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Stability of a functional equation connected with Reynolds operator

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Abstract

Let G be a commutative semigroup, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $F : G \to \mathbb{K}^n$. Generalizing the stability of the functional equation $F(x \circ g(y)) - F(x)F(y) = 0$ with bounded difference (Najdecki in J. Inequal. Appl. 2007:79816, 2007), we prove the stability of the above functional equation with unbounded differences. We also give a more precise description for bounded components of $F = (f_1, f_2, \dots, f_n)$.

MSC: 39B82

Keywords: bounded solution; exponential function; involution; Reynolds operator; stability

1 Main results

Throughout this paper, $\langle G, \circ \rangle$ is a commutative semigroup with an identity e, $\mathbb{R}$ the set of real numbers, $\mathbb{C}$ the set of complex numbers, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\epsilon \geq 0$, and $g : G \to G$ and $\phi : G \to [0, \infty)$ are given functions. For $(a_1, a_2, \dots, a_n), (b_1, b_2, \dots, b_n) \in \mathbb{K}^n$, we define $(a_1, a_2, \dots, a_n)(b_1, b_2, \dots, b_n) = (a_1 b_1, a_2 b_2, \dots, a_n b_n)$. A function $\sigma : G \to G$ is said to be *an involution* if $\sigma(x \circ y) = \sigma(x) \circ \sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$. A function $m : G \to \mathbb{K}^n$ is called *an exponential function* provided that $m(x \circ y) = m(x)m(y)$ for all $x, y \in G$.

Generalizing the result of Ger and Šemrl [1], Najdecki [2] proved the stability of the functional equation

$$F\bigl(x \circ g(y)\bigr) - F(x)F(y) = 0 \tag{1.1}$$

in the class of functions $F : G \to \mathbb{K}^n$. The particular cases of (1.1) are the exponential equation $f(xy) = f(x)f(y)$ (see Aczél and Dhombres [3] and Baker [4]) and the equation

$$f\bigl(xf(y)\bigr) = f(x)f(y) \tag{1.2}$$

for all $x, y \in \mathbb{K} \setminus \{0\}$, where $f : \mathbb{K} \setminus \{0\} \to \mathbb{K} \setminus \{0\}$ (see Brzdęk [5], Brzdęk, Najdecki and Xu [6] and Chudziak and Tabor [7] for related equations). As mentioned in [2, 5], (1.2) arises in averaging theory applied to the turbulent fluid motion and is connected with the Reynolds operator (see Marias [8]), the averaging operator and the multiplicatively symmetric operator (see [3]). Moreover, the equation (1.2) is connected with a description of some associative operations, *i.e.*, the binary operation $\circ : (\mathbb{K} \setminus \{0\}) \times (\mathbb{K} \setminus \{0\}) \to \mathbb{K} \setminus \{0\}$ defined by $x \circ f(y) = xf(y)$ is associative if and only if f satisfies (1.2) (see [5] for more details). We also refer the reader to Belluot, Brzdęk and Ciepliński [9] and Brzdęk and Ciepliński [10] for some recent developments on the issues of stability and superstability for functional equations.

The main result of Najdecki [2] is the following.

Theorem 1.1 *Let $F : G \to \mathbb{K}^n$, $F = (f_1, f_2, \ldots, f_n)$ satisfy*

$$\| F(x \circ g(y)) - F(x)F(y) \| \leq \epsilon \tag{1.3}$$

for all $x, y \in G$ with any norm $\| \cdot \|$ in $\mathbb{K}^n$. Then there exist ideals $I, J \subset \mathbb{K}^n$ such that $\mathbb{K}^n = I \oplus J$, PF is bounded and QF satisfies (1.1), *where $P : \mathbb{K}^n \to I$, $Q : \mathbb{K}^n \to J$ are the natural projections.*

In this paper, generalizing the above result we consider the functional inequalities

$$\| F(x \circ g(y)) - F(x)F(y) \| \leq \phi(y), \tag{1.4}$$

$$\| F(x \circ g(y)) - F(x)F(y) \| \leq \phi(x) \tag{1.5}$$

for all $x, y \in G$ with any norm $\| \cdot \|$ in $\mathbb{K}^n$ (see [6] for related results).

Throughout this paper we denote

$$L = \{j : f_j \text{ is bounded}, \ j = 1, 2, \ldots, n\},$$

$$K = \{j : f_j \text{ is unbounded}, \ j = 1, 2, \ldots, n\},$$

where $F = (f_1, f_2, \ldots, f_n)$.

