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A sharpened and generalized version of Aczél-Vasić-Pečarić inequality and its application

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Abstract

In this paper, we present a sharpened and generalized version of Aczél-Vasić-Pečarić inequality. As an application, an integral type of Aczél-Vasić-Pečarić inequality is obtained.

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Keywords: Aczél's inequality; Aczél-Vasić-Pečarić inequality; Hölder's inequality

1 Introduction

In 1956, Aczél [1] established the following inequality, which is of wide application in the theory of functional equations in non-Euclidean geometry.

Theorem A *If a_i, b_i $(i = 1,2,\dots,n)$ are positive numbers such that $a_1^2 - \sum_{i=2}^n a_i^2 > 0$ or $b_1^2 - \sum_{i=2}^n b_i^2 > 0$, then*

$$\left(a_1^2 - \sum_{i=2}^n a_i^2\right)\left(b_1^2 - \sum_{i=2}^n b_i^2\right) \le \left(a_1b_1 - \sum_{i=2}^n a_ib_i\right)^2. \tag{1}$$

Later, in 1959, Popoviciu [2] gave a generalization of the above inequality.

Theorem B *Let $p > 1$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and let a_i, b_i $(i = 1,2,\dots,n)$ be positive numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^q - \sum_{i=2}^n b_i^q > 0$. Then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}}\left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \le a_1b_1 - \sum_{i=2}^n a_ib_i. \tag{2}$$

In 1982, Vasić and Pečarić [3] presented the following reversed version of inequality (2).

Theorem C *Let $p < 1$ $(p \neq 0)$, $\frac{1}{p} + \frac{1}{q} = 1$, and let a_i, b_i $(i = 1,2,\dots,n)$ be positive numbers such that $a_1^p - \sum_{i=2}^n a_i^p > 0$ and $b_1^q - \sum_{i=2}^n b_i^q > 0$. Then*

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}}\left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \ge a_1b_1 - \sum_{i=2}^n a_ib_i. \tag{3}$$

Recently inequalities (2) and (3) were generalized and refined in many different ways; see, for example, [4–10] and [11]. In [12], Wu established an interesting generalization of Aczél-Popoviciu inequality (2) as follows.

Theorem D *Let $p, q > 0$, $a_i, b_i > 0$ ($i = 1, 2, \dots, n$), let k ($1 \le k < n$) be a positive integer such that $\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p > 0$ and $\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q > 0$. Then*

$$\begin{aligned}
&\left(\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} \\
&\quad \le 2^{\max\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(\sum_{i=1}^{k} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q\right)^{\frac{1}{q}} - \left(\sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} \\
&\qquad - \frac{2^{\max\{1-\frac{1}{p}-\frac{1}{q},0\}}}{\max\{p,q,1\}} \left(\sum_{i=1}^{k} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q\right)^{\frac{1}{q}} \left(\frac{\sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p} - \frac{\sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}\right)^2,
\end{aligned} \tag{4}$$

and equality holds if and only if

$$\frac{\sum_{i=1}^{k} a_i^p}{\sum_{i=k+1}^{n} a_i^p} = \frac{\sum_{i=1}^{k} b_i^q}{\sum_{i=k+1}^{n} b_i^q} = 2$$

for $\frac{1}{p} + \frac{1}{q} < 1$, or

$$\frac{\sum_{i=1}^{k} a_i^p}{\sum_{i=k+1}^{n} a_i^p} = \frac{\sum_{i=1}^{k} b_i^q}{\sum_{i=k+1}^{n} b_i^q}$$

for $\frac{1}{p} + \frac{1}{q} = 1$.

The main purpose of this work is to give a sharpened and generalized version of Aczél-Vasić-Pečarić inequality (3). Moreover, a new Aczél-Vasić-Pečarić type integral inequality is established.

2 A sharpened and generalized version of Aczél-Vasić-Pečarić inequality

We begin this section with some lemmas, which will be used in the sequel.

Lemma 2.1 [13] *If $x > -1$, $\alpha \ge 1$ or $\alpha < 0$, then*

$$(1 + x)^{\alpha} \ge 1 + \alpha x. \tag{5}$$

The inequality is reversed for $0 < \alpha < 1$. The sign of equality holds if and only if $x = 0$ or $\alpha = 1$.

