&\quad\le \delta\beta^{x-1} + \sqrt{\pi}E\beta^2\left(e^{(2-x)\eta_0}|_{\eta_0=|\xi_0|^\alpha\cos\frac{\alpha\pi}{2}}\right) + E\beta^x\xi_0^{-p} \\
&\quad\le \delta^{1-\gamma+\gamma x}E^{\gamma-\gamma x} + \sqrt{\pi}\delta^{2\gamma-(1-\frac{x}{2})\cos\frac{\alpha\pi}{2}}E^{1-2\gamma+(1-\frac{x}{2})\cos\frac{\alpha\pi}{2}}\left(\ln\frac{E}{\delta}\right)^{(\frac{px}{2}-p)\cos\frac{\alpha\pi}{2}} \\
&\qquad + \delta^{\gamma x}E^{1-\gamma x}\left[\frac{1}{2}\ln\left(\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-p}\right)\right]^{-\frac{p}{\alpha}}.
\end{aligned}$$

Here, for $x = 0$, we obtain

$$\begin{aligned}
&\left\|u^{\beta,\delta}(0,\cdot) - u(0,\cdot)\right\| \\
&\quad\le \delta^{1-\gamma}E^{\gamma} + \sqrt{\pi}\delta^{2\gamma-\cos\frac{\alpha\pi}{2}}E^{1-2\gamma+\cos\frac{\alpha\pi}{2}}\left(\ln\frac{E}{\delta}\right)^{-p\cos\frac{\alpha\pi}{2}} \\
&\qquad + E\left[\frac{1}{2}\ln\left(\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-p}\right)\right]^{-\frac{p}{\alpha}}.
\end{aligned}$$

□

4 Numerical example

The purpose of this section is to present a numerical example and illustrate the accuracy and efficiency of the proposed method. Here, the proposed methods will be implemented in Matlab. In our numerical experiment, we fix the interval $0 \le t \le 1$.

To illustrate the behavior of a modified kernel regularization method, we construct the test problems with a given function f at $x = 0$, then we compute a data function g at $x = 1$ according to $\hat{g}(\xi) = e^{-l(\xi)}\hat{f}(\xi)$, which is well-posed. We usually think that the computed data g is exact. The discrete noisy data g^δ is obtained by adding a random noise to the exact data g, that is,

$$g^\delta = g + \varepsilon\,\mathrm{randn}\big(\mathrm{size}(g)\big),$$

where

$$g = \big(g(x_1), g(x_2), \ldots, g(x_n)\big)^T, \quad x_i = (i-1)\Delta x, \Delta x = \frac{1}{n-1}, i = 1, 2, \ldots, n.$$

Then the total noise δ can be measured in the sense of the root mean square error according to

$$\delta := \|g^\delta - g\|_{l^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(g_i^\delta - g_i)^2}.$$

Here, the function randn(·) generates arrays of random numbers whose elements are normally distributed with mean 0, variance $\sigma^2 = 1$ and standard deviation $\sigma = 1$, the function randn(size(g)) returns an array of random entries that has the same size as g.

Example 1 Consider a smooth function

$$f(t) = \begin{cases} \frac{1}{t^{3/2}} e^{-\frac{1}{4t}}, & t > 0, \\ 0, & \text{others}. \end{cases}$$

The comparison of the computational effects with $\alpha = 0.3$, $\varepsilon = 0.001$, at $x = 0.8, 0.5, 0.2, 0$ are shown in Figure 1.

The comparison of the computational effects with $\alpha = 0.4$, $x = 0$, for $\varepsilon = 0.00001, 0.0001, 0.001, 0.01$ are shown in Figure 2.

The comparison of the computational effects with $\alpha = 0.6$, $x = 0$, $\varepsilon = 0.0001$ for $\gamma = 0.6, 0.7, 0.8, 0.9$ are shown in Figure 3.

