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Hamilton's gradient estimates and Liouville theorems for porous medium equations

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Abstract

Let (M^n, g) be an n-dimensional Riemannian manifold. In this paper, we derive a local gradient estimate for positive solutions of the porous medium equation

$$u_t = \Delta(u^p), \quad 1 < p < 1 + \frac{1}{\sqrt{n-1}},$$

posed on (M^n, g) with the Ricci curvature bounded from below. Moreover, we also obtain a Liouville type theorem. In particular, the results obtained in this paper generalize those in (Zhu in J. Math. Anal. Appl. 402:201-206, 2013).

MSC: Primary 35B45; secondary 35K55

Keywords: porous medium equation; Hamilton's gradient estimate; Liouville type theorem

1 Introduction

In this paper we study the porous medium equation

$$u_t = \Delta(u^p) \tag{1.1}$$

with $p > 1$, which is a nonlinear extension of the classical heat equation. As is typical of nonlinear problems, the mathematical theory of the porous medium equation is based on a priori estimates. In 1979, Aronson and Bénilan obtained a celebrated second-order differential inequality of the form [2]

$$\sum_i \frac{\partial}{\partial x^i}\left(pu^{p-2}\frac{\partial u}{\partial x^i}\right) \geq -\frac{k}{t}, \quad k := \frac{n}{n(p-1)+2},$$

which applies to all positive smooth solutions of (1.1) defined on the whole Euclidean space on the condition that $p > 1 - \frac{2}{n}$. For various values of $p > 1$, it has appeared in different applications to model diffusive phenomena; see [2–4] and the references therein.

In [5], Hamilton proved the following results.

Theorem A (Hamilton [5]) *Let (M^n, g) be an n-dimensional compact Riemannian manifold with* $\mathrm{Ric}(M^n) \geq -K$*, where K is a non-negative constant. Suppose that u is a positive solution to the heat equation*

$$u_t = \Delta u \tag{1.2}$$

with $u < M$ for all $(x,t) \in M^n \times (0,\infty)$. Then

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K\right) \log \frac{M}{u}. \tag{1.3}$$

Hamilton's estimate tells us that when the temperature is bounded we can compare the temperature of two different points at the same time. For the study of gradient estimates of the equation (1.1), see [6–10] and the references therein. In [1], Zhu applied similar techniques that were used for the heat equation, he derived the following Hamilton type estimate for equation (1.1).

Theorem B (Zhu [1]) *Let (M^n, g) be an n-dimensional Riemannian manifold with* $\mathrm{Ric}(M^n) \geq -K$*, where K is a non-negative constant. Suppose that u is a positive solution to the porous medium equation* (1.1) *in $Q_{R,T} := B_{x_0}(R) \times [t_0 - T, t_0] \subset M^n \times (-\infty,\infty)$. Let $v = \frac{p}{p-1}u^{p-1}$. Then for $1 < p < 1 + \frac{1}{\sqrt{2n+1}}$ and $v \leq M$,*

$$v^{\frac{1}{4}\frac{2-p}{p-1}} |\nabla v| \leq C M^{1+\frac{1}{4}\frac{2-p}{p-1}} \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K}\right) \tag{1.4}$$

in $Q_{R,T}$, where $C = C(p,n)$ is a constant depending only on p and n.

In this paper, by introducing a new parameter and using a lemma in [11], we generalize Theorem B as follows.

Theorem 1.1 *Let (M^n, g) be an n-dimensional Riemannian manifold with* $\mathrm{Ric}(M^n) \geq -K$*, where K is a non-negative constant. Suppose that u is a positive solution to the porous medium equation* (1.1) *in $Q_{R,T} := B_{x_0}(R) \times [t_0 - T, t_0] \subset M^n \times (-\infty,\infty)$. Let $v = \frac{p}{p-1}u^{p-1}$. Then for $1 < p < 1 + \frac{1}{\sqrt{n-1}}$ and $v \leq M$,*

$$v^{\frac{1}{2(p-1)}} |\nabla v| \leq C M^{1+\frac{1}{2(p-1)}} \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K}\right) \tag{1.5}$$

in $Q_{R,T}$, where $C = C(p,n)$ is a constant depending only on p and n.

As an application, we get the following Liouville type theorem.

