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Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters

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Abstract

In the article, we present the best possible parameters $\lambda = \lambda(p)$ and $\mu = \mu(p)$ on the interval $[0, 1/2]$ such that the double inequality

$$\begin{aligned} &G^p\bigl[\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a\bigr]A^{1-p}(a,b) \\ &\quad < E(a,b) < G^p\bigl[\mu a + (1-\mu)b, \mu b + (1-\mu)a\bigr]A^{1-p}(a,b) \end{aligned}$$

holds for any $p \in [1,\infty)$ and all $a, b > 0$ with $a \neq b$, where $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$ and $E(a,b) = [2\int_0^{\pi/2}\sqrt{a\cos^2\theta + b\sin^2\theta}\,d\theta/\pi]^2$ are the arithmetic, geometric and special quasi-arithmetic means of a and b, respectively.

MSC: 26E60; 33E05

Keywords: quasi-arithmetic mean; complete elliptic integral; Gaussian hypergeometric function; arithmetic mean; geometric mean

1 Introduction

Let $r \in (0,1)$. Then the Legendre complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ [1, 2] of the first and second kinds are defined as

$$\mathcal{K}(r) = \int_0^{\pi/2}\frac{dt}{\sqrt{1-r^2\sin^2(t)}}, \qquad \mathcal{E}(r) = \int_0^{\pi/2}\sqrt{1-r^2\sin^2(t)}\,dt,$$

respectively. It is well known that the function $r \to \mathcal{K}(r)$ is strictly increasing from $(0,1)$ onto $(\pi/2,\infty)$ and the function $r \to \mathcal{E}(r)$ is strictly decreasing from $(0,1)$ onto $(1,\pi/2)$, and they satisfy the formulas (see [3, Appendix E, pp. 474,475])

$$\begin{aligned} &\frac{d\mathcal{K}(r)}{dr} = \frac{\mathcal{E}(r) - r'^2\mathcal{K}(r)}{rr'^2}, &&\frac{d\mathcal{E}(r)}{dr} = \frac{\mathcal{E}(r) - \mathcal{K}(r)}{r}, \\ &\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right) = (1+r)\mathcal{K}(r), &&\mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E}(r) - r'^2\mathcal{K}}{1+r}, \end{aligned}$$

where $r' = \sqrt{1-r^2}$.

The complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ are the particular cases of the Gaussian hypergeometric function [4–10]

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}\quad(-1<x<1),$$

where $(a)_0=1$ for $a\neq 0$, $(a)_n=a(a+1)(a+2)\cdots(a+n-1)=\Gamma(a+n)/\Gamma(a)$ is the shifted factorial function and $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt$ $(x>0)$ is the gamma function [11–18]. Indeed,

$$\mathcal{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^2}{(n!)^2}r^{2n},$$
$$\mathcal{E}(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2})_n(\frac{1}{2})_n}{(n!)^2}r^{2n}.$$

Recently, the bounds for the complete elliptic integrals have attracted the attention of many researchers. In particular, many remarkable inequalities and properties for $\mathcal{K}(r)$, $\mathcal{E}(r)$ and $F(a,b;c;x)$ can be found in the literature [19–52].

In 1998, a class of quasi-arithmetic mean was introduced by Toader [53] which is defined by

$$M_{p,n}(a,b)=p^{-1}\left(\frac{1}{\pi}\int_0^{\pi}p\big(r_n(\theta)\,d\theta\big)\right)=p^{-1}\left(\frac{2}{\pi}\int_0^{\pi/2}p\big(r_n(\theta)\,d\theta\big)\right),$$

where $r_n(\theta)=(a^n\cos^2\theta+b^n\sin^2\theta)^{1/n}$ for $n\neq 0$, $r_0(\theta)=a^{\cos^2\theta}b^{\sin^2\theta}$, and p is a strictly monotonic function. It is well known that many important means are the special cases of the quasi-arithmetic mean. For example,