Lemma 2.2 [14] *Let $a_{rj} > 0$ ($r = 1, 2, \dots, n$, $j = 1, 2, \dots, m$), let $\lambda_1 \ne 0$, $\lambda_j < 0$ ($j = 2, 3, \dots, m$), and let $\tau = \max\{\sum_{j=1}^{m} \frac{1}{\lambda_j}, 1\}$. Then*

$$\sum_{r=1}^{n} \prod_{j=1}^{m} a_{rj} \ge n^{1-\tau} \prod_{j=1}^{m} \left(\sum_{r=1}^{n} a_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}. \tag{6}$$

The sign of equality holds if and only if the m sets $(a_{r1}), (a_{r2}), \dots, (a_{rm})$ *are proportional for* $\sum_{j=1}^{m} \frac{1}{\lambda_j} \le 1$, *or* $a_{1j} = a_{2j} = \cdots = a_{nj}$, $j = 1, 2, \dots, m$, *for* $\sum_{j=1}^{m} \frac{1}{\lambda_j} > 1$.

Lemma 2.3 *Let* $x > 1$, $y > 1$, *and let* $p < 0$, $q < 0$. *Then*

$$xy + \left(1 - x^p\right)^{\frac{1}{p}} \left(1 - y^q\right)^{\frac{1}{q}} \ge 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(1 - \min\left\{p^{-1}, q^{-1}\right\} \left(x^p - y^q\right)^2\right), \tag{7}$$

and equality holds if and only if $x^p = y^q = \frac{1}{2}$.

Proof Case (I). When $p < q < 0$, it implies that $\frac{1}{q} < 0$, $\frac{1}{p} - \frac{1}{q} > 0$. By applying Lemma 2.2 and Lemma 2.1, we have

$$\begin{aligned}
& xy + \left(1 - x^p\right)^{\frac{1}{p}} \left(1 - y^q\right)^{\frac{1}{q}} \\
&\quad = \left(x^p\right)^{\frac{1}{q}} \left(y^q\right)^{\frac{1}{q}} \left(x^p\right)^{\frac{1}{p}-\frac{1}{q}} + \left(1 - y^q\right)^{\frac{1}{q}} \left(1 - x^p\right)^{\frac{1}{q}} \left(1 - x^p\right)^{\frac{1}{p}-\frac{1}{q}} \\
&\quad \ge 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(x^p + \left(1 - y^q\right)\right)^{\frac{1}{q}} \left(y^q + \left(1 - x^p\right)\right)^{\frac{1}{q}} \left(x^p + \left(1 - x^p\right)\right)^{\frac{1}{p}-\frac{1}{q}} \\
&\quad = 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(1 - \left(x^p - y^q\right)^2\right)^{\frac{1}{q}} \\
&\quad \ge 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(1 - q^{-1} \left(x^p - y^q\right)^2\right).
\end{aligned} \tag{8}$$

Case (II). When $q < p < 0$, it implies that $\frac{1}{p} < 0$, $\frac{1}{q} - \frac{1}{p} > 0$. By using Lemma 2.2 and Lemma 2.1, we obtain

$$\begin{aligned}
& xy + \left(1 - x^p\right)^{\frac{1}{p}} \left(1 - y^q\right)^{\frac{1}{q}} \\
&\quad = \left(y^q\right)^{\frac{1}{p}} \left(x^p\right)^{\frac{1}{p}} \left(y^q\right)^{\frac{1}{q}-\frac{1}{p}} + \left(1 - x^p\right)^{\frac{1}{p}} \left(1 - y^q\right)^{\frac{1}{p}} \left(1 - y^q\right)^{\frac{1}{q}-\frac{1}{p}} \\
&\quad \ge 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(y^q + \left(1 - x^p\right)\right)^{\frac{1}{p}} \left(x^p + \left(1 - y^q\right)\right)^{\frac{1}{p}} \left(y^q + \left(1 - y^q\right)\right)^{\frac{1}{q}-\frac{1}{p}} \\
&\quad = 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(1 - \left(x^p - y^q\right)^2\right)^{\frac{1}{p}} \\
&\quad \ge 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(1 - p^{-1} \left(x^p - y^q\right)^2\right).
\end{aligned} \tag{9}$$