Corollary 1.2 *Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with non-negative Ricci curvature. Let u be a positive ancient solution to the porous medium equation* (1.1) *with $1 < p < 1 + \frac{1}{\sqrt{n-1}}$ such that $u(x,t) = o([d(x) + \sqrt{|t|}]^{\frac{2}{2p-1}})$ near infinity. Then u must be a constant.*

Remark 1.3 Note that $\frac{1}{\sqrt{2n+1}} < \frac{1}{\sqrt{n-1}}$. Therefore, the results obtained in this paper generalize those of Zhu in [1].

2 Proof of Theorem 1.1

Let $v = \frac{p}{p-1}u^{p-1}$. From (1.1), by simple calculations, it is easy to see that

$$v_t = (p-1)v\Delta v + |\nabla v|^2. \tag{2.1}$$

We define

$$w = \frac{|\nabla v|^2}{v^\beta},$$

where β is a constant to be determined. Then we have

$$\begin{aligned} w_t &= \frac{2v_i v_{it}}{v^\beta} - \beta\frac{v_i^2 v_t}{v^{\beta+1}} \\ &= 2\frac{v_i[(p-1)v\Delta v + |\nabla v|^2]_i}{v^\beta} - \beta\frac{v_i^2[(p-1)v\Delta v + |\nabla v|^2]}{v^{\beta+1}} \\ &= 2(p-1)\frac{v_i^2 v_{jj}}{v^\beta} + 2(p-1)\frac{v_i v_{jji}}{v^{\beta-1}} + 4\frac{v_{ij}v_i v_j}{v^\beta} - \beta(p-1)\frac{v_i^2 v_{jj}}{v^\beta} - \beta\frac{v_i^2 v_j^2}{v^{\beta+1}}, \end{aligned} \tag{2.2}$$

$$w_j = \frac{2v_i v_{ij}}{v^\beta} - \beta\frac{v_i^2 v_j}{v^{\beta+1}}, \tag{2.3}$$

and hence

$$w_{jj} = \frac{2v_{ij}^2}{v^\beta} + \frac{2v_i v_{ijj}}{v^\beta} - 4\beta\frac{v_{ij}v_i v_j}{v^{\beta+1}} - \beta\frac{v_i^2 v_{jj}}{v^{\beta+1}} + \beta(\beta+1)\frac{v_i^2 v_j^2}{v^{\beta+2}}. \tag{2.4}$$

By virtue of (2.2) and (2.4),

$$\begin{aligned} (p-1)v\Delta w - w_t &= 2(p-1)\frac{v_{ij}^2}{v^{\beta-1}} + 2(p-1)\frac{v_i v_{ijj}}{v^{\beta-1}} - 4\beta(p-1)\frac{v_i v_j v_{ij}}{v^\beta} \\ &\quad - \beta(p-1)\frac{v_i^2 v_{jj}}{v^\beta} + \beta(\beta+1)(p-1)\frac{v_i^2 v_j^2}{v^{\beta+1}} \\ &\quad - 2(p-1)\frac{v_i^2 v_{jj}}{v^\beta} - 2(p-1)\frac{v_i v_{jji}}{v^{\beta-1}} - 4\frac{v_{ij}v_i v_j}{v^\beta} \\ &\quad + \beta(p-1)\frac{v_i^2 v_{jj}}{v^\beta} + \beta\frac{v_i^2 v_j^2}{v^{\beta+1}} \\ &= 2(p-1)\frac{v_{ij}^2}{v^{\beta-1}} - 2(p-1)\frac{v_i^2 v_{jj}}{v^\beta} + 2(p-1)\frac{R_{ij}v_i v_j}{v^{\beta-1}} \\ &\quad - 4\bigl[1+\beta(p-1)\bigr]\frac{v_{ij}v_i v_j}{v^\beta} + \beta\bigl[(\beta+1)(p-1)+1\bigr]\frac{v_i^2 v_j^2}{v^{\beta+1}}, \end{aligned} \tag{2.5}$$

where, in the second equality, we use the Ricci formula: $v_{ijj} - v_{jji} = R_{ij}v_j$.

In order to prove Theorem 1.1, we need the following lemma (*cf.* Lemma A.1 in [11]).

Lemma 2.1 *Let $A = (a_{ij})$ be a nonzero $n \times n$ symmetric matrix. Then for $a, b \in \mathbb{R}$,*

$$\max_{A\in\mathcal{S}(n);|e|=1}\left(\frac{aA + b(\operatorname{tr} A)I_n}{|A|}(e,e)\right)^2 = (a+b)^2 + (n-1)b^2, \tag{2.6}$$

where I_n is an identity matrix.