$$M_{1/x,2}(a,b)=\frac{\pi}{2\int_0^{\pi/2}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}}=\begin{cases}\pi a/[2\mathcal{K}(\sqrt{1-(b/a)^2})], & a\geq b,\\ \pi b/[2\mathcal{K}(\sqrt{1-(a/b)^2})], & a<b,\end{cases}$$

is the arithmetic-geometric mean of Gauss [54–60],

$$M_{x,2}(a,b))=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta=\begin{cases}2a\mathcal{E}(\sqrt{1-(b/a)^2})/\pi, & a\geq b,\\ 2b\mathcal{E}(\sqrt{1-(a/b)^2})/\pi, & a<b,\end{cases}$$

is the Toader mean [61–70], and

$$M_{x,0}(a,b))=\frac{2}{\pi}\int_0^{\pi/2}a^{\cos^2\theta}b^{\sin^2\theta}\,d\theta$$

is the Toader-Qi mean [71–74].

Let $p=\sqrt{x}$ and $n=1$. Then $M_{p,n}(a,b)$ reduces to a special quasi-arithmetic mean

$$E(a,b)=M_{\sqrt{x},1}(a,b))=\begin{cases}4a[\mathcal{E}(\sqrt{1-b/a})]^2/\pi^2, & a\geq b,\\ 4b[\mathcal{E}(\sqrt{1-a/b})]^2/\pi^2, & a<b.\end{cases}\qquad(1.1)$$

Let

$$A(a,b)=\frac{a+b}{2},\qquad G(a,b)=\sqrt{ab},$$
$$M_p(a,b)=\left(\frac{a^p+b^p}{2}\right)^{1/p}(p\neq 0),\qquad M_0(a,b)=\sqrt{ab},$$

be the arithmetic, geometric and pth power means of a and b, respectively. Then it is well known that the inequality

$$G(a,b)=M_0(a,b)<A(a,b)=M_1(a,b) \tag{1.2}$$

holds for all $a,b>0$ with $a\neq b$, and the double inequality

$$\frac{\pi}{2}M_{3/2}\left(1,r'\right)<\mathcal{E}(r)<\frac{\pi}{2}M_2\left(1,r'\right) \tag{1.3}$$

holds for all $r\in(0,1)$ (see [75, 19.9.4]).

From (1.1)-(1.3) we clearly see that

$$G(a,b)<E(a,b)<A(a,b)$$

for all $a,b>0$ with $a\neq b$.

Let $p\in[1,\infty)$ and

$$f(x;p;a,b)=G^p\left[xa+(1-x)b,xb+(1-x)a\right]A^{1-p}(a,b).$$

Then it is not difficult to verify that the function $x\to f(x;p;a,b)$ is strictly increasing on $[0,1/2]$ for fixed $p\in[1,\infty)$ and $a,b>0$ with $a\neq b$. Note that

$$\begin{aligned} f(0;p;a,b)&=G^p(a,b)A^{1-p}(a,b)\leq G(a,b)\\ &<E(a,b)<A(a,b)=f(1/2;p;a,b) \end{aligned} \tag{1.4}$$

for all $p\in[1,\infty)$ and $a,b>0$ with $a\neq b$.

Motivated by inequalities (1.4) and the monotonicity of the function $x\to f(x;p;a,b)$ on the interval $[0,1/2]$, in the article, we shall find the best possible parameters $\lambda=\lambda(p),\mu=\mu(p)$ on the interval $[0,1/2]$ such that the double inequality

$$\begin{aligned} &G^p\left[\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a\right]A^{1-p}(a,b)\\ &\quad<E(a,b)<G^p\left[\mu a+(1-\mu)b,\mu b+(1-\mu)a\right]A^{1-p}(a,b) \end{aligned}$$

holds for any $p\in[1,\infty)$ and all $a,b>0$ with $a\neq b$.