Theorem 1.2 *Let $F : G \to \mathbb{K}^n$, $F = (f_1, f_2, \ldots, f_n)$ satisfy* (1.4) *for all $x, y \in G$ with any norm $\| \cdot \|$ in $\mathbb{K}^n$. Assume that one of the following two conditions is fulfilled.*

(i) *g is an involution,*

(ii) *for each $j \in K$, there exists a sequence x_n, $n = 1, 2, 3, \ldots$ (possibly depending on j) such that*

$$\frac{|f_j(x_n)|}{1 + \phi(x_n)} \to \infty \quad \text{as } n \to \infty. \tag{1.6}$$

Then there exist ideals $I, J \subset \mathbb{K}^n$ such that $\mathbb{K}^n = I \oplus J$, PF is bounded and QF satisfies (1.1), *where $P : \mathbb{K}^n \to I$, $Q : \mathbb{K}^n \to J$ are the natural projections. Moreover, $Q(F \circ g^{-1})$ is exponential provided g is bijective.*

Remark The case (ii) of Theorem 1.2 includes Theorem 1.1.

Theorem 1.3 *Let $F : G \to \mathbb{K}^n$, $F = (f_1, f_2, \ldots, f_n)$ satisfy* (1.5) *for all $x, y \in G$ with any norm $\| \cdot \|$ in $\mathbb{K}^n$. Assume that g is an involution. Then there exist ideals $I, J \subset \mathbb{K}^n$ such that $\mathbb{K}^n = I \oplus J$, PF is bounded, QF satisfies* (1.1), *where $P : \mathbb{K}^n \to I$, $Q : \mathbb{K}^n \to J$ are the natural projections.*

If we replace $\|\cdot\|$ by the usual norm $\|\cdot\|_u$ on $\mathbb{K}^n$ defined by

$$\|(a_1, a_2, \dots, a_n)\|_u = \sqrt{|a_1|^2 + |a_2|^2 + \cdots + |a_n|^2},$$

we can estimate PF (in Theorem 1.2 and Theorem 1.3) as follows.

Theorem 1.4 *The following two statements are valid.*

(a) *If* $F : G \to \mathbb{K}^n$, $F = (f_1, f_2, \dots, f_n)$ *satisfies* (1.4), *then* PF *satisfies*

$$\|PF(y)\|_u \le \frac{\sqrt{|L|}}{2}\bigl(1 + \sqrt{1 + 4\phi(y)}\bigr) \tag{1.7}$$

for all $y \in G$, *where* $|L|$ *denotes the number of the elements of* L. *In particular, if* $|L| = 1$ *and* G *is a group, then* PF *satisfies either*

$$\frac{1}{2}\bigl(1 + \sqrt{1 - 4\phi(y)}\bigr) \le \|PF(y)\|_u \le \frac{1}{2}\bigl(1 + \sqrt{1 + 4\phi(y)}\bigr) \tag{1.8}$$

for all $y \in B := \{y \in G : \phi(y) < \frac{1}{4}\}$, *or*

$$\|PF(y)\|_u \le \frac{1}{2}\bigl(1 - \sqrt{1 - 4\phi(y)}\bigr) \tag{1.9}$$

for all $y \in B$.

(b) *If* $F : G \to \mathbb{K}^n$, $F = (f_1, f_2, \dots, f_n)$ *satisfies* (1.5), *then* PF *satisfies* (1.7). *In particular if* G *is a group,* g *is surjective and* $|L| = 1$, *then* PF *satisfies* (1.8) *or* (1.9).

2 Proofs

Let $g : G \to G$ and $\phi : G \to [0, \infty)$ be given. We first consider the stability of the functional equation

$$f\bigl(x \circ g(y)\bigr) - f(x)f(y) = 0 \tag{2.1}$$

in the class of functions $f : G \to \mathbb{K}$, *i.e.*, we investigate both bounded and unbounded functions $f : G \to \mathbb{K}$ satisfying the functional inequalities

$$\bigl|f\bigl(x \circ g(y)\bigr) - f(x)f(y)\bigr| \le \phi(y), \tag{2.2}$$

$$\bigl|f\bigl(x \circ g(y)\bigr) - f(x)f(y)\bigr| \le \phi(x) \tag{2.3}$$

for all $x, y \in G$.