Notice that

$$\nabla v \nabla w = 2\frac{v_{ij}v_i v_j}{v^{\beta}} - \beta\frac{v_i^2 v_j^2}{v^{\beta+1}}. \tag{2.7}$$

It follows from (2.5) that, for any constant ε,

$$\begin{aligned}
&(p-1)v\Delta w - w_t + \varepsilon\nabla v\nabla w \\
&\quad = 2(p-1)\frac{v_{ij}^2}{v^{\beta-1}} - 2(p-1)\frac{v_i^2 v_{jj}}{v^{\beta}} + 2(p-1)\frac{R_{ij}v_i v_j}{v^{\beta-1}} \\
&\qquad + \bigl[2\varepsilon - 4\bigl(1+\beta(p-1)\bigr)\bigr]\frac{v_{ij}v_i v_j}{v^{\beta}} + \beta\bigl[(\beta+1)(p-1)+1-\varepsilon\bigr]\frac{v_i^2 v_j^2}{v^{\beta+1}} \\
&\quad = 2(p-1)\frac{|A|^2}{v^{\beta-1}} - 2(p-1)\frac{\operatorname{tr} A}{|A|}w|A| + 2(p-1)v\operatorname{Ric}(e,e)w \\
&\qquad + \bigl[2\varepsilon - 4\bigl(1+\beta(p-1)\bigr)\bigr]\frac{A(e,e)}{|A|}w|A| + \beta\bigl[(\beta+1)(p-1)+1-\varepsilon\bigr]v^{\beta-1}w^2 \\
&\quad = 2(p-1)\frac{|A|^2}{v^{\beta-1}} + \left(\bigl[2\varepsilon - 4\bigl(1+\beta(p-1)\bigr)\bigr]\frac{A(e,e)}{|A|} - 2(p-1)\frac{\operatorname{tr} A}{|A|}\right)w|A| \\
&\qquad + 2(p-1)v\operatorname{Ric}(e,e)w + \beta\bigl[(\beta+1)(p-1)+1-\varepsilon\bigr]v^{\beta-1}w^2 \\
&\quad = 2(p-1)\left[\frac{|A|}{v^{\frac{\beta-1}{2}}} + \frac{1}{4(p-1)}\left(\bigl[2\varepsilon - 4\bigl(1+\beta(p-1)\bigr)\bigr]\frac{A(e,e)}{|A|} - 2(p-1)\frac{\operatorname{tr} A}{|A|}\right)wv^{\frac{\beta-1}{2}}\right]^2 \\
&\qquad - \frac{1}{8(p-1)}\left(\bigl[2\varepsilon - 4\bigl(1+\beta(p-1)\bigr)\bigr]\frac{A(e,e)}{|A|} - 2(p-1)\frac{\operatorname{tr} A}{|A|}\right)^2 v^{\beta-1}w^2 \\
&\qquad + 2(p-1)v\operatorname{Ric}(e,e)w + \beta\bigl[(\beta+1)(p-1)+1-\varepsilon\bigr]v^{\beta-1}w^2 \\
&\quad \geq -\frac{1}{8(p-1)}\left|\bigl[2\varepsilon - 4\bigl(1+\beta(p-1)\bigr)\bigr]\frac{A(e,e)}{|A|} - 2(p-1)\frac{(\operatorname{tr} A)I_n(e,e)}{|A|}\right|^2 v^{\beta-1}w^2 \\
&\qquad + 2(p-1)v\operatorname{Ric}(e,e)w + \beta\bigl[(\beta+1)(p-1)+1-\varepsilon\bigr]v^{\beta-1}w^2,
\end{aligned} \tag{2.8}$$

where $A_{ij} = (v_{ij})$ and $e = \nabla v/|\nabla v|$. By virtue of Lemma 2.1, we have