2 Lemmas

Lemma 2.1 (see [3, Theorem 1.25]) *Let $-\infty<a<b<+\infty$, $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing* (*decreasing*) *on* (a,b), *then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.2 *The inequality*

$$\frac{1}{4p}+\left(\frac{2\sqrt{2}}{\pi}\right)^{4/p}<1$$

holds for all $p\in[1,\infty)$.

Proof Let

$$f(p)=\frac{1}{4p}+\left(\frac{2\sqrt{2}}{\pi}\right)^{4/p}. \tag{2.1}$$

Then simple computations lead to

$$\lim_{p\to\infty}f(p)=1, \tag{2.2}$$

$$\begin{aligned} f'(p)&=\frac{4}{p^2}\log\left(\frac{\sqrt{2}\pi}{4}\right)\left[\left(\frac{2\sqrt{2}}{\pi}\right)^{4/p}-\frac{1}{16\log(\frac{\sqrt{2}\pi}{4})}\right] \\ &\geq\frac{4}{p^2}\log\left(\frac{\sqrt{2}\pi}{4}\right)\left[\left(\frac{2\sqrt{2}}{\pi}\right)^{4}-\frac{1}{16\log(\frac{\sqrt{2}\pi}{4})}\right] \\ &=\frac{1024\log(\frac{\sqrt{2}\pi}{4})-\pi^4}{4\pi^4p^2}>0 \end{aligned} \tag{2.3}$$

for $p\in[1,\infty)$.

Therefore, Lemma 2.2 follows easily from (2.1)-(2.3). □

Lemma 2.3 *The following statements are true*:

(1) *The function* $r\mapsto[\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,1)$.
(2) *The function* $r\mapsto[\mathcal{K}(r)-\mathcal{E}(r)]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,\infty)$.
(3) *The function* $r\mapsto[\mathcal{E}(r)+(1-r^2)\mathcal{K}(r)]/(1-r^2)$ *is strictly increasing from* $(0,1)$ *onto* (π,∞).
(4) *The function* $r\mapsto[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]/(1+r^2)$ *is strictly decreasing from* $(0,1)$ *onto* $(1,\pi/2)$.
(5) *The function* $r\mapsto r^2[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]/[(1+r^2)^2(\mathcal{K}(r)-\mathcal{E}(r))]$ *is strictly decreasing from* $(0,1)$ *onto* $(0,2)$.

Proof Parts (1) and (2) can be found in the literature [3, Theorem 3.21(1) and Exercise 3.43(11)].

For part (3), let $f_1(r) = [\mathcal{E}(r) + (1-r^2)\mathcal{K}(r)]/(1-r^2)$. Then simple computations lead to

$$f_1(0^+) = \pi, \qquad f_1(1^-) = \infty, \tag{2.4}$$

$$f_1'(r) = \frac{r}{(1-r^2)^2}\left[\frac{2}{r^2}\left(\mathcal{E}(r) - \left(1-r^2\right)\mathcal{K}(r)\right) + \left(1-r^2\right)\mathcal{K}(r)\right]. \tag{2.5}$$

It follows from part (1) and (2.5) that

$$f_1'(r) > 0 \tag{2.6}$$

for all $r \in (0,1)$. Therefore, part (3) follows from (2.4) and (2.6).

For part (4), let $f_2(r) = [2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]/(1+r^2)$, then one has

$$f_2(0^+) = \frac{\pi}{2}, \qquad f_1(1^-) = 1, \tag{2.7}$$

$$f_2'(r) = \frac{r}{(1+r^2)^2}\left[\left(1-r^2\right)\frac{\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)}{r^2} - 2\mathcal{E}(r)\right]. \tag{2.8}$$

From part (1) and (2.8) we clearly see that

$$f_2'(r) < -\frac{r}{(1+r^2)} < 0 \tag{2.9}$$

for all $r \in (0,1)$. Therefore, part (4) follows from (2.7) and (2.9).