Lemma 2.1 *Assume that* $g = \sigma$ *is an involution and* $f : G \to \mathbb{K}$ *is an unbounded function satisfying the inequality* (2.2). *Then* f *is exponential and satisfies* (2.1). *In particular, if* G *is* 2*-divisible, then* f *has the form*

$$f(x) = m\left(\frac{x \circ \sigma(x)}{2}\right) \tag{2.4}$$

for all $x \in G$, *where* $m : G \to \mathbb{K}$ *is an exponential function.*

Proof Choose a sequence $x_n \in G$, $n = 1, 2, 3, \ldots$, such that $|f(x_n)| \to \infty$ as $n \to \infty$. Putting $x = x_n$, $n = 1, 2, 3, \ldots$, in (2.2), dividing the result by $|f(x_n)|$ and letting $n \to \infty$ we have

$$f(y) = \lim_{n\to\infty} \frac{f(x_n \circ \sigma(y))}{f(x_n)} \tag{2.5}$$

for all $y \in G$. Multiplying both sides of (2.5) by $f(x)$ and using (2.2) and (2.5) we have

$$\begin{aligned} f(y)f(x) &= \lim_{n\to\infty} \frac{f(x_n \circ \sigma(y))f(x)}{f(x_n)} = \lim_{n\to\infty} \frac{f(x_n \circ \sigma(y) \circ \sigma(x))}{f(x_n)} \\ &= \lim_{n\to\infty} \frac{f(x_n \circ \sigma(y \circ x))}{f(x_n)} = f(y \circ x) \end{aligned} \tag{2.6}$$

for all $x, y \in G$. Thus, f is an exponential function, say $f = m$. From (2.2) and (2.6) we have

$$\bigl|f(x)\bigr|\bigl|f\bigl(\sigma(y)\bigr) - f(y)\bigr| \le \phi(y) \tag{2.7}$$

for all $x, y \in G$. Since f is unbounded, from (2.7) we have

$$f\bigl(\sigma(y)\bigr) = f(y) \tag{2.8}$$

for all $y \in G$. Replacing y by $\sigma(y)$ in (2.6) and using (2.8) we get the equation (2.1). In particular, if G is 2-divisible, then we can write

$$\begin{aligned} f(x) &= f\left(\frac{x}{2} \circ \frac{x}{2}\right) = f\left(\frac{x}{2} \circ \sigma\left(\frac{x}{2}\right)\right) \\ &= f\left(\frac{x}{2} \circ \frac{\sigma(x)}{2}\right) = m\left(\frac{x \circ \sigma(x)}{2}\right) \end{aligned} \tag{2.9}$$

for all $x \in G$. This completes the proof. □

Lemma 2.2 *Let $f : G \to \mathbb{K}$ be an unbounded function satisfying* (2.2). *Assume that there exists a sequence x_n, $n = 1, 2, 3, \ldots$, satisfying*

$$\lim_{n\to\infty} \frac{|f(x_n)|}{1 + \phi(x_n)} = \infty. \tag{2.10}$$

Then f satisfies (2.1).

Proof Note that (2.10) implies

$$\lim_{n\to\infty} \frac{1}{|f(x_n)|} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{\phi(x_n)}{|f(x_n)|} = 0.$$

Putting $y = x_n$, $n = 1, 2, 3, \ldots$, in (2.2) and dividing the result by $|f(x_n)|$ we have

$$\left|f(x) - \frac{f(x \circ g(x_n))}{f(x_n)}\right| \le \frac{\phi(x_n)}{|f(x_n)|} \tag{2.11}$$

for all $x \in G$, $n = 1, 2, 3, \ldots$. Letting $n \to \infty$ in (2.11) we have

$$f(x) = \lim_{n\to\infty} \frac{f(x \circ g(x_n))}{f(x_n)} \tag{2.12}$$

for all $x \in G$. Multiplying both sides of (2.12) by $f(y)$ and using (2.2) and (2.12) we have

$$\begin{aligned} f(x)f(y) &= \lim_{n\to\infty} \frac{f(x \circ g(x_n))f(y)}{f(x_n)} = \lim_{n\to\infty} \frac{f(x \circ g(x_n) \circ g(y))}{f(x_n)} \\ &= \lim_{n\to\infty} \frac{f(x \circ g(y) \circ g(x_n))}{f(x_n)} = f\bigl(x \circ g(y)\bigr) \end{aligned} \tag{2.13}$$

for all $x, y \in G$. This completes the proof. □

Lemma 2.3 *Assume that g is bijective and $f : G \to \mathbb{K}$ is an unbounded function satisfying the inequality* (2.2). *Then $f \circ g^{-1}$ is an exponential function.*