$$\begin{aligned}
&(p-1)v\Delta w - w_t + \varepsilon\nabla v\nabla w \\
&\quad \geq -\frac{1}{8(p-1)}\left|\bigl[2\varepsilon - 4\bigl(1+\beta(p-1)\bigr)\bigr]\frac{A(e,e)}{|A|} - 2(p-1)\frac{(\operatorname{tr} A)I_n(e,e)}{|A|}\right|^2 v^{\beta-1}w^2 \\
&\qquad + 2(p-1)v\operatorname{Ric}(e,e)w + \beta\bigl[(\beta+1)(p-1)+1-\varepsilon\bigr]v^{\beta-1}w^2 \\
&\quad \geq -\frac{1}{8(p-1)}\Bigl(\bigl[2\varepsilon - 4\bigl(1+\beta(p-1)\bigr) - 2(p-1)\bigr]^2 + 4(n-1)(p-1)^2 \\
&\qquad - 8(p-1)\beta\bigl[(\beta+1)(p-1)+1-\varepsilon\bigr]\Bigr)v^{\beta-1}w^2 + 2(p-1)v\operatorname{Ric}(e,e)w \\
&\quad = -\frac{1}{8(p-1)}f(\beta,\varepsilon)v^{\beta-1}w^2 + 2(p-1)v\operatorname{Ric}(e,e)w,
\end{aligned} \tag{2.9}$$

where

$$f(\beta,\varepsilon) = \bigl[2\varepsilon - 4\bigl(1+\beta(p-1)\bigr) - 2(p-1)\bigr]^2 + 4(n-1)(p-1)^2 - 8(p-1)\beta\bigl[(\beta+1)(p-1)+1-\varepsilon\bigr].$$

For the purpose of showing that the coefficient of $v^{\beta-1}w^2$ is positive, we minimize the function $f(\beta,\varepsilon)$ by letting

$$\varepsilon = p$$

and

$$\beta = -\frac{p-\varepsilon+2}{2(p-1)} = -\frac{1}{p-1},$$

such that

$$f = -4 + 4(n-1)(p-1)^2.$$

Then (2.9) becomes

$$\begin{aligned}(p-1)v\Delta w - w_t + p\nabla v\nabla w &\geq \frac{1-(n-1)(p-1)^2}{2(p-1)}v^{\beta-1}w^2 - 2(p-1)Kvw \\ &= \alpha v^{\beta-1}w^2 - 2(p-1)Kvw, \end{aligned} \tag{2.10}$$

where $\alpha = \frac{1-(n-1)(p-1)^2}{2(p-1)}$.

We first recall the well-known smooth cutoff function ψ which originated with Li and Yau [12], satisfying the following:

(1) The cutoff function ψ satisfies $\psi = \psi(d(x,x_0),t) \equiv \psi(r,t)$ and $\psi(r,t) = 1$ in $Q_{R/2,T/2}$ with $0 \leq \psi \leq 1$.
(2) The function ψ is decreasing as a radial function in the spatial variables.
(3) $|\partial_r\psi|/\psi^a \leq C_a/R$ and $|\partial_r^2\psi|/\psi^a \leq C_a/R^2$ for $a \in (0,1)$.
(4) $|\partial_t\psi|/\psi^{\frac{1}{2}} \leq C/T$.

By virtue of (2.10), we have

$$\begin{aligned}&\bigl[(p-1)v\Delta - \partial_t\bigr](\psi w) \\ &\quad = \psi\bigl[(p-1)v\Delta - \partial_t\bigr]w + (p-1)vw\Delta\psi - w\psi_t \\ &\qquad + 2(p-1)v\frac{1}{\psi}\nabla\psi\nabla(\psi w) - 2(p-1)vw\frac{|\nabla\psi|^2}{\psi} \\ &\quad \geq \alpha v^{\beta-1}\psi w^2 - 2(p-1)Kv\psi w - p\nabla v\nabla(\psi w) \\ &\qquad + pw\nabla v\nabla\psi + (p-1)vw\Delta\psi - w\psi_t \\ &\qquad + 2(p-1)v\frac{1}{\psi}\nabla\psi\nabla(\psi w) - 2(p-1)vw\frac{|\nabla\psi|^2}{\psi}. \end{aligned} \tag{2.11}$$

Next we will apply the maximum principle to ψw in a closed set. Assume ψw achieves its maximum at the point (x_1,t_1) and assume $(\psi w)(x_1,t_1) > 0$ (otherwise the proof is trivial), which implies $t_1 > 0$. Then at the point (x_1,t_1)

$$(\Delta - \partial_t)(\psi w) \leq 0, \quad \nabla(\psi w) = 0,$$

and (2.11) becomes

$$\alpha v^{\beta-1}\psi w^2 \leq -pw\nabla v\nabla\psi + 2(p-1)vw\frac{|\nabla\psi|^2}{\psi} - (p-1)vw\Delta\psi + w\psi_t + 2(p-1)Kv\psi w. \tag{2.12}$$

That is,

$$2\psi w^2 \leq -p\gamma v^{1-\beta}w\nabla v\nabla\psi + 2(p-1)\gamma v^{2-\beta}w\frac{|\nabla\psi|^2}{\psi} - (p-1)\gamma v^{2-\beta}w\Delta\psi + \gamma v^{1-\beta}w\psi_t + 2(p-1)\gamma K v^{2-\beta}\psi w, \tag{2.13}$$

where $\alpha = \frac{2}{\gamma}$ and $\gamma = \frac{1-(n-1)(p-1)^2}{4(p-1)}$.