For part (5), let $f_3(r) = r^2[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]/[(1+r^2)^2(\mathcal{K}(r) - \mathcal{E}(r))]$, then $f_3(r)$ can be rewritten as

$$f_3(r) = \frac{2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)}{1+r^2} \times \frac{1}{\frac{\mathcal{K}(r)-\mathcal{E}(r)}{r^2}} \times \frac{1}{1+r^2}. \tag{2.10}$$

Therefore, part (5) follows easily from parts (2) and (4) together with (2.10). □

Lemma 2.4 *The function*

$$g(r) = \frac{r^2\mathcal{K}(r)}{(1+r^2)[\mathcal{K}(r) - \mathcal{E}(r)]}$$

is strictly decreasing from $(0,1)$ *onto* $(1/2, 2)$.

Proof Let $g_1(r) = r^2\mathcal{K}(r)$ and $g_2(r) = (1+r^2)[\mathcal{K}(r) - \mathcal{E}(r)]$. Then we clearly see that

$$g_1(0^+) = g_2(0^+) = 0, \qquad g(r) = \frac{g_1(r)}{g_2(r)}, \tag{2.11}$$

$$g(1^-) = \frac{1}{2}, \tag{2.12}$$

$$\frac{g_1'(r)}{g_2'(r)} = \frac{1}{2 - \frac{3\mathcal{E}(r)}{\frac{\mathcal{E}(r)+(1-r^2)\mathcal{K}(r)}{1-r^2}}}. \tag{2.13}$$

From Lemma 2.3(3), (2.11) and (2.13) we know that

$$g(0^+) = \lim_{r\to 0^+} \frac{g_1'(r)}{g_2'(r)} = 2 \tag{2.14}$$

and the function $g_1'(r)/g_2'(r)$ is strictly decreasing on $(0,1)$.

Therefore, Lemma 2.4 follows easily from Lemma 2.1, (2.11), (2.12) and (2.14) together with the monotonicity of the function $g_1'(r)/g_2'(r)$. □

Lemma 2.5 *Let $u \in [0,1]$, $r \in (0,1)$, $p \in [1,\infty)$ and*

$$h(u,p;r) = \frac{1}{2} p \log\left[1 - \frac{4ur^2}{(1+r^2)^2}\right] - \log\left[\frac{4(2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r))^2}{\pi^2(1+r^2)}\right]. \tag{2.15}$$

Then one has

(1) *$h(u,p;r) > 0$ for all $r \in (0,1)$ if and only if $u \le 1/4p$;*
(2) *$h(u,p;r) < 0$ for all $r \in (0,1)$ if and only if $u \ge 1 - (2\sqrt{2}/\pi)^{4/p}$.*

Proof It follows from (2.15) that

$$h(u,p;0^+) = 0, \tag{2.16}$$

$$h(u,p;1^-) = \frac{p}{2}\log(1-u) + \log\left(\frac{\pi^2}{8}\right), \tag{2.17}$$

$$\begin{aligned}\frac{\partial h(u,p;r)}{\partial r} &= \frac{2(1-r^2)[\mathcal{K}(r) - \mathcal{E}(r)]}{r(1+r^2)[2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]} - \frac{4pur(1-r^2)}{(1+r^2)[(1+r^2)^2 - 4ur^2]} \\ &= \frac{2(1-r^2)[2(\mathcal{K}(r) - \mathcal{E}(r)) + p(2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r))]}{(1+r^2)[(1+r^2)^2 - 4ur^2][2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r)]}\left[h_1(p;r) - 2u\right],\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}h_1(p;r) &= \frac{(1+r^2)^2[\mathcal{K}(r) - \mathcal{E}(r)]}{r^2[2(\mathcal{K}(r) - \mathcal{E}(r)) + p(2\mathcal{E}(r) - (1-r^2)\mathcal{K}(r))]} \\ &= \frac{1}{g(r) + (p-1)f_3(r)},\end{aligned} \tag{2.19}$$

where $f_3(r)$ and $g(r)$ are defined by (2.10) and Lemma 2.4, respectively.