Proof Choose a sequence $x_n \in G$, $n = 1, 2, 3, \ldots$, such that $|f(x_n)| \to \infty$ as $n \to \infty$. Putting $x = x_n$, $n = 1, 2, 3, \ldots$, in (2.2), dividing the result by $|f(x_n)|$, replacing y by $g^{-1}(y)$ and letting $n \to \infty$ we have

$$f\bigl(g^{-1}(y)\bigr) = \lim_{n\to\infty} \frac{f(x_n \circ y)}{f(x_n)} \tag{2.14}$$

for all $y \in G$. Multiplying both sides of (2.14) by $f(g^{-1}(x))$ and using (2.2) and (2.14) we have

$$\begin{aligned} f\bigl(g^{-1}(y)\bigr)f\bigl(g^{-1}(x)\bigr) &= \lim_{n\to\infty} \frac{f(x_n \circ y)f(g^{-1}(x))}{f(x_n)} \\ &= \lim_{n\to\infty} \frac{f(x_n \circ y \circ x)}{f(x_n)} = f\bigl(g^{-1}(y \circ x)\bigr) \end{aligned} \tag{2.15}$$

for all $x, y \in G$. Thus, $f \circ g^{-1}$ is an exponential function. This completes the proof. □

Proof of Theorem 1.2 Since every two norms in K^n are equivalent, from (1.4) there exists $\alpha > 0$ such that

$$\begin{aligned} \bigl|f_j\bigl(x \circ g(y)\bigr) - f_j(x)f_j(y)\bigr| &\le \bigl\|F\bigl(x \circ g(y)\bigr) - F(x)F(y)\bigr\|_u \\ &\le \alpha\bigl\|F\bigl(x \circ g(y)\bigr) - F(x)F(y)\bigr\| \le \alpha\phi(y) \end{aligned} \tag{2.16}$$

for all $x, y \in G$ and all $j \in \{1, 2, \ldots, n\}$. For the case (i), by Lemma 2.1, f_j satisfies (2.1) for all $j \in K$. For the case (ii), by Lemma 2.2, f_j satisfies (2.1) for all $j \in K$. Let $I = \{(a_1, a_2, \ldots, a_n) : a_i = 0 \text{ for } i \in K\}$, $J = \{(a_1, a_2, \ldots, a_n) : a_i = 0 \text{ for } i \in L\}$. Then it follows that $\mathbb{K}^n = I \oplus J$, PF is bounded and QF satisfies (1.1). If g is bijective, then by Lemma 2.3, $f_j \circ g^{-1}$ are exponential function for all $j \in K$, which implies $Q(F \circ g^{-1})$ is an exponential function. This completes the proof. □

Lemma 2.4 *Assume that $g = \sigma$ is an involution and $f : G \to \mathbb{K}$ is an unbounded function satisfying the inequality* (2.3). *Then f satisfies* (2.1). *In particular, if G is 2-divisible, then f*

has the form

$$f(x) = m\left(\frac{x \circ \sigma(x)}{2}\right) \tag{2.17}$$

for all $x \in G$, where $m : G \to \mathbb{K}$ is an exponential function.

Proof Choose a sequence $y_n \in G$, $n = 1, 2, 3, \ldots$, such that $|f(y_n)| \to \infty$ as $n \to \infty$. Putting $y = y_n$, $n = 1, 2, 3, \ldots$, in (2.3), dividing the result by $|f(y_n)|$ and letting $n \to \infty$ we have

$$f(x) = \lim_{n\to\infty} \frac{f(x \circ \sigma(y_n))}{f(y_n)}. \tag{2.18}$$

Putting $x = e$ in (2.3) and replacing y by $\sigma(y)$ in the result we have

$$\left|f(y) - f(e)f\left(\sigma(y)\right)\right| \le \phi(e) \tag{2.19}$$

for all $x, y \in G$. Multiplying both sides of (2.18) by $f(y)$ and using (2.3), (2.18), and (2.19) we have

$$\begin{aligned} f(y)f(x) &= \lim_{n\to\infty} \frac{f(y)f(x \circ \sigma(y_n))}{f(y_n)} = \lim_{n\to\infty} \frac{f(y \circ \sigma(x \circ \sigma(y_n)))}{f(y_n)} \\ &= \lim_{n\to\infty} \frac{f(e)f(\sigma(y) \circ x \circ \sigma(y_n))}{f(y_n)} = f(e)f\left(\sigma(y) \circ x\right) \end{aligned} \tag{2.20}$$

for all $x, y \in G$. Putting $x = e$ in (2.20) we have

$$f(y) = f\left(\sigma(y)\right) \tag{2.21}$$

for all $y \in G$. From (2.19) and (2.21) we have

$$\left|f(y)\right|\left|1 - f(e)\right| \le \phi(e) \tag{2.22}$$

for all $y \in G$. Since f is unbounded, from (2.22) we have $f(e) = 1$. Thus, f satisfies (2.1). This completes the proof. □