From Figures 1-3 (see also Tables 1-3), we can find that the numerical results near the boundary $x = 1$ are better than the ones close to $x = 0$, and the smaller the parameter ε,

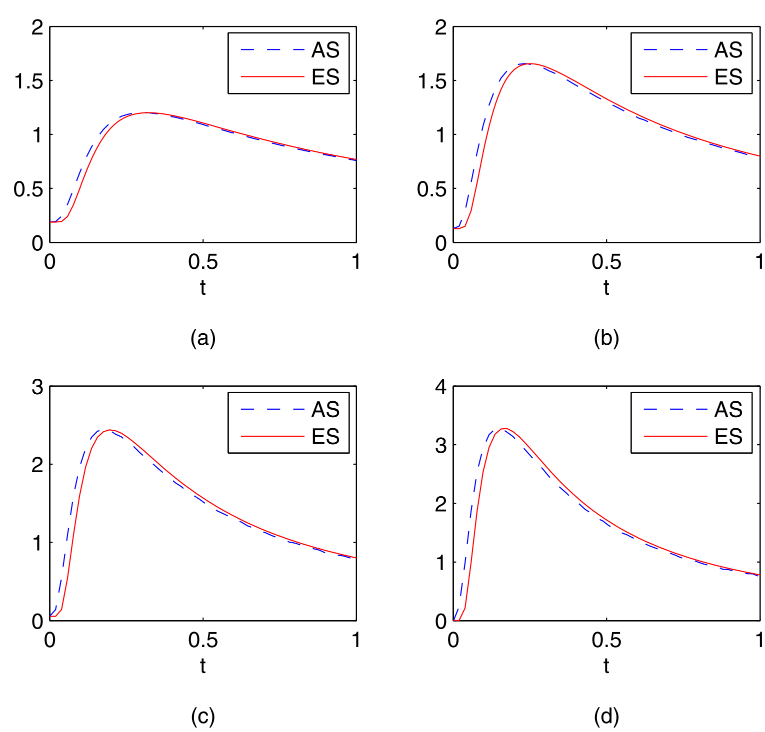

Figure 1 The exact solution (ES) and its approximation solution (AS). $\alpha = 0.3$, $\varepsilon = 0.001$ and **(a)** $x = 0.8$; **(b)** $x = 0.5$; **(c)** $x = 0.2$; **(d)** $x = 0$.