Case (III). When $p = q$, $p < 0$, $q < 0$. From Lemma 2.2 and Lemma 2.1 we have

$$\begin{aligned}
& xy + \left(1 - x^p\right)^{\frac{1}{p}} \left(1 - y^q\right)^{\frac{1}{q}} \\
&\quad = \left(y^p\right)^{\frac{1}{p}} \left(x^p\right)^{\frac{1}{p}} + \left(1 - x^p\right)^{\frac{1}{p}} \left(1 - y^p\right)^{\frac{1}{p}} \\
&\quad \ge 2^{\min\{1-\frac{2}{p},0\}} \left(y^p + \left(1 - x^p\right)\right)^{\frac{1}{p}} \left(x^p + \left(1 - y^p\right)\right)^{\frac{1}{p}} \\
&\quad = 2^{\min\{1-\frac{2}{p},0\}} \left(1 - \left(x^p - y^p\right)^2\right)^{\frac{1}{p}} \\
&\quad = 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(1 - \left(x^p - y^q\right)^2\right)^{\frac{1}{p}} \\
&\quad \ge 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(1 - p^{-1} \left(x^p - y^q\right)^2\right).
\end{aligned} \tag{10}$$

Combining inequalities (8)-(10) yields inequality (7). The condition of equality in (7) follows immediately from Lemma 2.2 and Lemma 2.1. The proof of Lemma 2.3 is completed. □

Lemma 2.4 *Let* $0 < x < 1$, $y > 1$, *and let* $p > 0$, $q < 0$. *Then*

$$xy + \left(1 - x^p\right)^{\frac{1}{p}} \left(1 - y^q\right)^{\frac{1}{q}} \geq 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(1 - \min\left\{p^{-1}, q^{-1}\right\} \left(x^p - y^q\right)^2\right), \tag{11}$$

and equality holds if and only if $x^p = y^q = \frac{1}{2}$ *for* $\frac{1}{p} + \frac{1}{q} < 1$, *or* $x^p = y^q$ *for* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof By using Lemma 2.2 and Lemma 2.1, we have

$$\begin{aligned}
&xy + \left(1 - x^p\right)^{\frac{1}{p}} \left(1 - y^q\right)^{\frac{1}{q}} \\
&\quad = \left(x^p\right)^{\frac{1}{q}} \left(y^q\right)^{\frac{1}{q}} \left(x^p\right)^{\frac{1}{p}-\frac{1}{q}} + \left(1 - y^q\right)^{\frac{1}{q}} \left(1 - x^p\right)^{\frac{1}{q}} \left(1 - x^p\right)^{\frac{1}{p}-\frac{1}{q}} \\
&\quad \geq 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(x^p + \left(1 - y^q\right)\right)^{\frac{1}{q}} \left(y^q + \left(1 - x^p\right)\right)^{\frac{1}{q}} \left(x^p + \left(1 - x^p\right)\right)^{\frac{1}{p}-\frac{1}{q}} \\
&\quad = 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(1 - \left(x^p - y^q\right)^2\right)^{\frac{1}{q}} \\
&\quad \geq 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(1 - q^{-1}\left(x^p - y^q\right)^2\right) \\
&\quad = 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(1 - \min\left\{p^{-1}, q^{-1}\right\} \left(x^p - y^q\right)^2\right).
\end{aligned} \tag{12}$$

In addition, the condition of equality for inequality (11) can easily be obtained by Lemma 2.1 and Lemma 2.2. The proof of Lemma 2.4 is completed. □

Theorem 2.5 *Let* $a_i > 0$, $b_i > 0$ $(i = 1, 2, \ldots, n)$, *let* $p \neq 0$, $q < 0$, *and let* k $(1 \leq k < n)$ *be a positive integer such that* $\sum_{i=1}^k a_i^p - \sum_{i=k+1}^n a_i^p > 0$ *and* $\sum_{i=1}^k b_i^q - \sum_{i=k+1}^n b_i^q > 0$. *Then*