Proof of Theorem 1.3 From (1.5), as in (2.16) there exists $\alpha > 0$ such that

$$\left|f_j\left(x \circ g(y)\right) - f_j(x)f_j(y)\right| \le \alpha\phi(x) \tag{2.23}$$

for all $x, y \in G$, $j \in \{1, 2, \ldots, n\}$. Applying Lemma 2.4 to (2.23) for each $j \in K$ we find that f_j satisfies (2.1) for all $j \in K$, which implies that QF satisfies (1.1). This completes the proof. □

Now, we investigate bounded functions satisfying each of (2.2) and (2.3) (see [4, 11–13] for bounded solutions of an exponential functional equation).

Lemma 2.5 *Let $f : G \to \mathbb{K}$ be a bounded function satisfying* (2.2). *Then f satisfies*

$$\bigl|f(y)\bigr| \le \frac{1}{2}\bigl(1 + \sqrt{1 + 4\phi(y)}\bigr) \tag{2.24}$$

for all $y \in G$. In particular, G is a group and let $B = \{y \in G : \phi(y) < \frac{1}{4}\}$, then f satisfies either

$$\frac{1}{2}\bigl(1 + \sqrt{1 - 4\phi(y)}\bigr) \le \bigl|f(y)\bigr| \le \frac{1}{2}\bigl(1 + \sqrt{1 + 4\phi(y)}\bigr) \tag{2.25}$$

for all $y \in B$, or

$$\bigl|f(y)\bigr| \le \frac{1}{2}\bigl(1 - \sqrt{1 - 4\phi(y)}\bigr) \tag{2.26}$$

for all $y \in B$.

Proof Let $M_f = \sup_{x \in G} |f(x)|$. Using the triangle inequality with (2.2) we have

$$\bigl|f(x)f(y)\bigr| \le \bigl|f\bigl(x \circ g(y)\bigr)\bigr| + \phi(y) \le M_f + \phi(y) \tag{2.27}$$

for all $x, y \in G$. Taking the supremum of the left hand side of (2.27) with respect to $x \in G$ we get $M_f|f(y)| \le M_f + \phi(y)$ for all $y \in G$. Thus, we have

$$M_f\bigl(\bigl|f(y)\bigr| - 1\bigr) \le \phi(y) \tag{2.28}$$

for all $y \in G$. From (2.28) we have

$$\bigl|f(y)\bigr|\bigl(\bigl|f(y)\bigr| - 1\bigr) \le \phi(y) \tag{2.29}$$

for all $y \in G$. Solving the inequality (2.29) we get (2.24). Now, we assume that G is a group. Replacing x by $x \circ g(y)^{-1}$ in (2.2) and using the triangle inequality we have

$$\bigl|f(x)\bigr| \le \bigl|f\bigl(x \circ g(y)^{-1}\bigr)f(y)\bigr| + \phi(y) \le M_f\bigl|f(y)\bigr| + \phi(y) \tag{2.30}$$

for all $x, y \in G$. Taking the supremum of the left hand side of (2.30) with respect to $x \in G$ we get $M_f \le M_f|f(y)| + \phi(y)$ for all $y \in G$. Thus, we have

$$M_f\bigl(1 - \bigl|f(y)\bigr|\bigr) \le \phi(y) \tag{2.31}$$

for all $y \in G$. From (2.28) and (2.31) we have

$$\bigl|f(y)\bigr|\bigl|1 - \bigl|f(y)\bigr|\bigr| \le M_f\bigl|1 - \bigl|f(y)\bigr|\bigr| \le \phi(y) \tag{2.32}$$

for all $y \in G$. For each fixed $y \in B$, solving the inequality (2.32) we get

$$\frac{1}{2}\bigl(1 + \sqrt{1 - 4\phi(y)}\bigr) \le \bigl|f(y)\bigr| \le \frac{1}{2}\bigl(1 + \sqrt{1 + 4\phi(y)}\bigr), \tag{2.33}$$

or

$$\left|f(y)\right| \le \frac{1}{2}\left(1 - \sqrt{1 - 4\phi(y)}\right). \tag{2.34}$$