From Lemma 2.3(5) and Lemma 2.4 together with (2.19) we clearly see that the function $r \to h_1(p;r)$ is strictly increasing on $(0,1)$ and

$$h_1(p;0^+) = \frac{1}{2p}, \tag{2.20}$$

$$h_1(p;1^-) = 2. \tag{2.21}$$

From Lemma 2.2 we know that $1 - (2\sqrt{2}/\pi)^{4/p} > 1/(4p)$. Therefore, we only need to divide the proof into three cases as follows.

Case 1 $u \le 1/(4p)$. Then Lemma 2.3(4), (2.18), (2.20) and the monotonicity of the function $r \to h_1(p;r)$ on the interval $(0,1)$ lead to the conclusion that the function $r \to h(u,p;r)$

is strictly increasing on $(0,1)$. Therefore, $h(u,p;r) > 0$ for all $r \in (0,1)$ follows from (2.16) and the monotonicity of the function $r \to h(u,p;r)$.

Case 2 $u \geq 1 - (2\sqrt{2}/\pi)^{4/p}$. Then from Lemma 2.2, Lemma 2.3(5), (2.17), (2.18), (2.20), (2.21) and the monotonicity of the function $r \to h_1(p;r)$ on the interval $(0,1)$ we clearly see that there exists $r_0 \in (0,1)$ such that the function $r \to h(u,p;r)$ is strictly decreasing on $(0,r_0)$ and strictly increasing on $(r_0,1)$, and

$$h\bigl(u,p;1^-\bigr) \leq 0. \tag{2.22}$$

Therefore, $h(u,p;r) < 0$ for all $r \in (0,1)$ follows from (2.16) and (2.22) together with the piecewise monotonicity of the function $r \to h(u,p;r)$ on the interval $(0,1)$.

Case 3 $1/(4p) < u < 1 - (2\sqrt{2}/\pi)^{4/p}$. Then (2.17) leads to

$$h\bigl(u,p;1^-\bigr) > 0. \tag{2.23}$$

It follows from Lemma 2.3(5), (2.18), (2.20), (2.21) and the monotonicity of the function $r \to h_1(p;r)$ on the interval $(0,1)$ that there exists $r^* \in (0,1)$ such that the function $r \to h(u,p;r)$ is strictly decreasing on $(0,r^*)$ and strictly increasing on $(r^*,1)$. Therefore, there exists $\lambda \in (0,1)$ such that $h(u,p;r) < 0$ for $r \in (0,\lambda)$ and $h(u,p;r) > 0$ for $r \in (\lambda,1)$. □

3 Main result

Theorem 3.1 *Let* $\lambda, \mu \in [0,1/2]$. *Then the double inequality*

$$\begin{aligned} &G^p\bigl[\lambda a + (1-\lambda)b, \lambda b + (1-\lambda)a\bigr]A^{1-p}(a,b) \\ &\quad < E(a,b) < G^p\bigl[\mu a + (1-\mu)b, \mu b + (1-\mu)a\bigr]A^{1-p}(a,b) \end{aligned}$$

holds for any $p \in [1,\infty)$ *and all* $a,b > 0$ *with* $a \neq b$ *if and only if* $\lambda \leq 1/2 - \sqrt{1-(2\sqrt{2}/\pi)^{4/p}}/2$ *and* $\mu \geq 1/2 - \sqrt{p}/(4p)$.