$$\begin{aligned}
&\left(\sum_{i=1}^k a_i^p - \sum_{i=k+1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^k b_i^q - \sum_{i=k+1}^n b_i^q\right)^{\frac{1}{q}} \\
&\quad \geq 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(\sum_{i=1}^k a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^k b_i^q\right)^{\frac{1}{q}} - \left(\frac{1}{n-k}\right)^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(\sum_{i=k+1}^n a_i b_i\right) \\
&\qquad - 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \min\left\{p^{-1}, q^{-1}\right\} \left(\sum_{i=1}^k a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^k b_i^q\right)^{\frac{1}{q}} \\
&\qquad \times \left(\frac{\sum_{i=k+1}^n a_i^p}{\sum_{i=1}^k a_i^p} - \frac{\sum_{i=k+1}^n b_i^q}{\sum_{i=1}^k b_i^q}\right)^2,
\end{aligned} \tag{13}$$

and equality holds if and only if $(2n-2k)^{-1} \sum_{i=1}^k a_i^p = a_{k+1}^p = a_{k+2}^p = \cdots = a_n^p$ *and* $(2n-2k)^{-1} \sum_{i=1}^k b_i^q = b_{k+1}^q = b_{k+2}^q = \cdots = b_n^q$ *for* $\frac{1}{p} + \frac{1}{q} < 1$, *or* $\frac{\sum_{i=1}^k a_i^p}{\sum_{i=1}^k b_i^q} = \frac{a_{k+1}^p}{b_{k+1}^q} = \frac{a_{k+2}^p}{b_{k+2}^q} = \cdots = \frac{a_n^p}{b_n^q}$ *for* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Case (I). When $p > 0$, $q < 0$. From the hypotheses of Theorem 2.5, we find that

$$\begin{aligned}
&0 < \left(\frac{\sum_{i=1}^k a_i^p - \sum_{i=k+1}^n a_i^p}{\sum_{i=1}^k a_i^p}\right)^{\frac{1}{p}} < 1, \\
&\left(\frac{\sum_{i=1}^k b_i^q - \sum_{i=k+1}^n b_i^q}{\sum_{i=1}^k b_i^q}\right)^{\frac{1}{q}} > 1.
\end{aligned}$$

Thus, by using Lemma 2.4 with a substitution

$$x = \left(\frac{\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p}\right)^{\frac{1}{p}}, \qquad y = \left(\frac{\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}\right)^{\frac{1}{q}}$$

in (11), we have

$$\begin{aligned}
&\left(\frac{\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p}\right)^{\frac{1}{p}} \left(\frac{\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}\right)^{\frac{1}{q}} + \left(\frac{\sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p}\right)^{\frac{1}{p}} \left(\frac{\sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}\right)^{\frac{1}{q}} \\
&\quad \geq 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \\
&\qquad \times \left[1 - \min\{p^{-1}, q^{-1}\}\left(\frac{\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p} - \frac{\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}\right)^2\right],
\end{aligned} \tag{14}$$

which implies

$$\begin{aligned}
&\left(\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} + \left(\sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} \\
&\quad \geq 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(\sum_{i=1}^{k} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q\right)^{\frac{1}{q}} \\
&\qquad \times \left[1 - \min\{p^{-1}, q^{-1}\}\left(\frac{\sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p} - \frac{\sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}\right)^2\right].
\end{aligned} \tag{15}$$

Hence, we obtain

$$\begin{aligned}
&\left(\sum_{i=1}^{k} a_i^p - \sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q - \sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} \\
&\quad \geq 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(\sum_{i=1}^{k} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q\right)^{\frac{1}{q}} - \left(\sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} \\
&\qquad - 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \min\{p^{-1}, q^{-1}\} \\
&\qquad \times \left(\sum_{i=1}^{k} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{k} b_i^q\right)^{\frac{1}{q}} \left(\frac{\sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p} - \frac{\sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}\right)^2,
\end{aligned} \tag{16}$$

where the equality holds if and only if $\frac{\sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p} = \frac{\sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q} = \frac{1}{2}$ for $\frac{1}{p} + \frac{1}{q} < 1$, or $\frac{\sum_{i=k+1}^{n} a_i^p}{\sum_{i=1}^{k} a_i^p} = \frac{\sum_{i=k+1}^{n} b_i^q}{\sum_{i=1}^{k} b_i^q}$ for $\frac{1}{p} + \frac{1}{q} = 1$.