It have been shown in [1] (see equations (2.6)-(2.10) in [1]) that

$$-p\gamma v^{1-\beta}w\nabla v\nabla\psi \leq \frac{1}{4}\psi w^2 + CM^{4-2\beta}\frac{1}{R^4}, \tag{2.14}$$

where we used the fact that $0 < v \leq M$ and $\beta \leq 2$,

$$2(p-1)\gamma v^{2-\beta}w\frac{|\nabla\psi|^2}{\psi} \leq \frac{1}{4}\psi w^2 + CM^{4-2\beta}\frac{1}{R^4}, \tag{2.15}$$

$$-(p-1)\gamma v^{2-\beta}w\Delta\psi \leq \frac{1}{4}\psi w^2 + CM^{4-2\beta}\left(\frac{1}{R^4} + K\frac{1}{R^2}\right), \tag{2.16}$$

$$\gamma v^{1-\beta}w\psi_t \leq \frac{1}{4}\psi w^2 + CM^{4-2\beta}\frac{1}{T^2}, \tag{2.17}$$

$$2(p-1)\gamma K v^{2-\beta}\psi w \leq \frac{1}{4}\psi w^2 + CM^{4-2\beta}K^2. \tag{2.18}$$

Substituting (2.14)-(2.18) into (2.13), we obtain

$$2\psi w^2 \leq \frac{5}{4}\psi w^2 + CM^{4-2\beta}\left(\frac{1}{R^4} + \frac{1}{T^2} + K^2\right), \tag{2.19}$$

which gives at the point (x_1, t_1)

$$\psi w^2 \leq CM^{4-2\beta}\left(\frac{1}{R^4} + \frac{1}{T^2} + K^2\right). \tag{2.20}$$

Therefore, for all $(x,t) \in Q_{R,T}$,

$$\begin{aligned}(\psi^2 w^2)(x,t) &\leq (\psi^2 w^2)(x_1,t_1) \leq (\psi w^2)(x_1,t_1) \\ &\leq CM^{4-2\beta}\left(\frac{1}{R^4} + \frac{1}{T^2} + K^2\right).\end{aligned}$$

Notice that $\psi = 1$ in $Q_{R/2,T/2}$ and $w = \frac{|\nabla v|^2}{v^\beta}$. Hence, we have

$$\frac{|\nabla v|}{v^{\frac{\beta}{2}}}(x,t) \leq CM^{1-\frac{\beta}{2}}\left(\sqrt{K} + \frac{1}{R} + \frac{1}{\sqrt{T}}\right). \tag{2.21}$$

One concludes the proof of Theorem 1.1 by letting $\beta = -\frac{1}{p-1}$.

3 Simple proof of Corollary 1.2

Suppose u is a positive ancient solution to the porous medium equation (1.1) such that $v(x,t) = o([d(x_0,x) + \sqrt{|t|}]^{\frac{2}{2p-1}})$ near infinity, where $v = \frac{p}{p-1}u^{p-1}$. Fixing (x_0,t_0) in space-time and using Theorem 1.1 for u on the cube $B(x_0,R) \times [t_0 - R^2, t_0]$, we obtain

$$v(x_0,t_0)^{\frac{1}{2(p-1)}} \left|\nabla v(x_0,t_0)\right| \le \frac{C}{R} \cdot o(R).$$

Letting $R \to \infty$, it follows that $|\nabla v(x_0,t_0)| = 0$. Since (x_0,t_0) is arbitrary, we see that v is a constant. Hence, u is also constant from $v = \frac{p}{p-1}u^{p-1}$. Thus ends the proof of Corollary 1.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Huang and Xu participated in gradient estimates in this paper. Zeng carried out the Laplacian operator and the applications of inequalities studies. All authors read and approved the final manuscript.

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