Proof Let $t \in [0,1/2]$, since $G^p[ta+(1-t)b, tb+(1-t)a]A^{1-p}(a,b)$ and $E(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b > 0$. Let $r \in (0,1)$ and $b/a = (1-r)^2/(1+r)^2$. Then (1.1) leads to

$$\begin{aligned} &E(a,b) = \frac{4(1+r)^2}{\pi^2(1+r^2)}A(a,b)\mathcal{E}^2\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{4}{\pi^2}A(a,b)\frac{[2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r)]^2}{1+r^2}, \\ &\log\bigl[G^p\bigl(ta+(1-t)b, tb+(1-t)a\bigr)A^{1-p}(a,b)\bigr] - \log E(a,b) \\ &\quad = \log\left[\frac{G^p(ta+(1-t)b, tb+(1-t)a)A^{1-p}(a,b)}{A(a,b)}\right] - \log\left[\frac{E(a,b)}{A(a,b)}\right] \\ &\quad = \frac{1}{2}p\log\left[1 - \frac{4(1-2t)^2r^2}{(1+r^2)^2}\right] - \log\left[\frac{4(2\mathcal{E}(r)-(1-r^2)\mathcal{K}(r))^2}{\pi^2(1+r^2)}\right]. \end{aligned} \tag{3.1}$$

Therefore, Theorem 3.1 follows easily from Lemma 2.5 and (3.1). □

Let $p = 1,2$, then Theorem 3.1 leads to Corollary 3.2 immediately.

Corollary 3.2 *Let* $\lambda_1, \mu_1, \lambda_2, \mu_2 \in [0,1/2]$. *Then the double inequalities*

$$H\big[\lambda_1 a + (1-\lambda_1)b, \lambda_1 b + (1-\lambda_1)a\big] < E(a,b) < H\big[\mu_1 a + (1-\mu_1)b, \mu_1 b + (1-\mu_1)a\big],$$
$$G\big[\lambda_2 a + (1-\lambda_2)b, \lambda_2 b + (1-\lambda_2)a\big] < E(a,b) < G\big[\mu_2 a + (1-\mu_2)b, \mu_2 b + (1-\mu_2)a\big]$$

hold for all $a,b>0$ *with* $a \neq b$ *if and only if* $\lambda_1 \leq 1/2 - \sqrt{1-8/\pi^2}/2 = 0.2823\ldots$, $\mu_1 \geq 1/2 - \sqrt{2}/8 = 0.3232\ldots$, $\lambda_2 \leq 1/2 - \sqrt{1-64/\pi^4}/2 = 0.2071\ldots$ *and* $\mu_2 \geq 1/4$.

Let $p \in [1,\infty)$, $r \in (0,1)$, $a = r$, $b = 1-r^2 = r'^2$, $\lambda = 1/2 - \sqrt{1-(2\sqrt{2}/\pi)^{4/p}}/2$ and $\mu = 1/2 - \sqrt{p}/(4p)$. Then (1.1) and Theorem 3.1 lead to Corollary 3.3 immediately.

Corollary 3.3 *The double inequality*

$$\begin{aligned}&\frac{\sqrt{2}\pi}{4}\left(1+r'^2\right)^{(1-p)/2}\left[4r'^2 + \left(\frac{8}{\pi^2}\right)^{2/p} r^4\right]^{p/4} \\ &\quad < \mathcal{E}(r) < \frac{\sqrt{2}\pi}{4}\left(1+r'^2\right)^{(1-p)/2}\left[\left(1+r'^2\right)^2 - \frac{r^4}{4p}\right]^{p/4}\end{aligned}$$

holds for all $r \in (0,1)$ *and* $p \in [1,\infty)$.

4 Results and discussion

In this paper, we provide the sharp bounds for the special quasi-arithmetic mean $E(a,b)$ in terms of the arithmetic mean $A(a,b)$ and geometric mean $G(a,b)$ with two parameters. As consequences, we present the best possible one-parameter harmonic and geometric means bounds for $E(a,b)$ and find new bounds for the complete elliptic integral of the second kind.

5 Conclusion

In the article, we derive a new bivariate mean $E(a,b)$ from the quasi-arithmetic mean and provide its sharp upper and lower bounds in terms of the concave combination of arithmetic and geometric means.

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Competing interests
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Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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