Now, assume that there exist a bounded function f and $y_1, y_2 \in B$ such that

$$\left|f(y_1)\right| \le \frac{1}{2}\left(1 - \sqrt{1 - 4\phi(y_1)}\right), \qquad \left|f(y_2)\right| \ge \frac{1}{2}\left(1 + \sqrt{1 - 4\phi(y_2)}\right). \tag{2.35}$$

Then from (2.31) we have

$$\left|f(y_2)\right|\left(1 - \left|f(y_1)\right|\right) \le M_f\left(1 - \left|f(y_1)\right|\right) \le \phi(y_1). \tag{2.36}$$

On the other hand, from (2.35) we have

$$\begin{aligned} \left|f(y_2)\right|\left(1 - \left|f(y_1)\right|\right) &\ge \frac{1}{2}\left(1 + \sqrt{1 - 4\phi(y_2)}\right)\left(1 - \frac{1}{2}\left(1 - \sqrt{1 - 4\phi(y_1)}\right)\right) \\ &> \frac{1}{2}\left(1 - \sqrt{1 - 4\phi(y_1)}\right)\left(1 - \frac{1}{2}\left(1 - \sqrt{1 - 4\phi(y_1)}\right)\right) = \phi(y_1), \end{aligned}$$

which contradicts (2.36). Thus, f satisfies (2.25) for all $y \in B$, or it satisfies (2.26) for all $y \in B$. This completes the proof. □

Lemma 2.6 *Let $f : G \to \mathbb{K}$ be a bounded function satisfying* (2.3). *Then f satisfies* (2.24) *for all $y \in G$. In particular, if G is a group and g is surjective, then f satisfies* (2.25) *for all $y \in B := \{y \in G : \phi(y) < \frac{1}{4}\}$, or satisfies* (2.26) *for all $y \in B$.*

Proof Using the triangle inequality with (2.3) we have

$$\left|f(x)f(y)\right| \le \left|f\left(x \circ g(y)\right)\right| + \phi(x) \le M_f + \phi(x) \tag{2.37}$$

for all $x, y \in G$. Taking the supremum of the left hand side of (2.37) with respect to $y \in G$ we get $M_f|f(x)| \le M_f + \phi(x)$ for all $x \in G$. Thus, we have

$$M_f\left(\left|f(x)\right| - 1\right) \le \phi(x) \tag{2.38}$$

for all $x \in G$. From (2.38) we get (2.24) as in the proof of Lemma 2.5. We assume that G is a group. For given $x, z \in G$, choosing $w \in G$ such that $g(w) = x^{-1} \circ z$, putting $y = w$ in (2.3) and using the triangle inequality we have

$$\left|f(z)\right| \le \left|f(x)f(w)\right| + \phi(x) \le \left|f(x)\right|M_f + \phi(x) \tag{2.39}$$

for all $x, z \in G$. Taking the supremum of the left hand side of (2.39) we get $M_f \le M_f|f(x)| + \phi(x)$ for all $x \in G$. Thus, we have

$$M_f\left(1 - \left|f(x)\right|\right) \le \phi(x) \tag{2.40}$$

for all $x \in G$. Now, the remaining parts of the proof are the same as those of Lemma 2.5. □

Proof of Theorem 1.4 From Lemma 2.5 and Lemma 2.6, for each $j \in L$ we have

$$\left|f_j(y)\right| \le \frac{1}{2}\left(1 + \sqrt{1 + 4\phi(y)}\right) \tag{2.41}$$

for all $y \in G$. Thus, from (2.41) we have

$$\left\|PF(y)\right\|_u = \sqrt{\sum_{j\in L}\left|f_j(y)\right|^2} \le \frac{\sqrt{|L|}}{2}\left(1 + \sqrt{1 + 4\phi(y)}\right)$$

for all $y \in G$, which gives (1.7). Now, if $|L| = 1$, say $L = \{j_0\}$ we have

$$\left\|PF(y)\right\|_u = \left|f_{j_0}(y)\right|$$

for all $y \in G$. Thus, the inequalities (1.8) and (1.9) follow immediately from (2.25) and (2.26). This completes the proof. □

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author is the only person who is responsible to this work.

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