On the other hand, by using Lemma 2.2, we obtain

$$\left(\sum_{i=k+1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=k+1}^{n} b_i^q\right)^{\frac{1}{q}} \leq \left(\frac{1}{n-k}\right)^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(\sum_{i=k+1}^{n} a_i b_i\right), \tag{17}$$

where the equality holds if and only if $a_{k+1} = a_{k+2} = \cdots = a_n$ and $b_{k+1} = b_{k+2} = \cdots = b_n$ for $\frac{1}{p} + \frac{1}{q} < 1$, or $\frac{a_{k+1}^p}{b_{k+1}^q} = \frac{a_{k+2}^p}{b_{k+2}^q} = \cdots = \frac{a_n^p}{b_n^q}$ for $\frac{1}{p} + \frac{1}{q} = 1$.

Combining the above two inequalities gives the desired result.

Case (II). When $p < 0$, $q < 0$. By the same method as in the above case (I) and using Lemma 2.3 and Lemma 2.2, we get that inequality (13) is also valid. The proof of Theorem 2.5 is completed. □

Remark 2.6 If we set $k = 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\frac{\sum_{i=2}^n a_i^p}{a_1^p} = \frac{\sum_{i=2}^n b_i^q}{b_1^q}$ in Theorem 2.5, then inequality (13) reduces to inequality (3).

If we set $k = 1$, then from Theorem 2.5 we obtain the following sharpened and generalized version of Aczél-Vasić-Pečarić inequality (3).

Corollary 2.7 *Let $p \neq 0$, $q < 0$, and let $a_i > 0$, $b_i > 0$, $a_1^p - \sum_{i=2}^n a_i^p > 0$, $b_1^q - \sum_{i=2}^n b_i^q > 0$ $(i = 1, 2, \ldots, n)$. Then*

$$\begin{aligned} &\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \\ &\quad \geq 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} a_1 b_1 - \left(\frac{1}{n-1}\right)^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \left(\sum_{i=2}^n a_i b_i\right) \\ &\qquad - 2^{\min\{1-\frac{1}{p}-\frac{1}{q},0\}} \min\{p^{-1}, q^{-1}\} a_1 b_1 \left[\sum_{i=2}^n \left(\frac{a_i^p}{a_1^p} - \frac{b_i^q}{b_1^q}\right)\right]^2, \end{aligned} \tag{18}$$

and equality holds if and only if $(2n-2)^{-\frac{1}{p}} a_1 = a_2 = \cdots = a_n$ and $(2n-2)^{-\frac{1}{p}} b_1 = b_2 = \cdots = b_n$ for $\frac{1}{p} + \frac{1}{q} < 1$, or $\frac{a_1^p}{b_1^q} = \frac{a_2^p}{b_2^q} = \cdots = \frac{a_n^p}{b_n^q}$ for $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, if we set $\frac{1}{p} + \frac{1}{q} \leq 1$, then from Corollary 2.7 we get the sharpened version of Aczél-Vasić-Pečarić inequality (3) as follows.

Corollary 2.8 *Let $p > 0$, $q < 0$, $\frac{1}{p} + \frac{1}{q} \leq 1$, and let $a_i > 0$, $b_i > 0$, $a_1^p - \sum_{i=2}^n a_i^p > 0$, $b_1^q - \sum_{i=2}^n b_i^q > 0$ $(i = 1, 2, \ldots, n)$. Then*

$$\begin{aligned} &\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \\ &\quad \geq a_1 b_1 - \left(\sum_{i=2}^n a_i b_i\right) - \frac{a_1 b_1}{q} \left[\sum_{i=2}^n \left(\frac{a_i^p}{a_1^p} - \frac{b_i^q}{b_1^q}\right)\right]^2, \end{aligned} \tag{19}$$

and equality holds if and only if $\frac{a_2^p}{b_2^q} = \frac{a_3^p}{b_3^q} = \cdots = \frac{a_n^p}{b_n^q}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

3 Application

As application of the above results, we establish here an integral type of Aczél-Vasić-Pečarić inequality.