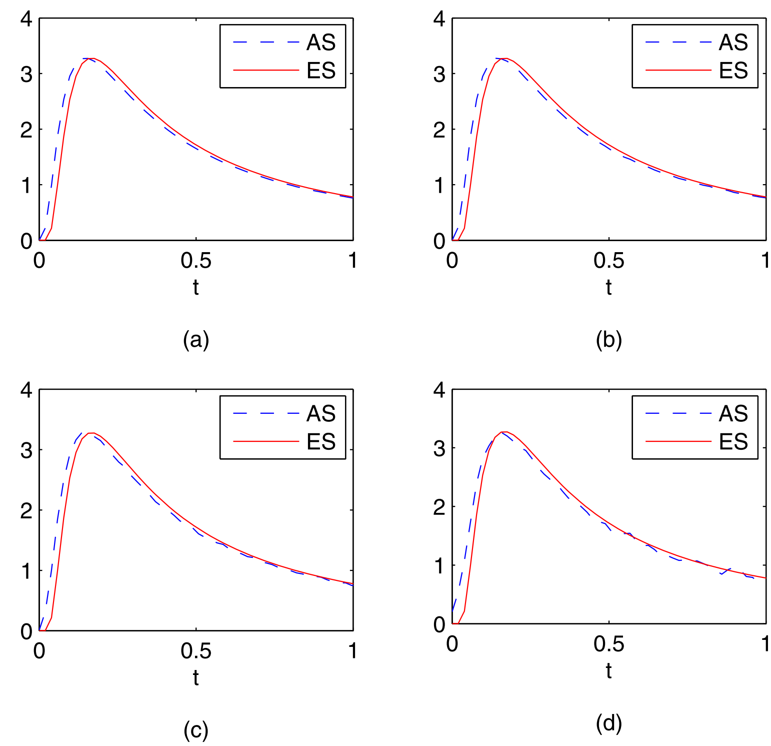

Figure 2 The exact solution (ES) and its approximation solution (AS). $\alpha = 0.4$, $x = 0$ and **(a)** $\varepsilon = 0.00001$; **(b)** $\varepsilon = 0.0001$; **(c)** $\varepsilon = 0.001$; **(d)** $\varepsilon = 0.01$.

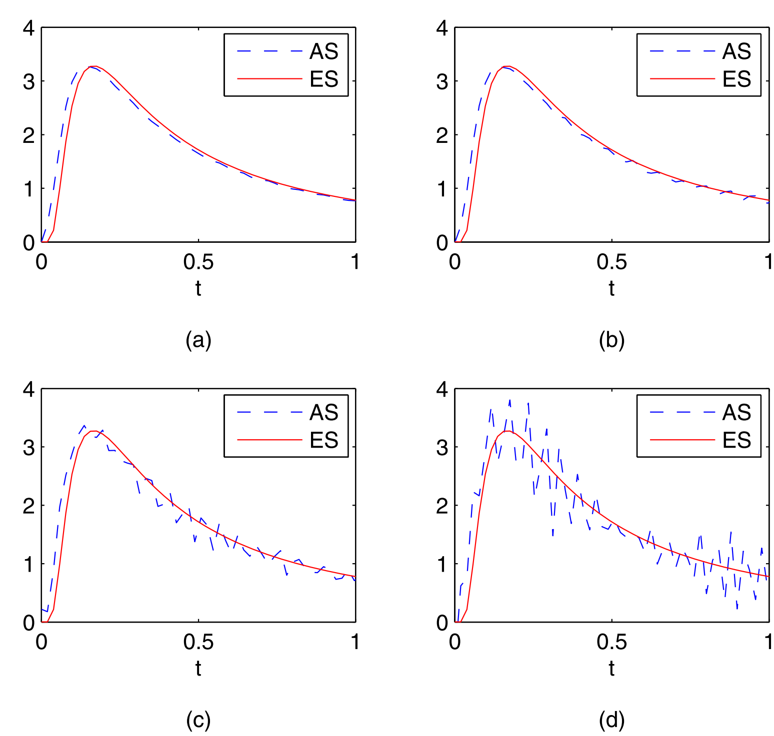

Figure 3 The exact solution (ES) and its approximation solution (AS). $\alpha = 0.6$, $x = 0$ and **(a)** $\gamma = 0.6$; **(b)** $\gamma = 0.7$; **(c)** $\gamma = 0.8$; **(d)** $\gamma = 0.9$.

Table 1 The executing time for the running time program with different x for Example 1

x	0.8	0.5	0.2	0
t	9.420000	9.526000	9.483000	9.389000

Table 2 The executing time for the running time program with different ε for Example 1

ε	0.00001	0.0001	0.001	0.01
t	9.241000	9.494000	9.531000	9.546000

Table 3 The executing time for the running time program with different γ for Example 1

γ	0.6	0.7	0.8	0.9
t	9.346000	9.508000	9.514000	9.523000

the better the computed approximation is. Moreover, for $x = 0$, we can see that the smaller the parameter γ, the better the computed approximation is.

5 Conclusion

In this paper, we propose a modified kernel method for solving time-fractional inverse diffusion problem by producing a stable approximation solution. For this regularization strategy, in the presence of noisy data, we establish and prove the convergence estimates for the cases $0 \leq x < 1$ under the *a priori* bound assumptions for the exact solution and the suitable choices of the regularization parameter. From the results of numerical simulations, it seems clear that the proposed method works well for the model problem with small measurement error.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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