Theorem 3.1 *Let $p > 0$, $q < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, let $A > 0$, $B > 0$, and let $f(x)$, $g(x)$ be positive Riemann integrable functions on $[a, b]$ such that $A^p - \int_a^b f^p(x)\,\mathrm{d}x > 0$ and $B^q - \int_a^b g^q(x)\,\mathrm{d}x > 0$.*

Then

$$\left(A^p - \int_a^b f^p(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \left(B^q - \int_a^b g^q(x)\,\mathrm{d}x\right)^{\frac{1}{q}} \geq AB - \int_a^b f(x)g(x)\,\mathrm{d}x - \frac{AB}{q}\left[\int_a^b \left(\frac{f^p(x)}{A^p} - \frac{g^q(x)}{B^q}\right)\mathrm{d}x\right]^2. \tag{20}$$

Proof For any positive integer n, we choose an equidistant partition of $[a,b]$ as

$$a < a + \frac{b-a}{n} < \cdots < a + \frac{b-a}{n}k < \cdots < a + \frac{b-a}{n}(n-1) < b,$$
$$x_i = a + \frac{b-a}{n}i, \quad i = 0,1,\ldots,n, \qquad \Delta x_k = \frac{b-a}{n}, \quad k = 1,2,\ldots,n.$$

Since

$$A^p - \int_a^b f^p(x)\,\mathrm{d}x > 0, \qquad B^q - \int_a^b g^q(x)\,\mathrm{d}x > 0,$$

we have

$$A^p - \lim_{n\to\infty}\sum_{k=1}^n f^p\left(a + \frac{k(b-a)}{n}\right)\frac{b-a}{n} > 0,$$

and

$$B^q - \lim_{n\to\infty}\sum_{k=1}^n g^q\left(a + \frac{k(b-a)}{n}\right)\frac{b-a}{n} > 0.$$

Hence, there exists a positive integer N such that

$$A^p - \sum_{k=1}^n f^p\left(a + \frac{k(b-a)}{n}\right)\frac{b-a}{n} > 0,$$

and

$$B^q - \sum_{k=1}^n g^q\left(a + \frac{k(b-a)}{n}\right)\frac{b-a}{n} > 0 \quad \text{for all } n > N.$$

By using Corollary 2.8, we obtain that for any $n > N$, the following inequality holds:

$$\begin{aligned}
&\left[A^p - \sum_{k=1}^n f^p\left(a + \frac{k(b-a)}{n}\right)\frac{b-a}{n}\right]^{\frac{1}{p}}\left[B^q - \sum_{k=1}^n g^q\left(a + \frac{k(b-a)}{n}\right)\frac{b-a}{n}\right]^{\frac{1}{q}} \\
&\quad\geq AB - \sum_{k=1}^n f^p\left(a + \frac{k(b-a)}{n}\right)g^q\left(a + \frac{k(b-a)}{n}\right)\left(\frac{b-a}{n}\right)^{\frac{1}{p}+\frac{1}{q}} \\
&\qquad - \frac{AB}{q}\left\{\sum_{k=1}^n\left[\frac{1}{A^p}f^p\left(a + \frac{k(b-a)}{n}\right)\frac{b-a}{n} - \frac{1}{B^q}g^q\left(a + \frac{k(b-a)}{n}\right)\frac{b-a}{n}\right]\right\}^2.
\end{aligned} \tag{21}$$

Since

$$\frac{1}{p}+\frac{1}{q}=1,$$

we have

$$\begin{aligned}
&\left[A^p-\sum_{k=1}^{n}f^p\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right]^{\frac{1}{p}}\left[B^q-\sum_{k=1}^{n}g^q\left(a+\frac{k(b-a)}{n}\right)\frac{b-a}{n}\right]^{\frac{1}{q}}\\
&\quad\geq AB-\sum_{k=1}^{n}f^p\left(a+\frac{k(b-a)}{n}\right)g^q\left(a+\frac{k(b-a)}{n}\right)\left(\frac{b-a}{n}\right)\\
&\qquad-\frac{AB}{q}\left\{\sum_{k=1}^{n}\left[\frac{1}{A^p}f^p\left(a+\frac{k(b-a)}{n}\right)-\frac{1}{B^q}g^q\left(a+\frac{k(b-a)}{n}\right)\right]\left(\frac{b-a}{n}\right)\right\}^2. \qquad (22)
\end{aligned}$$

In view of the hypotheses that $f(x)$, $g(x)$ are positive Riemann integrable functions on $[a,b]$, we conclude that $f(x)g(x)$, $f^p(x)$ and $g^q(x)$ are also integrable on $[a,b]$. Passing the limit as $n\to\infty$ in both sides of inequality (22), we obtain inequality (20). The proof of Theorem 3.1 is completed. □

Competing interests
The author declares that he has have no competing interests.

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